

3. Principles of QM

Axiomatic principles

State vector axiom: State vector at t is ket $|\psi(t)\rangle$, or $|\psi\rangle$, bra state.

Probability axiom: Given a system in state $|\psi\rangle$, a measurement will find it in state $|\phi\rangle$ with probability amplitude $\langle\phi|\psi\rangle$.

Hermitian operator axiom: Physical observable is represented by a linear and Hermitian operator.

Measurement axiom: Measurement of a physical observable results in eigenvalue of observable. Observable \hat{A} , we have $\hat{A}|a\rangle = a|a\rangle$, where a is eigenvalue and $|a\rangle$ is eigenvector. Measurement of the physical quantity represented by \hat{A} collapses the state $|\psi\rangle$ before measurement into an eigenstate $|a\rangle$ of \hat{A} .

Time evolution axiom: $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$, w/o consider x or p .

Vector space

State vector is neither in position nor momentum space.

$$\text{Basis vectors: } |0\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, |n\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \text{ (in } n\text{th pos).}$$

Linearity: Because the SE is linear, given two states $|\psi_1(t)\rangle$ and $|\psi_2(t)\rangle$, $|\psi(t)\rangle = c_1|\psi_1(t)\rangle + c_2|\psi_2(t)\rangle$ is also a sol. (c 's are complex).

Properties of a vector space

Dual vector space $c|\psi\rangle$ is mapped to $c^* \langle\psi|$. Given a vector,

$$|\psi\rangle = \begin{bmatrix} \vdots \\ \vdots \\ \alpha \\ \vdots \end{bmatrix}, \text{ the dual vector is } \langle\psi| = [\cdots \quad \alpha^* \quad \cdots].$$

Dual basis vectors are $\langle 0| = [1 \quad 0 \quad \cdots]$, \cdots , $\langle n| = [0 \quad \cdots 1]$.

Inner product: $\langle\phi|\psi\rangle = c$, where c is complex.

$\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^* \rightarrow \langle\psi|\psi\rangle$ is real, positive, and finite for a normalizable ket vector. Can choose $\langle\psi|\psi\rangle = 1$. $\langle\psi_m|\psi_n\rangle = \delta_{mn}$

Operators

A matrix operator \hat{A} acting on a state vector $|\psi\rangle$ transforms it into another state vector $|\phi\rangle$, $\hat{A}|\psi\rangle = |\phi\rangle$. It is linear.

Properties of operators

Hermitian conjugate (Hermitian adjoint) operator in the dual space

Hermitian adjoint operator \hat{A}^\dagger acts on the dual vector $\langle\psi|$ from the right as $\langle\psi|\hat{A}^\dagger\rangle$, where $\hat{A}^\dagger = (\hat{A})^{T*}$.

$$(\hat{A}|\psi\rangle)^\dagger = |\psi\rangle^\dagger \hat{A}^\dagger = \langle\psi|\hat{A}^\dagger \quad \langle\psi| = |\psi\rangle^\dagger \quad \langle\psi|^\dagger = |\psi\rangle \\ (\hat{A}\hat{B})^\dagger = (\hat{A}\hat{B})^{T*} = (\hat{B}^T \hat{A}^T)^* = \hat{B}^{T*} \hat{A}^{T*} = \hat{B}^\dagger \hat{A}^\dagger, \quad (c\hat{A})^\dagger = c^* \hat{A}^\dagger$$

Outer product operators

$$|\psi\rangle\langle\phi| \quad [|\psi\rangle\langle\phi||\chi\rangle = |\psi\rangle\langle\phi|\chi\rangle]$$

Matrix elements of operators

$$\langle\phi|\hat{A}|\psi\rangle \text{ (complex num)}$$

Hermitian equiv to complex conj $\langle\phi|\hat{A}|\psi\rangle^\dagger = \langle\psi|\hat{A}^\dagger|\phi\rangle = \langle\phi|\hat{A}|\psi\rangle^*$

Hermitian operators: $\hat{A}^\dagger = \hat{A}$, so given $\hat{A}|\phi\rangle$ in the vector space, we have $\langle\psi|\hat{A}^\dagger = \langle\phi|\hat{A}$ in the dual vector space.

Matrix elements of a Hermitian operator

$$\langle\phi|\hat{A}|\psi\rangle^\dagger = \langle\phi|\hat{A}|\psi\rangle^* = \langle\psi|\hat{A}^\dagger|\phi\rangle = \langle\psi|\hat{A}|\phi\rangle$$

Hermitian operator, real expectation vals: $\langle\psi|\hat{A}|\phi\rangle^* = \langle\psi|\hat{A}|\phi\rangle \equiv \langle\hat{A}\rangle$

Same result whether \hat{A} acts to right or left: $\langle\phi|\hat{A}|\psi\rangle = \langle\phi|\hat{A}^\dagger|\psi\rangle$

Eigenvalues and eigenvector of Hermitian operators

$$\hat{A}|a_n\rangle = a_n|a_n\rangle, \text{ where } a_n \text{ is real}$$

With normalized eigenvectors, $\langle a_m|a_n\rangle = \delta_{mn}$. Use Gram-Schmidt if system has degenerate eigenvectors.

Completeness of eigenvector of a Hermitian operator

The set of $|a_n\rangle$ is complete if $\sum_n |\langle a_n|\psi\rangle|^2 = 1$.

$$\sum_n |a_n\rangle\langle a_n| = 1 \text{ (identity operator)}$$

Continuous spectra of a Hermitian operator

Gram-Schmidt orthogonalization procedure

Position and momentum representation

Operators and wavefunction in position representation

Operators and wavefunction in momentum representation

Commuting operators

Non-commuting operators and the general uncertainty principle

Time evolution of expectation value of an operator and Ehrenfest's theorem

The simple harmonic oscillator

Raising and lowering operators

Normalized number state vectors

Wavefunctions in position representation

Classical simple harmonic oscillator

The quantum simple harmonic oscillator and coherent state

4. Three-dimensional systems

Three-dimensional infinite square well

The Schrödinger equation in spherical coordinates

Orbital angular momentum

Spherical harmonics