UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **COMP0017**

ASSESSMENT PATTERN : **COMP0017A6UA**

MODULE NAME : **Computability and Complexity Theory**

LEVEL: : **Undergraduate**

DATE : **24 April 2019**

TIME : **14:30**

TIME ALLOWED : **2 hrs 30 mins**

This paper is suitable for candidates who attended classes for this module in the following academic year(s):

Year
2018-19

EXAMINATION PAPER CANNOT BE REMOVED FROM THE EXAM HALL. PLACE EXAM PAPER AND ALL COMPLETED SCRIPTS INSIDE THE EXAMINATION ENVELOPE

Hall Instructions	.
Standard Calculators	N
Non-Standard Calculators	N

TURN OVER

Computability and Complexity Theory, COMP0017

Main Summer Examination Period, 2018/19

There are TWO questions. Answer ALL TWO questions.

Marks for each part of each question are indicated in square brackets.

Calculators are NOT permitted.

1. In the sequel, let code$(-)$ be a computable injective function that encodes Turing machines into strings from $\{0,1\}^*$.

   a. (i) Define what it means for a language to be *decidable* and to be *recognisable*.

   (ii) Let $L \subseteq \{0,1\}^*$ be the language

   $$L = \{x \in \{0,1\}^* \mid x \text{ contains at least one occurrence of the symbol } 1\}$$

   For example $0000 \notin L$ but $00100 \in L$. Give the complete definition (as a tuple) of a Turing Machine that *decides* L. You may define the transition function using diagrams. (For full marks, pay attention to not using more states than needed.)

   (iii) Modify the Turing machine that you defined in the previous exercise into one that only *recognises* the language L.

   [15 marks]

   b. For each of the following languages, say (without proving it) if it is: (1) decidable; (2) undecidable but recognisable; (3) unrecognisable. Moreover, for each language, say whether Rice's theorem applies to that language.

   $$L_1 = \{y \in \{0,1\}^* \mid y = \text{code}(M) \text{ for some TM } M \text{ and } M \text{ does not halt on code}(M)\}$$
   $$L_2 = \{y \in \{0,1\}^* \mid y = \text{code}(M) \text{ for some TM } M \text{ and } M \text{ halts on at least one input.}\}$$
   $$L_3 = \{y \in \{0,1\}^* \mid y = \text{code}(M) \text{ for some TM } M \text{ and } M \text{ halts on input } 0.\}$$
   $$L_4 = \{y \in \{0,1\}^* \mid y = \text{code}(M) \text{ for some TM } M \text{ and } M \text{ has five states}\}$$
   $$L_5 = \{y \in \{0,1\}^* \mid y = \text{code}(M) \text{ for some TM } M \text{ and } M \text{ halts on strings of odd length.}\}$$

[Question 1 cont. over page]

[20 marks]

c. Consider the language

$$L = \{y \in \{0,1\}^\star \mid y = \text{code}(M) \text{ for some } \textbf{TM } M \text{ and } M \text{ halts on code}(M)\}.$$

Also recall the language $HALT$, defined as

$$HALT = \{\langle y, x\rangle \in \{0,1\}^\star \mid y = \text{code}(M) \text{ for some } \textbf{TM } M \text{ and } M \text{ halts on input } x.\}$$

- Prove that $HALT$ mapping-reduces to L, notation $HALT \leq L$.
- What does this mean for the decidability of L?
- Consider the complement L^- of L. Based on the fact that $HALT \leq L$, what can we infer on the decidability/recognisability of L^-?

[15 marks]

[TOTAL=50 marks]

2. a. Define the *Hamiltonian Circuit Problem* (HCP) for undirected graphs.

[4 marks]

b. Consider the *Multiplicative Circuit Problem* (MCP). An instance (X, f, k) consists of a set X (consisting of n nodes, say), a symmetric function $f : X \times X \rightarrow \mathbb{N}$ (i.e. $f(y, x) = f(x, y)$) and a threshhold $k \in \mathbb{N}$.
(X, f, k) is a yes instance if there is an enumeration of distinct elements $(x_0, x_1, \ldots, x_{n-1})$, covering X such that $f(x_0, x_1) \times f(x_1, x_2) \times \ldots \times f(x_{n-2}, x_{n-1}) \times f(x_{n-1}, x_0) \leq k$, it is a no instance otherwise. For each of the following functions $f : X \rightarrow \mathbb{N}$ find the minimum k such that (X, f, k) is a yes instance of GCP.

1.

f	0	1	2	3
0	1	2	3	2
1	2	1	5	0
2	3	5	1	1
3	2	0	1	1

2.

f	0	1	2	3
0	0	2	5	2
1		0	1	3
2			0	4
3				0

[6 marks]

c. In order terms, expressing your answer as a function of $|X|$, m and k where m is the maximum of $\{f(x, y) : x, y \in X\}$, how much space does it take to store an instance (X, f, k). You should use a binary encoding for any integers you need to store.

[7 marks]

d. Let $\leq_p$ denote p-time reduction. Prove that $\leq_p$ is transitive.

[4 marks]

[Question 2 cont. over page]

e. By writing a suitable non-deterministic algorithm or otherwise, prove that MCP belongs to NP.

[9 marks]

f. Define a p-time reduction $HCP \leq_p MCP$ and show that your reduction is correct.

[10 marks]

g. Assuming answers to previous questions and assuming that HCP is NP-complete, what if anything can you conclude about the complexity of MCP?

[10 marks]

[TOTAL=50 marks]