## 3. PRINCIPLES OF QM

### Axiomatic principles

**State vector axiom:** State vector at $t$ is ket $\psi(t)$, or $|\psi\rangle$.

**Probability axiom:** Given a system in state $|\psi\rangle$, a measurement will find it in state $|\phi\rangle$ with probability amplitude $\langle\phi|\psi\rangle$.

**Hermitian operator axiom:** Physical observable is represented by a linear and Hermitian operator.

**Measurement axiom:** Measurement of a physical observable results in eigenvalue of observable. Observable $\widehat{A}$, we have $\widehat{A}|a\rangle = a|a\rangle$, where $a$ is eigenvalue and $|a\rangle$ is eigenvector. Measurement of the physical quantity represented by $\widehat{A}$ collapses the state $|\psi\rangle$ before measurement into an eigenstate $|a\rangle$ of $\widehat{A}$.

**Time evolution axiom:** $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \widehat{H}|\psi(t)\rangle$, w/o consider $x$ or $p$.

### Vector space

State vector is neither in position nor momentum space. Basis vectors: $|0\rangle, |1\rangle, |n\rangle$

**Linearity** : Because the SE is linear, given two states $|\psi_1(t)\rangle$ and $|\psi_2(t)\rangle$, $|\psi(t)\rangle = c_1|\psi_1(t)\rangle + c_2|\psi_2(t)\rangle$ is also a sol. ($c$'s are complex).

**Properties of a vector space**

**Dual vector space** $c|\psi\rangle$ is mapped to $c * \langle\psi|$. Given a vector, $|\psi\rangle = \begin{bmatrix}\vdots\\ \alpha\\ \vdots\end{bmatrix}$, the dual vector is $\langle\psi| = \begin{bmatrix}\cdots & \alpha^* & \cdots\end{bmatrix}$.

Dual basis vectors are $\langle 0| = \begin{bmatrix}1 & 0 & \cdots\end{bmatrix}, \cdots, \langle n|\begin{bmatrix}0 & \cdots & 1\end{bmatrix}$.

**Inner product** : $\langle\phi|\psi\rangle = c$, where $c$ is complex.

$\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^* \to \langle\psi|\psi\rangle$ is real, positive, and finite for a normalizable ket vector. Can choose $\langle\psi|\psi\rangle = 1$. $\langle\psi_m|\psi_n\rangle = \delta_{mn}$

### Operators

A matrix operator $\widehat{A}$ acting on a state vector $|\psi\rangle$ transforms it into another state vector $|\phi\rangle$, $\widehat{A}|\psi\rangle = |\phi\rangle$. It is linear.

**Properties of operators**

**Hermitian conjugate (Hermitian adjoint) operator in the dual space**

Hermitian adjoint operator $\widehat{A}^\dagger$ acts on the dual vector $\langle\psi|$ from the right as $\langle\psi|\widehat{A}^\dagger\rangle$, where $\widehat{A}^\dagger = (\widehat{A})^{T*}$.

$$(\widehat{A}|\psi\rangle)^\dagger = |\psi\rangle^\dagger\widehat{A}^\dagger = \langle\psi|\widehat{A}^\dagger \qquad \langle\psi| = |\psi\rangle^\dagger \qquad \langle\psi|^\dagger = |\psi\rangle$$

$$(\widehat{A}\widehat{B})^\dagger = (\widehat{A}\widehat{B})^{T*} = (\widehat{B}^T\widehat{A}^T)^* = \widehat{B}^{T*}\widehat{A}^{T*} = \widehat{B}^\dagger\widehat{A}^\dagger, \quad (c\widehat{A})^\dagger = c^*\widehat{A}^\dagger$$

**Outer product operators** : $|\psi\rangle\langle\phi| \qquad [|\psi\rangle\langle\phi|]\chi\rangle = |\psi\rangle\langle\phi|\chi\rangle$

**Matrix elements of operators**

$$\langle\phi|\widehat{A}|\psi\rangle \text{ (complex num)}$$

Hermitian equiv to complex conj $\langle\phi|\widehat{A}|\psi\rangle^\dagger = \langle\psi|\widehat{A}^\dagger|\phi\rangle = \langle\phi|\widehat{A}|\psi\rangle^*$

**Hermitian operators** : $\widehat{A}^\dagger = \widehat{A}$, so given $\widehat{A}|\phi\rangle$ in the vector space, we have $\langle\psi|\widehat{A}^\dagger = \langle\phi|\widehat{A}$ in the dual vector space.

**Matrix elements of a Hermitian operator**

$$\langle\phi|\widehat{A}|\psi\rangle^\dagger = \langle\phi|\widehat{A}|\psi\rangle^* = \langle\psi|\widehat{A}^\dagger|\phi\rangle = \langle\psi|\widehat{A}|\phi\rangle$$

Hermitian operator, real expectation vals: $\langle\psi|\widehat{A}|\phi\rangle^* = \langle\psi|\widehat{A}|\phi\rangle \equiv \langle\widehat{A}\rangle$

Same result whether $\widehat{A}$ acts to right or left: $\langle\phi|\widehat{A}|\psi\rangle = \langle\phi|\widehat{A}^\dagger|\psi\rangle$

**Eigenvals and eigenvecs of Hermitian operators** : $\widehat{A}|a_n\rangle = a_n|a_n\rangle$

Normalized eigvecs $\langle a_m|a_n\rangle = \delta_{mn}$. Gram-Schmidt, degenerate evec.

**Completeness of eigenvector of a Hermitian operator** Set $|a_n\rangle$ is complete if $\sum_n|\langle a_n|\psi\rangle|^2 = 1$. $\sum_n|a_n\rangle\langle a_n| = 1$ (identity operator)

**Continuous spectra of a Hermitian operator** Hermitian operator $\widehat{A}$, $\widehat{A}|a\rangle = a|a\rangle$, where $a$ is continuous.

$\int da'\langle a'|\widehat{A}|a\rangle = a\int da'\langle a'|a\rangle = \int da' a'\langle a'|a\rangle \to \langle a'|a\rangle = \delta(a'-a)$

Continuous condition: $\int da|a\rangle\langle a| = 1$

**Gram-Schmidt orthogonalization procedure** Eigval (like energy level) is $n$-fold degenerate: $n$ states w same eigval.

Orthogonal eigenstates $\to$ no degeneracy.

1. Normalize each state and define $\alpha_i = \frac{\alpha_i}{\sqrt{\langle\alpha_i|\alpha_i\rangle}}$. 2. $|\alpha_1'\rangle = |\alpha_1\rangle$.

3. $|\alpha_2'\rangle = \frac{|\alpha_2\rangle - |\alpha_1\rangle\langle\alpha_1|\alpha_2\rangle}{\sqrt{\langle\alpha_2|\alpha_2\rangle - \langle\alpha_1|\alpha_2\rangle\langle\alpha_2|\alpha_1\rangle}} = \frac{|\alpha_2\rangle - |\alpha_1\rangle\langle\alpha_1|\alpha_2\rangle}{\sqrt{1-\langle\alpha_1|\alpha_2\rangle\langle\alpha_2|\alpha_1\rangle}}$

4. Subtract components of $|\alpha_3\rangle$ along $|\alpha_1\rangle$ and $|\alpha_2\rangle$, $|\alpha_3\rangle - |\alpha_1\rangle\langle\alpha_1|\alpha_3\rangle - |\alpha_2\rangle\langle\alpha_2|\alpha_3\rangle$, normalize and promote to $|\alpha_3'\rangle$. ...

### Position and momentum representation

$\widehat{\vec r}|\vec r\rangle = \vec r|\vec r\rangle \quad \langle\vec r'|\vec r\rangle = \delta^3(\vec r' - \vec r), \int d^3\vec r|\vec r\rangle\langle\vec r| = 1, \langle\vec r'|\widehat{\vec r}|\vec r\rangle = \vec r\delta^3(\vec r' - \vec r)$

$\widehat{\vec p}|\vec p\rangle = \vec p|\vec p\rangle \quad \langle\vec p'|\vec p\rangle = \delta^3(\vec p' - \vec p), \int d^3\vec p|\vec p\rangle\langle\vec p| = 1$

State vector $|\psi(t)\rangle$ in position space (scalar): $\langle\vec r|\psi(x,t)\rangle \equiv \psi(\vec r, t)$

$\langle\psi|\widehat{\vec p}|\psi\rangle = \frac{d}{dt}\langle\psi|\widehat{\vec r}|\psi\rangle m$

Representation of momentum operator in position space: $\widehat{\vec p} = -i\hbar\vec\nabla$.

$\langle x|\widehat p|x'\rangle = -i\hbar\frac{\partial}{\partial x}\delta(x-x') = -i\hbar\frac{\partial}{\partial x}\langle x|x'\rangle$.

$\widehat p = -i\hbar\frac{\partial}{\partial x}$ is Hermitian, $\frac{\partial}{\partial x}$ is not.

$\langle x|\widehat p|p\rangle = p\langle x|p\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|p\rangle$. The solution is $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}px}$.

In 3D, $\langle\vec r|\vec p\rangle = \frac{1}{(2\pi\hbar)^{3/2}}e^{\frac{i}{\hbar}\vec p\vec r}$.

We can write the normalized wavefunction of definite position in momentum space, $\langle p|x\rangle = \langle x|p\rangle^*$. So, $\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}px}$ (particle moving to the left, or with momentum $-p$, in the momentum space). $[x,p] = i\hbar$

**Operators and wavefunction in position representation** Position and momentum operators in pos space: $\widehat{\vec r} = \vec r$, $\widehat{\vec p} = -i\hbar\vec\nabla$.

$\widehat{\vec r}$ is Hermitian and $\langle\phi|\widehat{\vec r}^\dagger|\psi\rangle = \langle\phi|\widehat{\vec r}|\psi\rangle$.

$\widehat O(\widehat{\vec r}, \widehat{\vec p}) = \widehat O(\vec r, -i\hbar\vec\nabla)$

The expectation val of the observable should be indep of representation. In state $\psi(t)$, $\langle\widehat O\rangle = \langle\psi(t)|\widehat O|\psi(t)\rangle$.

Insert $\int d^2\vec r|\vec r\rangle\langle\vec r| = 1$ to get $\langle\widehat O\rangle = \int d^2\vec r\langle\psi(t)|\vec r\rangle\langle\vec r|\widehat O|\psi(t)\rangle$

$\psi(\vec r, t) = \langle\vec r|\psi(t)\rangle, \qquad \psi(\vec r,t)^* = \langle\vec r|\psi(t)\rangle^* = \langle\psi(t)|\vec r\rangle$,

$\langle\vec r|\widehat O|\psi(t)\rangle = \widehat O(\vec r, -i\hbar\vec\nabla)\psi(\vec r,t), \langle\widehat O\rangle = \int d^3\vec r\psi(\vec r,t)^*\vec O(\vec r, -i\hbar\vec\nabla)\psi(\vec r,t)$

**Operators and wavefunction in momentum representation** $\widehat{\vec r} = i\hbar\vec\nabla_{\vec p}$, or in 1D, $\widehat x = i\hbar\frac{\partial}{\partial p}$, $\widehat{\vec p} = \vec p$, where $\vec p^* = \vec p$.

$\widehat O(\widehat{\vec r}, \widehat{\vec p}) = \widehat O(i\hbar\vec\nabla_{\vec p}, \vec p)$

$\langle\widehat O\rangle = \langle\psi(t)|\widehat O|\psi(t)\rangle \to \langle\widehat O\rangle = \int d^2\vec p\langle\psi(t)|\vec p\rangle\langle\vec p|\widehat O|\psi(t)\rangle$.

$\psi(\vec p, t) = \langle\vec p|\psi(t)\rangle, \qquad \psi(\vec p,t)^* = \langle\vec p\psi(t)\rangle^* = \langle\psi(t)|\vec p\rangle$

$\langle\vec p|\widehat O|\psi(t)\rangle = \widehat O(i\hbar\vec\nabla_{\vec p}, \vec p), \langle\widehat O\rangle = \int d^3\vec p\psi(\vec p,t)^*\widehat O(i\hbar\vec\nabla_{\vec p}, \vec p)\psi(\vec p,t)$.

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$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \widehat H|\psi(t)\rangle$, where $\widehat H = \frac{\widehat{\vec p}^2}{2m} + V(\widehat{\vec r}, t)$ becomes

$i\hbar\frac{\partial\psi(\vec r,t)}{\partial t} = -\frac{\hbar^2}{2m}\vec\nabla^2\psi(\vec r,t) + V(\vec r,t)\psi(\vec r,t)$

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### Commuting operators

If $[\widehat A, \widehat B] = 0$ and the states are nondegenerate, $|\psi\rangle$ is a simultaneous eigenstate of $\widehat A$ and $\widehat B$.

$|\psi\rangle = |ab\rangle$, and $\widehat A|ab\rangle = a|ab\rangle$, $\widehat B|ab\rangle = b|ab\rangle$

### Non-commuting operators and the general uncertainty principle

$(\Delta A)^2(\Delta B)^2 \geq (\frac{1}{2i}\langle[\widehat A, \widehat B]\rangle)^2$

Cannot construct simulatneous eigenstates (which correspond to definite eigenvalues) of non-commuting observables.

### Time evolution of expectation value of an operator and Ehrenfest's theorem

Ehrenfest's theorem: how observable $\widehat O$'s expectation value in state $|\psi(t)\rangle$ evolves in time, $\frac{d}{dt}\langle\widehat O\rangle = \langle\frac{\partial\widehat O}{\partial t}\rangle + \frac{i}{\hbar}\langle[\widehat H, \widehat O]\rangle$. If operator has no explicit time dep, $\frac{d}{dt}\langle\widehat O\rangle = \frac{1}{i\hbar}\langle[\widehat O, \widehat H]\rangle$.

For $\widehat O = \widehat{\vec p}$ and a Hamiltonian that is TI, $\frac{d}{dt}\langle\widehat{\vec p}\rangle = -\langle\vec\nabla V(\widehat{\vec r})\rangle$, which is just Newton's Second Law! $\to$ QM contains all of classical mech.

### The simple harmonic oscillator

$\widehat H = \frac{\widehat p^2}{2m} + \frac{1}{2}m\omega^2\widehat x^2$

**Raising and lowering operators** Lowering op: $\widehat a = \sqrt{\frac{m\omega}{2\hbar}}(\widehat x + \frac{i}{m\omega}\widehat p)$, Raising op: $\widehat a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\widehat x - \frac{i}{m\omega}\widehat p)$.

$[\widehat a, \widehat a^\dagger] = 1 \qquad \widehat x = \sqrt{\frac{\hbar}{2m\omega}}(\widehat a^\dagger + \widehat a)$, $\widehat p = i\sqrt{\frac{m\omega\hbar}{2}}(\widehat a^\dagger - \widehat a)$

$\widehat H = (\widehat N + \frac{1}{2})\hbar\omega$, where $\widehat N = \widehat a^\dagger\widehat a$. Now $\widehat N$ is Hermitian, and $\widehat N|n\rangle = n|n\rangle$.

$[\widehat N, \widehat a] = -\widehat a$, $[\widehat N, \widehat a^\dagger] = \widehat a^\dagger$

$\widehat N(\widehat a|n\rangle) = (n-1)(\widehat a|n\rangle)$, $\widehat N(\widehat a^\dagger|n\rangle) = (n+1)(\widehat a^\dagger|n\rangle)$

**Normalized number state vectors** Energy levels are not degenerate, so $|n-1\rangle = c_n\widehat a|n\rangle \to c_n = \frac{1}{\sqrt n} \to \widehat a|n\rangle = \sqrt n|n-1\rangle$.

$|n+1\rangle = d_n\widehat a^\dagger|n\rangle \to d_n = \frac{1}{\sqrt{n+1}} \to \widehat a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

Ground state: $|0\rangle$, excited state: $|n\rangle = \frac{(\widehat a^\dagger)^n}{\sqrt{n!}}|0\rangle$, $n = 0, 1, 2, \ldots$

$\langle n'|\widehat x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n'|(\widehat a^\dagger + \widehat a)|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n+1}\delta_{n',n+1} + \sqrt n\delta_{n',n-1})$

$\langle n'|\widehat p|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\langle n'|(\widehat a^\dagger - \widehat a)|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}(\sqrt{n+1}\delta_{n',n+1} - \sqrt n\delta_{n',n-1})$

**Wavefunctions in position representation** $E_n = (n+\frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$

The stationary wavefunctions of definite energy: $\psi_n(x) = \langle x|n\rangle$

$\langle x'|\widehat a^\dagger|x''\rangle = \delta(x'-x'')\frac{1}{\sqrt2\sigma}(x'' - \sigma^2\frac{\partial}{\partial x''})$, where $\sigma \equiv \sqrt{\frac{\hbar}{m\omega}}$

$\xi = \frac{x}{\sigma}, \qquad \langle x|n\rangle = \frac{1}{\sqrt{\sqrt\pi n!2^n\sigma}}(\xi - \frac{\partial}{\partial\xi})^n e^{-\frac{1}{2}\xi^2}$

$\langle x|0\rangle = (\frac{m\omega}{\pi\hbar})^{1/4}e^{-\frac{m\omega}{2\hbar}x^2}, \qquad \langle x|1\rangle = \sqrt2(\frac{m^3\omega^3}{\pi\hbar^3})^{1/4}xe^{-\frac{m\omega}{2\hbar}x^2}$

**Classical simple harmonic oscillator** Hamiltonian of a simple harmonic is

$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2. \qquad \dot x = \frac{\partial H}{\partial p} = \frac{p}{m}, \qquad \dot p = -\frac{\partial H}{\partial x} = -m\omega^2x$

Define $\sqrt{\hbar\omega}\alpha = \sqrt{\frac{m\omega^2}{2}}x + \frac{i}{\sqrt{2m}}p$, so $x = \sqrt{\frac{2\hbar}{m\omega}}\alpha_R$ and $p = \sqrt{2m\hbar\omega}\alpha_I$

Rewrite Hamiltonian, $H = \hbar\omega|\alpha|^2, \qquad \dot\alpha = -i\omega\alpha$. The sol is $\alpha = \alpha_0e^{-i\omega t}$.

**The quantum simple harmonic oscillator and coherent state** Coherent state, superpos of stat states $|n\rangle$: $|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^\infty\frac{\alpha^n}{\sqrt{n!}}|n\rangle$

$P(n) = |\langle n|\alpha\rangle|^2 = |\alpha_n|^2 = \frac{\langle n\rangle^n e^{-\langle n\rangle}}{n!}$, where $\langle n\rangle = \langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2$.

## 4. 3D SYSTEMS

### Three-dimensional infinite square well

$-\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})\psi(x,y,z) = E\psi(x,y,z)$ for $0 \leq x \leq l_x, \ldots$ while $\psi(x,y,z) = 0$ outside.

Separation of vars: $\psi(x,y,z) = \psi_1(x)\psi_2(y)\psi_3(z)$

$\to$ SE becomes $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_1(x) = E_1\psi_1(x), \ldots$, where $E = E_1 + E_2 + E_3$.

$\psi_{n_xn_yn_z}(x,y,z) = \sqrt{\frac{8}{l_xl_yl_z}}\sin\left(\frac{n_x\pi}{l_x}x\right)\sin\left(\frac{n_y\pi}{l_y}z\right)\sin\left(\frac{n_z\pi}{l_z}z\right)$

$E_{n_xn_yn_z} = \frac{\hbar^2\pi^2}{2m}(\frac{n_x^2}{l_x^2} + \frac{n_y^2}{l_y^2} + \frac{n_z^2}{l_z^2})$, with $n_x, n_y, n_z = 1, 2, \ldots$.

Wave vector: $\vec k = (k_x, k_y, k_z) = (\frac{n_x\pi}{l_x}, \frac{n_y\pi}{l_y}, \frac{n_z\pi}{l_z})$

### The Schrödinger equation in spherical coordinates

$i\hbar\frac{\partial\psi(\vec r,t)}{\partial t} = -\frac{\hbar^2}{2m}\vec\nabla^2\psi(\vec r,t) + V(\vec r)\psi(\vec r,t)$, where $\vec r = (r,\theta,\phi)$,

$\psi(\vec r,t) = \psi(r,\theta,\phi,t)$ and $\vec\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$.

For a TI and central potential, potential depends only on $r$, $V(\vec r) = V(r)$.

$\frac{1}{R(r)}[\frac{d}{dr} - \frac{2mr^2}{\hbar^2}(V(r) - E)] = -\frac{1}{Y(\theta,\phi)}[\frac{1}{\sin\theta}\frac{d}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2Y(\theta,\phi)}{d\phi^2}]$

Each side must be constant and equal (let it be $l(l+1)$).

$\frac{1}{\sin\theta}\frac{d}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2Y(\theta,\phi)}{d\phi^2} = -l(l+1)Y(\theta,\phi)$

$\frac{d}{dr} - \frac{2mr^2}{\hbar^2}(V(r) - E) = l(l+1)R(r)$

### Orbital angular momentum

$\widehat L_x = \widehat y\widehat p_z - \widehat z\widehat p_y, \widehat L_y = \widehat z\widehat p_x - \widehat x\widehat p_z, \widehat L_z = \widehat x\widehat p_y - \widehat y\widehat p_x$

$[\widehat L_i, \widehat L_j] = i\hbar\epsilon_{ijk}\widehat L_k$, with $i = 1, 2, 3$ representing the $x$, $y$, and $z$ components, and $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, which is -1 for odd perms of indices, and vanishes when repeated.

$\widehat L^2 = \widehat L_x^2 + \widehat L_y^2 + \widehat L_z^2$, $[\widehat{L^2}, \widehat L_i] = 0$

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In pos rep, $\widehat{\vec L} = \widehat{\vec r}\times\widehat{\vec p} = -i\hbar\vec r\times\vec\nabla$. In sph coords,

$\widehat{\vec L} = -i\hbar r\widehat r\times(\frac{\partial}{\partial r}\widehat r + \frac{1}{r}\frac{\partial}{\partial\theta}\widehat\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\widehat\phi = -i\hbar(\widehat\phi\frac{\partial}{\partial\theta} - \widehat\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi})$

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$\widehat r = \sin\theta\cos\psi\widehat x + \sin\theta\sin\phi\widehat y + \cos\theta\widehat z$

$\widehat\theta = \cos\theta\cos\phi\widehat x + \cos\theta\sin\phi\widehat y - \sin\theta\widehat z \qquad \widehat\phi = -\sin\phi\widehat x - \cos\phi\widehat y$

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$\widehat L_x = i\hbar(\sin\theta\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}) \qquad \widehat L_y = i\hbar(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi})$

$\widehat L_z = -i\hbar\frac{\partial}{\partial\phi} \qquad \widehat L^2 = -\hbar^2[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}]$

$\widehat L^2Y(\theta,\phi) = l(l+1)\hbar^2Y(\theta,\phi)$

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$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr} - V_{\text{eff}}(r)R(r) = ER(r)$, $V_{\text{eff}}(r) = V(r) + \frac{l(l+1)\hbar^2}{2mr^2}$, centrifugal

**Spherical harmonics** Find sols to angular eqn. Sep vars $Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$.
$\frac{1}{\Theta}[\sin\theta\frac{d}{d\theta}+l(l+1)\sin^2\theta=-\frac{1}{\Theta}\frac{d^2\Phi}{d\phi^2}=\text{constant}=m^2$
$\Phi(\phi)=e^{im\phi}$, periodic in $\phi$ w period $2\pi$ gives constraint $m=0,\pm1,\pm2,\cdots$
$\Theta(\theta)$ can be written in terms of $x\equiv\cos\theta$ as
$(1-x^2)\frac{d^2P(x)}{dx^2}-2x\frac{dP(x)}{dx}+(l(l+1)-\frac{m^2}{1-x^2})P(x)=0$

Associated Legendre functions: $P_l^{m_l}(x)=(1-x^2)^{|m_l|/2}(\frac{d}{dx})^{|m_l|}P_l(x)$,
where $P_l(x)$ is the $l^{th}$ Legendre polynomial given by the Rodrigues formula
$P_l(x)=\frac{1}{2^l l!}(\frac{d}{dx})^l(x^2-1)^l$, with $l$ taking values $l=0,1,2,...$
and for each $l$, $m_l$ takes $2l+1$ values $m_l=-l,-l+1,...,l-1,l$.

Spherical harmonics, normalized angular wave functions:
$Y_l^m(\theta,\phi)=\epsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}e^{im\phi}P_l^m(\cos\theta)$, where $\epsilon=(-1)^m$ for $m\ge0$ and $\epsilon=1$ for $m\le0$.

$\hat{L}^2Y_l^{m_l}=l(l+1)\hbar^2Y_l^{m_l}$, $\hat{L}_zY_l^{m_l}=m\hbar Y_l^{m_l}$

The Legendre polynomials are normalized s.t. they satisfy the ortho relation
$\int_{-1}^1 P_{l'}P_l(x)dx=\int_0^\pi P_{l'}(\theta)P_l(\theta)\sin\theta d\theta=\frac{2}{2l+1}\delta_{l'l}$

$P_0^0(\theta)=1, P_1^1(\theta)=\sin\theta, P_1^0(\theta)=\cos\theta$, with $P_l^{-m_l}(x)=P_l^{m_l}(x)$

$\int_{-1}^1 P_{l'}^{m'}(x)P_l^m(x)dx=\int_0^\pi P_{l'}^{m'}(\theta)P_l^m(\theta)\sin\theta d\theta=\frac{(l+m)!}{(2l+1)(l-m)!}\delta_{l'l}\delta_{m',m}$
Satisfy the orthogonality relation
$\int_0^{2\pi}d\phi\int_0^\pi d\theta\sin\theta Y_{l'}^{m_l'*}(\theta,\phi)Y_l^{m_l}(\theta,\phi)=\delta_{l'l}\delta_{m_l'm_l}$

$\hat{L}^2|lm_l\rangle=l(l+1)\hbar^2|lm_l\rangle$, $\hat{L}_z|lm_l\rangle=m\hbar|lm_l\rangle$
$\hat{L}_+=L_x+iL_y$, $\hat{L}_-L_x-iL_y$, $L_x=\frac12(L_-+L_+)$, $\langle L_x^2\rangle=\frac12\langle L^2-L_z^2\rangle$
$L_\pm|lm\rangle=\hbar\sqrt{(l\mp m)(l\pm m+1)}|l,m\pm1\rangle$
Spherical harmonics are the wavefunctions in pos rep, $Y_l^{m_l}(\theta,\phi)=\langle\vec r|lm_l\rangle$

**Parity of the spherical harmonics**
$\hat P\psi(x,y,z)=\psi(-x,-y,-z)$, $\hat P\psi(r,\theta,\phi)=\psi(r,\pi-\theta,\phi+\theta)$
For the Legendre polynomials, $\hat PP_l^{m_l}(\theta)=(-1)^{l-|m_l|}P_l^{m_l}(\theta)$
$\to$ even for $l+|m_l|$ even and odd for $l+|m_l|$ odd.
Azimuthal part of the wavefunction, $\hat Pe^{im_l\phi}=e^{im_l(\phi+\pi)}=(-1)^{m_l}e^{im_l\phi}$.
The spherical harmonics are products of two, and $\hat PY_l^{m_l}(\theta,\phi)=$
$Y_l^{m_l}(\pi-\theta,\phi+\pi)=(-1)^{l-|m_l|+m_l}Y_l^{m_l}(\theta,\phi)=(-1)^lY_l^{m_l}(\theta,\phi)$

## The hydrogen atom

Coulomb's law, $\hat V=-\frac{e^2}{4\pi\epsilon_0}\frac1r$
Let $u(r)\equiv rR(r)$, Radial eq: $-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+[-\frac{e^2}{4\pi\epsilon_0}\frac1r+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}]u=Eu$

**The radial wave function**
$\kappa\equiv\frac{\sqrt{-2mE}}{\hbar}$. Divide by $E$, $\frac{1}{\kappa^2}\frac{d^2u}{dr^2}=[1-\frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{(\kappa r)}+\frac{l(l+1)}{(\kappa r)^2}]u$
Introduce $\rho\equiv\kappa r$, $\rho_0\equiv\frac{me^2}{2\pi\epsilon\hbar^2\kappa}$, $\frac{d^2u}{d\rho^2}=[1-\frac{\rho_0}{\rho}+\frac{l(l+1)}{\rho^2}]u$

As $\rho\to\infty$, the constant term in the brackets dominates, so $\frac{d^2u}{d\rho^2}=u$.
General sol is $u(\rho)=Ae^{-\rho}+Be^\rho$, but $B=0\to u(\rho)=Ae^{-\rho}$ for large $\rho$.

As $\rho\to0$, centriugal term dominates, $\frac{d^2u}{d\rho^2}=\frac{l(l+1)}{\rho^2}u$
The general sol is $u(\rho)=C\rho^{l+1}+D\rho^{-l}$, but $\rho^{-l}$ blows up as $\rho\to0$, so $D=0$. Thus, $u(\rho)\approx Cp^{l+1}$ for small $\rho$.

Peel off the asymptotic behavior, let $u(\rho)=\rho^{l+1}e^{-\rho}v(\rho)$
$\frac{du}{d\rho}=\rho^le^{-\rho}[(l+1-\rho)v+\rho\frac{dv}{d\rho}]$
$\frac{d^2u}{d\rho^2}=\rho^le^{-\rho}\{[-2l-2+\rho+\frac{l(l+1)}{\rho}]v+2(l+1-\rho)\frac{dv}{d\rho}+\rho\frac{d^2v}{d\rho^2}\}$
Radial eq in terms of $v(\rho)$, $\rho\frac{d^2v}{d\rho^2}+2(l+1-\rho)\frac{dv}{d\rho}+[\rho_0-2(l+1)]v=0$

Assume $v(p)$ can be expressed as a power series in $\rho$: $v(\rho)=\sum_{j=0}^\infty c_j\rho^j$.
$\frac{dv}{d\rho}=\sum_{j=0}^\infty jc_j\rho^{j-1}=\sum_{j=0}^\infty(j+1)c_{j+1}\rho^j$,
$\frac{d^2v}{d\rho^2}=\sum_{j=0}^\infty j(j+1)c_{j+1}\rho^{j-1}$
$j(j+1)c_{j+1}+2(l+1)(j+1)c_{j+1}-2jc_j+[\rho_0-2(l+1)]c_j=0$
$c_{j+1}=\frac{2(j+l+1)-\rho_0}{(j+1)(j+2l+2)}c_j$

For large $j$ (corresponding to large $\rho$), $c_{j+1}=\frac{2j}{j(j+1)}c_j=\frac{2}{j+1}c_j$
If this were exact, $c_j=\frac{2^j}{j!}c_0$, $v(\rho)=c_0\sum_{j=0}^\infty\frac{2^j}{j!}\rho^j=c_0e^{2\rho}$, and hence $u(\rho)=c_0\rho^{l+1}e^\rho$, which blows up at large $\rho$
$\exists c_{j_{max}+1}=0$, so $2(j_{max}+l+1)-\rho_0=0$.

Define principle quantum number, $n\equiv j_{max}+l+1$, so $\rho_0=2n$
$E=-\frac{\hbar^2\kappa^2}{2m}=-\frac{me^3}{8\pi^2\epsilon_0^2\hbar^2\rho_0^2}$
Bohr formula: $E_n=-[\frac{m}{2\hbar^2}(\frac{e^2}{4\pi\epsilon})^2]\frac{1}{n^2}=\frac{E_1}{n^2}=\frac{-13.6\text{ eV}}{n^2}$, $n=1,2,3,...$
$\kappa=(\frac{me^2}{4\pi\epsilon_0\hbar^2})\frac1n=\frac{1}{an}$, Bohr radius: $a\equiv\frac{4\pi\epsilon_0\hbar^2}{me^2}=0.529\times10^{-10}\text{m}$
$\psi_{nlm}(r,\theta,\phi)=R_{nl}(r)Y_l^m(\theta,\phi)$, $\psi_{100}(r,\theta,\phi)=\frac{1}{\sqrt{\pi a^3}}e^{-r/a}$

For arbitrary $n$, $l=0,1,...,n-1$, so $d(n)=2\sum_{l=0}^{n-1}(2l+1)=2n^2$
$v(\rho)=L_{n-l-1}^{2l+1}(2\rho)$, where $L_{q-p}^p(x)\equiv(-1)^p(\frac{d}{dx})^pL_q(x)$ is an associated Laguerre polynomial. $L_q(x)\equiv e^x(\frac{d}{dx})^q(e^{-x}x^q)$ is the $q$th Lag. poly.
Normalized hydrogen wavefunctions:
$$\psi_{nlm}=\sqrt{(\frac{2}{na})^3\frac{(n-l-1)!}{2n[(n+1)!]^3}}e^{-r/na}(\frac{2r}{na})^l[L_{n-l-1}^{2l+1}(2r/na)Y_l^m(\theta,\phi)$$
Wavefunctions are mutually orthogonal.
$\int\psi*_{n'l'm_l'}\psi_{nlm_l}r^2\sin\theta drd\theta d\phi=\delta_{n'n}\delta_{l'l}\delta_{m_l'm_l}$

**Spectrum** Transitions: $E_\gamma=E_i-E_f=-13.6eV(\frac{1}{n_i^2}-\frac{1}{n_f^2})$
Planck formula, $E_\gamma=h\nu$, wavefunction is $\lambda=c/\nu$.
Rydberg: $\frac1\lambda=R(\frac{1}{n_f^2}-\frac{1}{n_i^2})$, $R\equiv\frac{m}{4\pi c\hbar^3}(\frac{e^2}{4\pi\epsilon_0})^2=1.097\times10^7\text{ m}^{-1}$

## General angular momentum

$\hat J=(\hat J_x,\hat J_y,\hat J_z)=(\hat J_1,\hat J_2,\hat J_3)$ $\quad\hat J^2=\hat J_x^2+\hat J_y^2+\hat J_z^2$
$[\hat J_i,\hat J_j]=i\hbar\epsilon_{ijk}\hat J_k$, $[\hat J^2,J_i]=0$
Take commuting set to be $\hat J^2$ and $\hat J_z$. Trade $\hat J_x$ and $\hat J_y$ for $\hat J_\pm=\hat J_x\pm i\hat J_y$
Commutation relations: $[\hat J_+,\hat J_-]=2\hbar\hat J_z$, $[\hat J_z,\hat J_\pm]=\pm\hbar\hat J_\pm$, $[\hat J^2,\hat J_\pm]=0$
$\hat J^2$ and $\hat J_z$ commute $\to$ we can simulaneously diagonalize them. Let the simultaneous eigenstate be $|ab\rangle$ s.t. $\hat J^2|ab\rangle=a|ab\rangle$, $\hat J_z|ab\rangle=b|ab\rangle$
$\hat J^2(\hat J_\pm|ab\rangle)=a(\hat J_\pm|ab\rangle)$ $\quad\hat J_z(\hat J_\pm|ab\rangle)=(b\pm\hbar)(\hat J_\pm|ab\rangle)$
$\hat J_+$ raises and $\hat J_-$ lowers the eigenvalue $b$ of $\hat J_z$. Assuming $|ab\rangle$ is normalized, $\hat J_\pm|ab\rangle=c_\pm|ab\pm\hbar\rangle$, where $c_\pm$ are normalization constants.

$\hat J_\pm\hat J_\mp=\hat J^2-\hat J_z^2\pm\hbar\hat J_z$
$0=\langle ab_{max}|\hat J_-\hat J_+|ab_{max}\rangle=a-b_{max}^2-\hbar b_{max}$, $0=a-b_{min}^2+\hbar b_{min}$
$b_{max}=\frac{-\hbar+\sqrt{\hbar^2+4a}}{2}$, $b_{min}=\frac{\hbar-\sqrt{\hbar^2+4a^2}}{2}$, $b_{max}=-b_{min}=j\hbar$, $j=0,\frac12,1,...$
$j\equiv\frac n2$, then $a=b_{max}^2+\hbar b_{max}=j^2\hbar^2+\hbar^2j=j(j+1)\hbar^2$
$\hat J_\pm|jm_j\rangle=\hbar\sqrt{(j\mp m_j)(j\pm m_j+1)}|jm_j\pm1\rangle$
$\langle j'm_j'|\hat J_\pm|jm_j\rangle=\hbar\sqrt{(j\mp m_j)(j\pm m_j+1)}\langle j'm_j'|jm_j\pm1\rangle=$
$\hbar\sqrt{(j\mp m_j)(j\pm m_j+1)}\delta_{j'j}\delta_{m_j'm_j\pm1}$

## Spin

**Classical orbital and spinning motion** Infinitesimal classical angular momentum corresponding to an infinite linear momentum $d\vec p=dm\vec v$ at position $\vec r$ from the axis of rotation is $d\vec L=\vec r\times d\vec p$
The total angular momentum is $\vec L=\int\vec r\times d\vec p=\int\vec r\times dm\vec v$
Point particle of mass $m$ at radius $r$ spinning w constant angular velocity $\omega$ about the $z$-axis, $\vec L=I\omega\hat z=m\omega r^2\hat z$
Considering a particle of mass $m$ and charge $q$ rotating with angular velocity $\omega$ at radius $r$ about the $z$-axis, the angular momentum $\vec L$ and the momentum dipole momentum $\vec\mu$ are given by $\vec L=m\omega r^2\hat z$, $\vec\mu=\frac q2\omega r^2\hat z$, where we used $\mu=I\pi r^2$ with current $I=\frac{q}{2\pi/\omega}=\frac{q\omega}{2\pi}$. Thus, $\vec\mu=\frac{q}{2m}\vec L$

**Spin** Electron: $j=\frac12$, $m_j=\pm\frac12$. Spin-$\frac12$: $s=\frac12$, use $\hat{\vec J}\to\hat{\vec S}$.
Basis vectors are $|\frac12,\frac12\rangle\equiv|1\rangle=\begin{bmatrix}1\\0\end{bmatrix}$, $|\frac12,-\frac12\rangle\equiv|2\rangle=\begin{bmatrix}0\\1\end{bmatrix}$

$\hat S_z$ and $\hat{\vec S}^2$ are diagonal, since simultaneously diagonalized. Matrix elements:
$\langle s'm_s'|\hat{\vec S}^2|sm_s\rangle=s(s+1)\hbar^2\delta_{s's}\delta_{m_s'm_s}$,
$\langle s'm_s'|\hat S_z|sm_s\rangle=m_s\hbar\delta_{s's}\delta_{m_s'm_s}$
$\hat{\vec S}^2=\frac34\hbar^2\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $\hat S_z=\frac\hbar2\begin{bmatrix}1&0\\0&-1\end{bmatrix}$. $\hat S_+=\hbar\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $\hat S_-=\frac\hbar2\begin{bmatrix}1&0\\0&-1\end{bmatrix}$,
$\hat S_x=\frac12(\hat S_++\hat S_-)$, $\hat S_y=\frac1{2i}(\hat S_+-\hat S_-)$, $\hat S_x=\frac\hbar2\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $\hat S_y=\frac\hbar2\begin{bmatrix}0&-i\\i&0\end{bmatrix}$

Spin angular momentum: $\vec S=\frac\hbar2\vec\sigma$. Pauli m: $\sigma_x=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $\sigma_y=\begin{bmatrix}0&-i\\i&0\end{bmatrix}$, $\sigma_z=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$
$[\hat S_i,\hat S_j]=i\hbar\epsilon_{ijk}\hat S_k$ and $[\sigma_i,\sigma_j]=2i\epsilon_{ijk}\sigma_k$
A general state of a spin-half system is given by a spinor,
$|\chi\rangle=\alpha|\frac12,\frac12\rangle+\beta|\frac12,\frac12\rangle=\begin{bmatrix}\alpha\\\beta\end{bmatrix}$, where $\alpha$ and $\beta$ are complex constants.
$a_x=\sin\theta\cos\phi, a_y=\sin\theta\sin\phi, a_z=\cos\theta$

**Magnetic moment of the electron** $\vec\mu=g\frac{q}{2m}\vec S$, gyromagentic factor (distribution of mass != charge). For the electron, $q=-e$, and $\vec\mu=-g\frac{e}{2m}\vec S$
$\hat{\vec\mu}=-g\frac{e}{2m}\hat{\vec S}=-\frac g2\frac{e\hbar}{2m}\vec\sigma=-\frac g2\mu_B\vec\sigma$, where $\mu_B=\frac{e\hbar}{2m}$ is Bohr magneton.

**Electron in a magnetic field** Intrinsic spin angular momentum $\to$ intrinsic magnetic moment. Energy from spin & external mag field: $\hat H=\hat V=-\hat{\vec\mu}\cdot\vec B$
For a magnetic field along the $z$-axis, $\vec B=B\hat z$, and
$\hat H=-\hat\mu_zB=-(-\frac g2\frac em\vec S)\dot B\hat z=\frac g2\frac{eB}{m}S_z=\omega_sS_z=\frac g2\frac{eB\hbar}{2m}\sigma_z$, where $\omega_s=\frac g2\frac{eB}{m}=\frac g2\omega_c$ is the spin precession (or Larmor) frequency and $w_c=\frac{eB}{m}$ is cyclotron frequency. $g\approx2$ but $g\ne2\to\omega_s\ne\omega_c$.

Rewrite Hamiltonian as $\hat H=\omega_sS_z$. In the bases in which $\hat{\vec S}$ and $\hat S_z$ are diagonalized, the eigenstates are given by
$\hat H|\frac12,\frac12\rangle=\omega_s\hat S_z|\frac12,\frac12\rangle=\frac12\hbar\omega_s|\frac12,\frac12\rangle$,
$\hat H|\frac12,-\frac12\rangle=\omega_s\hat S_z|\frac12,-\frac12\rangle=-\frac12\hbar\omega_s|\frac12,-\frac12\rangle$
Interaction of electron spin w external magentic field $\to$ energies $\pm\frac12\hbar\omega_s$.
Spin-up $|\frac12,\frac12\rangle$ & spin-down state $|\frac12,-\frac12\rangle$, with a gap of $\hbar\omega_s$ btwn them.
Consider $\vec B=B_x\hat e_x+B_y\hat e_y+B_z\hat e_z$.
$\hat H=(\frac g2\frac em\vec S)\cdot\vec B=\frac g2\frac{e\hbar}{2m}\begin{bmatrix}B_z&B_x-iB_y\\B_x+iB_y&-B_z\end{bmatrix}$
Eigenvals of matrix $\begin{vmatrix}B_z-\lambda&B_x-iB_y\\B_x+iB_y&-B_z-\lambda\end{vmatrix}=0$, which gives $\lambda=\pm B$,
where $B=|\vec B|$. Therefore, eigenvals of $\hat H$ are $\pm\frac g2\frac{e\hbar B}{2m}=\pm\frac12\hbar\omega_s$.

**The Stern-Gerlach experiment**
Force on electron w spin-up: $\vec F_1=-\vec\nabla V_1=\frac12\hbar\vec\nabla\omega_s=\frac g2\frac{e\hbar}{2m}\frac{\partial B(z)}{\partial z}$
Force on electron w spin-down: $\vec F_2=-\vec\nabla V_2=-\frac12\hbar\vec\nabla\omega_s=-\frac g2\frac{e\hbar}{2m}\frac{\partial B(z)}{\partial z}$
Electrons deflected up/down depending on whether spin-up/spin-down along $\vec B$.

**Spin precession** $|\chi(0)\rangle=\begin{bmatrix}a\\b\end{bmatrix}$, $|a|^2+|b|^2=1$ and $a=\cos\frac\alpha2$, $b=\sin\frac\alpha2$
$|\chi(0)\rangle=\cos\frac\alpha2|\frac12\frac12\rangle+\sin\frac\alpha2|\frac12-\frac12\rangle=\begin{bmatrix}\cos\frac\alpha2\\\sin\frac\alpha2\end{bmatrix}$, $|\chi(t)\rangle=\begin{bmatrix}e^{-\frac i2\omega_st}\cos\frac\alpha2\\e^{\frac i2\omega_st}\sin\frac\alpha2\end{bmatrix}$
$\langle\hat S_z\rangle=|e^{-\frac i2\omega_st}\cos\frac\alpha2|^2\frac\hbar2-|e^{-\frac i2\omega_st}\sin\frac\alpha2{}^2\frac\hbar2=(\cos^2\frac\alpha2-\sin^2\frac\alpha2)\frac\hbar2$
$\langle\hat S_x\rangle=\frac\hbar2\sin\alpha\cos\omega_st$, $\quad\langle\hat S_y\rangle\frac\hbar2\sin\alpha\sin\omega_st$, $\langle\hat S_z\rangle=\frac\hbar2\cos\alpha$
Angle $\alpha\rangle\pi-\alpha$ for spin-down. Spin-up, $\hat S_z$ eigenval is $\frac\hbar2$, $|\hat{\vec S}^2|$ is $\frac{\sqrt3\hbar}{2}$.
Space quantization: angular momentum along any fixed direction take only discrete $(2j+1)$ values.

## Addition of angular momentum

$\hat{\vec J}_1$, $|j_1,m_{j1}\rangle$. $\hat{\vec J}_2$, $|j_2,m_{j2}\rangle$. $\hat{\vec J}=\hat{\vec J}_1+\hat{\vec J}_2$. $\hat{\vec J}^2$ & $\hat{\vec J}_z$: sim diag set. $|j,m_j\rangle$

**Triplet and singlet states of a system of two spin-halves**
$|j_1,m_{j1}\rangle\otimes|j_2,m_{j2}\rangle$
The triplet states ($j=1$ multiplet): $|1,1\rangle=|\frac12,\frac12\rangle\otimes|\frac12,\frac12\rangle$,
$|1,0\rangle=\frac{1}{\sqrt2}(|\frac12,\frac12\rangle\otimes|\frac12,-\frac12\rangle+|\frac12,-\frac12\rangle\otimes|\frac12,\frac12\rangle)$,
$|1,-1\rangle=|\frac12,-\frac12\rangle\otimes|\frac12,-\frac12\rangle$
Singlet state ($j=0$): $|0,0\rangle=\frac1{\sqrt2}(|\frac12,\frac12\rangle\otimes|\frac12,-\frac12\rangle-|\frac12,-\frac12\rangle\otimes|\frac12,\frac12\rangle)$
$s=1,0$ out of $s_1$ and $s_2$ as $\frac12\otimes\frac12=1\oplus0$
$\hat{\vec J}^2=\hat{\vec J}_1^2\otimes1+1\otimes\hat{\vec J}_2^2+2\hat J_{1z}\otimes\hat J_{2z}+\hat J_{1+}\otimes\hat J_{2-}+\hat J_{1-}\otimes\hat J_{2+}$
Spin angular momentum, interchan. use $\hat{\vec S}$ for $\hat{\vec J}$, and $s$ and $m_s$ for $j$ and $m_j$.

**Addition of general angular momentum**
$|j_1+j_2,j_1+j_2\rangle=|j_1,m_{j1}\rangle\otimes|j_2,m_{j2}\rangle$
$j=j_1\otimes j_2=j_1+j_2\oplus j_1+j_2-1\oplus j_1+j_2-2\oplus\cdots\oplus|j_1-j_2|$

**Clebsch-Gordon coefficients**
Complete states: $\sum_{m_{j1},m_{j2}}|j_1,m_{j1};j_2,m_{j2}\rangle\langle j_1,m_{j1};j_2,m_{j2}|=1$
$|j,m_j\rangle=\sum_{m_j=m_{j1}+m_{j2}}\langle j_1,m_{j1};j_2,m_{j2}|j,m_j\rangle|j_1,m_{j1};j_2,m_{j2}\rangle$
where $\langle j_1,m_{j1};j_2,m_{j2};j,m_j\rangle$ are Clebsch-Gordon coefficients.