COMP0147_18-19

Sheet 4

~~**Due Monday 20 February.**~~
~~Hand in solutions to questions 2, 4, 5, 6~~

$\{1,2\} \cup \{3,4\} = \{1,2,3,4\}$.
Eq. class of $\{1,2\}$ $= [\{1,2\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}]$

1. Let $X = \{1,2,3,4,5\}$ and $Y = \{3,4\}$. Define a relation $\mathcal{R}$ on the power set $P(X)$ of X by

$$A \mathcal{R} B \text{ iff } A \cup Y = B \cup Y.$$

Reflexivity: $A\mathcal{R}A \Rightarrow$ since $A \cup Y = A \cup Y$.
Symmetry: if $A\cup Y = B \cup Y$ then $B \cup Y = A \cup Y$; $A\mathcal{R}B \Rightarrow B\mathcal{R}A$.
Transitivity: $A\mathcal{R}B \Rightarrow A\cup Y = B\cup Y$, $B\mathcal{R}C \Rightarrow B \cup Y = C \cup Y$ $\Rightarrow A\cup Y = C \cup Y$, $A\mathcal{R}C$

   (a) Prove that R is an equivalence relation.
   (b) What is the equivalence class of $\{1,2\}$?

2. Let X be $\mathbb{Z}\times\mathbb{Z}$, i.e. X is the set of all ordered pairs of the form (x,y) with $x,y \in \mathbb{Z}$. Define the relation R on X as follows:

$$(x_1,x_2)R(y_1,y_2) \quad \text{iff} \quad x_1^2 + x_2^2 = y_1^2 + y_2^2.$$

Is it an equivalence relation?

Yes.

Reflexivity: $(x_1,x_2)R(x_1,x_2) \Rightarrow x_1^2+x_2^2 = x_1^2+x_2^2$
Symmetry: $(x_1,x_2)R(y_1,y_2) \Rightarrow x_1^2+x_2^2 = y_1^2+y_2^2$; $(y_1,y_2)R(x_1,x_2) \Rightarrow y_1^2+y_2^2 = x_1^2+x_2^2$
Transitivity: $(x_1,x_2)R(y_1,y_2)$, $x_1^2+x_2^2 = y_1^2+y_2^2$; $(y_1,y_2)R(z_1,z_2)$, $y_1^2+y_2^2 = z_1^2+z_2^2$; $x_1^2+x_2^2 = z_1^2+z_2^2$; $(x_1,x_2)R(z_1,z_2)$

3. Let $X = \mathbb{R}\times\mathbb{R}$. Define the relation R on X as follows:

$$(x_1,y_1)R(x_2,y_2) \quad \text{iff} \quad y_1 - y_2 = 2(x_1 - x_2).$$

   (a) Is it an equivalence relation?
   (b) If it is, what is the equivalence class of the point $(3,1)$?

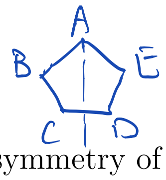
Yes.

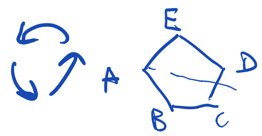
Reflexivity: $(x_1,y_1)R(x_1,y_1)$ $y_1 - y_1 = 2(x_1 - x_1) \Rightarrow 0=0$ ✓.
Symmetry: $(x_1,y_1)R(x_2,y_2)$ $y_1-y_2 = 2(x_1-x_2) \Rightarrow -(y_1-y_2) = -2(x_1-x_2)$ $\Rightarrow (y_2-y_1) = 2(x_2-x_1) \Rightarrow (x_2,y_2) \cup (x_1,y_1)$

Transitivity: $(x_1,y_1)R(x_2,y_2)$ $\Rightarrow y_1-y_2 = 2(x_1-x_2)$; $(x_2,y_2) \cup (x_3,y_3)$ $\Rightarrow y_2-y_3 = 2(x_2-x_3)$; + ; $y_1 - y_3 = 2(x_1-x_3)$ $\Downarrow$ $(x_1,y_1)R(x_3,y_3)$

$(3,1)$ $1-y_2 = 2(3-x_2) = 6-2x_2$ $2x_2 = 5+y_2$ $2x_2-5 = y_2$ $(x, 2x-5)$ $x \in \mathbb{R}$.

4. Let P be a regular pentagon. Let R be the rotation of P by $\frac{2\pi}{5}$ anticlockwise and let F be the reflection of P in the vertical line of symmetry. Represent R and F by permutations and hence calculate





$$FR^2FRF^3R^3F,$$

C D E B E B

expressing this first as a permutation and then as a symmetry of P.

R of P

5. Let G be a group and let H_1 and H_2 be subgroups of G. Show that $H_1 \cap H_2$ is a subgroup of G.

$H_1 \subseteq G$
$H_2 \subseteq G$
$H_1 \cap H_2 \subseteq G$.
$\forall h_1, h_2 \in H_1 \cap H_2$.
$h_1, h_2 \in H_1$
$h_1, h_2 \in H_2$.

$h_1 h_2 \in H_1$
$h_1 h_2 \in H_2$
$h_1 h_2 \in H_1 \cap H_2$. Closure ✓
Inheriet. Associativity ✓
$\forall \varepsilon$ is also included since $\varepsilon \in H_1$, $\varepsilon \in H_2$

$(h_1)^{-1} \in H_1$
$(h_1)^{-1} \in H_2$
$(h_1)^{-1} \in H_1 \cap H_2$
Invertibility ✓.

6. Let $G = \{1_G, g, h, k\}$ be a group with 4 elements and suppose G is not cyclic. Using Lagrange's Theorem show that g, h and k all have order 2 and write down a table for the group operation.

7. Let G be a finite group with no subgroups apart from $\{1_G\}$ and G.

   (a) Show that G is cyclic. 1_G is $\varepsilon \Leftarrow$ neutral element

   (b) Show that the number of elements in G is either 1 or a prime number.

Using Lagrange's Theorem, we know that, $|H|$ divides $|G|$.

However, only subgroup $H = \{1_G\}$

$|H| = 1$.

$|G|$ is a prime number. / 1.