1. (a)

$x \in (A\cup B)\setminus(B\setminus A)$

$x \in A\cup B \vee x \notin B\setminus A$

$(x\in A \wedge x\in B) \vee x \notin B$

$\therefore x \in A$

$(A\cup B)\setminus(B\setminus A) \supseteq A$

$y \in A$ $\quad y \notin B$

$y \in A\cup B$ $\quad y \notin B\setminus A$

$\therefore y \in (A\cup B)\setminus(B\setminus A)$

$\therefore A \supseteq (A\cup B)\setminus(B\setminus A)$

$\therefore A = (A\cup B)\setminus(B\setminus A)$

(b) $n \in \mathbb{N}$

$n\to\infty \quad \frac{1}{n}\to 0$

$\bigcap_{n\in\mathbb{N}} B_n = B_1\cap B_2\cap B_3 \cdots \cap B_n$

$B_1 = (0,1] \quad B_2 \subseteq B_1$

$B_2 = (0,\frac{1}{2}] \quad B_1\cap B_2 = B_2$

$B_n = (0,\frac{1}{n}]$ Assume $\cdots B_{n-1}\cap B_n = B_n$

$B_{n+1} = (0,\frac{1}{n+1}] \quad \frac{1}{n+1} < \frac{1}{n}$

$\therefore B_n \cap B_{n+1} = B_{n+1}$

$\therefore B_1 \cdots \cap B_{n+1} = B_{n+1}$

$\because \frac{1}{n}\to 0. \quad \therefore (0,0) = \phi.$

$\therefore \bigcap_{n\in\mathbb{N}} B_n = \phi.$

(c). $n\in\mathbb{N}$

$n=1. \quad T_1 = [0,1+1]\times(-1,1+1]$

$n=2 \quad T_2 = [0,1+\frac{1}{2}]\times(-\frac{1}{2},1+\frac{1}{2}].$

$[0,2] \supseteq [0,\frac{3}{2}]$

$(-1,2] \supseteq (-\frac{1}{2},\frac{3}{2}].$

$\therefore T_1\cap T_2 = T_2$

Assume $T_1\cap T_2\cap\cdots\cap T_n = T_n.$

$T_{n+1} = [0,1+\frac{1}{n+1}]\times[-\frac{1}{n+1},1+\frac{1}{n+1}].$

$\because \frac{1}{n} > \frac{1}{n+1}$ smaller bigger $(n\in\mathbb{N}).$

$-\frac{1}{n} < -\frac{1}{n+1}$

$\underline{1+\frac{1}{n} > 1+\frac{1}{n+1}} \quad \underline{-\frac{1}{n} < -\frac{1}{n+1}}$

$\therefore T_n\cap T_{n+1} = T_{n+1}.$

$\therefore T_1\cap T_2 \cdots \cap T_2\cap T_{n+1} = T_{n+1}$

$\therefore n\to\infty \quad \frac{1}{n}\to 0^+ \quad -\frac{1}{n}\to 0^-$ $-\frac{1}{n}$ never reaches 0.

$\therefore \bigcap_{n\in\mathbb{N}} T_n = T_\infty = [0,1]\times[0,1]$

2. injective: $f(x)=y$ has at most one solution. $\forall y$ in Y

every element in codomain Y has at most one correspondence in domain X

Surjective $f(x)=y$ has at least one solution $\forall y$ in Y.

— — — — has at least one — — — —

bijective $f(x)=y$ has one solution $\forall y$ in Y

— — one to one unique — —

(b) (i) $f(x) = \begin{cases} \sqrt{x}, & x>0 \\ 0, & x\le 0 \end{cases}$ left $f(x)$ is not injective. $f(x)=0$ $x=0,-1,-2\ldots$ No.

$f: \mathbb{R} \to \mathbb{R}$ right $f(x)$ is not surjective $f(x)=-1$ no solution. No

two sided $f(x)$ is not bijective. No.

(ii) left inverse no. $f(x)$ not injective $f(x)=0$ $x=0, 1 \cdots$

right inverse yes. $f(x)$ is surjective.

$$f^{-1}(x) = x^2 \quad (x \geq 0).$$

two sided no. $f(x)$ not bijective

3. (a)

(b) $\begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 3 & 5 \\ 4 & 7 \\ 5 & 2 \\ 6 & 6 \\ 7 & 1 \\ 8 & 9 \\ 9 & 8 \end{pmatrix}$

(i) $\sigma = (1\ 4\ 7)(2\ 3\ 5)(8\ 9)$

(ii) $\sigma = (1\ 4)(4\ 7)(2\ 3)(3\ 5)(8\ 9)$

(iii) $\operatorname{order}(\sigma) = 6$

(iv) $\sigma^{173} = \sigma^{6 \times 28 + 5} = \sigma^5 = \sigma^{-1}$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 5 & 2 & 1 & 3 & 6 & 4 & 9 & 8 \end{pmatrix}$$

(v) $\operatorname{sgn}(\sigma) = 1.$

4.

(a)

p_i	q_i	s_i	t_i
355		1	0
79	4	0	1
39	2	1	-4
1	39	-2	9
0			

$\gcd(355, 79) = 1$ $\quad 1 = 355 \times (-2) + 79 \times 9$

(b) $\cancel{*}$ $\quad a * b = (a \times b) \bmod n.$

(c) $\mathbb{Z}_9^* = \{x \mid x \equiv 1, 2, 4, 5, 7, 8 \pmod 9\}.$

(d)
$$\begin{array}{r|l} 2 & 504 \\ 2 & 252 \\ 2 & 126 \\ 7 & 63 \\ & 9 \end{array}$$

$504 = 2^3 \times 7 \times 9.$

$504 \times \frac{1}{2} \times \frac{6}{7} \times \frac{8}{9}$

$= 252 \times \frac{6}{7} \times \frac{8}{9} = 252 \times \frac{48}{63} = 4 \times 48$

$= 192$

(e) $3^{285} \pmod{14}$

$\phi(14) = 14 \times \frac{1}{2} \times \frac{6}{7} = 6.$

$3^{6} = 1 \pmod{14}.$

$3^{285} = 3^{6\times 47+3} = 3^{3} \pmod{14} = 27 \pmod{14}$

$= \underbrace{13}.$

5.(a) $\gcd(a,p) = 1,\ p \to a|$

$1 = ar + pq \quad abr + pbq = b \quad p|ab \to p|b$

(b) $x^{13} \equiv 5 \pmod{24}$

$x = \sqrt[13]{5} \pmod{24} = 5^{13^{-1}} \pmod{24}$

y

$y \equiv 13^{-1} \pmod{\phi(24)} = 13^{-1} \pmod{8}$

$\phi(24) = 24 \times \frac{1}{2} \times \frac{2}{3} = 8$

$\phi(8) = 8 \times \frac{1}{2} = 4.$

$\therefore 13^{4} \equiv 1 \pmod{8}.$

$13 \equiv 5 \pmod{8}$

$13^{2} \equiv 1 \pmod{8}$

$13^{3} = 13^{-1} = 5 \pmod{8}$

$\therefore y \equiv 5 \pmod{8}$

$\therefore x \equiv 5^{5} \pmod{24} = 5 \pmod{24}.$

6(a) $\begin{cases} x+y-2z = a. \\ x+y+w = 2. \\ bx+y = 1. \\ y+2w = -1. \end{cases}$

$2x+2y+2w = 4$

$y + 2w = -1$

$2x + y = 5.$

$(b-2)x = -4. \quad x = -\frac{4}{b-2}. \quad (b=2)$ no solution.

$x+y = 5-x = 5+\frac{4}{b-2} = \frac{5b-10+4}{b-2} = \frac{5b-6}{b-2}$

$\frac{5b-6}{b-2} - 2z = a$

$z = \frac{5b-6-ab+2a}{2(b-2)}$

$= \frac{-ab+2a+5b-6}{2(b-2)}$

if $b=2$ no solution

$b \neq 2$ one solution. $x = \frac{4}{b-2}$. $y = \frac{5b-6+4}{b-2} = \frac{5b-2}{b-2}$

$w = \frac{2b-4}{b-2} - \frac{5b-6}{b-2} = \frac{-3b+2}{b-2}$

$z = \frac{-ab+2a+5b-6}{2(b-2)}$

(b). eigenvalue λ

eigenvector $v := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$\lambda v = Av$.

7. (a). $(A - \lambda E)v = 0 \qquad v \neq 0$

$\begin{pmatrix} 1-\lambda & 1 & -2 \\ 4 & -\lambda & 2 \\ 0 & 1 & -1-\lambda \end{pmatrix} = 0$

$(1-\lambda)\cdot(-\lambda)\cdot(-1-\lambda) + 4\cdot 1\cdot -2 +$

$-2\cdot(1-\lambda) - (-1-\lambda)\cdot 4$

$= -\lambda\cdot(\lambda^2-1) - 8 - 2 + 2\lambda + 4\lambda + 4$

$= -\lambda^3 + \lambda - 10 + 6\lambda + 4$

$= -\lambda^3 + 7\lambda - 6 = -(\lambda^3 - 7\lambda + 6)$

$= -(\lambda-1)(\lambda^2+\lambda-6)$

$= -(\lambda-1)(\lambda+3)(\lambda-2)$

$\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = -3.$

① $\lambda_1 = 1$. $\begin{pmatrix} 0 & 1 & -2 \\ 4 & -1 & 2 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$\begin{cases} y - 2z = 0 \\ 4x - y + 2z = 0 \end{cases}$

let $y = 2 \quad z = 1$

$4x = y - 2z = 0 \quad x = 0$

② $\lambda_2 = 2$.

$\begin{pmatrix} -1 & 1 & -2 \\ 4 & -2 & 2 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$\begin{cases} -x + y - 2z = 0 \\ 4x - 2y + 2z = 0 \\ y - 3z = 0 \end{cases}$

$-4x + 4y - 8z = 0$

$2y - 6z = 0$

$$v = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

let $y = 3$ $z = 1.$

$x = 1.$

$$v = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

(3) $\lambda_3 = -3$

$$\begin{pmatrix} 4 & 1 & -2 \\ 4 & 3 & 2 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$4x + y - 2z = 0$

$4x + 3y + 2z = 0$

$y + 2z = 0$

$2y + 4z = 0$

let $y = -2$ $z = 1.$

$x = 1$

$$v = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$