P.E.S. College of Engineering,

Mandya – 571 401, Karnatak

(An Autonomous Institution affiliated to VTU, Belagavi)

DEPARTMENT OF PHYSICS

Unit - I

Modern Physics and Quantum Mechanics

Notes

Engineering Physics : 2018 – 19 Scheme

Syllabus:

Unit – I : Modern Physics and Quantum Mechanics: 10 hrs

Modern Physics - Black body radiation spectrum. Statements of Wien's law, Rayleigh-Jean's law, Stefan-Boltzmann's law and Planck's law (Qualitative). Wave-Particle duality, deBroglie concept of matter waves and their characteristic properties, definations of Phase velocity, group velocity and Particle velocity; Relation between them. Expression for deBroglie wavelength using group velocity concept. *Quantum Mechanics* - Heisenberg's uncertainty principle and its applications (Non-existence of electrons in the nucleus). Wave function, properties, Physical significance of wave function and Normalization. Time-independent one dimensional Schrodinger's wave equation. Eigen functions and Eigen values. Applications of Schrodinger wave equation: 1. Free Particle and 2. Particle in one dimensional potential well of infinite height and finite width. Numerical Problems.

Modern physics

Introduction:
At the end of 19^{th} century and in the beginning of 20^{th} century, many new phenomena such as photoelectric effect, Compton Effect, pair production, Zeeman Effect, radiation effects, nuclear radiations etc., were discovered. Since classical mechanics fails to explain the above phenomena, a new physics known as modern physics was developed on the basis of quantum theory of radiation. In order to explain the distribution of energy in the blackbody radiation Planck introduced the concept of quantum theory of radiation in 1900.

Radiation:
It is a process of transmission of energy from one place to another without the aid of any intervening medium.

Heat radiation can pass through vacuum just as light. Heat radiation is also called thermal radiation.

OR

"Radiation is the emission of electromagnetic waves by matter when supplied with appropriate amount of energy".

Blackbody radiation

Black body: when thermal radiation is incident on the surface of blackbody, it absorbs a part of the incident energy. The amount of energy absorbed depends on

1. Nature of the surface
2. Surface area
3. Temperature of the surface

"A body which absorbs radiation of all wavelengths incident on it is called a perfect blackbody". Such a body cannot reflect or transmit a radiation and therefore it appears black. Thus r = 0, t = 0 & a = 1. After absorbing the radiation the blackbody gets heated & starts emitting radiation of all possible wavelengths. The radiation emitted is independent of the nature of the body & depends only on the temperature of the blackbody. This radiation is known as blackbody radiation. Thus a perfect blackbody is a good absorber as well as good emitter".
In practice a perfect blackbody doesn't exist. An artificial blackbody designed by Fery is as shown in the figure.

It consists of hollow double walled sphere with a fine hole and a pointed projection in front of the hole. The space between the two walls is evacuated. The inner wall is coated with lampblack. The pointed projection prevents direct reflection of the incident radiation. Any radiation entering through the hole is completely absorbed after multiple reflections. That is at each reflection 98% of the incident radiation is absorbed. Thus after a few reflections at the inner surface, the radiation is completely absorbed. Hence this device acts as a blackbody absorber. It can also be used as blackbody radiator by heating it in a suitable bath to high temperature.

Emissive power and Absorption power

Emissive power: The emissive power (E) of a surface is the amount of radiation of all wavelengths emitted by unit area of the surface in unit time.

OR

The emissive power (e_λ) of a body at a temperature (T) for a wavelength (λ) is defined as the energy emitted per second per unit surface area of the body within a unit wavelength range.
The emissive power of a perfect blackbody is maximum and it is denoted by E_λ.

Absorptive power: The absorptive power (A) of a surface is the fraction of the incident radiation of all wavelengths which is absorbed by it.

$$i.e., \qquad A = \frac{\text{Amount of energy (radiation) absorbed}}{\text{Amount of radiation incident}}$$

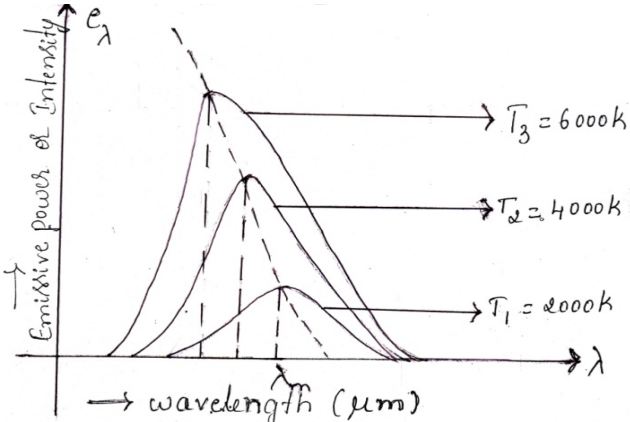
OR

The absorptive power (a_λ) of a body at a temperature (T) and for a wavelength (λ) is defined as the ratio of amount of energy absorbed by the surface in a given time to the amount of energy incident on the same surface in the same time.
The absorptive power of a perfect blackbody is maximum and it is equal to 1 i.e., $A_\lambda = 1$.
According to Kirchhoff's law, the ratio of the emissive power (e_λ) to the absorptive power (a_λ) for a given wavelength (λ) at a given temperature (T) is the same for all bodies and is equal to the emissive power of a perfect blackbody at that temperature.

$$ie., \qquad \frac{e_\lambda}{a_\lambda} = E_\lambda$$

Distribution of energy in blackbody radiation

The distribution of energy among the different wavelength of blackbody radiation was studied by ***Lummer*** and ***Pringsheim*** in 1809. The curve in the figure shows the variation of intensity of radiation with wavelength for different temperature of the blackbody ($T_1 < T_2 < T_3$). As a result, an examination of these curves leads us to the following conclusions.



1. At a given temperature, the energy distribution is not uniform in the spectrum of blackbody radiation.
2. At a given temperature, the blackbody emits continuous range of wavelengths of radiation. As the wavelength increases the intensity of radiation increases reaches a maximum value at a particular wavelength λ_m and then decreases with further increase in wavelength.
3. As the temperature increases, the wavelength λ_m corresponds to maximum energy density decreases or shift towards shorter wavelengths region. The points on the dotted line represent λ_m at various temperatures. It is found that This shows Wien's displacement law. i.e., $\lambda_m \propto 1/T$
4. For all wavelengths, an increase in temperature causes an increase in the energy emission or intensity.
5. For a given temperature, the area under the curve represents the total energy emitted for the complete spectrum. The area under the curve increases with increase in temperature. It is found that the area is directly proportional to the 4th power of absolute temperature of the blackbody i.e., $E \propto T^4$.

   This represents Stefan-Boltzmann law.

Laws of Blackbody radiation

Stefan's law or (Stefan-Boltzmann law) :

Stefan-Boltzmann law of radiation states that "the emissive power (E) of a blackbody is directly proportional to the fourth power of its absolute temperature (T)".

$$E \propto T^4 \qquad \text{or} \qquad \mathrm{E} = \sigma \mathrm{T}^4$$

Where σ is a constant known as Stefan's constant. Its value is 5.67×10^{-8} $Wm^{-2}k^{-4}$. This law was suggested experimentally by Stefan and later derived by Boltzmann and hence it is called as Stefan-Boltzmann law.

Wien's displacement law:

Wien's law states that "the wavelength λ_m corresponds to maximum energy is inversely proportional to the absolute temperature T of the blackbody".

$$\text{i.e., } \lambda_m \propto \frac{1}{T} \quad or \quad \lambda_m T = b \quad \text{is a consant}$$

Where b is called the Wien's constant and it is equal to 2.892×10^{-3} mK. It shows that, as temperature increases λ_m decreases. Thus if λ_{m1}, λ_{m2}, λ_{m3} etc., are the wavelengths corresponds to the maximum energy densities at temperatures T_1, T_2, T_3 etc., then according to Wien's law we have,

$$\lambda_{m_1} T_1 = \lambda_{m_2} T_2 = \lambda_{m_3} T_2 = ..., = b$$

This is known as Wien's displacement law.

Wien's law of Radiation:

Wien has also shown that the "maximum energy E_m is directly proportional to fifth power of the absolute temperature T".

$$\text{i.e., } \quad E_m \propto T^5 \quad \text{or} \quad E_m = \text{constant x } T^5 \quad \text{or} \quad E_m T^{-5} = \text{constant}$$

This is known as Wien's law of radiation.

Wien's distribution law:

The energy density emitted by a blackbody in the wavelength range λ and λ + dλ in a given temperature T is given by,

$$e_\lambda \, d\lambda = C_1 \lambda^{-5} e^{\left(\frac{-C_2}{\lambda T}\right)} d\lambda$$

where C_1 & C_2 are the constants. This is known as Wien's distribution law.

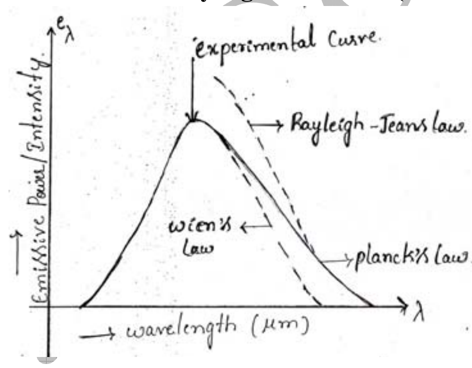
Wien's law holds good only in shorter wavelength region with the experimental results at high temperature of the source. It fails to explain the gradual decrease in the intensity of radiation with the wavelength beyond the wavelength λ_m corresponds to the peak value.

Rayleigh-Jeans Law:
Rayleigh and Jeans gave a mathematical expression for the distribution of energy in blackbody radiation.
According to Rayleigh-Jeans law the energy density or intensity emitted by a blackbody in the wavelength range λ and $\lambda + d\lambda$ is given by,

$$e_\lambda d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda \quad \text{or} \quad e_\lambda d\lambda = 8\pi kT\,\lambda^{-4} d\lambda$$

This equation doesn't shows any peak in the energy value but the energy goes on decreases with increase in wavelength. This law holds good only in the longer wavelength region of the experimental results.

Drawbacks of Wien's law and Rayleigh-Jeans law (ultraviolet catastrophe)



The Wien's law is valid only for the shorter wavelength region at high temperature of the source. It doesn't holds good in the longer wavelength region and it fails to explain the gradual decrease in the intensity of radiation whose wavelength are larger than the wavelength λ_m corresponds to the peak value.

Rayleigh-Jeans law holds good only in the longer wavelength region and it doesn't holds good in the shorter wavelength region. According to Rayleigh-Jeans law,

$$e_\lambda d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda$$

As $\lambda \to 0$, $e_\lambda \to \infty$ thus at lower wavelength region, as the wavelength decreases the energy density increases and when the wavelength reaches ultraviolet region the energy density becomes large. Such a large increase in the energy density in the lower wavelength region below ultraviolet doesn't occur experimentally. This discrepancy is known as "ultraviolet catastrophe".

Planck's radiation law:

In 1900 Max Planck proposed a new theory called quantum theory which successfully explained the energy distribution among different wavelengths in the blackbody radiation. The Planck's theory is based on the following postulates (or) assumptions.

1. A blackbody consists of a large number of oscillating particles (atoms). These particles can vibrate in all possible energies.
2. An oscillator can have a discrete set of energies which are integral multiples of a finite quantum of energies. ie., $E_n = nh\nu$
   where n is an integer (n = 1, 2, 3, 4…,), h is a Planck's constant and ν is the frequency of the oscillating particle.
3. The atomic oscillator can emit or absorb energy in discrete units only by making a transition from one quantum state to another.

Thus, the energy of a quantum is directly proportional to the frequency, the quantum traverse in space with the velocity of light.

According to Planck's, the energy density emitted from a blackbody at a temperature T for all wavelength range λ and λ+dλ is given by,

$$e_{\lambda} d\lambda = \frac{8\pi hc}{\lambda^5}\left[\frac{1}{e^{\left(\frac{hc}{\lambda kT}\right)} - 1}\right] d\lambda$$

where k is the Boltzmann constant. According to this equation, the distribution of energy in blackbody radiation agrees well with the experimental observations in all wavelength range. This equation is known as Planck's law or Planck's radiation law.

Matter Waves

Dual Nature of Matter:

The wave theory of electromagnetic radiation satisfactorily explains the phenomena reflection, refraction, interference, diffraction and polarization. But it failed to explain the phenomena of Photoelectric Effect and Compton Effect.

On the other hand they were explained on the basis of quantum theory of radiation. According to which a beam of light of frequency ν consists of small packets each having energy hν called photon or quanta.

Sometimes these photons behave like a waves and sometimes like a corpuscles i.e., particles. Thus radiation have dual nature i.e., wave and particle or quantum nature.

Matter waves and their characteristics properties

In 1924 Louis de Broglie suggested that the particles like protons, electrons, & neutrons in motion exhibit characteristic properties of waves. Thus a moving particle can be associated with a wave or a wave can guide the motion of the particle. Hence the waves associated with the moving particles are known as de-Broglie waves or matter waves.

According to de-Broglie hypothesis, a particle of mass 'm' moving with velocity 'v' is associated with the wave. This wave is called matter wave. The wavelength of matter wave is,

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

1. Matter waves are the waves associated with moving particles.
2. Lighter the particles, greater is the wavelength associated with it, because here $\lambda \propto 1/m$
3. Greater the velocity of the particle, smaller is the wavelength associated with the particle. $\lambda \propto 1/v$
4. Matter waves are not electromagnetic waves. Since they don't depends on the charge of the particle.
5. The velocity of the matter waves is not constant. But it depends on the velocity of the particle.
6. Light wave has got same velocity, for all wavelengths. But in case of matter waves, the velocity is inversely proportional to the wavelength.
7. It is not possible to determine the exact position and momentum of a particle simultaneously.
8. Matter waves are also called as de-Broglie waves (or) pilot waves.

Phase velocity (v_{Phase} or v_p)

A point marked on a wave can be regarded as representing a particular phase for the wave at that point. The velocity with which such a point would propagate is known as phase velocity (or) wave velocity.

It is represented by

$$v_{phase} \quad \text{or} \quad v_p = \frac{\omega}{k}$$

where, ω is angular frequency and k is the propagation constant or wave number

Group Velocity (v_{group} or v_g)

The velocity with which the resultant envelops of the group of waves travels is called group velocity.

It is denoted by v_g or v_{group} and is equal to the particle velocity v.

$$v_{group} \quad or \quad v_g = \frac{d\omega}{dk}$$

Relation between Group Velocity v_g and Phase Velocity v_p

The equations for group velocity and phase velocity are given by,

$$v_g = \frac{dw}{dk} \qquad \rightarrow (1)$$

$$v_p = \frac{\omega}{k} \qquad \rightarrow (2)$$

where ω is the angular frequency of the wave and k is the propagation constant or wave vector.

$$\therefore \omega = v_p k \qquad \rightarrow (3)$$

$$\frac{dw}{dk} = \frac{d(v_p k)}{dk} = v_p + k\frac{dv_p}{dk} \qquad \rightarrow (4)$$

$$\text{But,} \quad k\frac{dv_p}{dk} = k\frac{dv_p}{d\lambda} \text{ x } \frac{d\lambda}{dk} \qquad \rightarrow (5)$$

$$\text{we know that,} \quad k = \frac{2\pi}{\lambda} \quad or \quad \lambda = \frac{2\pi}{k} \quad \text{and} \quad \frac{dk}{d\lambda} = 2\pi\left(\frac{-1}{\lambda^2}\right) \quad or \quad \frac{d\lambda}{dk} = -\frac{\lambda^2}{2\pi}$$

$$\therefore \text{ eqn (5) becomes,} \quad k\frac{dv_p}{dk} = \frac{2\pi}{\lambda} \text{ x } \frac{-\lambda^2}{2\pi} \text{ x } \frac{dv_p}{d\lambda}$$

$$or \quad k\frac{dv_p}{dk} = -\lambda \text{ x } \frac{dv_p}{d\lambda}$$

On substituting the values in equation (4) we get,

$$v_g = v_p - \lambda\frac{dV_p}{d\lambda} \quad or \quad v_{group} = v_{phase} - \lambda\left(\frac{dv_{phase}}{d\lambda}\right)$$

This is the relation between group velocity and phase velocity.

Relation between Group Velocity v_g and Particle Velocity v

The group velocity of a wave packet is given by,

$$v_g = \frac{d\omega}{dk} \qquad \rightarrow (1)$$

where ω = 2πν and ν is the wave frequency

But from quantum theory of radiation, E = hν ∴ ν = E/h

$$\therefore \omega = 2\pi \frac{E}{h} \qquad \rightarrow (2)$$

On differentiating eqn. (2) we get,

$$d\omega = \frac{2\pi}{h} dE \qquad \rightarrow (3)$$

Also, we know that, k = 2π/λ, and λ = h/P

$$\therefore k = 2\pi \frac{p}{h} \qquad \rightarrow (4)$$

On differentiating eqn. (4) we get,

$$dk = \frac{2\pi}{h} dp \qquad \rightarrow (5)$$

Dividing equation (3) by equation (5), we have,

$$\frac{d\omega}{dk} = \frac{\frac{2\pi}{h} dE}{\frac{2\pi}{h} dp} = \frac{dE}{dp} \qquad \rightarrow (6)$$

But we know that, if P is the momentum of the particle, then

$$E = \frac{P^2}{2m}$$

On differentiating the above equation, we get,

$$dE = \frac{2P}{2m} dP \qquad \text{or} \qquad \frac{dE}{dP} = \frac{P}{m} \qquad \rightarrow (7)$$

From equations (1), (6) and (7) we get,

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dP} = \frac{P}{m} = \frac{mv}{m} = v$$

where, $P = mv$ and $v = v_{particle}$ is the velocity of the particle.

$$v_g = v \qquad \text{or} \qquad v_{group} = v_{particle}$$

$\therefore$ The de-Broglie wave group associated with a particle travels with a velocity equal to the velocity of the particle itself.

Relation between group velocity v_g, phase velocity v_p & velocity of light c

The equation for phase velocity is,

$$v_p = \frac{\omega}{k} \qquad \rightarrow (1)$$

$$But, \quad \omega = 2\pi\nu = 2\pi \frac{E}{h} \qquad \because E = h\nu$$

$$\& \quad k = \frac{2\pi}{\lambda} = 2\pi \frac{P}{h} \qquad \because \lambda = \frac{h}{p}$$

By substituting the values of *w* and k in eqn. (1) we get,

$$v_p = \frac{\omega}{k} = \frac{E}{P} \qquad \rightarrow (2)$$

Since $E = mc^2$ and $P = m$v, equn. (2) becomes

$$v_p = \frac{mc^2}{mv} = \frac{c^2}{v}$$

$$\text{or} \qquad v_p \text{ x } v = c^2 \qquad \text{or } v_{phase} \text{ x } v_{particle} = c^2 \rightarrow (3)$$

Since group velocity is equal to particle velocity i.e., $v_g = v$, Equation (3) becomes,

$$v_p \text{ x } v_{group} = c^2 \qquad \text{or} \qquad v_{phase} \text{ x } v_{group} = c^2 \rightarrow (4)$$

This is the relation of velocity of light with phase velocity & group velocity. Since V_{group} is the same as $V_{particle}$, & the velocity of a material particle can never be greater than or even equal to c, it implies that, the phase velocity is always equal to or greater than c.

Expression for de-Broglie wavelength using group velocity

According to de-Broglie's theory, it is assumed that a material particle in motion is associated with a system of plane waves; the superposition of these waves gives rise to a wave packet. In such a wave motion, the group velocity is to be considered. The group velocity is given by,,

$$V_g = \frac{d\omega}{dk} \qquad \rightarrow (1)$$

But we know that, $w = 2\pi\nu$ and $k = 2\pi/\lambda$

$$\therefore d\omega = 2\pi\, d\nu \quad \& \quad dk = 2\pi\, d\left(\frac{1}{\lambda}\right)$$

$$\therefore \frac{d\omega}{dk} = \frac{d\nu}{d\left(\frac{1}{\lambda}\right)}$$

Then from equation (1), group velocity becomes,

$$v_g = \frac{d\nu}{d\left(\frac{1}{\lambda}\right)}$$

Since, we know that the group velocity v_g is same as the particle velocity 'v', we have

$$\therefore v = \frac{d\nu}{d\left(\frac{1}{\lambda}\right)}$$

$$or \qquad d\left(\frac{1}{\lambda}\right) = \frac{d\nu}{v} \qquad \rightarrow (2)$$

Let m be the mass of the particle, v be its velocity then E is its total energy. If V is the potential energy of the particle, then

$$E = \tfrac{1}{2} m\text{v}^2 + V \qquad \rightarrow (3)$$

But from quantum theory of radiation we have $E = h\nu$, therefore equation (3) becomes

$$h\nu = \tfrac{1}{2} m\text{v}^2 + V \qquad \rightarrow (4)$$

Let the particle be moving in a field of constant potential, i.e., V is a constant.
Thus on differentiation equation (4) it becomes

$$hd\nu = \frac{1}{2} m(2\text{v})d\text{v}$$

$$hd\nu = m\text{v}d\text{v}$$

$$\frac{d\nu}{\text{v}} = \frac{m}{h} d\text{v} \qquad \rightarrow (5)$$

From equation (1) & equation (5) we have,

$$d\left(\frac{1}{\lambda}\right) = \left(\frac{m}{h}\right) d\text{v} \qquad \rightarrow (6)$$

On integrating equation (6), we get

$$\frac{1}{\lambda} = \frac{m}{h}\text{v} + \text{constan}$$

Let the momentum of the particle be P = mv.
Also considering the constant of integration as zero, we have

$$\frac{1}{\lambda} = \frac{P}{h} \quad or \quad \lambda = \frac{h}{P} = \frac{h}{m\text{v}}$$

The above equation is the de-Broglie's equation.

Problems:

1. Calculate the surface temperature of the sun & moon. Given that λ_m= 4753Å and 14µm respectively.

$$we\ know\ that, \lambda_m T = \text{constant}\ (b) = 2.898 \times 10^{-3}\, mK$$

$$For\ sun \quad \lambda_m = 4753\overset{0}{A} = 4753x10^{-10}\, m, \text{ Surface temperature of the sun is,}$$

$$T_S = \frac{b}{\lambda_m} = \frac{2.898 \times 10^{-3}}{4753 \times 10^{-10}} = 6097.20 = 6097\ K$$

$$\text{For moon } \lambda_m = 14\mu m = 14 \times 10^{-6}\, m, \text{ and surface temperature of moon is,}$$

$$T_M = \frac{b}{\lambda_m} = \frac{2.898 \times 10^{-3}}{14 \times 10^{-6}} = 207\ K$$

2. A body at 1500K emits maximum energy of wavelength 2000nm. If the sun emits maximum energy of wavelength 550nm, find the temperature of the sun?

Given, we know that

$$\lambda_m T = constant$$

$T_1 = 1500\ K$

$\lambda_1 = 2000\,nm$

$\lambda_2 = 550\,nm$

$$\therefore\ \lambda_{m_1} T_1 = \lambda_{m_2} T_2 = \ldots, = b$$

$$\lambda_{m_1} T_1 = \lambda_{m_2} T_2$$

$$T_2 = \frac{\lambda_1 T_1}{\lambda_2} = \frac{2000\times10^{-9}\,\text{x}1500}{550\times10^{-9}} = 5454.545\,\text{K}$$

Temperature of the sun = 5455 K

3. Deduce the temperature at which a blackbody loses thermal energy at the rate of 1watt/cm^2.

Soln. E = 1 watt/cm^2 = 1x10^4 W/m^2

T =?

$$we\ \ know\ that\ \ E = \sigma T^4$$

$$\therefore T = \left(\frac{E}{\sigma}\right)^{\frac{1}{4}} == \left(\frac{1\times10^4}{5.67\times10^{-8}}\right)^{\frac{1}{4}} = 648\ K$$

$\therefore$ The temperature of the blackbody is 648 K.

4. Compare the radiant emittance of a blackbody at 200 K & 2000 K. Given σ = 5.67x10^{-8} Wm^{-2}K^{-4}.

Solution, T_1 =200 K, T_2 =2000 K, $\frac{E_1}{E_2}$ =?

$$\text{we know that}\quad E = \sigma T^4 \ \therefore\ \ E_1 = \sigma T_1^{\,4}\ \&\ E_2 = \sigma T_2^{\,4}$$

$$\therefore \frac{E_1}{E_2} = \frac{T_1^4}{T_2^4} = \left(\frac{T_1}{T_2}\right)^4 = \left[\frac{200}{2000}\right]^4$$

$$\frac{E_1}{E_2} = \left(\frac{1}{10}\right)^4 = \left(10^{-1}\right)^4 = 10^{-4} \quad \therefore\ \frac{E_1}{E_2} = 10^{-4}$$

5. Evaluate the surface temperature of the star which is radiating 1.6x10^5 times more energy per unit area per second, compare to sun & surface temperature of the sun is 6000 K.

$Given,\ E_1 = 1.6\times10^5 E_2$ and $\frac{E_1}{E_2} = 1.6\times10^5, T_1 = ?$ if $T_2 = 6000$ K

we know that $E = \sigma T^4$

$$\therefore\ E_1 = \sigma T_1^4 \ \&\ E_2 = \sigma T_2^4$$

$$\therefore \frac{E_1}{E_2} = \frac{T_1^4}{T_2^4} = \left(\frac{T_1}{T_2}\right)^4 \quad \text{or} \quad \frac{T_1}{T_2} = \left(\frac{E_1}{E_2}\right)^{\frac{1}{4}}$$

$$T_1 = T_2\left(\frac{E_1}{E_2}\right)^{\frac{1}{4}} = 6000\left(1.6\times10^5\right)^{\frac{1}{4}} = 6000\times20$$

$$T_1 = 1{,}20{,}000\ \ K$$

Surface temperature of the star is 1, 20,000 K.

6. The temperature of a furnace is 2500 K. Find the heat radiated by it per second per square meter area of its surface. Given σ = 5.67×10^{-8} $W/m^2/K^4$

Solution

T = 2500 K, E =?

we know that

$$E = \sigma T^4 = \left[5.67\times10^{-8}\right]\times\left[2500\right]^4 = 2.21\times10^6 W/m^2 \ \ or\ \ J/s/m^2$$

7. The surface temperature of a hot body is 1227°C. Find the wavelength at which it radiates maximum energy.

$Given,\quad T = 1227+273 = 1500\ \ K,\quad \lambda_m = ?$

we know that,

$$\lambda_m T = constant = b$$

$$\lambda_m = \frac{b}{T} = \frac{2.89\times10^{-3}}{1500} = 1.9266\times10^{-6} m = 1926.6\times10^{-9} m$$

$$\lambda_m = 1926\ \ nm$$

8. Calculate the energy radiated by unit area of a blackbody in one second when its temperature is 1000 K.

$Given\ ,\ \ T = 1000\ \ \ K\ ,\ \ E = ?$

weknowthat

$$E = \sigma T^4 = 5.67\times10^{-8}\times(1000)^4 = 5.67\times10^{-8}\left(10^3\right)^4 = 5.67\times10^{-8}\times10^{12} = 5.67\times10^4 W/m^2$$

9. If the group velocity of a particle is 3×10^6 m/s, calculate its phase velocity. (Given c = 3×10^8 m/s)

$Given,\ \ V_g = 3\times10^6 m/s,\ \ c = 3\times10^8\ m/s,\quad v_p = ?$

$$w.k.t; \qquad V_{group} V_{phase} = c^2$$

$$or \quad V_{phase} = \frac{c^2}{V_{group}} = \frac{(3\times10^8)^2}{(3\times10^6)} = \frac{9\times10^{16}}{3\times10^6} = 3\times10^{10}\ m/s$$

10. Calculate the de-Broglie wavelength associated with a proton moving with a velocity equal to 1/20 th of the velocity of light.

$$Given, \quad v = \frac{1}{20}\times c = \frac{1}{20}\times 3\times10^8, \quad m_p = 1.67\times10^{-27}\,kg, \quad \lambda_p = ?$$

$$w.k.t, \quad \lambda_p = \frac{h}{mv}$$

$$\lambda_p = \frac{6.63\times10^{-34}}{1.67\times10^{-27}\times\frac{1}{20}\times3\times10^8} = \frac{20\times6.63\times10^{-34}}{1.67\times10^{-27}\times3\times10^8} = 2.646\times10^{-14}\,m$$

11. Find the amount of energy radiated by the surface of area 200 sq.m in an hour if it is maintained at a temperature of 1500 K

$$Given, \quad T = 1500\ K, \quad E = ?, \quad E_1 = ?$$

$$w.k.t; \quad E = \sigma T^4$$

$$E = 5.67\times10^{-8}\times[1500]^4 = 287043.75\ \text{W/m}^2$$

$$E_1 = 287043.75\times200\times60\times60$$

$$E_1 = 2.0667\times10^{11}\,W/m^2$$

12. Find the kinetic energy and group velocity of an electron with de-Broglie wavelength of 0.2 nm.

$$Given, \quad \lambda = 0.2\ nm = 0.2\times10^{-9}\,m, \quad K.E = ?, \quad V_{group} = ?, \quad m_e = 9.1\times10^{-31}\,kg$$

$$w.k.t; \quad \lambda = \frac{h}{p}$$

$$\therefore\ p = \frac{h}{\lambda} = \frac{6.63\times10^{-34}}{0.2\times10^{-9}} = 3.315\times10^{-24}\ kg-m/s$$

$$K.E = E = \frac{p^2}{2m} = \frac{(3.315\times10^{-24})^2}{2\times9.1\times10^{-31}} = 6.038\times10^{-18}\ J$$

$$V_g\ or\ V_{group} = \frac{p}{m} = \frac{3.315\times10^{-24}}{9.1\times10^{-31}} = 3642857.143 = 3.64\times10^6\,m/s$$

Questions
QUIZ

1. The wavelength of maximum intensity is inversely proportional to the absolute temperature of the body emitting radiation. This is called ………………..law.
2. Each quantum carries an energy of ……………………
3. Phase velocity of matter waves is always ………………… than velocity of light.
4. The law which describes the blackbody radiation completely is ………………… law.
5. The wavelength associated with a particle of mass m, moving with a velocity v, is given by …………………..
6. The law which failed to account for shorter wavelength region of blackbody radiation spectrum is …………………………
7. In a blackbody radiation spectrum, the Wien's distribution law is applicable only for …………………. Wavelength region.
8. The product of phase velocity and group velocity is equal to……………
9. Group velocity of a wave packet is equal to ……………………

SHORT ANSWER QUESTIONS

1. State Stefan's law of blackbody radiation.
2. State Wien's displacement law.
3. What is Planck's law of radiation?
4. What are the limitations of Wien's law?
5. List out the characteristics of matter waves.
6. What are matter waves?
7. Define phase velocity and group velocity.

LONG ANSWER QUESTIONS

1. Explain the energy distribution in the spectrum of a blackbody.
2. Explain Wien's law & Rayleigh-Jeans law. Mention their drawbacks.
3. Describe the ultraviolet catastrophe.
4. State and explain Planck's law of radiation.
5. What is de-Broglie concept of matter wave? And explain the characteristics of matter wave.
6. Deduce an expression for de-Broglie wavelength using group velocity concept.
7. Define phase velocity and group velocity & obtain a relation between the two.
8. Obtain the relation between group velocity and particle velocity.
9. Derive the relation between group velocity, phase velocity and velocity of light.

Question paper questions

1. What are matter waves? Derive the expression for de Broglie wavelength using the concept of matter waves?

2. State Stefan's law of radiation. Explain salient features of blackbody radiation spectrum.

3. Give an account of the attempts made through three laws to explain the blackbody radiation spectrum.

4. Explain Planck's law, Wien's law, Rayleigh-jeans law and Stefan-Boltzmann law

5. Define group velocity and particle velocity. Show that group velocity is equal to particle velocity.

6. What is ultraviolet catastrophe? Explain the energy distribution in the spectrum of a black body.

Problems:

1. Compare the energy of photon with that of an electron when both are associated with a wavelength of 0.2 nm.

2. Estimate the amount of energy radiated by the unit surface area of a blackbody in one hour maintained at a temperature of 1500 K

3. Calculate the de Broglie wavelength of a 1000 kg automobile travelling at 100 m/s and a 0.1 kg bullet travelling at 500 m/s.

4. Find the temperature at which the emissive power of a blackbody is four times its emissive power at temperature 1500 K.

5. A fast moving neutron is found to a have an associated de Broglie wavelength of $2x10^{-12}$ m. Find its kinetic energy and group velocity of the de Broglie waves using the relativistic change in mass. (Mass of neutron = $1.675x10^{-27}$ kg)

6. Calculate the de Broglie wavelength associated with an electron with a kinetic energy of 2000 eV.

Quantum Mechanics

The branch of mechanics that deals with the mathematical description of the motion and interaction of subatomic particles, incorporating the concepts of quantisation of energy, wave – particle duality, the uncertainty principle and the correspondence principles.

Heisenberg's uncertainty principle:

According to this principle "It is impossible to determine precisely and simultaneously the values of both the members of the pair of physical variables. Which describe the motion of the atomic system". Such variables are called canonically conjugate variables.

Example: Position and momentum, energy and time etc.,

Statement: "it is impossible to determine simultaneously both position and momentum of a moving particle accurately at same time. The product of uncertainty in these quantities is always greater than or equal to h/4π".

If Δx and ΔP_x are the uncertainties in the measurement of position and momentum of a particle, then

$$\Delta x \,.\, \Delta p_x \geq \frac{h}{4\pi}$$

If Δx is small, ΔP_x will be large and vice versa. That is if one quantity is measured accurately, the other quantity becomes less accurate.

Similarly the other uncertainty relations for other physical variables pair are,

$$\Delta E . \Delta t \geq \frac{h}{4\pi}$$

$$\Delta L . \Delta \theta \geq \frac{h}{4\pi}$$

Applications of Uncertainty Principle:

Non-existence of electrons in the nucleus and its implications- non-relativistic approach

According to theory of relativity, the energy E of particle is expressed as,

$$E = mc^2 \quad \rightarrow (1)$$

where m is the relativistic mass of a particle moving the a velocity v and the expression for it in terms of rest mass can be written as,

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \rightarrow (2)$$

By squaring on both sides, we get

$$m^2 = \frac{m_o^{\,2}}{1 - \frac{v^2}{c^2}} = \frac{m_o^{\,2} c^2}{c^2 - v^2}$$

$\therefore\ m^2c^2 - m^2v^2 = m_o^{\,2}c^2 \quad \rightarrow (3)$

Eqn.(3) is multiplied by c^2 on both sides, then we get

$m^2c^4 - m^2v^2c^2 = m_o^{\,2}c^4$

But w.k.t. $P = mv$ and $E = mc^2$, $\therefore$ The above equation becomes,

$E^2 = P^2c^2 + m_o^{\,2}c^4$ or $E^2 = c^2(P^2 + m_o^{\,2}c^2)$

or $E = c(P^2 + m_o^{\,2}c^2)^{1/2} \quad \rightarrow (4)$

This is the expression for the total energy of a particle in terms of momentum and its rest mass energy.

Heisenberg's uncertainty principle states that,

$$\Delta x.\Delta p_x \geq \frac{h}{4\pi} \quad \text{or} \quad \Delta p_x \geq \frac{h}{4\pi.\Delta x} \quad \rightarrow (5)$$

We know that the diameter of the nucleus is of the order of 10^{-14} m. If an electron is to exist inside the nucleus, then the uncertainty in its position Δx must not exceed the size of the nucleus,

$$i.e., \quad \Delta x \leq 10^{-14}\, m$$

Using Δx in equation (5) we have,

$$\Delta p_x \geq \frac{h}{4\pi.\Delta x} \geq \frac{6.63\times10^{-34}}{4\pi\times10^{-14}} \geq 0.5\times10^{-20} \text{ Ns}$$

$$\therefore \quad p_x \geq 0.5\times10^{-20}\ Ns$$

This is the uncertainty in momentum of an electron and it is equal to the momentum of the electron inside the nucleus,

Using momentum value P_x and rest mass m_0 of the electron in an equation (4) we get,

$$E \geq (3\times10^8)\left[(0.2\times10^{-20})^2 + (9.11x10^{-31})^2 x(3x10^8)^2\right]^{1/2}$$

$$E \geq 1.5\times10^{-12} J \quad or \quad E \geq 9.4 \text{ MeV}$$

An electron may exist inside the nucleus if its energy is equal to or greater than 9.4MeV. But, the experimental investigations on β-decay say that kinetic energy of the β–particles is 3 to 4 MeV. This clearly indicates that, electrons cannot exist within the nucleus.

$$n \rightarrow p + e^{-1} + \nu \ (energy) \qquad \text{or} \qquad p \rightarrow n + e^{+1} + \nu \ (energy)$$

<u>Wave Function</u>:

In general, a wave is characterised by periodic variation in some physical quantity.

For example – pressure varies periodically in sound waves whereas electric and magnetic fields vary periodically in an electromagnetic wave. Similarly whose periodic variations make up the matter wave is called wave function.

Wave function in quantum mechanics accounts for the wave like properties of particle and is obtained by solving a fundamental equation called Schrödinger's equation.

The wave functions vary with respect to both position co-ordinates of the physical system and the time (x, y, z & t) is called total wave function.

It is denoted by the capital form of Greek letter 'Ψ'. If the wave function has variation only with position (x, y, z) it is denoted by the lower case Greek letter 'ψ'.

The function ψ itself does not have any physical significance hence it is not an experimentally measurable quantity. The probability of finding a particle at some point in space at time 't' is a positive value between 0 & 1.

But ψ can be positive or negative or complex. Hence ψ is not an observable quantity.

Physical Significance of Wave Function:

The physical significance of ψ could be realised through its probabilistic nature in quantum mechanics in terms of probability density.

Probability Density:

In classical mechanics, the square of wave amplitude associated with electromagnetic radiation is interpreted as measure of intensity. This suggests there will be a similar interpretation for de-Broglie waves associated with electron or any particle.

Let τ be a volume inside which a particle is present, but where exactly the particle is situated inside τ is not known

"If ψ is the wave function associated with the particle then the probability of finding the particle in certain volume $d\tau$ of τ is equal to $|\psi|^2\, d\tau$. So $|\psi|^2$ is called the probability density".

$$|\psi|^2\, d\tau$$

This interpretation was first given by Max Born in 1926.

If the value of $|\psi|^2$ is large, then the probability of finding the particle at the point at that time is more. If $|\psi|^2 = 0$, then the probability of finding the particle is zero or less.

Therefore the total wave function can be represented by the equation,

$$\Psi = Ae^{i(kx-\omega t)} \rightarrow (1)$$

where A is a constant, ω is angular frequency of the wave
The complex conjugate of Ψ is given by,

$$\Psi^* = Ae^{-i(kx-\omega t)} \rightarrow (2)$$

From equation (1) and (2), $\Psi\Psi^*$ is real and positive quantity which is called the probability density.

$$\text{i.e,}\quad |\psi|^2 = \psi\psi^* = A^2$$

Therefore $|\Psi|^2 dx$ is the probability density in 1- dimension,

And $|\Psi|^2$ dv is the probability density in 3-dimension.

Normalization:

According to Born's interpretation the probability of finding the particle within an element of volume is $|\Psi|^2$ dv, since the particle is certainly present somewhere inside the volume dv.

Therefore "The integral of the square of the wave function over the entire volume in space must be equal to unity" and mathematically it is represented as,

$$\int_{-\infty}^{\infty} |\psi|^2\, dv = 1$$

Where, the wave function satisfying the above relation is the normalized wave function.

Very often Ψ is not a normalized wave function. If this function Ψ is multiplied by a constant *A*, then the new wave function *A*Ψ is also a solution of the wave equation. Hence the new wave function is a normalized wave function.

$$\int_{-\infty}^{\infty} |A\psi^*||A\psi| d\text{v} = 1 \quad \text{or} \quad A^2 \int_{-\infty}^{\infty} \psi\psi^* d\text{v} = 1$$

$$|A|^2 = \frac{1}{\int_{-\infty}^{\infty} \psi^*\psi \, d\text{v}}$$

Where $|A|^2$ is known as normalizing constant, the quantity $|A\Psi|^2$ represents probability.

Therefore the process of constructing AΨ from Ψ is called normalization of the wave function.

Properties of Wave Functions

Physically acceptable wave function Ψ must satisfy the following conditions,

1. Ψ is single valued everywhere

If Ψ has more than one value at any point, it would mean more than one value of probability of finding the particle at that point which is obviously ridiculous. Therefore Ψ must be single valued everywhere.

2. Ψ is finite everywhere

If Ψ is infinite at a point there will be large probability of finding the particle at that point this violet the uncertainty principle therefore Ψ must be finite or zero value at any point and hence $|\Psi|^2$ represents probability.

3. Ψ and its first derivatives dΨ/dx with respect to its variables are continuous everywhere

This is necessary from Schrödinger's equation itself which shows that dΨ/d*x* must be finite everywhere. Further, the existence of is continuous function, which implies that the function of Ψ is also continuous everywhere.

Time Independent One Dimensional Schrondinger's Wave Equation

Based on de-Broglie idea of matter waves, Schrödinger developed a mathematical theory for a particle of mass 'm' moving with a velocity 'v' along x-direction associated with a wave of wavelength,

$$\lambda = \frac{h}{p}$$

Where, p = mv is the momentum of the particle.

Let a wave function Ψ describing the de-Broglie wave travelling in +ve x-direction is given by,

$$\psi = Ae^{i(kx-\omega t)} \quad \rightarrow (1)$$

Where Ψ is a total wave function, A is a constant and ω is angular frequency of wave.

Let us differentiate Ψ (in equation 1) twice with respect to 'x' then

$$\frac{d\psi}{dx} = A(ik)e^{i(kx-\omega t)}$$

$$\frac{d^2\psi}{dx^2} = A(ik)^2 e^{i(kx-\omega t)}$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi \quad or \quad \frac{d^2\psi}{dx^2} + k^2\psi = 0 \qquad \rightarrow (2) \quad \because i^2 = -1$$

$$But \quad k = \frac{2\pi}{\lambda} \quad and \quad \lambda = \frac{h}{mv}$$

$$\therefore k = \frac{2\pi m v}{h} \quad \text{or} \quad k^2 = \frac{4\pi^2 m^2 v^2}{h^2}$$

Hence equation (2) becomes,

$$\frac{d^2\psi}{dx^2} + \frac{4\pi^2 m^2 v^2}{h^2}\psi = 0 \quad \rightarrow (3)$$

The total energy E of the particle is the sum of kinetic energy T and potential energy V,

$$\therefore E = T + V$$

$$\text{But} \quad T = \frac{1}{2}mv^2 \quad \therefore \frac{1}{2}mv^2 = mv^2 \quad \text{or} \quad mv^2 = 2(\text{E} - \text{V})$$

Substitute this value of mv^2 in equation (3) we get

$$\frac{d^2\psi}{dx^2}+\frac{8\pi^2 m}{h^2}(E-V)\psi=0 \quad \rightarrow (5)$$

This is known as time independent 1 - dimensional Schrödinger equation.

Equation (5) can also be extended for 3-dimensional space as,

$$\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}+\frac{d^2\psi}{dz^2}+\frac{8\pi^2 m}{h^2}(E-V)\psi=0 \rightarrow (6)$$

$$or \quad \nabla^2\psi+\frac{8\pi^2 m}{h^2}(E-V)\psi=0 \quad \rightarrow (7)$$

$$where \quad \nabla^2=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$$

Equation (6) and (7) are the 3-dimensional time independent Schrödinger wave equation, where Ψ is Ψ(x, y, z).

EIGEN FUNCTIONS AND EIGEN VALUES

"Eigen functions are those wave functions of quantum mechanics which possess the properties that they are single valued, finite everywhere & also their first derivatives with respect to their variables are continuous everywhere".

When the Eigen functions are operated by quantum mechanical operators on physical quantities like momentum, energy etc of a system. The possible values are observed & these values are called Eigen values".

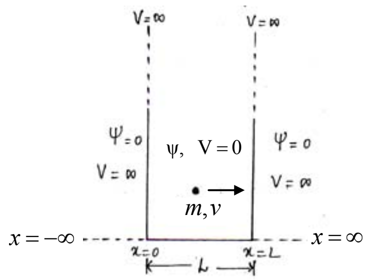
Ex: 1. If an operator say *d/dx* operates on a wave function $\Psi = e^{ax}$, then

$$\frac{de^{ax}}{dx}=ae^{ax}=a\Psi$$

That is it produces the wave function multiplied by a constant. Such values obtained for a physical observable are called Eigen values.

Here 'a' is the Eigen value & $\Psi = e^{ax}$ is the Eigen function.

Applications of Schrödinger wave equation to particle trapped in a one dimensional square potential well
(Derivation of energy Eigen values and Eigen function)



Consider a particle of mass 'm' moving with a speed 'v' along x-axis is confined to a box of length 'L' with perfectly rigid walls at x = 0 & x = L as shown in the figure.

The particle does not lose energy when it collides with the walls so that its total energy remains constant. The potential energy V of the particle is constant within the box which can be taken to be zero for convenience.

$$\therefore V = 0 \quad for \quad 0 < x < L \quad \rightarrow (1)$$

The potential energy of the particle is infinite on and beyond the walls of the box.

$$\therefore V = \infty \quad for \quad x \leq 0 \quad and \quad x \geq L \rightarrow (2)$$

As the particle does not exist on the walls and beyond them, the wave function Ψ is zero.

$$\therefore \psi = 0 \quad for \quad x \leq 0 \quad x \geq L \rightarrow (3)$$

The wave function Ψ exists within the box only.

∴ The Schrödinger's time independent wave equation is,

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0 \quad \rightarrow (4)$$

For the particle exists inside the box, V = 0

∴ Equation (4) becomes

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}E\psi = 0 \quad \rightarrow (5)$$

$$\text{let} \quad \frac{8\pi^2 mE}{h^2} = k^2 \rightarrow (6)$$

∴Equation (5) becomes

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \quad \rightarrow (7)$$

This is the second order differential equation. The general solution of this equation is given by,

$$\psi = A\sin kx + B\cos kx \quad \rightarrow (8)$$

where, A & B are arbitrary constants, which are to be evaluated by using boundary conditions.

From the first boundary conditions, $\Psi = 0$ at $x = 0$, $\therefore$ Equation (8) becomes,

$$0 = A\sin 0 + B\cos 0$$

$$\text{Since, } \sin 0 = 0 \ \ \& \ \ \cos 0 = 1, \ \text{ we have } B = 0$$

$\therefore$ Equation (8) becomes,

$$\psi = A\sin kx \quad \rightarrow (9)$$

From second boundary conditions, $\Psi = 0$ at $x = L$, $\therefore$ Equation (8) becomes,

$$0 = A\sin kL + B\cos kL$$

Since B = 0, the above equation becomes,

$$A\sin kL = 0$$

$$\because A \neq 0, \ \sin kl = 0 \quad \text{for all values of} \quad kL = n\pi \ \text{ where } n = 1, 2, 3, \ldots\ldots$$

$$\therefore k = \frac{n\pi}{L} \quad \rightarrow (10)$$

By substituting the value of k in equation (9) and in equation (6) we get general wave function called Eigen wave function and Eigen energy equation.

$$\psi_n = A\sin\left(\frac{n\pi}{L}\right)x \rightarrow (11)$$

This is known as Eigen function or Eigen wave function.

From equation (6) we get,

$$\frac{8\pi^2 mE}{h^2} = \frac{n^2\pi^2}{L^2} \quad \text{or} \quad E = \frac{n^2h^2}{8mL^2}$$

$$\text{In general} \quad E_n = \frac{n^2h^2}{8mL^2} \quad \rightarrow (12)$$

This is the expression for Eigen values or Eigen energy values.

Thus, we see that in a potential well the particle cannot have an arbitrary energy, but it can have only discrete energy values corresponding to n = 1, 2, 3 ... are the Eigen values.

According to equation (11) if n = 0, $\Psi_n = 0$, which means that the particle doesn't present inside the box, which is not true. $\therefore$ The value of $E_n = 0$ for n = 0 is not acceptable. Hence the lowest allowed energy corresponding to n = 1 is called the 'zero-point energy or ground state energy'. Thus zero-point or ground state of energy of the particle in an infinite potential well is given by,

$$E_1 = \frac{h^2}{8mL^2}$$

The energy states corresponding to n >1 are called excited states.

Normalisation:

To evaluate A in Eigen function Ψ_n, one has to perform the normalization of the wave function.

The allowed solutions of the Schrödinger equation are the Eigen functions, according to the equation.

$$\psi_n = A\sin\left(\frac{n\pi}{L}\right)x \quad \rightarrow (13)$$

The complex conjugate of equation (13) is,

$$\psi_n^* = A\sin\left(\frac{n\pi}{L}\right)x \quad \rightarrow (14)$$

To find the value of A, we use the normalization condition.

$$i.e., \quad \int_{-\infty}^{\infty} \psi_n^* \psi_n \, dx = \int_{-\infty}^{\infty} |\psi|^2 \, dx = 1$$

In this case, the particle exists only within the box of length (L). $\therefore$ The above equation can be written as,

$$\int_0^L |\psi_n|^2 \, dx = 1$$

By substituting the values of Ψ_n and Ψ_n^* in the above equation, we get

$$\int_0^L A^2 \sin^2 \frac{n\pi}{L} x \, dx = 1$$

$$A^2 \int_0^L \frac{1}{2}\left(1 - \cos\frac{2n\pi}{L}\right)x \, dx = 1 \qquad \left[\because \sin^2 A = \frac{1-\cos 2A}{2}\right]$$

$$\frac{A^2}{2}\left[\int_0^L dx - \int_0^L \cos\frac{2n\pi}{L} x dx\right] = 1$$

$$\frac{A^2}{2}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi}{L}\right)x\right]_0^L = 1$$

$$\frac{A^2}{2}\left[L - \frac{L}{2n\pi}\sin(2n\pi) - 0 + 0\right] = 1 \qquad \text{Here, for any value of n, } \sin 2n\pi = 0$$

$$\therefore \frac{A^2 L}{2} = 1 \quad \text{or} \quad A^2 = \frac{2}{L} \quad \text{or} \quad A = \sqrt{\frac{2}{L}}$$

Thus, by substituting the value of A in equation (13) we get normalized wave functions or Eigen function of a particle in one dimensional infinite potential well.

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}\right)x \rightarrow (15)$$

The first three eigen functions Ψ_1, Ψ_2, Ψ_3 together with the probability densities $|\psi_1|^2$, $|\psi_2|^2$, $|\psi_3|^2$ and eigen values E_1, E_2, E_3 are as shown in figure (a), (b) & (c) respectively for n = 1, 2 & 3.

For n = 1, this is the ground state and the particle is normally found in this state.

$$\text{Eigen function} \quad \psi_1 = A \sin\left(\frac{\pi}{L}\right)x$$

$\Psi_1 = 0$ for both $x = 0$ and $x = L$ and Ψ_1 has maximum value A for $x = L/2$

At x = 0 and $x = L$, $|\Psi_1|^2 = 0$ it means that the particle does not exist at the walls.

$|\Psi_1|^2$ is maximum at $x = L/2$, it means that the particle exist at the centre of the well.

$$E_1 = \frac{h^2}{8mL^2}$$

This is the energy eigen energy value for ground sate.

For first excited state, n = 2

$$\therefore \quad \psi_2 = A\sin\left(\frac{2\pi}{L}\right)x$$

$\Psi_2 = 0$ for $x = 0$, L/2 and L and Ψ_2 reaches maximum value for $x = L/4$ and 3L/4.

At $x = 0$, L/2 and L, $|\Psi_2|^2 = 0$ it means that the particle does not exist at 0, L/2 and L.

$|\Psi_2|^2$ is maximum at x = L/4 and 3L/4

Energy Eigen values can be calculated by using equation,

$$E_2 = \frac{4h^2}{8mL^2} \quad \text{or } E_2 = 4E_1$$

This is the equation to calculate the energy of the particle in first excited state.

For second excited state, n = 3

$$\psi_3 = A\sin\left(\frac{3\pi}{L}\right)x$$

$\Psi_3 = 0$ for x = 0, L/3, 2L/3 and L, and Ψ_3 reaches maximum value for x = L/6, L/2 and 5L/6.

At x = 0, L/3, 2L/3 and L, $|\Psi_3|^2 = 0$ it means that the particle does not exist at 0, L/3, 2L/3 and L.

$|\Psi_3|^2$ is maximum at x = L/6, L/2 and 5L/6.

Energy Eigen values can be calculated by using equation,

$$E_3 = \frac{9h^2}{8mL^2} \quad \text{or} \quad E_3 = 9E_1$$

This is the equation to calculate the energy of the particle in second excited state.

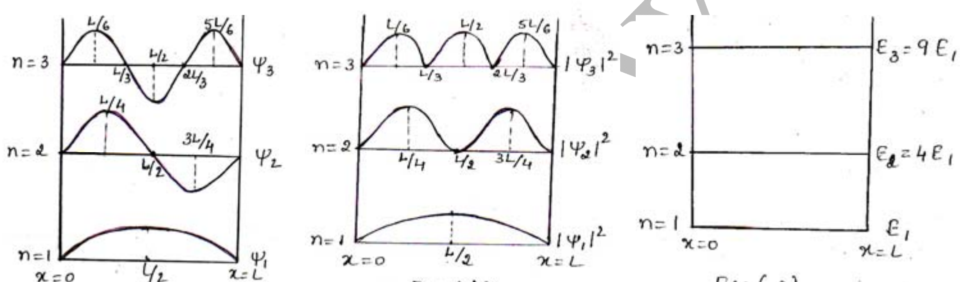


Fig (a)
Wave Functions

Fig (b)
Probability Functions

Fig (c)
Energy Levels.

Free particle

Energy Eigen values for a free particle:

Free particle means, it is not under the influence of any kind of field or force. Thus it has zero potential, i.e., V = 0.

Hence Schrödinger's equation becomes,

$$\frac{d^2\psi}{dx^2} = -\frac{8\pi^2 m}{h^2}(E-V)\psi \; = 0$$

Since V = 0, $$\frac{d^2\psi}{dx^2} = -\frac{8\pi^2 m}{h^2}E\psi \; = 0$$

The above equation holds good for a particle for which the potential v=0 over the entire space (No boundaries at all).

We know that in the case of particle in an infinite potential well, the condition V=0 holds good only over a infinite width 'L' and outside region, V= ∞ ,

Since for the free particle, V = 0 holds good everywhere, we can extend the case of particle in an infinite potential well to the free particle's case, by treating the width of the well to be infinity, i.e., by allowing L = ∞,

We have the equation for energy Eigen values for a particle in an infinite potential well as,

$$E = \frac{n^2 h^2}{8mL^2}$$

Where, n = 1, 2, 3, ...

Rearranging the above equation, we have,

$$n = \frac{2L}{h}\sqrt{2Em}$$

Here, we see that for a particle with constant energy E but confined in the well, n depends solely on 'L'. Hence as L→ ∞, n→∞. In the particle is no more confined in any sort of well but free, at which time it also follows that n = ∞ which essentially means that a free particle can have any energy i.e., the energy Eigen values or the possible values of energy are infinite in number. Keeping in the mind the energy level representation, we say that the permitted energy values are continues. All these mean, there is no discreteness in the allowed energy values. In other word, there is no quantization of energy in case of a free particle and the problem is dealt in classical mechanics. Thus a free particle is a classical entity.

Problems

1. If the uncertainty in the position of an electron is 4×10^{-10} m, calculate the uncertainty in its momentum.

Given, $\Delta x = 4 \times 10^{-10} m, \quad \Delta P_x = ?$

$$\text{w.}k.t \quad \Delta x . \Delta p_x \geq \frac{h}{4\pi}$$

$$\therefore \Delta p_x \geq \frac{h}{4\pi\, \Delta x} \geq \frac{6.63 \times 10^{-34}}{4 \times 3.14 \times 4 \times 10^{-10}} \geq 1.318 \times 10^{-25} kgs^{-1}$$

2. In a simultaneous measurement of position and velocity of an electron moving with a speed of 6×10^5 m/s. Calculate the highest accuracy with which its position

could be determined if the inherent error in the measurement of the velocity is 0.01% for the speed stated.

$Given,\ v=6\times10^{5} m/s,\ \%\ of\ error\ in\ velocity = 0.01\%,\ \ \Delta x = ?$

$$percentage\ \ of\ uncerta\ \text{int}\ y\ \ in\ \ velocity\ = \frac{uncerta\ \text{int}\ y\ in\ velocity}{velocity}\times 100$$

$$uncertaini\ ty\ \ in\ \ velocity\ = \frac{percentage\ \ of\ uncertaini\ ty\ \ in\ \ velocity}{100}\times velocity$$

$$\Delta v = \frac{\%\ of\ error\ in\ the\ measurment\ \ of\ velocity}{100}\times v$$

$$\Delta v = \frac{0.01}{100}\times 6\times 10^{5} = 60\, ms^{-1}$$

$$\text{w}.k.t;\quad \Delta x\,.\Delta p_x \geq \frac{h}{4\pi}$$

$$\Delta x \geq \frac{h}{4\pi.m.\Delta v_x} \geq \frac{6.63\times 10^{-34}}{4\times 3.142\times 9.1\times 10^{-31}\times 60} \geq 9.66\times 10^{-7}\, m$$

3. An electron has a speed of 300m/s accurate to 0.01% with what fundamental accuracy can we locate the position of the electron.

Given, V = 300 m/s, $\Delta x = ?$, $\Delta v = ?$ and % of accuracy in speed = 0.01

$$\Delta v = 0.01\%\ of\ v \quad \text{or} \quad \Delta v = \frac{0.01}{100}\times v = \frac{0.01}{100}\times 300 = 0.03\ m/s$$

$$\Delta v = 3\times 10^{-2}\, m/s$$

w.k.t

$$\Delta x.\Delta p_x \geq \frac{h}{4\pi} \quad \text{or} \quad \Delta x \geq \frac{h}{4\pi\,\Delta p_x} = \frac{h}{4\pi.m.\Delta v}$$

$$\therefore \Delta x \geq \frac{6.63\times10^{-34}}{4\times3.142\times9.1\times10^{-31}\times3\times10^{-2}} = 1.932\times10^{-3}\, m$$

$\therefore$ The maximum accuracy with which the electron can be located is 1.932 x 10^{-3} m.

4. The speed of electron is measured to within an uncertainty of 1 x 10^{4} m/s. What is the minimum space required by the electron to be confined in an atom.

Given, $\Delta v = 1\times 10^{4}\, m/s,\ \Delta x = ?$

w.k.t; $\Delta x.\Delta p_x \geq \frac{h}{4\pi}$

$$\therefore \; \Delta x \geq \frac{h}{4\pi\,\Delta p_x} = \frac{h}{4\pi.m.\Delta v} \geq \frac{6.63\times10^{-34}}{4\times3.142\times9.1\times10^{-31}\times1\times10^{4}} = 5.797\times10^{-9}\,m$$

$$\Delta x \geq 57.97\times10^{-10}\,m = 57.97\,\overset{0}{A}$$

5. The position and momentum of 1keV electron are simultaneously determined and if its position is located within 1Å. What is the minimum percentage of uncertainty in its momentum?

Given, E = 1keV, E = $1\text{x}10^{3}$ eV , E = $1\text{x}10^{3}\text{x}1.602\text{x}10^{-19}$ J, E = $1.602\text{x}\,10^{-16}$ J

$\Delta x = 1\overset{0}{A} = 1\times10^{-10}\,m$ $\quad \Delta P = ?$, $\frac{\Delta P}{P}\times100 = ?$

$$w.k.t; \quad \Delta x.\Delta p_x \geq \frac{h}{4\pi}$$

$$\therefore \quad \Delta P_x \geq \frac{h}{4\pi\,\Delta x} \geq \frac{6.63\times10^{-34}}{4\times3.142\times10^{-10}} = 5.275\times10^{-25}\ \text{k}gms^{-1}$$

we have the equation for momentum,

$$P = \sqrt{2mE} = \sqrt{2\times9.1\times10^{-31}\times1.602\times10^{-16}}$$

$$P = 1.707\times10^{-23}\,kg\ ms^{-1}$$

$$Percentage\ of\ uncertainity\ in\ momentum = \frac{Uncertainity\ in\ momentum}{momentum}\times100$$

$$Percentage\ of\ uncertainity\ in\ momentum = \frac{5.275\times10^{-25}}{1.707\times10^{-23}}\times100 = 3.087 = 3.1$$

6. The inherent uncertainty in the measurement of time spent by Iridium -191 nuclei in the excited state is found to be $1.4\text{x}10^{-10}$s. Estimate the uncertainty that results in its energy in the excited state.

$Given,\ \Delta t = 1.4\times10^{-10}\,s, \quad \Delta E = ?$

w.k.t; $\Delta E.\Delta t \geq \frac{h}{4\pi}$

$$\therefore \quad \Delta E \geq \frac{h}{4\pi.\Delta t} \geq \frac{6.63\times10^{-34}}{4\times3.142\times1.4\times10^{-10}} = 3.768\times10^{-25}\,J$$

$$\Delta E = \frac{3.768\times10^{-25}}{1.602\times10^{-19}}\,eV = 2.353\times10^{-6}\,eV$$

7. The average time that an atom retains excess excitation energy before re-emitting it in the form of electromagnetic radiation is 10^{-8}s. Calculate the limit of

accuracy with which the excitation energy of the emitted radiation can be determined.

(OR)

What is the minimum uncertainty in the energy state of an atom if an electron remains in this state for 10^{-8} seconds?

$Given, \quad \Delta t = 1.4\times10^{-10}\,s, \quad \Delta E = ?$

$$\text{w.k.t}; \quad \Delta E.\Delta t \geq \frac{h}{4\pi}$$

$$\therefore \quad \Delta E \geq \frac{h}{4\pi.\Delta t} \geq \frac{6.63\times10^{-34}}{4\times3.142\times10^{-8}} = 5.275\times10^{-27}J$$

$$\Delta E = \frac{5.275\times10^{-27}}{1.602\times10^{-19}}eV = 3.292\times10^{-8}eV$$

8. An electron is confined to a box of length 10^{-8}m. Calculate the minimum uncertainty in its velocity.

$Given, \quad \Delta v = ? \quad \Delta x = 10^{-8}m$

$$w.k.t; \quad \Delta x.\Delta p_x \geq \frac{h}{4\pi}$$

$$or \quad \Delta P_x \geq \frac{h}{4\pi\,\Delta x} \quad \text{or} \quad m.\Delta v_x = \frac{h}{4\pi.\Delta x}$$

$$\therefore \quad \Delta v_x = \frac{h}{4\pi.m.\Delta x} \geq \frac{6.63\times10^{-34}}{4\times3.142\times10^{-8}\times9.1\times10^{-31}} = 5797.03 \;\; m/s$$

$$\Delta v = 5800 \;\; m/s$$

9. Calculate the zero point energy for an electron in a box of width 10Å.

$Given, \; \text{L} = 10\,\overset{0}{\text{A}} = 10\text{x}10^{-10}\,\text{m}, \; \text{E}_1 \text{ or } \text{E}_0 = ?$

$$w.k.t; \quad E_n = \frac{n^2h^2}{8mL^2}$$

for ground state n = 1

$$E_1 = \frac{h^2}{8mL^2} = \frac{\left(6.63\times10^{-34}\right)^2}{8\times9.1\times10^{-31}\times\left(10\times10^{-10}\right)^2} = 6.038\times10^{-20}J$$

$$E_1 = \frac{6.038\times10^{-20}}{1.602\times10^{-19}}eV = 0.376\,eV$$

10. An electron is bound in a one dimensional potential well of width 1Å, but of infinite height. Find the energy value for the electron in the ground state.

$$Given,\ L = 1\overset{0}{A} = 1x10^{-10}\,m, \quad E_1 \text{ or } E_0 = ?$$

$$w.k.t; \quad E_n = \frac{n^2h^2}{8mL^2}$$

for ground state n = 1

$$E_1 = \frac{h^2}{8mL^2} = \frac{\left(6.63\times10^{-34}\right)^2}{8\times9.1\times10^{-31}\times\left(1\times10^{-10}\right)^2} = 6.038\times10^{-18}\,J$$

$$E_1 = \frac{6.038\times10^{-18}}{1.602\times10^{-19}}\,eV = 37.69\,eV$$

11. An electron is trapped in a one-dimensional box of length 0.1 nm. Calculate the energy required to excite the electron from its ground state to the 2nd excited state.

$$Given,\ L = 0.1\,nm = 0.1x10^{-9}\,m,\ E_1 \text{ or } E_0 = ?$$

$$w.k.t; \quad E_n = \frac{n^2h^2}{8mL^2}$$

for ground state n = 1

$$E_1 = \frac{h^2}{8mL^2} = \frac{\left(6.63\times10^{-34}\right)^2}{8\times9.1\times10^{-31}\times\left(0.1\times10^{-9}\right)^2} = 6.038\times10^{-18}\,J$$

$$E_1 = \frac{6.038\times10^{-18}}{1.602\times10^{-19}}\,eV = 37.691\,eV$$

for 2nd excited state n = 3

$$E_3 = \frac{9h^2}{8mL^2} = \frac{9\times\left(6.63\times10^{-34}\right)^2}{8\times9.1\times10^{-31}\times\left(0.1\times10^{-9}\right)^2} = 339.219\,eV$$

$$E_3 = 9\times E_1 = 9\times37.691 = 339.219\,eV$$

The energy required to excite the electron from its ground state to the 2nd excited state is,

$$\therefore E = E_3 - E_1 = \left(339.219 - 37.691\right)eV$$

$$E = 301.528\,eV$$

12. Calculate the lowest energy of the system consisting of three electrons in a one - dimensional potential box of length 1Å.

$$Given, \quad L = 1\overset{0}{A} = 1x10^{-10}\,m, \; E_1 \text{ or } E_0 = ?$$

$$w.k.t; \quad E_n = \frac{n^2h^2}{8mL^2}$$

for lowest energy n = 1

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.63\times10^{-34})^2}{8\times9.1\times10^{-31}\times(1\times10^{-10})^2} = 1.8114\times10^{-17}\,J$$

$$E_1 = \frac{1.8114\times10^{-17}}{1.602\times10^{-19}}\,eV = 113.07\;eV$$

13. An electron is constrained to a one-dimensional box of side 1nm. Calculate the first 3-eigen values in electron volt.

$$Given, \quad L = 1\,nm = 1x10^{-9}\,m, \; E_1 \text{ or } E_0 = ?, \; E_2 = ?, \; E_3 = ?$$

$$w.k.t; \quad E_n = \frac{n^2h^2}{8mL^2}, \quad \text{for ground state } n = 1$$

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.63\times10^{-34})^2}{8\times9.1\times10^{-31}\times(1\times10^{-9})^2} = 6.038\times10^{-20}\,J$$

$$E_1 = \frac{6.038\times10^{-20}}{1.602\times10^{-19}}\,eV = 0.376\;eV$$

For 2nd & 3rd excited state n = 2 & n = 3

$$E_2 = 4\times E_1 = 4\times 0.376 = 1.504\,eV$$

$$E_3 = 9\times E_1 = 9\times 0.376 = 3.384\,eV$$

14. An electron is trapped in one-dimensional infinite potential box of width 0.1nm. Calculate its wavelengths and energies corresponding to first two excited states.

$$Given, \quad L = 0.1\,nm = 0.1x10^{-9}\,m, \; E_2 = ?, \; E_3 = ?, \; \lambda_2 = ?, \; \lambda_3 = ?$$

$$w.k.t; \quad E_n = \frac{n^2 h^2}{8mL^2}$$

for first excited state n = 2

$$E_2 = \frac{4h^2}{8mL^2} = \frac{4\left(6.63\times10^{-34}\right)^2}{8\times9.1\times10^{-31}\times\left(0.1\times10^{-9}\right)^2} = 2.415\times10^{-17}\,J$$

$$E_2 = \frac{2.415\times10^{-17}}{1.602\times10^{-19}}eV = 150.76\,eV$$

for second excited state n = 3

$$E_3 = \frac{9h^2}{8mL^2} = \frac{9\left(6.63\times10^{-34}\right)^2}{8\times9.1\times10^{-31}\times\left(0.1\times10^{-9}\right)^2} = 5.434\times10^{-17}\,J$$

$$E_3 = \frac{5.4354\times10^{-17}}{1.602\times10^{-19}}eV = 339.215\,eV$$

w.k.t

$$\lambda_n = \frac{2L}{n}$$

For first excited state n = 2

$$\lambda_2 = \frac{2L}{2}$$

$$\lambda_2 = L = 0.1\times10^{-9}\,m = 0.1\,nm$$

For second excited state n = 3

$$\lambda_3 = \frac{2L}{3} = \frac{2\times0.1\times10^{-9}}{3}$$

$$\lambda_3 = 6.666\times10^{-11}\,m$$

$$\lambda_3 = 0.066\times10^{-9}\,m$$

$$\lambda_3 = 0.066\,nm$$

<u>Question paper questions</u>

1. Assuming the time independent Schrödinger's wave equation, discuss the solution for a particle in one dimensional potential well of infinite height,

2. What is a wave function? Explain its physical significance.

3. Solve Schrödinger wave equation for allowed energy values in case of a particle in a potential box.

4. State and explain Heisenberg uncertainty principle illustrate it using gamma ray microscope.

5. State and explain Heisenberg uncertainty principle. Show that the electron doesn't exist inside the nucleus.

6. Explain the significance of wave function. Set up time independent Schrödinger's wave equation.

Problems

1. An electron as a speed of 5.2×10^5 m/s accurate to 0.01%, with what accuracy the position of electron can be located?

2. An electron is confined to a one dimensional box of width 1 nm. Calculate the first three Eigen values in eV.

3. An electron is bound in a one dimensional box of width 4×10^{-10} m. compute the energy and de-Broglie wavelength of ground and first excited states.

4. Find the energy of an electron in the ground state, when it is trapped in an infinite potential well of width 2 $\mathring{A}$.

5. In a measurement that involved a maximum uncertainty of 0.003 %, the speed of an electron was found to be 800 m/s. calculate the corresponding uncertainty involved in determining its position.

6. An electron is bound in one dimensional potential well of width 1Å but of infinite height. Find its energy values in ground state and first two exited states.