

Formal Logic : Syntax, Semantics., proof system (Inference)
 how to write what it means deduction: proofs.
 (how to reason)

Propositional Logic.

prop := $p \mid q \mid r \mid$

$f_m ::= prop \mid \neg f_m \mid f_m \circ f_m$
 an operand $\wedge, \vee, \rightarrow$

Literal: prop or its negation $p \mid \neg p$

Main Connective: $(p \wedge q) \bigcirc (\neg (q \rightarrow r))$

the connective with the largest scope (most outside, highest level)

Semantics

$v \rightarrow$ evaluation. Truth \top false. \perp

$v(\neg \phi) = \top \Leftrightarrow v(\phi) = \perp$

$v(\phi \wedge \psi) = \top \Leftrightarrow v(\phi) = v(\psi) = \top$

$v(\phi \vee \psi) = \top \Leftrightarrow v(\phi) = \top \text{ or } v(\psi) = \top$

$v(\phi \rightarrow \psi) = \top \Leftrightarrow v(\neg \phi) = \top \text{ or } v(\psi) = \top$

Validity, Satisfiability, Equivalence.

ϕ is valid if $v(\phi) = \top$ for all types of valuations v . (always true)

ϕ is satisfiable if $\exists v \ v(\phi) = \top$. (true at least once)

ϕ and ψ is logically eq. iff $\forall v \ v(\phi) = v(\psi) \Rightarrow \phi \equiv \psi$

All valid formulae are satisfiable

Predicate Logic

Language $L(C, F, P)$

C constant symbols.

F function symbols f^n (n -ary)

P a nonempty predicate symbol set P^n (n -ary)
(relation)

$tm ::= _ ; v \in \text{Var} \mid c \in C \mid f^n(tm \dots tm) : f^n \in F$.

$$3 + (x \times 2) \Leftrightarrow +(3, \times(x, 2))$$

$2,3 \in C \quad x \in \text{Var.}$
 $+x \in f^2.$

atom := $P^n(t_1, \dots, t_m) : P \in \mathcal{P}.$

$x+y < 2 \times y - 1 < \text{is a } P^2$

fm := atom | $\neg fm$ | $(fm \wedge fm)$ | $\exists v fm: v \in \text{Var.}$

$(fm \wedge fm)$

$(fm \rightarrow fm)$

$\forall x \phi$ as abbrev. $\neg \exists x \neg \phi$ $\forall x \phi \Leftrightarrow \neg \exists x \neg \phi$

L-structure:

(D, I) D is any non-empty set (domain)
 I constants, funcs, predicates

I_C maps constant symbols in C to elements of D

If — — n-ary funcs $f \in \mathcal{F}$ to n-ary functions over D

I_P — — n-ary predicate symbols $p \in \mathcal{P}$ to n-ary relations over D

interpretation
 domain \downarrow $A: \text{Var} \rightarrow D$ is a variable assignment. e.g.
 $S = [D, I]$ $[c]^{S,A} = I(c).$
 structure \uparrow $[x]^{S,A} = A(x)$

$I_P(=) = \{(d, d) : d \in D\}$

$[f(t_0, \dots, t_{n-1})]^{S,A} = I(f)[t_0]^{S,A}, \dots, [t_{n-1}]^{S,A}$

Rep. $S, A \models R(t_0, \dots, t_{n-1}) \Leftrightarrow ([t_0]^{S,A}, \dots, [t_{n-1}]^{S,A}) \in I(R)$

$S, A \models \neg \phi \Leftrightarrow S, A \not\models \phi.$

$S, A \models (\phi \vee \psi) \Leftrightarrow S, A \models \phi \text{ or } S, A \models \psi$

$S, A \models \exists x \phi \Leftrightarrow S, A[x \mapsto d] \models \phi \text{ for some } d \in D.$

Validity

$S = (D, I)$ be a L-structure, ϕ a formula.

ϕ is valid in S , for all $A: \text{Var} \rightarrow D$

we have $S, A \models \phi$

$S \models \phi.$

ϕ is valid. for all L-structures S we have

$S \models \phi.$

Satisfiability.

ϕ is satisfiable in S if $\exists A: \text{Var} \rightarrow D$ such that $S, A \models \phi$.

ϕ is satisfiable if $\exists A, S \models A \models \phi$.

ϕ is not valid iff $\neg\phi$ is satisfiable.

Propositional Proof System.

a system for determining validity of formula.

Obvious one: write down truth table for ϕ .
problem - exponential time.

$\phi: p_1 \vee p_2 \dots \vee p_{50}$

$\geq 2^{50}$ possibilities.

Better: manipulate and analyse the syntax of formula
to see if anything can falsify it.

Problem: how to make sure syntactical changes

make semantic sense?

make sure the proof system is sound and complete.

$\vdash \phi \Leftrightarrow \phi$ is valid.

$\vdash \phi \Leftrightarrow$ there is a proof of ϕ

Soundness $\vdash \phi \Rightarrow \vdash \phi$ System can prove only valid things

completeness $\vdash \phi \Rightarrow \vdash \neg \phi$

if sth. is valid.

the system can prove it

$\vdash \phi \Leftrightarrow \vdash \neg \phi$

Axiomatic Proof System.

Fix a prop. lang with only \rightarrow and \neg . (no double \neg).

I. $(p \rightarrow (q \rightarrow p))$

II. $((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$

$\neg p \wedge (q \rightarrow r)$

$\neg p \wedge (\neg q \wedge r)$



$$\begin{aligned}
 & (\neg p \wedge \neg q \wedge r) \rightarrow ((\neg p \wedge q) \wedge (\neg p \wedge r)) \\
 & \quad \text{D} \\
 & \neg(\neg p \wedge \neg q \wedge r) \wedge (\neg p \vee \neg q) \wedge \neg p \wedge r \\
 & \quad (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q) \wedge \neg p \wedge r \\
 & \quad \text{true.}
 \end{aligned}$$

$$\begin{aligned}
 \text{III. } & (\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p) \\
 & \neg(p \wedge \neg q) \wedge (\neg q \wedge p) \\
 & (\neg p \vee q) \wedge (\neg q) \wedge p \\
 & \quad \text{II} \\
 & \quad \text{true.}
 \end{aligned}$$

Inference Rule:

$$\begin{array}{l}
 \text{Modus Ponens. if proved } \phi \text{ and } \phi \rightarrow \psi. \\
 \frac{\phi \quad (\phi \rightarrow \psi)}{\psi}. \quad \text{then deduce } \psi.
 \end{array}$$

$$\begin{array}{l}
 \text{Modus Tollens. if proved } \neg q \text{ and } p \rightarrow q. \\
 \frac{\neg q. \quad (\phi \rightarrow \psi)}{\neg \phi}. \quad \text{then deduce } \neg \phi.
 \end{array}$$

Proof.

a proof is a seq. of fmlas.

$$\phi_0 \cdots \phi_n.$$

s.t. for $i \leq n$. ϕ_i is either *axiom
* obtained by
modus ponens
from $\phi_j \phi_k$

$$\begin{array}{c}
 \overline{\phi_j.} \\
 \overline{\phi_k = \phi_j \rightarrow \phi_i} \quad (j, k < i) \\
 \text{e.g. } \frac{\phi_k = \phi_j \rightarrow \phi_i}{\phi_i} \quad \text{if } \vdash \phi_j \quad \text{if } \vdash \phi_i
 \end{array}$$

If such a proof exists, ϕ_n is a theorem.

$\vdash \phi_n$.

Ex. $\vdash (p \rightarrow p)$

AxII. $(p \rightarrow ((p \rightarrow p) \rightarrow p)) \rightarrow (((p \rightarrow (p \rightarrow p)) \rightarrow (p \rightarrow p)))$

AxI. $(p \rightarrow ((p \rightarrow p) \rightarrow p))$

L. $(p \rightarrow (p \rightarrow p)) \rightarrow (\neg p \rightarrow p)$

Ax I. $(\neg p \rightarrow (p \rightarrow p)).$

L. $(\neg p \rightarrow p).$

Proofs with other conn

IV. $p \rightarrow \neg \neg p$ and $\neg \neg p \rightarrow p$

V. $(p \vee q) \Leftrightarrow (\neg p \rightarrow q) \Leftrightarrow p \vee q$

VI. $(p \wedge q) \Leftrightarrow \neg \neg p \rightarrow \neg \neg q \quad | \rightarrow \neg(\neg p \vee \neg q) \\ | \Leftrightarrow p \wedge q.$

Proofs with assumptions

$\vdash \phi$

if there's a proof of ϕ using assumptions from T.

$\phi_0 \dots \phi_n.$

ϕ_i is either

* axiom.

* an assumption.

* obtained from $\phi_j \phi_k$.

$(j, k < i)$

Ex. $\vdash p$

$\vdash (q \rightarrow q) \vdash p$

$\neg q \rightarrow q.$

$\neg \neg (q \rightarrow q) \rightarrow \neg \neg (q \rightarrow q) \text{ Ax.IV.}$

$\vdash \neg \neg (q \rightarrow q). \text{ MP.}$

$\neg \neg (q \rightarrow q) \rightarrow (\neg p \rightarrow \neg (q \rightarrow q))$
 L. $\neg p \rightarrow \neg (q \rightarrow q)$
 $\neg (\neg p \rightarrow \neg (q \rightarrow q))$
 $\rightarrow (\neg (q \rightarrow q) \rightarrow p)$ Ax II.
 L. $\neg (q \rightarrow q) \rightarrow p$
 $\neg (q \rightarrow q)$ Assum.
 L. P

$\neg \neg (q \rightarrow q) \rightarrow (p \rightarrow \neg (q \rightarrow q))$ Ax I.
 L. $p \rightarrow \neg (q \rightarrow q)$
 $\neg (p \rightarrow \neg (q \rightarrow q)) \rightarrow (\neg (q \rightarrow q) \rightarrow \neg p)$ Ax III.
 L. $\neg (q \rightarrow q) \rightarrow \neg p$
 $\neg (q \rightarrow q)$ Assumption.
 L. $\neg p$

Soundness ✓.

check all axioms

check if ϕ and $\neg \phi$. then ϕ is valid. Modus Ponens

all provable formulas are valid.

$$\vdash \phi \Rightarrow \models \phi.$$

Completeness.

$$\models \phi \Rightarrow \vdash \phi. \text{ (all. valid formula are provable)}$$

Propositional Tableaux.

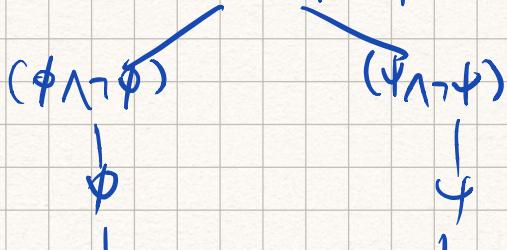
Given formula ϕ , tableau. can tell us whether it is satisfiable or not.

* decomposing formula according to certain rules
to the point only literals are left

↓
atomic formula
or its negation.

* tableau. (closed or not).
if tableau closes ϕ is unsatisfiable
tableau never closes ϕ is satisfiable

$$((\phi \wedge \neg \phi) \vee (\psi \wedge \neg \psi))$$



$\neg\phi$ $\neg\psi$
 \oplus \oplus
 closes both \rightarrow not satisfiable.

Tableau T is a BT. (node labelled by formula)

Every formula in Tab, except literals.

gets expanded. and ticket.

if branch T_B contains p and $\neg p$.
 it's closed.

if every branch of T is closed then
 T is closed.

a formula. $\phi \wedge \psi$.

$\phi \wedge \psi$ is true iff ϕ and ψ is true.

$$\begin{array}{c} \neg\neg\phi \\ | \\ \phi \end{array}$$

$$\begin{array}{c} \neg(\phi \vee \psi) \\ | \\ \neg\phi \\ | \\ \neg\psi \end{array}$$

$$\neg(\phi \rightarrow \psi)$$

$$\begin{array}{c} \phi \wedge \psi \\ | \\ \phi \\ | \\ \psi \end{array}$$

B formula. $\phi \vee \psi$

$\phi \vee \psi$ is true iff ϕ or ψ is true.

$$\begin{array}{c} \phi \vee \psi \\ | \quad | \\ \phi \quad \psi \end{array}$$

$$\begin{array}{c} \neg(\phi \wedge \psi) \\ | \quad | \\ \neg\phi \quad \neg\psi \end{array}$$

$$\begin{array}{c} \phi \rightarrow \psi \\ | \quad | \\ \neg\phi \quad \psi \end{array}$$

Example. $p \vee r$ B.

$$\begin{array}{c} \neg q \\ | \\ X \end{array}$$

$$\neg((\neg q \rightarrow r) \rightarrow s) \alpha.$$

$$((p \rightarrow r) \vee (q \rightarrow r)). \quad \beta.$$

$$\neg(((p \rightarrow r) \vee (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)) \quad \alpha.$$

$$\neg(p \rightarrow r) \quad \alpha.$$

$$\begin{array}{c} q \rightarrow r \\ \hline \neg r \end{array} \quad \beta.$$

$$\neg(p \wedge q) \quad \beta.$$

$$\neg((p \rightarrow r) \vee (q \rightarrow r)) \quad \alpha.$$

$$p \vee q. \quad \beta.$$

$$\neg(((p \wedge q) \rightarrow r) \rightarrow ((p \rightarrow r) \vee (q \rightarrow r))) \quad \alpha.$$

$$\begin{array}{c} p \\ \hline p \rightarrow r \end{array} \quad \beta.$$

$$\neg(q \rightarrow r). \quad \alpha$$

$$\neg(((p \wedge q) \rightarrow r)). \quad \alpha.$$

* a tableau is complete if either ticked (expanded)

* if ϕ is at the root or is a literal, or is a closed, tableau, ϕ is satisfiable.

ϕ is satisfiable. $\Leftrightarrow \neg\phi$ is not valid.

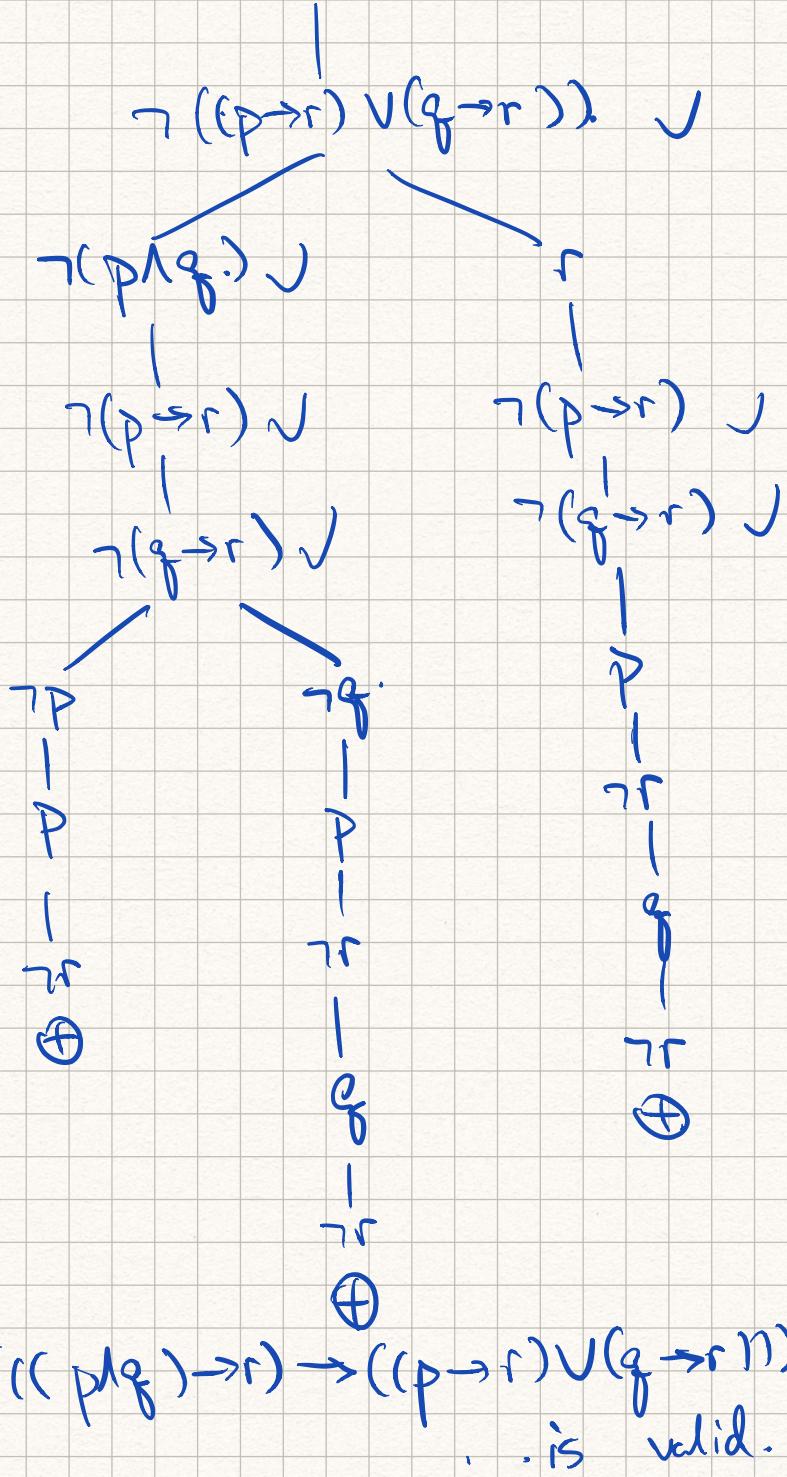
ϕ is valid $\Leftrightarrow \neg\phi$ is not satisfiable.

test ϕ is valid \Leftrightarrow test $\neg\phi$ is not satisfiable

Ex,1. is $((p \wedge q) \rightarrow r) \rightarrow ((p \rightarrow r) \vee (q \rightarrow r))$ valid?

is $\neg(((p \wedge q) \rightarrow r) \rightarrow ((p \rightarrow r) \vee (q \rightarrow r)))$ not satisfiable?

$$\begin{array}{c} | \\ ((p \wedge q) \rightarrow r) \end{array} \quad \downarrow \quad \alpha.$$



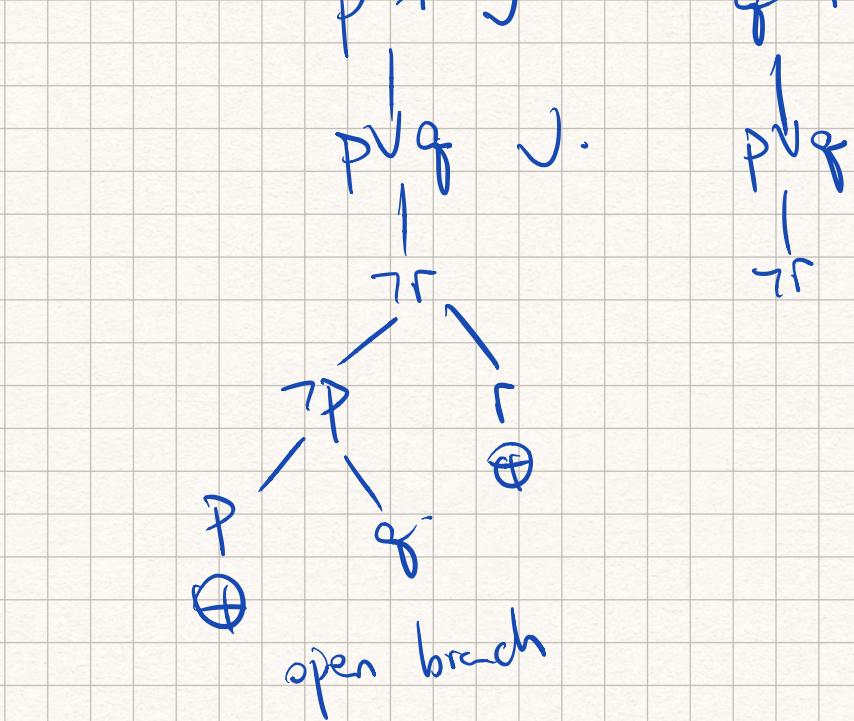
Ex 2. Is $((p \rightarrow r) \vee (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$ valid.

↪ Is $\neg((p \rightarrow r) \vee (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$ satisfiable.

$$((p \rightarrow r) \vee (q \rightarrow r)) \vee \alpha.$$

$$\neg((p \vee q) \rightarrow r) \quad \checkmark$$

$$\begin{array}{c}
 \neg \rightarrow r \quad | \\
 | \\
 \neg r \quad \beta
 \end{array}$$



$\therefore ((p \rightarrow r) \vee (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$ is not valid.

DNF

Disjunctive Normal form.

disjunctions of conjunction literals.

$$(p \wedge \neg q \wedge r) \vee (p \wedge q) \vee \dots \vee (x \wedge x \wedge x)$$

Each formula has an equivalent formula in DNF.

Conjunctive Normal Form.

$$(p \vee q) \wedge (\neg p \vee \neg r) \wedge q \wedge \dots \wedge \lambda(x \vee x \vee x)$$

Converting to DNF.

truth table / logical equivalences / tautology

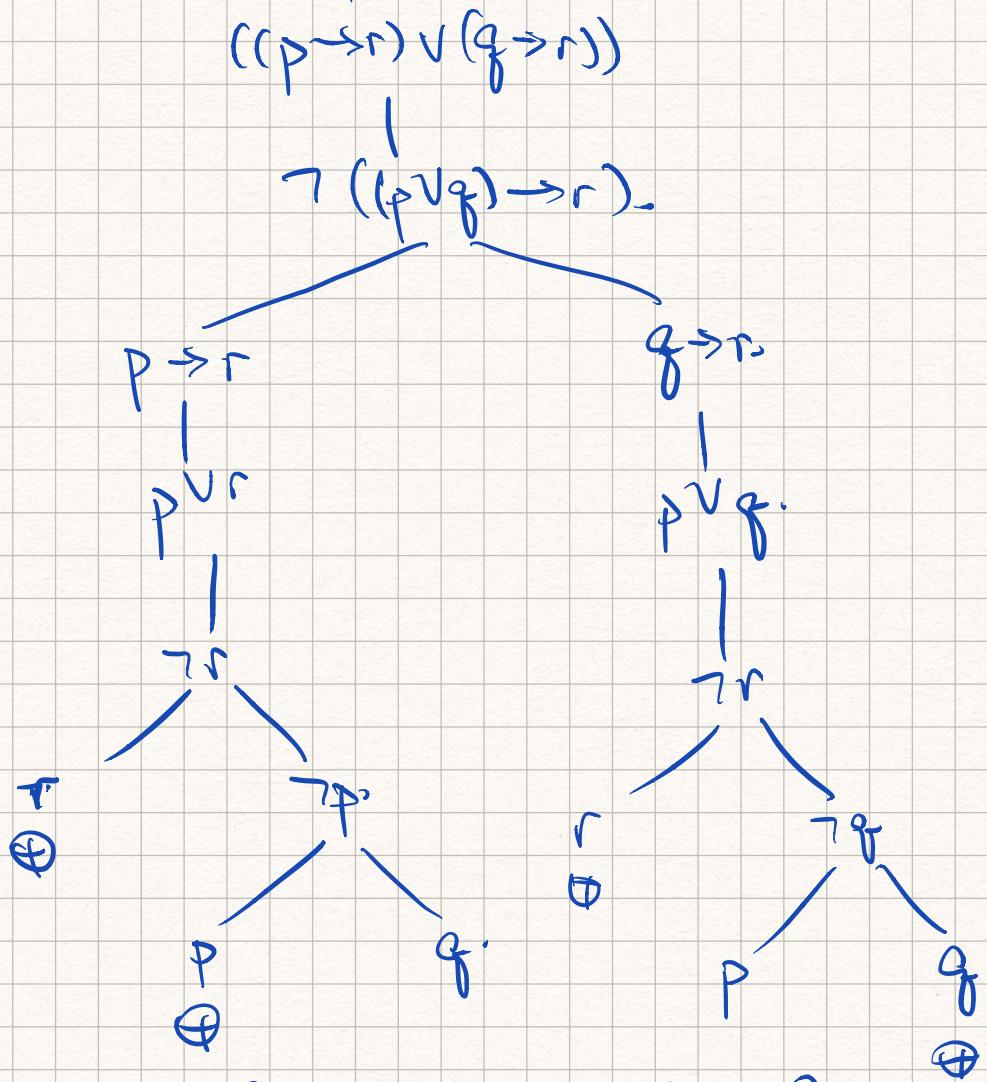
- ϕ at root of T.
- Expand until T is completed.
- open branch Θ of T let,

$$C_\Theta = \lambda \{ \text{literals in } \Theta \}.$$

then

$$\phi \equiv \bigvee C_\Theta$$

$$\neg (((p \rightarrow r) \vee (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r))$$



DNF: $\neg((p \rightarrow r) \vee (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$

$$(\neg p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r).$$

First Order Tableaux

- A literel. is an atom or its negation.
- closed term is a term contains no vars.
- Same kind of tableau construction.

2. β remain the same.
 γ formula.

choose a const. p .

add rule at each leaf below the node.

$$\begin{array}{c} \exists x \phi & \neg \forall x \phi \\ | & | \\ \phi(p/x). & \neg \phi(p/x) \\ \underbrace{\exists x \neg \phi \equiv \neg \forall x \phi.}_{\text{ }} \end{array}$$

γ formula

pick any closed
term $t.$
don't tick node.

$$\begin{array}{c} \forall x \phi. & \neg \exists x \phi \\ | & | \\ \phi(t/x). & \neg \phi(t/x) \end{array}$$

until you used
every available const/closed term

$$\forall x \neg p(x). \quad \gamma$$

$$H(a) \rightarrow F(a) \quad \beta.$$

$\neg \exists y p(y).$ ~~X~~ (double negation first) α rule

$$\neg (\forall x \neg p(x) \Rightarrow \exists y p(y)) \quad \alpha.$$

$$\neg (\forall x \neg p(x) \vee \exists x \forall y \neg (x < y)) \quad \alpha.$$

$$G(a) \rightarrow H(a). \quad \beta$$

$$\neg \neg (\forall x (G(x) \rightarrow H(x)) \wedge \forall x (H(x) \rightarrow F(x)) \wedge G(a) \wedge \neg \exists x (G(x) \wedge F(x))). \quad \alpha.$$

$$\neg \exists x (G(x) \wedge F(x)) \quad \gamma$$

$$\neg \forall y \neg (c < y) \quad \delta.$$

$$\forall x (H(x) \rightarrow F(x)) \quad \gamma'.$$

$$G(a) \quad X,$$

$$\neg p(c) \quad X.$$

$$\forall x (G(x) \rightarrow H(x)) \quad \gamma.$$

$$\neg H(a). \quad X.$$

$$\forall x (G(x) \rightarrow H(x)) \wedge \forall x (H(x) \rightarrow F(x)) \wedge G(a) \wedge \neg \exists x (G(x) \wedge F(x))$$

$$\neg G(a). \quad X \quad \alpha.$$

$$\neg (G(a) \wedge F(a)) \quad \text{p.}$$

$$\exists y P(y). \quad 8.$$

Ex.3.

$$(\forall x \neg P(x) \rightarrow \exists y P(y)) \text{ valid?}$$

$$\neg (\forall x \neg P(x) \rightarrow \exists y \neg P(y))$$

|

$$\forall x \neg P(x) \quad J$$

$$\exists y P(y). \quad J$$

$$\neg P(c). \quad J$$

$$\neg \neg P(c).$$

|

⊕

$$\text{Ex4. } \neg (\forall x (Gx \rightarrow Hx) \wedge \forall x (Hx \rightarrow Fx) \wedge Ga) \wedge$$

$$\neg \exists x (Gx \wedge Fx) \text{ valid?}$$

$$\forall x (Gx \rightarrow Hx) \wedge \forall x (Hx \rightarrow Fx) \wedge Ga \wedge$$

$$\neg \exists x (Gx \wedge Fx).$$

|

$$\forall x (Gx \rightarrow Hx)$$

|

$$\forall x (Hx \rightarrow Fx)$$

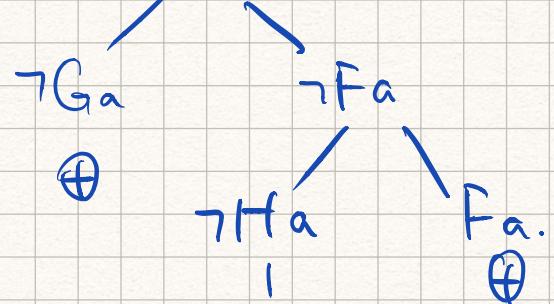
|

$$Ga.$$

|

$$\neg \exists x (Gx \wedge Fx)$$

$$\neg (Ga \wedge Fa) \quad J$$



$$\begin{array}{c} Ga \rightarrow Ha \\ \diagdown \quad \diagup \\ \neg Ga \quad Ha. \\ \oplus \qquad \oplus \end{array}$$

$$\perp, \neg(\forall x(Gx \rightarrow Hx) \wedge \forall x(Hx \rightarrow \neg Fx))$$

$\wedge G \in \perp \exists x(Gx \wedge Fx)$ is valid

Infinite Tableau.

$$\text{Ex.5. } \neg(\forall x \neg f(x) \vee \exists x \forall y \neg(x < y))$$

for infinite

tableau it's
satisfiable.

$$D\{c, d, e, \dots\}$$

$$I(f) = \{c\}.$$

$$I(<) = \{c, d, (d, e), \neg \forall y \neg(c < y)\} \cup \dots$$

$$\begin{array}{c} \top \\ | \\ \neg \forall x \neg f(x) \vee \\ | \\ \neg \exists x \forall y \neg(x < y). \end{array}$$

$$\begin{array}{c} \top \\ | \\ \neg \neg f(c) \vee \\ | \\ f(c) \end{array}$$

$$\begin{array}{c} \top \\ | \\ \neg \neg (c < d). \end{array}$$

$$\begin{array}{c} c < d. \\ | \\ \neg \forall y \exists (d < y). \\ | \\ \neg \exists (d < e) \vee \\ | \\ d < e. \\ | \\ \end{array}$$

Alternative.

Tchkean as lists.