COMP0009 Revision Notes

Formal Logic : Syntax, Semantics, proof system (Inference)
how to write / what it means / deduction. proofs. (how to reason)

Propositional Logic.

prop ::= p | q | r |

fm ::= prop | ¬ fm | fm ∘ fm

∘ : an operand ∧, ∨ or →

Literal: prop or its negation p / $\neg p$

Main Connective: $(p \wedge q) \vee \neg(q \to r)$
the connective with the largest scope (most outside, highest level)

Semantics

$v \to$ evaluation. $\top$ truth $\bot$ false.

$v(\neg\phi) = \top \iff v(\phi) = \bot$

$v(\phi \wedge \psi) = \top \iff v(\phi) = v(\psi) = \top.$

$v(\phi \vee \psi) = \top \iff v(\phi) = \top$ or $v(\psi) = \top.$

$v(\phi \to \psi) = \top \iff v(\neg\phi) = \top$ or $v(\psi) = \top$

Validity, Satisfiability, Equivalence.

ϕ is valid if $v(\phi) = \top$ for all types of valuations v. (always true)

ϕ is satisfiable if $\exists v$ $v(\phi) = \top$. (true at least once)

ϕ and ψ is logically eq. iff $\forall v$ $v(\phi) = v(\psi) \Rightarrow \phi \equiv \psi$

All valid fmla are satisfiable

Predicate Logic

Language $L(C, F, P)$

C constant symbols.

F function symbols f^n (n-ary)

P a nonempty predicate (relation) symbol set P^n (n-ary)

$tm ::= v : v \in Var \mid c : c \in C \mid f^n(tm \ldots tm) : f^n \in F.$

$3 + (x \times 2) \iff +(3, \times(x, 2)).$

$2, 3 \in C \quad x \in Var.$

$+, \times \in f^2.$

atom $:= p^n(tm, \ldots, tm) : p^n \in P.$

$x+y < 2 \times y - 1 \qquad <$ is a p^2

fm $:=$ atom $| \neg fm | (fm \vee fm) | \exists v fm : v \in Var.$

$(fm \wedge fm)$

$(fm \to fm)$

$\forall x \phi$ as abbr. $\neg \exists x \neg \phi \qquad \forall x \phi \Leftrightarrow \neg \exists x \neg \phi.$

L-structure.

(D, I) D is any non-empty set (domain)
I constants, funcs, predicates

I_c maps constant symbols in C to elements of D.
I_f — — n-ary funcs $f \in F$ to n-ary functions over D
I_p — — n-ary predicate symbols $p \in P$ to n-ary relations over D (subset of D^n)

e.g. $I_p(=) = \{(d, d) : d \in D\}$.

interpretation

domain

$A : Var \to D$ is a variable assignment.

$S = [D, I]$

structure

$[c]^{S,A} = I(c).$

$[x]^{S,A} = A(x)$

$[f(T_0 \cdots T_{n-1})]^{S,A} = I(f)([T_0]^{S,A} \cdots [T_{n-1}]^{S,A})$

REP. $S, A \models R(T_0 \cdots T_{n-1}) \Leftrightarrow ([T_0]^{S,A} \cdots [T_{n-1}]^{S,A}) \in I(R)$

$S, A \models \neg \phi \Leftrightarrow S, A \not\models \phi.$

$S, A \models (\phi \vee \phi') \Leftrightarrow S, A \models \phi$ or $S, A \models \phi'$

$S, A \models \exists x \phi \Leftrightarrow S, A[x \mapsto d] \models \phi$ for some $d \in D.$

Validity

$S = (D, I)$ be a L-structure, ϕ a formula

ϕ is valid in S. for all $A : Var \to D$ we have $S, A \models \phi$

$S \models \phi.$

ϕ is valid. for all L-structures S we have $S \models \phi.$

$\models \phi.$

Satisfiability.

ϕ is satisfiable in S if $\exists A : Var \to D.$ such that $S, A \models \phi.$

ϕ is satisfiable if $\exists A$. s.t. $A \models \phi$.

ϕ is not valid iff $\neg\phi$ is satisfiable.

Propositional Proof System.

a system for determining validity of fmla.

Obvious one: write down truth table for ϕ.

problem - exponential tie.

$\phi: p_1 \vee p_2 \cdots \vee p_{50}$

2^{50} possibilities.

Better: manipulate and analyse the syntax of formula to see if anything can falsify it.

Problem: how to make sure syntactical changes $\Updownarrow$ make semantic sense?

make sure the proof system is sound and complete.

$\models \phi \Leftrightarrow \phi$ is valid.

$\vdash \phi \Leftrightarrow$ there is a proof of ϕ