COMP0009 Revision Notes

Formal Logic : Syntax, Semantics,, proof system (Inference)
how to write — what it means — deduction. proofs. (how to reason)

Propositional Logic.

$prop ::= p \mid q \mid r \mid$

$fm ::= prop \mid \neg fm \mid fm \circ fm$

$\circ$: an operand $\wedge, \vee$ or $\to$

Literal: prop or its negation $p \,/\, \neg p$

Main Connective: $(p \wedge q) \vee \neg(q \to r)$
the connective with the largest scope (most outside, highest level)

Semantics

$v \to$ evaluation. $\top$ truth $\perp$ false.

$v(\neg\phi) = \top \iff v(\phi) = \perp$

$v(\phi \wedge \psi) = \top \iff v(\phi) = v(\psi) = \top$.

$v(\phi \vee \psi) = \top \iff v(\phi) = \top$ or $v(\psi) = \top$.

$v(\phi \to \psi) = \top \iff v(\neg\phi) = \top$ or $v(\psi) = \top$

Validity, Satisfiability, Equivalence.

ϕ is valid if $v(\phi) = \top$ for all types of valuations v. (always true)

ϕ is satisfiable if $\exists v$ $v(\phi) = \top$. (true at least once)

ϕ and ψ is logically eq. iff $\forall v$ $v(\phi) = v(\psi) \Rightarrow \phi \equiv \psi$

All valid fmla are satisfiable

Predicate Logic

Language $L(C, F, P)$

C constant symbols.

F function symbols f^n (n-ary)

P a nonempty predicate (relation) symbol set p^n (n-ary)

$tm ::= v : v \in Var \mid c : c \in C \mid f^n(tm \ldots tm) : f^n \in F.$

$3 + (x \times 2) \iff +(3, \times(x, 2)).$

$2, 3 \in C \quad x \in Var.$

$+, \times \in f^2.$

atom $:= p^n(tm, \ldots, tm) : p^n \in P.$

$x+y < 2 \times y - 1 \quad <$ is a p^2

$fm ::= atom \mid \neg fm \mid (fm \vee fm) \mid \exists v fm : v \in Var.$

$(fm \wedge fm)$

$(fm \to fm)$

$\forall x \phi$ as abbr. $\neg \exists x \neg \phi \qquad \forall x \phi \Leftrightarrow \neg \exists x \neg \phi.$

L-structure.

(D, I) D is any non-empty set (domain)
I constants, funcs, predicates

I_c maps constant symbols in C to elements of D.
I_f — — n-ary funcs $f \in F$ to n-ary functions over D
I_p — — n-ary predicate symbols $p \in P$ to n-ary relations over D (subset of D^n)

e.g. $I_p(=) = \{(d, d) : d \in D\}$

interpretation

domain

$S = [D, I]$

structure

$A : Var \to D$ is a variable assignment.

$[c]^{S,A} = I(c).$

$[x]^{S,A} = A(x)$

$[f(\tau_0 \cdots \tau_{n-1})]^{S,A} = I(f)([\tau_0]^{S,A} \cdots [\tau_{n-1}]^{S,A})$

REP. $S, A \models R(\tau_0 \cdots \tau_{n-1}) \Leftrightarrow ([\tau_0]^{S,A} \cdots [\tau_{n-1}]^{S,A}) \in I(R)$

$S, A \models \neg \phi \Leftrightarrow S, A \not\models \phi.$

$S, A \models (\phi \vee \phi') \Leftrightarrow S, A \models \phi$ or $S, A \models \phi'$

$S, A \models \exists x \phi \Leftrightarrow S, A[x \mapsto d] \models \phi$ for some $d \in D.$

Validity

$S = (D, I)$ be a L-structure, ϕ a formula.

ϕ is valid in S. for all $A : Var \to D$ we have $S, A \models \phi$

$S \models \phi.$

ϕ is valid. for all L-structures S we have $S \models \phi.$

$\models \phi.$

Satisfiability.

ϕ is satisfiable in S if $\exists A : Var \to D$. such that $S, A \models \phi.$

ϕ is satisfiable if $\exists A, S$ $S, A \models \phi$.

ϕ is not valid iff $\neg\phi$ is satisfiable.

Propositional Proof System.

a system for determing validity of fmla.

Obvious one: write down truth table for ϕ.

problem - exponential tie.

$\phi: p_1 \vee p_2 \cdots \vee p_{50}$

2^{50} possibilities.

Better: manipulate and analyse the syntax of formula to see if anything can falsify it.

problem: how to make sure syntactical changes $\Updownarrow$ make semantic sense?

make sure the proof ~~system~~ is sound and complete.

$\models\phi \Leftrightarrow \phi$ is valid.

$\vdash\phi \Leftrightarrow$ there is a proof of ϕ

Soundness $\vdash\phi \Rightarrow \models\phi$ System can prove only valid things

completeness $\models\phi \Rightarrow \vdash\phi$. if sth. is valid. the system can prove it.

$\models\phi \Leftrightarrow \vdash\phi$

Axiomatic Proof System.

Fix a prop. lang with only $\rightarrow$ and $\neg$. (no double $\neg$).

I. $(p \rightarrow (q \rightarrow p))$

II. $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$.

$\Updownarrow$

$\neg p \wedge (q \rightarrow r)$

$\Updownarrow$

$\Updownarrow$

$\neg p \wedge (\neg q \wedge r)$

$$\Updownarrow$$

$$(\neg p \wedge \neg q \wedge r) \to ((\neg p \wedge q) \wedge (\neg p \wedge r))$$

$$\Updownarrow$$

$$\neg(\neg p \wedge \neg q \wedge r) \wedge ((p \vee \neg q) \wedge \neg p \wedge r)$$

$$(p \vee q \vee \neg r) \wedge (p \vee \neg q) \wedge \neg p \wedge r$$

$$\Updownarrow$$

true.

III. $(\neg p \to \neg q) \to (q \to p)$.

$$\neg(p \wedge \neg q) \wedge (\neg q \wedge p)$$

$$(\neg p \vee q) \wedge (\neg q) \wedge p$$

$$\Updownarrow$$

true.

Inference Rule.

Modus Ponens.

$$\frac{\phi \quad (\phi \to \psi)}{\psi}.$$

if proved ϕ and $\phi \to \psi$. then deduce ψ.

modus tolles.

$$\frac{\neg q. \quad (p \to q)}{\neg p}.$$

if proved $\neg q$ and $p \to q$. then deduce $\neg p$.

Proof.

a proof is a seq. of fmlas.

$\phi_0 \cdots \phi_n$.

s.t. for $i \leq n$. ϕ_i is either * axiom

* obtained by modus ponens from ϕ_j ϕ_k $(j, k < i)$

e.g. $\boxed{\begin{array}{l}\phi_j. \\ \phi_k = \phi_j \to \phi_i\end{array}}$

$$\Downarrow$$

ϕ_i

If such a proof exists, ϕ_n is a theorem.

$\vdash \phi_n$.

Ex. $\vdash (p \to p)$

Ax II. $(p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p))$

Ax I. $(p \to ((p \to p) \to p))$

$\angle$. $(p \to (p \to p)) \to (p \to p)$

Ax I. $(p \to (p \to p))$.

$\angle$. $\vdash p \to p$.

Proofs with other conn

IV. $p \to \neg\neg p$ and $\neg\neg p \to p$

V $(p \vee q) \Leftrightarrow (\neg p \to q)$. $\Leftrightarrow p \vee q$

VI $(p \wedge q) \Leftrightarrow \neg(p \to \neg q)$ | $\to \neg(\neg p \vee \neg q)$ $\Leftrightarrow p \wedge q$.

Proofs with assumptions

$\Gamma \vdash \phi$

if there's a proof of ϕ using assumptions from Γ.

$\phi_0 \cdots \phi_n$.

ϕ_i is either
* axiom.
* an assumption.
* obtained from $\phi_j \phi_k$. $(j, k < i)$

Ex. $\bot \vdash p$

$\{\neg(q \to q)\} \vdash p$

- $\angle$ $q \to q$.
- $\angle$ $(q \to q) \to \neg\neg(q \to q)$ Ax. IV.
- $\angle$ $\neg\neg(q \to q)$. MP.

$\neg\neg(q\to q)\to(\neg p\to\neg\neg(q\to q))$

$\neg p\to\neg\neg(q\to q)$

$(\neg p\to\neg\neg(q\to q))\to(\neg(q\to q)\to p)$ Ax III.

$\neg(q\to q)\to p$

$\neg(q\to q)$ Assum.

p

$\neg\neg(q\to q)\to(p\to\neg\neg(q\to q))$ Ax. I.

$p\to\neg\neg(q\to q)$

$(p\to\neg\neg(q\to q))\to(\neg(q\to q)\to\neg p)$ Ax III.

$\neg(q\to q)\to\neg p$

$\neg(q\to q)$ Assumption.

$\neg p$

Soundness ✓.

check all axioms

check if ϕ and $\phi\to\psi$. then ψ is valid. Modus Ponens

$\Downarrow$

all provable fmlas are valid.

$$\vdash\phi\Rightarrow\models\phi.$$

Completeness.

$\models\phi\Rightarrow\vdash\phi.$ (all. valid fmla are provable)

Propositional Tableaux.

given fmla ϕ, tableau. can tell us whether it is satisfiable or not.

* decomposing fmla according to certain rules to the point only literals are left.

$\Downarrow$

atomic fmla. or its negation.

* tableau. (closed or not).

if tableau closes ϕ is unsatisfiable

tableau never closes ϕ is satisfiable

$((\phi\wedge\neg\phi)\vee(\psi\wedge\neg\psi))$

$(\phi\wedge\neg\phi)$ $(\psi\wedge\neg\psi)$

ϕ ψ

$\neg\phi$ $\qquad\qquad$ $\neg\psi$

$\oplus$ $\qquad\qquad$ $\oplus$

closes both $\to$ not satisfiable.

Tableau T is a BT. (node labelled by fmla)

Every fmla in tab, except literals.

gets expanded. and ticked.

if branch T_B contains p and $\neg p$.

it's closed.

if every branch of T is closed then

T is closed.

α formula. $\phi \wedge \psi$.

$\phi\wedge\psi$ is true iff ϕ and ψ is true.

$$\phi\wedge\psi \;|\; \phi \;|\; \psi.$$

$$\neg\neg\phi \;|\; \phi \qquad \neg(\phi\vee\psi) \;|\; \neg\phi \;|\; \neg\psi \qquad \neg(\phi\to\psi) \;|\; \phi \;|\; \neg\psi.$$

β formula. $\phi\vee\psi$

$\phi\vee\psi$ is true iff ϕ or ψ is true.

$$\phi\vee\psi \;/\backslash\; \phi \quad \psi.$$

$$\neg(\phi\wedge\psi) \;/\backslash\; \neg\phi \quad \neg\psi. \qquad \phi\to\psi \;/\backslash\; \neg\phi \quad \psi.$$

Example. $p\vee r$ $\quad\beta$.

$\neg q$ $\quad\times$

$\neg((p\vee q)\to r)$ $\quad\alpha$.

$\neg(p \vee q) \to r$ ...

$((p \to r) \vee (q \to r))$. β.

$\neg(((p \to r) \vee (q \to r)) \to ((p \vee q) \to r))$ α.

$\neg(p \to r)$ α.

$q \to r$ β.

$\neg r$ $\times$

$\neg(p \wedge q)$ β.

$\neg((p \to r) \vee (q \to r))$ α.

$p \vee q$. β.

$\neg(((p \wedge q) \to r) \to ((p \to r) \vee (q \to r)))$ α.

p $\times$

$p \to r$ β.

$\neg(q \to r)$. α

$\neg(((p \wedge q) \to r)$. α.

* a tableau is complete is either ticked. (expanded) or is a literal.

* if ϕ is at the root of a complete open tableau, ϕ is satisfiable.

ϕ is satisfiable. $\Leftrightarrow$ $\neg\phi$ is not valid.

ϕ is valid $\Leftrightarrow$ $\neg\phi$ is not satisfiable.

test ϕ is valid $\Leftrightarrow$ test $\neg\phi$ is not satisfiable

Ex.1. is $((p \wedge q) \to r) \to ((p \to r) \vee (q \to r)))$ valid?

is $\neg(((p \wedge q) \to r) \to ((p \to r) \vee (q \to r)))$ not satisfiable?

α.

$((p \wedge q) \to r)$ ✓

$\neg((p\to r)\vee(q\to r))$ ✓

Left branch:

$\neg(p\wedge q)$ ✓

$\neg(p\to r)$ ✓

$\neg(q\to r)$ ✓

- $\neg p$, p, $\neg r$, $\otimes$
- $\neg q$, p, $\neg r$, q, $\neg r$, $\otimes$

Right branch:

r

$\neg(p\to r)$ ✓

$\neg(q\to r)$ ✓

p, $\neg r$, q, $\neg r$, $\otimes$

$\therefore$ $(((p\wedge q)\to r)\to((p\to r)\vee(q\to r)))$ is valid.

Ex 2. Is $(((p\to r)\vee(q\to r))\to((p\vee q)\to r))$ valid.

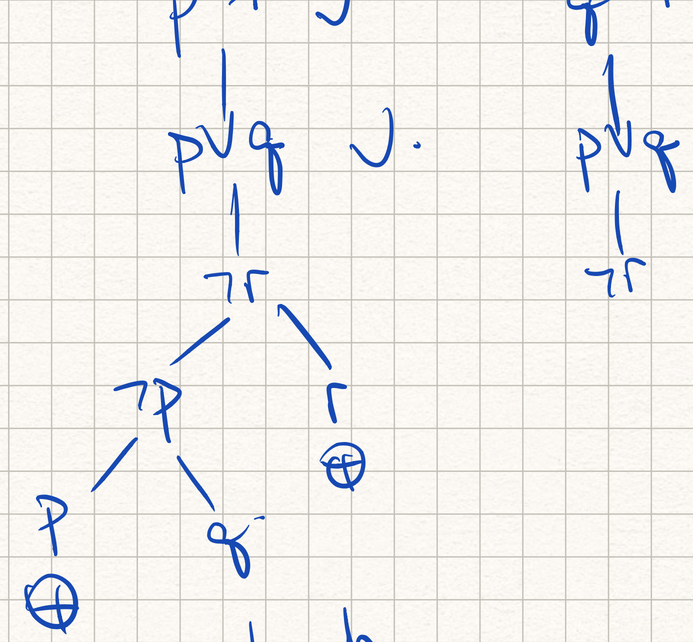
$\Leftrightarrow$ is $\neg(((p\to r)\vee(q\to r))\to((p\vee q)\to r))$ ✓ satisfiable

$((p\to r)\vee(q\to r))$ ✓ α

$\neg((p\vee q)\to r)$ ✓

β

- $p\to r$
- $q\to r$



open branch

$\therefore\ ((p\to r)\vee(q\to r)) \to ((p\vee q)\to r))$ is not valid

DNF

Disjunctive Normal form.

disjunctions of conjunction literals.

$$(p\wedge\neg q\wedge r)\vee(p\wedge q) \ \cdots\ \vee\ \cdots\ (x\wedge x\wedge x)$$

Each fmla. has an equivalent fmla in DNF.

Conjunctive Normal Form.

$$(p\vee q)\wedge(\neg p\vee\neg r)\wedge q\ \cdots\ \wedge\ \cdots\ \wedge(x\vee x\vee x)$$

Converting to DNF.

truth table / logical equivalences / tableau

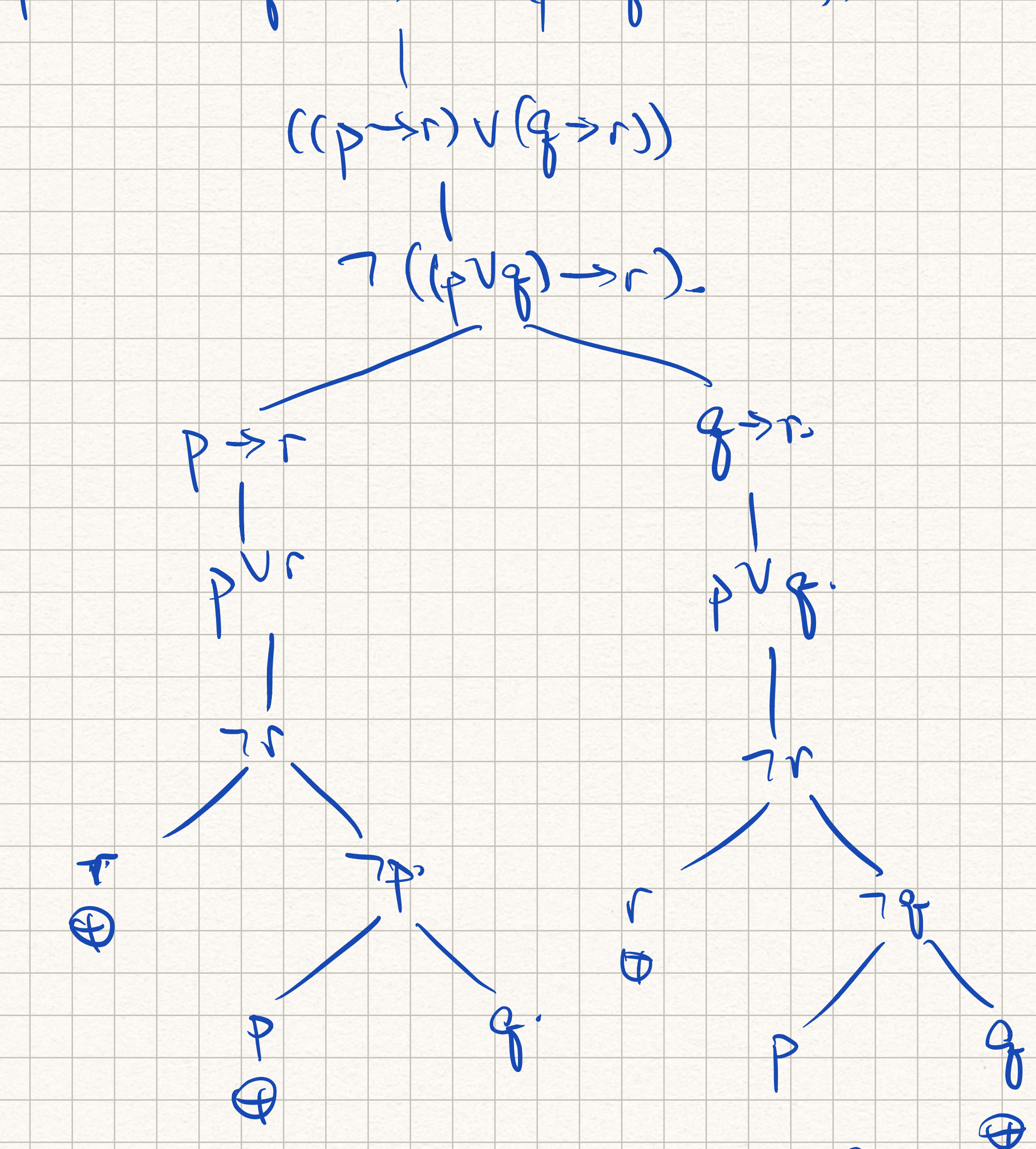
- φ at root of T.
- Expand until T is completed.
- open branch θ of T let.

$$C_\theta = \bigwedge\{\text{literal in } \theta\}.$$

then

$$\varphi \equiv \bigvee C_\theta$$

$$\neg(((p\to r)\vee(q\to r)) \to ((p\vee q)\to r))$$



DNF: $\neg(((p \to r) \vee (q \to r)) \to ((p \vee q) \to r))$

$(\neg p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r)$.

First Order Tableaus

- A literal is an atom or its negation.
- closed term is a term contains no vars.
- Same kind of tableau construction.

α, β remain the same.

δ formula.

choose a const. p.

add fmla at each leaf below the node.

$$\exists x\,\phi \qquad\qquad \neg\forall x\,\phi$$
$$| \qquad\qquad\qquad |$$
$$\phi(p/x). \qquad \neg\phi(p/x)$$

$$\underbrace{\exists x \neg\phi \equiv \neg\forall x\,\phi.}$$

γ formula

pick any closed term t.

don't tick node.

$$\forall x\,\phi. \qquad\qquad \neg\exists x\,\phi$$
$$| \qquad\qquad\qquad |$$
$$\phi(t/x). \qquad \neg\phi(t/x).$$

$$\underbrace{\neg\exists x\,\phi \equiv \forall x \neg\phi.}$$

$\forall x \neg p(x).$ $\quad\gamma$

$H(a) \to F(a)$ $\quad\beta.$

$\neg\neg\exists y\; p(y).$ $\quad\delta$

$\neg(\forall x \neg p(x) \to \neg\exists y\; p(y))$ $\quad\alpha.$

$\neg(\forall x \neg q(x) \vee \exists x \forall y \neg(x<y))$ $\quad\alpha.$

$G(a) \to H(a).$ $\quad\beta$

$\neg\neg(\forall x(G(x) \to H(x)) \wedge \forall x(H(x) \to F(x)) \wedge G(a)$
$\quad \wedge \neg\exists x(G(x) \wedge F(x))).$ $\quad\alpha.$

$\neg\exists x(G(x) \wedge F(x))$ $\quad\gamma$

$\neg\forall y \neg(c<y)$ $\quad\delta.$

$\forall x(H(x) \to F(x))$ $\quad\gamma.$

$G(a)$ $\quad\times.$

$\neg p(c)$ $\quad\times.$

$\forall x(G(x) \to H(x))$ $\quad\gamma.$

$\neg H(a).$ $\quad\times.$

$\forall x(G(x) \to H(x)) \wedge \forall x(H(x) \to F(x)) \wedge G(a) \wedge \neg\exists x(G(x) \wedge F(x))$ $\quad\alpha.$

$\neg G(a).$ $\quad\times$

$\neg(G(a) \wedge F(a))$ β.

$\exists y\, p(y)$. δ.

Ex. 3.

$(\forall x \neg p(x) \to \neg \exists y\, p(y))$ valid?

$\neg(\forall x \neg p(x) \to \neg\exists y\, p(y))$

|

$\forall x \neg p(x)$ ✓

|

$\exists y\, p(y)$. ✓

|

$p(c)$.

|

$\neg p(c)$.

|

$\otimes$