THE COMPLETED STANDARD L-FUNCTION OF MODULAR FORMS ON G_2

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ABSTRACT. Modular forms on the split exceptional group G_2 over $\mathbb{Q}$ are a special class of automorphic forms on this group, which were introduced by Gan, Gross, and Savin. If π is a cuspidal automorphic representation of $G_2(\mathbb{A})$ corresponding to a level one, even weight modular form φ on G_2, we define an associated completed standard L-function, $\Lambda(\pi, Std, s)$. Assuming that a certain Fourier coefficient of φ is nonzero, we prove the functional equation $\Lambda(\pi, Std, s) = \Lambda(\pi, Std, 1-s)$. The proof proceeds via a careful analysis of a Rankin-Selberg integral due to Gurevich and Segal.

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1. INTRODUCTION

1.1. **History.** Let G_2 denote the split exceptional linear algebraic group over $\mathbb{Q}$ of Dynkin type G_2, and suppose that π is a cuspidal automorphic representation of $G_2(\mathbb{A})$. The study of L-functions associated to π has a substantial history. Piatetski–Shapiro, Rallis and Schiffmann [11] first studied such an L-function by constructing a Rankin-Selberg integral for the tensor product L-function of π and a cuspidal automorphic representation on GL_2. Their result applies to π that are globally generic, i.e., those π that admit a nonvanishing Whittaker coefficient. Then Ginzburg [4] proved that the partial standard L-function $L^S(\pi, Std, s)$ has a meromorphic continuation with at most a simple pole, again for generic π, by using an appropriate Rankin-Selberg integral.

Ginzburg and Hundley [5], Gurevich and Segal [9] and then Segal [14] constructed Rankin integrals that apply to cuspidal representations π that are not necessarily generic. In fact, it is proved in [14] that the partial standard L-function of an arbitrary such π admits meromorphic continuation in s. However, bounding the poles of the L-function $L^S(\pi, Std, s)$ in a left half-plane and proving a precise functional equation relating its values at s to the values at $1-s$ are difficult problems in general. This is, in part, because of the difficulty of analyzing local L-functions and local zeta integrals at ramified finite places and at the Archimedean place.

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1.2. **Statement of theorems.** The purpose of this paper is to make a refined study of the L-functions $L(\pi, Std, s)$ for certain non-generic cuspidal automorphic representations $\pi = \pi_f \otimes \pi_\infty$ of $G_2(\mathbb{A})$. In more detail, recall that Gan, Gross and Savin [3] have defined *modular forms* on G_2. Said briefly, these are automorphic forms on $G_2(\mathbb{A})$ that correspond to $\pi = \pi_f \otimes \pi_\infty$ where π_∞ is a certain quaternionic discrete series representation of $G_2(\mathbb{R})$. Let K denote a maximal compact subgroup of $G_2(\mathbb{R})$, so that $K \simeq (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm 1\}$ with the first copy of $\mathrm{SU}(2)$ being the "long root" $\mathrm{SU}(2)$ and the second being the "short root" $\mathrm{SU}(2)$. Then for $\ell \geq 2$, there is a discrete series representation $\pi_{\ell,\infty}$ of $G_2(\mathbb{R})$ whose minimal K-type is $Sym^{2\ell}(\mathbb{C}^2) \boxtimes \mathbf{1}$, as a representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. The representations $\pi_{\ell,\infty}$ are not generic.

If $\ell \geq 2$ is even, we define the archimedean L-factor

$$L_\infty(\pi_{\ell,\infty}, s) = \Gamma_{\mathbb{C}}(s + \ell - 1)\Gamma_{\mathbb{C}}(s + \ell)\Gamma_{\mathbb{C}}(s + 2\ell - 1)\Gamma_{\mathbb{R}}(s + 1).$$

Here[1] $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Suppose now that $\pi = \pi_f \otimes \pi_{\ell,\infty}$ with π_f unramified at every finite place and $\ell \geq 2$ is even. Then the L-function $L(\pi, Std, s)$ and the completed L-function $\Lambda(\pi, Std, s) = L_\infty(\pi_{\ell,\infty}, s)L(\pi, Std, s)$ are defined. Our main results concern these L-functions.

To state the results, recall that such a π has an associated weight ℓ, level one cuspidal modular form φ_π on G_2. It is proved in [3] that these φ_π have Fourier coefficients $a_{\varphi_\pi}(T)$, where T is cubic ring such that $T \otimes \mathbb{R} \simeq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Theorem 1.1. *Suppose that φ is a level one cuspidal modular form on G_2 of even weight $\ell > 0$ that generates the cuspidal automorphic representation π. Further, assume that the Fourier coefficient of φ corresponding to the split cubic ring $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is nonzero. Then*

$$\Lambda(\pi, Std, s) = \Lambda(\pi, Std, 1 - s).$$

The proof of this theorem rests on a refined analysis of the Rankin integral in [9]. In fact, a Dirichlet series for the L-function $L(\pi, Std, s)$ follows from our proof of Theorem 1.1.

Corollary 1.2. *Let the assumptions be as in Theorem 1.1, and let $a_\varphi(T)$ denote the Fourier coefficient of φ corresponding to the cubic ring T. Then*

$$\sum_{T \subseteq \mathbb{Z}^3, n \geq 1} \frac{a_\varphi(\mathbb{Z} + nT)}{[\mathbb{Z}^3 : T]^{s-\ell+1} n^s} = a_\varphi(\mathbb{Z}^3) \frac{L(\pi, Std, s - 2\ell + 1)}{\zeta(s - 2\ell + 2)^2 \zeta(2s - 4\ell + 2)}.$$

Here the sum is over the subrings T of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and integers $n \geq 1$.

In [12], Pollack has given a streamlined account of the integrals of [9] and [14] by simplifying some of the computations. He also used this analysis of the Rankin-Selberg integral to give a Dirichlet series for the standard L-function of modular forms on G_2 outside of the primes $p = 2, 3$, and began some calculations of the archimedean zeta integral associated to the global Rankin-Selberg convolution. Theorem 1.1 and Corollary 1.2 bring to completion the work that began in [12].

1.3. **Special values.** From the archimedean factor $L_\infty(\pi_{\ell,\infty}, s)$, one can verify that the integers $\{1, 3, 5, \ldots, \ell - 1\}$ are *critical* for $L(\pi, Std, s)$ in the sense of Deligne, i.e., that $L_\infty(\pi_{\ell,\infty}, s)$ and $L_\infty(\pi_{\ell,\infty}, 1 - s)$ are finite at these integers. It would be highly interesting to obtain a special value result in the direction of Deligne's conjecture for these L-values. While such a result is out of reach via our methods, we can obtain what can be considered the most basic nontrivial special value result, namely, an analysis of the trivial zeros of the L-function $L(\pi, Std, s)$. This is a direct corollary of the functional equation of the completed L-function.

[1]It seems worthwhile to point out that Gross and Savin [8] had previously defined the archimedean L-factor for representations of the *compact* group $G_2^c(\mathbb{R})$, and our infinite factor agrees with the one in [8, pg 168] with $k_1 = 0$ and $k_2 = \ell - 2$ in their notation.

In more detail, the completed L-function $\Lambda(\pi, Std, s)$ is finite and non-zero for $\mathrm{Re}(s) \gg 0$, and also for $\mathrm{Re}(s) \ll 0$ by using the functional equation. However, the archimedean factor $L_\infty(\pi_{\ell,\infty}, s)$ has poles at sufficiently negative integers, i.e., negative integers of sufficiently large absolute value. These poles must be compensated for by zeros of the standard L-function. We obtain:

Corollary 1.3. *Let the assumptions be as in Theorem 1.1. Then $L(\pi, Std, s)$ vanishes to order 3 at sufficiently negative even integers, and vanishes to order 4 at sufficiently negative odd integers.*

1.4. **Outline of the proof of Theorem 1.1.** As mentioned, our proof of Theorem 1.1 is based on a refined analysis of the Rankin-Selberg integral in [9], following the work of [12]. Let G denote the split group Spin(8). If φ is a weight ℓ modular form on G_2, then by definition, φ is a $\mathbf{V}_\ell$-valued automorphic function on $G_2(\mathbb{A})$, where $\mathbf{V}_\ell = Sym^{2\ell}(\mathbb{C}^2)$ (see [12] for modular forms on G_2). We define a normalized Eisentstein series $E^*_\ell(g, s)$ on $G(\mathbb{A})$ that is also valued in $\mathbf{V}_\ell$. The Rankin integral that we consider is

$$I_\ell(\varphi, s) = \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \{\varphi(g), E^*_\ell(g, s)\}_K \, dg,$$

where $\{\cdot, \cdot\}_K$ is a K-equivariant pairing $\mathbf{V}_\ell \otimes \mathbf{V}_\ell \to \mathbb{C}$.

We obtain Theorem 1.1 by proving

(1) $I_\ell(\varphi, s)$ is equal to $a_\varphi(\mathbb{Z}^3)\Lambda(\pi, Std, s-2)$, up to a nonzero constant, and
(2) the Eisenstein series $E^*_\ell(g, s)$ satisfies the functional equation $E^*_\ell(g, s) = E^*_\ell(g, 5-s)$.

To prove the first statement, one must analyze local integrals $I_p(s)$ for finite primes p which will be defined in (3.3) and an archimedean integral $I^*(s;\ell)$ defined in (8.1), and prove that these local integrals are equal to the corresponding local L-factors up to some simple factors. For $p \geq 5$, these local integrals $I_p(s)$ are analyzed in [9]. To carry out the computation of these integrals for $p = 2, 3$, we follow the method of [12] and use some results on cubic rings. The analysis of the integral $I^*(s;\ell)$ was begun in [12], where the computation was reduced to an integral $J'(s)$ over the space of real binary cubics of a general form which was previously considered by Shintani [15]. We evaluate the integral $J'(s)$ explicitly in terms of the Γ-function, thereby proving that $I^*(s;\ell) = L(\pi_{\ell,\infty}, s-2)$ up to a nonzero constant.

To prove the second statement, we use Langlands' functional equation of the Eisenstein series. The main difficulty in applying this result is that the Eisenstein series $E^*_\ell(g, s)$ is not spherical at the archimedean place. Thus a careful analysis must be made of certain archimedean intertwining operators.

1.5. **Outline of the paper.** We now give an outline of paper section-by-section. In Section 2, we setup some of our notation used in various parts of the paper. In Section 3, we give an overview of the Rankin-Selberg integral and present our strategy to calculate the non-archimedean L-factor. In Section 4, we prove some needed results about the relationship between binary cubic forms and cubic rings. In Section 5, we compute a Fourier coefficient of the so-called approximate basic function. In Section 6, we complete the unramified computation. This involves computing the function $\Phi_{p,\chi}(t, g)$ defined in (3.4), which is related to the inducing section of the Eisenstein series, and making certain Hecke operator calculations. In Section 7, we prove the functional equation of the Eisenstein series $E^*_\ell(g, s)$, and in Section 8, we compute the archimedean zeta integral $I^*(s;\ell)$.

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2. Setups

2.1. **Octonions and reductive groups.** We use the notation in [12, Section 2]. To review some of this notation, fix the representation V_3 of SL_3 and its dual representation $V_3^\vee$. The space V_3 has a standard basis $\{e_1, e_2, e_3\}$ and $V_3^\vee$ has the dual basis $\{e_1^*, e_2^*, e_3^*\}$. We let Θ denote the split octonions in the Zorn model (see [12, Section 2.1]).

Using the quadratic norm on Θ, we can define the group $G' = \mathrm{SO}(\Theta)$. Now, let G denote the algebraic group $\mathrm{Spin}(\Theta)$. This is the group of triples $(g_1, g_2, g_3) \in \mathrm{SO}(\Theta)^3$ satisfying $(g_1x_1, g_2x_2, g_3x_3) = (x_1, x_2, x_3)$ for all $x_1, x_2, x_3 \in \Theta$. Here $(x_1, x_2, x_3) = \mathrm{tr}_\Theta(x_1(x_2x_3))$. We fix a map $G \to G'$ as $(g_1, g_2, g_3) \mapsto g_1$. This map induces an isomorphism on Lie algebras.

Denote by Θ_0 the standard maximal lattice in the Zorn model. Then Θ_0 consists of the matrices $\left(\begin{smallmatrix} a & v \\ \phi & d \end{smallmatrix}\right)$ where $a, d \in \mathbb{Z}$, v is in the $\mathbb{Z}$-span of the e_1, e_2, e_3 and ϕ is in the $\mathbb{Z}$-span of e_1^*, e_2^*, e_3^*. We denote $K_f = \prod_p K_p$, where K_p is the hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$ that is specified as the stabilizer of $(\Theta_0 \otimes \mathbb{Z}_p)^3$ inside of $G(\mathbb{Q}_p)$. That K_p is hyperspecial follows from [6, Section 4] and [1, Proposition 5.4]. Similarly, we let $G_2(\mathbb{Z}_p)$ be the stabilizer of $\Theta_0 \otimes \mathbb{Z}_p$ inside of $G_2(\mathbb{Q}_p)$; this is a hyperspecial maximal compact subgroup.

We define a maximal compact subgroup of $G'(\mathbb{R})$, K'_∞ as follows. Given $v \in V_3$, let $\widetilde{v} \in V_3^\vee$ be given by the mapping $e_j \mapsto e_j^*$ and extending by linearity. Similarly, if $\phi \in V_3^\vee$, let $\widetilde{\phi}$ in V_3 be given by the mapping $e_j^* \mapsto e_j$ and extending by linearity. Define a quadratic form q_{maj} on $\Theta \otimes \mathbb{R}$ by $q_{maj}\left(\left(\begin{smallmatrix} a & v \\ \phi & d \end{smallmatrix}\right)\right) = a^2 + d^2 + (v, \widetilde{v}) + (\widetilde{\phi}, \phi)$, where $(\,,)$ denotes the evaluation pairing between V_3 and $V_3^\vee$. One defines $K'_\infty \subseteq G'(\mathbb{R})$ as the subgroup that also preserves the quadratic form q_{maj}. We now let $K_\infty \subseteq G(\mathbb{R})$ be the inverse image of K'_∞ under the map $G \to G'$.

Put another way, define $\iota : \Theta \to \Theta$ as

$$\iota\left(\begin{pmatrix} a & v \\ \phi & d \end{pmatrix}\right) = \begin{pmatrix} d & -\widetilde{\phi} \\ -\widetilde{v} & a \end{pmatrix}.$$

Thus if $x = \left(\begin{smallmatrix} a & v \\ \phi & d \end{smallmatrix}\right)$, then $q_{maj}(x) = (x, \iota(x))$. Conjugation by ι induces a Cartan involution on G' and on $G_2 \subseteq G$ (see [12, Claim 2.1]).

2.2. **Setup for K_∞.** The Lie algebra $Lie(K_\infty) \otimes \mathbb{C} = Lie(K'_\infty) \otimes \mathbb{C}$ is a sum of four copies of $\mathfrak{sl}_2$. We now make this explicit since these $\mathfrak{sl}_2$'s will be used in Section 7.

We let

$$\{b_1, b_2, b_3, b_4, b_{-4}, b_{-3}, b_{-2}, b_{-1}\} = \{e_1, e_3^*, \epsilon_2, e_2^*, -e_2, \epsilon_1, -e_3, -e_1^*\} \tag{2.1}$$

in order. Here $\epsilon_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\epsilon_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. With the basis $\{b_1, b_2, b_3, b_4, b_{-4}, b_{-3}, b_{-2}, b_{-1}\}$ of Θ, one has $(b_i, b_j) = (b_{-i}, b_{-j}) = 0$ and $(b_i, b_{-j}) = \delta_{ij}$. The involution ι satisfies $\iota(b_j) = b_{-j}$ and $\iota(b_{-j}) = b_j$ for $j = 1, 2, 3, 4$. Define

$$u_1 = \frac{1}{\sqrt{2}}(b_1 + b_{-1}), \quad u_2 = \frac{1}{\sqrt{2}}(b_2 + b_{-2}),$$
$$v_1 = \frac{1}{\sqrt{2}}(b_3 + b_{-3}), \quad v_2 = \frac{1}{\sqrt{2}}(b_4 + b_{-4}),$$

and

$$u_{-1} = \frac{1}{\sqrt{2}}(b_1 - b_{-1}), \quad u_{-2} = \frac{1}{\sqrt{2}}(b_2 - b_{-2}),$$
$$v_{-1} = \frac{1}{\sqrt{2}}(b_3 - b_{-3}), \quad v_{-2} = \frac{1}{\sqrt{2}}(b_4 - b_{-4}).$$

With this notation, we can specify the four copies $\mathfrak{sl}_2$'s in $Lie(K_\infty) \otimes \mathbb{C}$. One copy of $\mathfrak{sl}_2$ is given by

- $e_\ell^+ = \frac{1}{2}(u_1 - iu_2) \wedge (v_1 - iv_2)$
- $h_\ell^+ = i(u_1 \wedge u_2 + v_1 \wedge v_2)$
- $f_\ell^+ = -\frac{1}{2}(u_1 + iu_2) \wedge (v_1 + iv_2)$.

The other $\mathfrak{sl}_2$ from the first SO(4) in $K'_\infty = S(O(4) \times O(4))$ is obtained by replacing v_2 with $-v_2$ in the above formulas. That is, it has the basis

- $e_\ell'^+ = \frac{1}{2}(u_1 - iu_2) \wedge (v_1 + iv_2)$
- $h_\ell'^+ = i(u_1 \wedge u_2 - v_1 \wedge v_2)$
- $f_\ell'^+ = -\frac{1}{2}(u_1 + iu_2) \wedge (v_1 - iv_2)$.

The third $\mathfrak{sl}_2$ is given by

- $e_\ell^- = \frac{1}{2}(u_{-1} - iu_{-2}) \wedge (v_{-1} - iv_{-2})$
- $h_\ell^- = -i(u_{-1} \wedge u_{-2} + v_{-1} \wedge v_{-2})$
- $f_\ell^- = -\frac{1}{2}(u_{-1} + iu_{-2}) \wedge (v_{-1} + iv_{-2})$.

The fourth $\mathfrak{sl}_2$ is given by

- $e_\ell'^- = \frac{1}{2}(u_{-1} - iu_{-2}) \wedge (v_{-1} + iv_{-2})$
- $h_\ell'^- = -i(u_{-1} \wedge u_{-2} - v_{-1} \wedge v_{-2})$
- $f_\ell'^- = -\frac{1}{2}(u_{-1} + iu_{-2}) \wedge (v_{-1} - iv_{-2})$.

The compatible Cartan involutions on G_2 and G, and the embedding $G_2 \subseteq G$ picks out a distinguished $\mathfrak{sl}_2$ of the above four, the image of the long root $\mathfrak{sl}_2$ of G_2. The long root $\mathfrak{sl}_2$ is given in [12, Section 4.1.1] or equivalently by combining Section 5.1 and section 4.2.4 of [13]. In the notation of [12, Section 2.2],

$$e_\ell = \frac{1}{4}(iE_{12} + v_1 - i\delta_3 - E_{23} + E_{32} - iv_3 + \delta_1 - iE_{21})$$

$f_\ell = -\overline{e_\ell}$ and $h_\ell = [e_\ell, f_\ell]$. This simplifies to

$$e_\ell = \frac{1}{4}(e_1 + e_1^* - i(e_3 + e_3^*)) \wedge (\epsilon_2 - \epsilon_1 - i(e_2 + e_2^*)).$$

With the above identification (2.1), one obtains that the long root $\mathfrak{sl}_2$ of G_2 maps into the third $\mathfrak{sl}_2$ of $Lie(K_\infty) \otimes \mathbb{C}$.

We denote by V_2 the representation of $Lie(K_\infty) \otimes \mathbb{C}$ which is the two-dimensional representation of the long root $\mathfrak{sl}_2$ and the trivial representation of the other $\mathfrak{sl}_2$'s. Let $\{x, y\}$ be a basis of V_2 on which h_ℓ acts as $1, -1$, respectively, and for which $f_\ell x = y$. The even symmetric powers in $Sym^{2\ell}(V_2)$ exponentiate to representations of K_∞ and have the basis $\{x^{2\ell}, x^{2\ell-1}y, \dots, xy^{2\ell-1}, y^{2\ell}\}$.

2.3. **Lie algebra definitions.** The maps $G_2 \to G \to G'$ induce $Lie(G_2) \to Lie(G') \simeq \wedge^2\Theta$. This embedding is the one specified in [12, Section 2.2]. We use notation from [12, Section 2.2].

The Heisenberg parabolic P_G of G is defined to be the one which stabilizes the line spanned by $E_{13} = e_3^* \wedge e_1$ in $\wedge^2\Theta$. The Heisenberg parabolic P of G_2 is similarly defined as the stabilizer of the line spanned by E_{13} in $Lie(G_2)$, thus $P_G \cap G_2 = P$.

2.4. **Binary cubic forms.** Let V_2 denote the defining representation of GL_2. The space $W = Sym^3(V_2) \otimes \det(V_2)^{-1}$ is the space of binary cubic forms. If $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ is a binary cubic and $g \in \mathrm{GL}_2$, then we define $g \cdot f$ to be the binary cubic $(g \cdot f)(x, y) = \det(g)^{-1} f((x, y)g)$.

We will also sometimes use a right action of GL_2 on the space of binary cubics. For $g = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \mathrm{GL}_2$, define $\widetilde{g} = \left(\begin{smallmatrix} s & -q \\ -r & p \end{smallmatrix}\right)$ so that $g\widetilde{g} = \det(g)$. We define $f \cdot g = \widetilde{g} \cdot f = \det(g)^2 f((x, y)g^{-1})$.

There is a GL_2-equivariant symplectic form on W defined as

$$\langle ax^3 + bx^2y + cxy^2 + dy^3, a'x^3 + b'x^2y + c'xy^2 + d'y^3 \rangle = ad' - \frac{bc'}{3} + \frac{cb'}{3} - da'.$$

One has

$$\langle g\cdot f, g\cdot f'\rangle = \det(g)\langle f, f'\rangle \quad \text{and} \quad \langle f, g\cdot f'\rangle = \langle f\cdot g, f'\rangle \quad \text{for all } f, f' \in W.$$

The space W has a GL_2-equivariant quartic form q, defined as follows. Suppose $v = ax^3 + bx^2y + cxy^2 + dy^3$. Then

$$\begin{aligned} q(v) &= \left(ad - \frac{bc}{3}\right)^2 + \frac{4}{27}ac^3 + \frac{4}{27}db^3 - \frac{4}{27}b^2c^2 \\ &= -\frac{1}{27}(-27a^2d^2 + 18abcd + b^2c^2 - 4ac^3 - 4db^3). \end{aligned}$$

2.5. **Characters of the Heisenberg parabolic.** Throughout the paper, we fix the standard additive character $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$. We will abusively denote the p-component of this additive character by ψ. Thus, if $x \in \mathbb{Q}_p$ and $x = x_0 + x_1$ with $x_0 \in \mathbb{Z}_p$ and $x_1 = m/p^r$, then $\psi(x) = e^{2\pi i x_1}$.

We let N denote the unipotent radical of the Heisenberg parabolic P of G_2 and let M denote the Levi subgroup of P that also stabilizes the line spanned by E_{31}. We identify M with GL_2 as in [12, Section 5.2]. We now recall this identification. Suppose that $g \in \mathrm{GL}_2$ has the matrix $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. Then the action of g on Θ is given by

- $e_1 \mapsto ae_1 + ce_3^*$
- $e_3^* \mapsto be_1 + de_3^*$
- $(ad-bc)\epsilon_1 \mapsto ad\epsilon_1 + abe_2^* - cde_2 - bc\epsilon_2$
- $(ad-bc)e_2^* \mapsto ac\epsilon_1 + a^2e_2^* - c^2e_2 - ac\epsilon_2$
- $(ad-bc)e_2 \mapsto -bd\epsilon_1 - b^2e_2^* + d^2e_2 + bd\epsilon_2$
- $(ad-bc)\epsilon_2 \mapsto -bc\epsilon_1 - abe_2^* + cde_2 + ad\epsilon_2$

On another note, a binary cubic form gives a character of N. Let Z denote the one-dimensional center of N. Denote by W the representation $Sym^3(V_2)\otimes\det(V_2)^{-1}$ of $M \simeq \mathrm{GL}_2$. The exponential map provides an identification $\exp : W \simeq N/Z$ as specified in [12, page 18]. Namely, to the binary cubic $u_1x^3 + u_2x^2y + u_3xy^2 + u_4y^3$, we associate the element $u_{12}E_{12} + \frac{u_2}{3}v_1 + \frac{u_3}{3}\delta_3 + u_4E_{23}$ of $Lie(G_2)$. Now, if $\omega \in W$, then $n \mapsto \psi(\langle\omega, \overline{n}\rangle)$ defines a character of N. Here $\overline{n}$ is the image of n in $N/Z \simeq W$ and ψ is our fixed additive character.

3. The Rankin-Selberg integral

In this section we give an overview of the calculations which will be done in the rest of the paper.

3.1. **The Eisenstein series.** We begin by defining various Eisenstein series on the group G that are related. Recall that we let P_G denote the Heisenberg parabolic of G. Denote by $\nu : P_G \to \mathrm{GL}_1$ the generating character so that $\delta_{P_G}(p) = |\nu(p)|^5$.

Denote by $E_\ell(g,s)$ the Eisenstein series of weight ℓ on G that is normalized with a flat section. More precisely, define $f_\ell(g,s)$ to be the unique section in $\mathrm{Ind}_{P_G(\mathbb{A})}^{G(\mathbb{A})}(|\nu|^s)$ which is valued in $Sym^{2\ell}(V_2)$ and satisfies

(1) $f_\ell(k_f, s) = x^\ell y^\ell$ for all $k_f \in K_f \subseteq G(\mathbb{A}_f)$
(2) $f_\ell(gk, s) = k^{-1}\cdot f_\ell(g,s)$ for all $g \in G(\mathbb{A})$ and $k \in K_\infty$.

The Eisenstein series is defined as

$$E_\ell(g,s) = \sum_{\gamma\in P_G(\mathbb{Q})\backslash G(\mathbb{Q})} f_\ell(\gamma g, s).$$

For our purposes, we define a normalized Eisenstein series as

$$E_\ell^*(g,s) = \Lambda(s-1)^2\Lambda(s)\Lambda(2s-4)\frac{\Gamma(s+\ell-1)\Gamma(s+\ell-2)}{\Gamma(s-1)\Gamma(s-2)}E_\ell(g,s). \tag{3.1}$$

It will be convenient for our calculations to define one more related Eisenstein series. Let Φ_f be the Schwartz-Bruhat function on $\wedge^2\Theta \otimes \mathbb{A}_f$ that is the characteristic function of $\wedge^2\Theta_0 \otimes \widehat{\mathbb{Z}}$. Note that Φ_f is stable by K_f. For $g \in G(\mathbb{A}_f)$, we define

$$f_{fte}(g, \Phi_f, s) = \int_{\mathrm{GL}_1(\mathbb{A}_f)} |t|^s \Phi_f(tg^{-1}E_{13})\, dt,$$

and

$$f(g, \Phi_f, s) = f_{fte}(g_f, \Phi_f, s) f_\ell(g_\infty, s).$$

Because Φ_f is K_f stable, it is immediate that $f(g, \Phi_f, s) = \zeta(s) f_\ell(g, s)$. We set

$$E(g, \Phi_f, s) = \sum_{\gamma \in P_G(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g, \Phi_f, s),$$

and

$$I(\varphi, \Phi, s) = \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \{\varphi(g), E(g, \Phi_f, s)\}_K\, dg.$$

3.2. **The unfolded Rankin integral.** We now explain how the Rankin integral $I(\varphi, \Phi_f, s)$ unfolds.

Let $v_E \in W$ be the binary cubic $x^2y + xy^2 = xy(x+y)$. Let χ be the character of $N(\mathbb{Q})\backslash N(\mathbb{A})$ determined by v_E as in Section 2 and set

$$\varphi_\chi(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi^{-1}(n)\varphi(ng)\, dn.$$

Define $\widetilde{v_E} = \epsilon_1 \wedge (e_1 + e_3^*)$ in $\wedge^2\Theta$. Let $N^{0,E} \subseteq N$ be the subgroup consisting of those $n \in N$ for which $\langle v_E, \overline{n}\rangle = 0$.

Theorem 3.1 ([9], [14])**.** *One has*

$$I(\varphi, \Phi_f, s) = \int_{N^{0,E}(\mathbb{A})\backslash G(\mathbb{A})} \{f(\gamma_0 g, \Phi_f, s), \varphi_\chi(g)\}_K\, dg \tag{3.2}$$

where $\gamma_0 \in G(\mathbb{Q})$ satisfies $\gamma_0^{-1}E_{13} = \widetilde{v_E}$.

The proof of this theorem is due to Gurevich and Segal, but the form in which we state is essentially Theorem 5.2 in [12].

3.3. **Local integrals.** In order to analyze $I(\varphi, \Phi_f, s)$, we must consider associated local integrals at every place of $\mathbb{Q}$. In this subsection, we describe these local integrals, and also deduce the main global theorems of the paper.

At the archimedean place, we compute

$$I(s;\ell) = \int_{N^{0,E}(\mathbb{R})\backslash G_2(\mathbb{R})} \{f_\ell(\gamma_0 g, s), \mathcal{W}_\chi(g)\}_K\, dg.$$

Here $\mathcal{W}_\chi$ is the generalized Whittaker function of [12, Section 4] and [13]. The integral $I(s;\ell)$ is computed in Section 8.

In view of the normalization of the Eisenstein series in (3.1), note that

$$\Gamma_\mathbb{R}(s-1)^2\Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(2s-4)\frac{\Gamma(s+\ell-1)\Gamma(s+\ell-2)}{\Gamma(s-1)\Gamma(s-2)} = 2^s\Gamma_\mathbb{R}(s-1)\Gamma_\mathbb{C}(s+\ell-1)\Gamma_\mathbb{C}(s+\ell-2).$$

Thus we suitably define a normalized archimedean zeta integral as

$$I^*(s;\ell) = 2^s\Gamma_\mathbb{R}(s-1)\Gamma_\mathbb{C}(s+\ell-1)\Gamma_\mathbb{C}(s+\ell-2)I(s;\ell),$$

In Theorem 8.1, we prove that $I^*(s;\ell) = L(\pi_{\ell,\infty}, s-2)$, up to a nonzero constant.

We now describe the local integrals that are computed at the finite places. Denote by Φ_p the characteristic function of $\wedge^2\Theta_0 \otimes \mathbb{Z}_p$ and $f_p(g, \Phi_p, s)$ the associated local inducing section, so that $f_p(1, \Phi_p, s) = \zeta_p(s)$. Let V_{π_p} denote the space of representation π_p, and write v_0 for a spherical vector. Suppose that $L : V_{\pi_p} \to \mathbb{C}$ is a (N, χ)-functional, i.e., $L(nv) = \chi(n)L(v)$ for all $n \in N(\mathbb{Q}_p)$ and $v \in V_{\pi_p}$. At a finite place p, we will compute

$$I_p(s) = I_p(L, s) := \int_{N^{0,E}(\mathbb{Q}_p)\backslash G_2(\mathbb{Q}_p)} f_p(\gamma_0 g, \Phi_p, s) L(gv_0)\, dg \tag{3.3}$$

$$= \int_{N(\mathbb{Q}_p)\backslash \mathrm{GL}_1(\mathbb{Q}_p)\times G_2(\mathbb{Q}_p)} |t|^s \Phi_{p,\chi}(t, g) L(gv_0)\, dt\, dg.$$

Here

$$\Phi_{p,\chi}(t, g) = \int_{N^{0,E}(\mathbb{Q}_p)\backslash N(\mathbb{Q}_p)} \chi(n)\Phi_p(tg^{-1}n^{-1}\widetilde{v_E})\, dn. \tag{3.4}$$

We will prove the following theorem.

Theorem 3.2. *One has*

$$I_p(L; s) = L(v_0)\frac{L(\pi_p, Std, s-2)}{\zeta_p(s-1)^2\zeta_p(2s-4)}. \tag{3.5}$$

See Subsection 3.4 for an explanation of how we prove Theorem 3.2. As a corollary of Theorem 3.2, we obtain Corollary 1.2:

Proof of Corollary 1.2. This follows from Theorem 3.2 exactly as in [12, Subsection 5.9]. □

By combining our calculation of the archimedean integral $I(s; \ell)$ with this result on $I_p(L; s)$, we obtain:

Theorem 3.3. *The integral $I_\ell(\varphi, s)$ is equal to $a_\varphi(\mathbb{Z}^3)\Lambda(\pi, Std, s-2)$, up to a nonzero constant.*

Proof. Taking into account the normalization of the Eisenstein series $E^*_\ell(g, s)$, this follows directly from Theorem 3.2 and Theorem 8.1 using the technique of "non-unique models". □

In Theorem 7.1, we prove that $E^*_\ell(g, s) = E^*_\ell(g, 5-s)$. Combining this functional equation with Theorem 3.3, Theorem 1.1 follows immediately.

3.4. **The integrals at the finite places.** In this subsection we explain the proof of Theorem 3.2, following the strategies in [9, 12, 14].

Denote by V_7 the perpendicular subspace to 1 in Θ, and set $V_7(\mathbb{Z}) = V_7 \cap \Theta_0$. Similarly, define $V_7(\mathbb{Z}_p) = V_7(\mathbb{Z}) \otimes \mathbb{Z}_p$. Note that because G_2 stabilizes 1, $V_7(\mathbb{Z}_p)$ is stabilized by $G_2(\mathbb{Z}_p)$. We write $r_7 : G_2 \to \mathrm{GL}(V_7)$ for the action map.

We now define two Hecke operators on G_2. First, for $t \in \mathrm{GL}_1(\mathbb{Q}_p)$ and $h \in G_2(\mathbb{Q}_p)$, let

$$\Delta(t, h) = \mathrm{char}(t \cdot r_7(h) \in End(V_7(\mathbb{Z}_p))).$$

We call it the approximate basic function; see [12, Section 5.3] for some remarks on this terminology. Define another Hecke operator on G_2 as

$$T = p^{-3} \times \mathrm{char}(g \in G_2(\mathbb{Q}_p) \quad p.r_7(g) \in End(V_7(\mathbb{Z}_p))).$$

For ease of notation, set $z = p^{-s}$. In order to prove (3.5), we will actually prove the following equality:

$$\int_{\mathrm{GL}_1(\mathbb{Q}_p)\times G_2(\mathbb{Q}_p)} |t|^{s+2}\Delta(t,g)L(gv_0)\,dt\,dg \tag{3.6}$$
$$= M(\pi_p,s)\int_{N(\mathbb{Q}_p)\backslash GL_1(\mathbb{Q}_p)\times G_2(\mathbb{Q}_p)} |t|^{s+1}\Phi_{p,\chi}(t,g)L(gv_0)\,dt\,dg,$$

where
$$M(\pi_p,s)=(1-pz)(1-z)N_0(\pi_p,s-1)\zeta_p(s)^2\zeta_p(2s-2),$$
and
$$N_0(\pi_p,s)v_0 = 1+(p^{-1}+1)z+\frac{z^2}{p}+(p^{-2}+p^{-1})z^3+\frac{z^4}{p^2}-\frac{z^2}{p}.$$
Proving (3.6) implies our desired relation between $I_p(L;s)$ and $L(\pi_p,Std,s)$ as in Theorem 3.2, this is essentially [9, Proposition 7.1]. For an explanation of this implication, see [12, Section 5.3].

Now, the equality of (3.6) has been proved for $p\geq 5$ in [9] and [12]. We will prove it for $p=2,3$ as well. To do so, we will compute the left-hand side and the right-hand side of (3.6) separately and show that they are the same. We will consider the left-hand side in Section 5 and the right-hand side in Section 6. In order to do this computation, we will find it useful to have facts about binary cubic forms and their relation to cubic rings, which we spell out in Section 4.

4. The arithmetic invariant theory of binary cubics

The purpose of this section is to give some needed results on the relationship between binary cubic forms and cubic rings. We refer to [3, Section 4] and [7] for a primer on this relationship.

Suppose that $f(x,y)=ax^3+bx^2y+cxy^2+dy^3$ is a binary cubic over some ring R. One associates to f the cubic R-algebra T with the basis $\{1,\omega,\theta\}$ and the multiplication table

- $\omega\theta=-ad$
- $\omega^2=-ac+a\theta-b\omega$
- $\theta^2=-bd+c\theta-d\omega.$

Suppose that $m=\left(\begin{smallmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{smallmatrix}\right)$ is a 2×2 matrix with R coefficients. Write T_m for the R lattice in T spanned by $1, m_{11}\omega+m_{12}\theta, m_{21}\omega+m_{22}\theta$. One can naturally ask what condition guarantees that T_m is closed under multiplication. This question is answered by the following proposition.

Proposition 4.1. *Suppose R has characteristic 0. Set $f'(x,y)=m\cdot f(x,y)$ with $f'(x,y)=a'x^3+b'x^2y+c'xy^2+d'y^3$. With the notation as above, the R-lattice T_m is closed under multiplication if and only if*

(1) $f'(x,y)$ has coefficients in R, i.e., $a',b',c',d'\in R$;

(2) and $\begin{pmatrix} b'\\ -c'\end{pmatrix}\equiv m\begin{pmatrix} b\\ -c\end{pmatrix} \pmod 3$.

To prove this proposition, we require a lemma. With the same notation as above, set $\omega''=m_{11}\omega+m_{12}\theta$ and $\theta''=m_{21}\omega+m_{22}\theta$. Let ω' and θ' be defined by
$$\begin{pmatrix}\omega'\\ \theta'\end{pmatrix}=m\begin{pmatrix}\omega\\ \theta\end{pmatrix}+\frac{m}{3}\begin{pmatrix} b\\ -c\end{pmatrix}+\frac{1}{3}\begin{pmatrix} -b'\\ c'\end{pmatrix}.$$
Finally, let $\omega_0=\omega+\frac{b}{3}$ and $\theta_0=\theta-\frac{c}{3}$, so that $\mathrm{tr}(\omega_0)=\mathrm{tr}(\theta_0)=0$.

Lemma 4.2. *The elements ω' and θ' have the multiplication table*

(1) $\omega'\theta'=-a'd'$
(2) $\omega'^2=-a'c'+a'\theta'-b'\omega'$
(3) $\theta'^2=-b'd'+c'\theta'-d'\omega'$.

Proof. This lemma is well-known. It essentially is the statement that the association of based cubic rings to binary cubic forms is equivariant for the action of GL_2. For completeness, we give some details regarding a proof.

First, instead of checking the multiplication table above of $1, \omega', \theta'$, we verify the equivalent multiplication table for $\begin{pmatrix} \omega_0' \\ \theta_0' \end{pmatrix} = m \begin{pmatrix} \omega_0 \\ \theta_0 \end{pmatrix}$.

Now, the trace 0 basis ω_0, θ_0 has multiplication table

- $\omega_0\theta_0 = \frac{b}{3}\theta_0 - \frac{c}{3}\omega_0 + \left(\frac{bc}{9} - ad\right)$
- $\omega_0^2 = a\theta_0 - \frac{b}{3}\omega_0 + \frac{2}{9}(b^2 - 3ac)$
- $\theta_0^2 = \frac{c}{3}\theta_0 - d\omega_0 + \frac{2}{9}(c^2 - 3bd)$.

We wish to prove that the multiplication table for ω_0', θ_0' has the same form, with a, b, c, d replaced by a', b', c', d', respectively.

To do this, first write the multiplication table of ω_0, θ_0 as

$$\begin{pmatrix} \omega_0 \\ \theta_0 \end{pmatrix} \begin{pmatrix} \omega_0 & \theta_0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3a & b \\ b & c \end{pmatrix}\theta_0 - \frac{1}{3}\begin{pmatrix} b & c \\ c & 3d \end{pmatrix}\omega_0 + \frac{1}{9}\begin{pmatrix} 2b^2 - 6ac & bc - 9ad \\ bc - 9ad & 2c^2 - 6bd \end{pmatrix}.$$

From this, one obtains

$$\begin{aligned} m\begin{pmatrix} \omega_0 \\ \theta_0 \end{pmatrix} \begin{pmatrix} \omega_0 & \theta_0 \end{pmatrix} m^t = {} & \frac{1}{3}\left(m_{21}^{-1} m \begin{pmatrix} 3a & b \\ b & c \end{pmatrix} m^t - m_{11}^{-1} m \begin{pmatrix} b & c \\ c & 3d \end{pmatrix} m^t\right)\omega_0' \\ & + \frac{1}{3}\left(m_{22}^{-1} m \begin{pmatrix} 3a & b \\ b & c \end{pmatrix} m^t - m_{12}^{-1} m \begin{pmatrix} b & c \\ c & 3d \end{pmatrix} m^t\right)\theta_0' \\ & + \frac{1}{9} m \begin{pmatrix} 2b^2 - 6ac & bc - 9ad \\ bc - 9ad & 2c^2 - 6bd \end{pmatrix} m^t. \end{aligned}$$

This is also

$$\begin{aligned} \begin{pmatrix} \omega_0' \\ \theta_0' \end{pmatrix} \begin{pmatrix} \omega_0' & \theta_0' \end{pmatrix} = {} & -\frac{1}{3}\det(m)^{-1}\left(m_{21} m \begin{pmatrix} 3a & b \\ b & c \end{pmatrix} m^t + m_{22} m \begin{pmatrix} b & c \\ c & 3d \end{pmatrix} m^t\right)\omega_0' \\ & + \frac{1}{3}\det(m)^{-1}\left(m_{11} m \begin{pmatrix} 3a & b \\ b & c \end{pmatrix} m^t + m_{12} m \begin{pmatrix} b & c \\ c & 3d \end{pmatrix} m^t\right)\theta_0' \\ & + \frac{1}{9} m \begin{pmatrix} 2b^2 - 6ac & bc - 9ad \\ bc - 9ad & 2c^2 - 6bd \end{pmatrix} m^t. \end{aligned}$$

The final expression gives

$$\begin{pmatrix} \omega_0' \\ \theta_0' \end{pmatrix} \begin{pmatrix} \omega_0' & \theta_0' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3a' & b' \\ b' & c' \end{pmatrix}\theta_0 - \frac{1}{3}\begin{pmatrix} b' & c' \\ c' & 3d' \end{pmatrix}\omega_0 + \frac{1}{9}\begin{pmatrix} 2b'^2 - 6a'c' & b'c' - 9a'd' \\ b'c' - 9a'd' & 2c'^2 - 6b'd' \end{pmatrix},$$

where we used the definition of the action of GL_2 on binary cubics and the equivariance of the Hessian of a binary cubic. The lemma then follows. $\square$

Let $\dagger$ denote the condition $\begin{pmatrix} b' \\ -c' \end{pmatrix} \equiv m \begin{pmatrix} b \\ -c \end{pmatrix} \pmod 3$ of Proposition 4.1. Then the statement of Proposition 4.1 follows immediately from:

Proposition 4.3. *The following statements are equivalent.*

(1) The R lattice spanned by $1, \omega'', \theta''$ is closed under multiplication.
(2) The R lattice spanned by $1, \omega', \theta'$ is closed under multiplication and $\dagger$ holds.
(3) $m \cdot f$ has coefficients in R and $\dagger$ holds.

Proof. From Lemma 4.2, it is clear that (2) and (3) are equivalent. It is also clear that (2) implies (1). To prove that (1) implies (2), we argue as follows. First, define

$$\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \frac{m}{3}\begin{pmatrix} b \\ -c \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -b' \\ c' \end{pmatrix},$$

so that $\omega' = \omega'' + \delta_1$ and $\theta' = \theta'' + \delta_2$. Observe that

$$\begin{aligned} \omega''\theta'' &= (\omega' - \delta_1)(\theta' - \delta_2) \\ &= A - \delta_1\theta'' - \delta_2\omega'' \end{aligned}$$

for some $A \in Frac(R)$. Thus if (1) holds, then $\delta_1, \delta_2 \in R$. Because $\delta_1, \delta_2 \in R$, the equalities $\omega' = \omega'' + \delta_1$, $\theta' = \theta'' + \delta_2$ imply that the R lattice spanned by $1, \omega', \theta'$ is closed under multiplication, so that (2) holds. The proposition now follows. $\square$

In case $R = \mathbb{Z}_p$, we can go further.

Proposition 4.4. *Let $R = \mathbb{Z}_p$. Then $Span_{\mathbb{Z}_p}(1, \omega', \theta')$ being closed under multiplication implies that † holds. Equivalently, $m \cdot f$ having coefficients in R implies that † holds.*

Of course, the proposition only has nontrivial content for $p = 3$, although we write the proof out for general p.

Proof. The idea of the proof is to use the Cartan decomposition of $m \in \mathrm{GL}_2(\mathbb{Q}_p) \cap M_2(\mathbb{Z}_p)$. That is, such an m is a product $k_1 t k_2$ with $k_1, k_2 \in \mathrm{GL}_2(\mathbb{Z}_p)$ and t diagonal in $\mathrm{GL}_2(\mathbb{Q}_p) \cap M_2(\mathbb{Z}_p)$.

We first claim that if $m = k \in \mathrm{GL}_2(\mathbb{Z}_p)$, then † automatically holds for this k. To see this, note that because $k \in \mathrm{GL}_2(\mathbb{Z}_p)$, $\mathrm{Span}_{\mathbb{Z}_p}(1, \omega, \theta) = \mathrm{Span}_{\mathbb{Z}_p}(1, \omega'', \theta'')$. Thus that † holds follows from the equivalence of (1) and (3) of Proposition 4.3.

Now suppose that $m = t = \mathrm{diag}(t_1, t_2)$ is diagonal in $M_2(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Z}_p)$. Then for this m, $b' = t_1 b$ and $c' = t_2 c$, so that it is clear that $t \cdot f$ has coefficients in $\mathbb{Z}_p$ implies that † holds.

Now suppose that $m = k_1 t k_2$ and that $m \cdot f$ has coefficients in $\mathbb{Z}_p$. It follows that $t k_2 \cdot f$ has coefficients in $\mathbb{Z}_p$. Thus, from what has been said, † holds for $m' := t k_2$. Applying k_1, it follows that † holds for m, as desired. $\square$

We note the following corollary of Lemma 4.2.

Corollary 4.5. *Let $R = \mathbb{Z}_p$, and let the binary cubic $f(x, y)$ correspond to the maximal order T is the étale $\mathbb{Q}_p$-algebra $T \otimes \mathbb{Q}_p$. Suppose that $m \in \mathrm{GL}_2(\mathbb{Q}_p)$ satisfies $m \cdot f$ has coefficients in $\mathbb{Z}_p$. Then $m \in M_2(\mathbb{Z}_p)$.*

Proof. Let $m = \left(\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{smallmatrix}\right)$ and T has the good basis $\{1, \omega, \theta\}$. We have $\omega' = m_{11}\omega + m_{12}\theta + \delta_1$ and $\theta' = m_{21}\omega + m_{22}\theta + \delta_2$. Lemma 4.2 implies, because $m \cdot f$ has coefficients in $\mathbb{Z}_p$, that $\mathrm{Span}_{\mathbb{Z}_p}\{1, \omega', \theta'\}$ is closed under multiplication. But then, because T is maximal by assumption, this means that $\omega', \theta' \in T$. It follows that all the coefficients m_{ij} are in $\mathbb{Z}_p$, as desired. $\square$

5. The Fourier coefficient of the approximate basic function

In this section, we explain the computation of the left-hand side of (3.6).

For $t \in \mathrm{GL}_1$ and $h \in \mathrm{GL}_2 \simeq M$ the Levi of the Heisenberg parabolic of G_2, define

$$D_\chi(t, h) = \int_{N(\mathbb{Q}_p)} \chi(n)\Delta(t, nh)\, dn.$$

Then, applying the Iwasawa decomposition, the left-hand side of (3.6) becomes

$$D(s) = \int_{\mathrm{GL}_1(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{Q}_p)} \delta_P^{-1}(h)|t|^{s+2} D_\chi(t, h) L(hv_0)\, dh\, dt.$$

For $p \geq 5$, the $D_\chi(t,h)$ is computed in [12, Proposition 5.7]. Below, we explain that the expression obtained for $D_\chi(t,h)$ in *loc cit* continues to hold for $p=2$ and 3.

We now recall various notations from [12] that we need to state the computation of $D_\chi(t,h)$. First, let $f_{\max}(x;y) = ax^3+bx^2y+cxy^2+dy^3$ be a binary cubic form corresponding to the maximal order $\mathcal{O}_E = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ in $\mathbb{Q}_p^3$, so that f_{max} is some $\mathrm{GL}_2(\mathbb{Z}_p)$ translate of $xy(x+y)$. Let $1, \omega, \theta$ be the good basis $\mathcal{O}_E$ associated to f_{max}. If $x = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$, then denote by $T(x)$ the $\mathbb{Z}_p$ module spanned by $1, \delta\omega - \beta\theta$, and $-\gamma\omega + \alpha\theta$. Hence $T(x) = T_{\widetilde{x}}$ in the notation of Section 4. Note that, by the results of Section 4, $T(x)$ is closed under multiplication implies $x \in M_2(\mathbb{Z}_p)$ and $\widetilde{x} \cdot f_{max} = f_{max} \cdot x$ has $\mathbb{Z}_p$ coefficients, and conversely.

For a general binary cubic form V with coefficients in $\mathbb{Z}_p$, define $N(V)$ to be the number of 0's of V in $\mathbf{P}^1(\mathbb{F}_p)$. Also, for an element $h \in \mathrm{GL}_2(\mathbb{Q}_p)$, define $\mathrm{val}(h) \in \mathbb{Z}$ to be the largest integer n so that $p^{-n}h \in M_2(\mathbb{Z}_p)$.

Proposition 5.1 (Proposition 5.7 in [12])**.** *Define $x_0(h)$ by $h = p^{val(h)}x_0(h)$, and set $\lambda = \det(h)/t$. Write $D'_\chi(\lambda, h) = D_\chi(t,h)$, i.e., D'_χ is the same function as D_χ, except expressed in terms of the new variables λ, h. Finally, define:*

$$\epsilon(x_0(h)) = \begin{cases} 1 & \text{if } \ x_0(h) \in \mathrm{GL}_2(\mathbb{Z}_p), \\ 2 & \text{if } \ x_0(h) \notin \mathrm{GL}_2(\mathbb{Z}_p). \end{cases}$$

Then

$$\begin{aligned} D'_\chi(\lambda; h) = {} & |\det(\lambda^{-1}h)|^{-1} \operatorname{char}(h \in M_2(\mathbb{Z}_p), \mathrm{val}(\lambda^{-1}h) \in \{0,1\}, T(x_0(h)) \text{ a ring}) \\ & \times \left\{ \begin{array}{ll} 1 & \text{if } \ \mathrm{val}(\lambda^{-1}h) = 0 \\ N(f_{max}) - \epsilon(x_0(h)) & \text{if } \ \mathrm{val}(\lambda^{-1}h) = 1 \end{array} \right\}. \end{aligned}$$

Proof. The proof from [12] carries over line-by-line (with one minor change, as we explain). To aid the reader in checking this, we give some of the omitted details from *loc cit* to make clear that they continue to hold for $p = 2, 3$.

First, we show that $D_\chi(t,h) \neq 0$ implies $h \in M_2(\mathbb{Z}_p)$. We have $D_\chi(t,h) = \int_N \psi(\langle \omega, n \rangle)\Delta(t, nh)\, dn$ for ω the element of W corresponding to f_{max}. By the change of variables $n \mapsto hnh^{-1}$, one finds that, up to positive constant coming from the change in measure,

$$D_\chi(t,h) = \int_N \psi(\langle \omega, hnh^{-1} \rangle)\Delta(t, hn) dn.$$

Now, Δ is right-invariant under $G_2(\mathbb{Z}_p)$, so if $u_0 \in G_2(\mathbb{Z}_p) \cap N$, then $\Delta(t, hnu_0) = \Delta(t, hn)$. By another change of variables $n \mapsto nu_0$ in $D_\chi(t,h)$, one finds that

$$D_\chi(t,h) = \psi(\langle \omega, hu_0h^{-1} \rangle) D_\chi(t,h).$$

Thus, for $D_\chi(t,h)$ to be nonzero, one must have $\langle \omega, hu_0h^{-1} \rangle \in \mathbb{Z}_p$ for every $u_0 \in N \cap G_2(\mathbb{Z}_p)$. It follows that $\omega \cdot h$ corresponds to a binary cubic form with $\mathbb{Z}_p$ integral coefficients, and thus $h \in M_2(\mathbb{Z}_p)$ by Corollary 4.5.

Let us remark upon the one aspect of the proof which is ever so slightly different from [12, Proof of Proposition 5.7]. In *loc cit*, one verifies that $D_\chi(t,h)$ nonzero implies $f_{max} \cdot x_0(h)$ has $\mathbb{Z}_p$ coefficients. Now, by Proposition 4.4, one concludes that $T(x_0(h))$ is a ring.

As mentioned, the rest of the proof of the proposition carries over line-by-line from [12, Proof of Proposition 5.7]. ∎

6. Non-archimedean zeta integral

In this section we compute the right-hand side of (3.6). In the case when $p \geq 5$, the calculation is done in [12], so the new work is for $p = 2$ and 3. Still, many computations are similar to the ones in the proof for the case $p \geq 5$.

6.1. **The computation of $\Phi_{p,\chi}$.** To compute the right-hand side of (3.6), we first compute the function $\Phi_{p,\chi}(t, g)$ in (3.4). This computation of $\Phi_{p,\chi}$ is different from the one in [12, Lemma 5.6].

Lemma 6.1. *Suppose that $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in the Heisenberg Levi, so that h takes e_1 to $ae_1 + ce_3^*$ and e_3^* to $be_1 + de_3^*$. Let $f_0(x, y) = x^2y + xy^2$. Set $\lambda = \det(h)/t$ and $h' = \frac{1}{\lambda}\widetilde{h} = \lambda^{-1}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Set $f_1(x, y) = h' \cdot f_0(x, y)$ with $f_i(x, y) = \alpha_i x^3 + \beta_i x^2 y + \gamma_i xy^2 + \delta_i y^3$ for $i = 0, 1$. Then $\Phi_{p,\chi}(t, h) = |\lambda| A_0(\lambda, h)$, where $A_0(\lambda, h)$ is the characteristic function of the following quantities:*

- *λ integral,*
- *h' integral,*
- *$h' \cdot f_0(x, y) = \det(h')^{-1} f_0((x, y)h')$ integral,*
- $\begin{pmatrix} \beta_1 \\ -\gamma_1 \end{pmatrix} \equiv h' \begin{pmatrix} \beta_0 \\ -\gamma_0 \end{pmatrix} \pmod 3$.

Before we prove Lemma 6.1, we state a corollary and give various notations that will be used in its proof. For a cubic ring T over $\mathbb{Z}_p$, the largest integer c so that $T = \mathbb{Z}_p + p^c T_0$ for a cubic ring T_0 over $\mathbb{Z}_p$ is called the *p-adic content* of T and denote by $c(T)$. If T corresponds to the binary cubic g, then the p-adic content of T is the largest integer c so that $p^{-c}g$ has coefficients in $\mathbb{Z}_p$. Let $x = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Q}_p)$. Recall that we denote $T(x)$ as the $\mathbb{Z}_p$-module spanned by spanned by $1, \delta\omega - \beta\theta, -\gamma\omega + \alpha\theta$, where $1, \omega, \theta$ is the good basis of the fixed maximal order.

Corollary 6.2. *One has*

$$A_0(\lambda, h) = \mathrm{char}(\lambda \in \mathbb{Z}_p, T(\lambda^{-1}h) \textit{ is a ring}) = \mathrm{char}\left(\lambda \in \mathbb{Z}_p, \lambda \mid p^{c(T(h))}\right). \tag{6.1}$$

Proof. This follows from Lemma 6.1 by an application of the results of Section 4. The condition $\lambda \in \mathbb{Z}_p$ corresponds to the first bullet point and $\lambda \mid p^{c(T(h))}$ corresponds to the last three bullet points in Lemma 6.1, respectively. $\square$

The characteristic function of Corollary 6.2 is now the same characteristic function as described in [12, pg. 21] for $p \geq 5$.

We now make some convenient notations that will be used in the proof of Lemma 6.1. Recall that Θ_0 is the split model of the integral octonions with $\mathbb{Z}$-basis $\{\epsilon_1, \epsilon_2, e_1, e_2, e_3, e_1^*, e_2^*, e_3^*\}$. Now, for α_i, γ_j in a ring R, we define

$$\{\alpha_1, \alpha_2, \alpha_3\}_{e_1} := \alpha_1 \epsilon_1 \wedge e_1 - \alpha_2 \epsilon_2 \wedge e_1 + \alpha_3 e_2^* \wedge e_3^*,$$
$$\{\alpha_1, \alpha_2, \alpha_3\}_{e_2} := \alpha_1 \epsilon_1 \wedge e_2 - \alpha_2 \epsilon_2 \wedge e_2 + \alpha_3 e_3^* \wedge e_1^*,$$

$$\{\alpha_1, \alpha_2, \alpha_3\}_{e_3} := \alpha_1 \epsilon_1 \wedge e_3 - \alpha_2 \epsilon_2 \wedge e_3 + \alpha_3 e_1^* \wedge e_2^*.$$

Similarly,

$$\{\gamma_1, \gamma_2, \gamma_3\}_{e_1^*} := \gamma_2 \epsilon_1 \wedge e_1^* - \gamma_1 \epsilon_2 \wedge e_1^* + \gamma_3 e_2 \wedge e_3,$$
$$\{\gamma_1, \gamma_2, \gamma_3\}_{e_2^*} := \gamma_2 \epsilon_1 \wedge e_2^* - \gamma_1 \epsilon_2 \wedge e_2^* + \gamma_3 e_3 \wedge e_1,$$
$$\{\gamma_1, \gamma_2, \gamma_3\}_{e_3^*} := \gamma_2 \epsilon_1 \wedge e_3^* - \gamma_1 \epsilon_2 \wedge e_3^* + \gamma_3 e_1 \wedge e_2$$

as elements of $\wedge^2_{\mathbb{Z}} \Theta_0 \otimes R$. This notation is useful because one has

$$[\{\alpha_1, \alpha_2, \alpha_3\}_{e_1}, \{\beta_1, \beta_2, \beta_3\}_{e_2}] = \{\alpha_2\beta_3 + \alpha_3\beta_2, \alpha_3\beta_1 + \alpha_1\beta_3, \alpha_1\beta_2 + \alpha_2\beta_1\}_{e_3^*}$$

and

$$[\{\gamma_1,\gamma_2,\gamma_3\}_{e_1^*},\{\delta_1,\delta_2,\delta_3\}_{e_2^*}] = \{\gamma_2\delta_3+\gamma_3\delta_2,\gamma_3\delta_1+\gamma_1\delta_3,\gamma_1\delta_2+\gamma_2\delta_1\}_{e_3}.$$

We also calculate that

$$\begin{aligned}[\{\alpha_1,\alpha_2,\alpha_3\}_{e_1},\{\gamma_1,\gamma_2,\gamma_3\}_{e_3^*}] &= (\alpha_1\gamma_1+\alpha_2\gamma_2+\alpha_3\gamma_3)e_1\wedge e_3^*,\\ [\{\alpha_1,\alpha_2,\alpha_3\}_{e_j},\{\beta_1,\beta_2,\beta_3\}_{e_j}] &= 0.\end{aligned} \tag{6.2}$$

For our later use, we record the following formulas. From the formulas in Subsection 2.5, one computes that under the action of $m = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{GL}_2 \simeq M$, one has

$$2\epsilon_1\wedge e_1+\epsilon_2\wedge e_1-e_2^*\wedge e_3^* \mapsto a\left(2\epsilon_1\wedge e_1+\epsilon_2\wedge e_1-e_2^*\wedge e_3^*\right)+c\left(2\epsilon_2\wedge e_3^*+\epsilon_1\wedge e_3^*+e_1\wedge e_2\right),$$

and

$$2\epsilon_1\wedge e_3^*+\epsilon_2\wedge e_3^*-e_1\wedge e_2 \mapsto b\left(\epsilon_1\wedge e_1+2\epsilon_2\wedge e_1+e_2^*\wedge e_3^*\right)+d\left(2\epsilon_1\wedge e_3^*+\epsilon_2\wedge e_3^*-e_1\wedge e_2\right).$$

In other words,

$$\{2,-1,-1\}_{e_1} \mapsto a\{2,-1,-1\}_{e_1}+c\{-2,1,1\}_{e_3^*},$$

and

$$\{-1,2,-1\}_{e_3^*} \mapsto b\{1,-2,1\}_{e_1}+d\{-1,2,-1\}_{e_3^*}.$$

Recall that

$$\widetilde{v_E} = \epsilon_1\wedge(e_1+e_3^*) = \epsilon_1\wedge e_1+\epsilon_1\wedge e_3^* = \{1,0,0\}_{e_1}+\{0,1,0\}_{e_3^*}.$$

We have

$$\begin{aligned}\widetilde{v_E} &= \{1,0,0\}_{e_1}+\{0,1,0\}_{e_3^*}\\ &= \frac{1}{3}\left(\{1,1,1\}_{e_1}+\{1,1,1\}_{e_3^*}\right)+\frac{1}{3}\left(\{2,-1,-1\}_{e_1}+\{-1,2,-1\}_{e_3^*}\right).\end{aligned}$$

The terms in the first set of parentheses are in the Lie algebra $\mathfrak{g}_2$, and correspond to the polynomial x^2y+xy^2 in the sense described in the first paragraph of [12, pg. 18].

If $m \in M \simeq \mathrm{GL}_2$, we can now compute $m\widetilde{v_E}$. Indeed, suppose $m = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. Set $f_0(x,y) = x^2y+xy^2$, and let

$$m\cdot f_0(x,y) = \det(m)^{-1}f_0((x,y)m) = \alpha x^3+\beta x^2y+\gamma xy^2+\delta y^3.$$

Recall the following notation from [12, Sect. 4.1]:

- $E_{12} = -e_1\wedge e_2^*$
- $v_1 = \{1,1,1\}_{e_1}$
- $\delta_3 = \{1,1,1\}_{e_3^*}$
- $E_{23} = -e_2\wedge e_3^*$.

With this notation, we have

$$\begin{aligned}m\widetilde{v_E} &= \left(\alpha E_{12}+\frac{1}{3}\beta v_1+\frac{1}{3}\gamma\delta_3+\delta E_{23}\right)\\ &\quad+\frac{1}{3}\left(a\{2,-1,-1\}_{e_1}+c\{-2,1,1\}_{e_3^*}+b\{1,-2,1\}_{e_1}+d\{-1,2,-1\}_{e_3^*}\right).\end{aligned}$$

Next, we present one final calculation before computing $\Phi_\chi(t,m)$. Suppose that $X = u_1E_{12}+u_2v_1+u_3\delta_3+u_4E_{23}$ is in the Lie algebra of N, the unipotent radical of the Heisenberg parabolic of G_2. Then we must compute $[X, \widetilde{v_E}]$. Then using (6.2), we obtain

$$\begin{aligned}[X, \widetilde{v_E}] &= [u_2 v_1 + u_3 \delta_3, \widetilde{v_E}] \\ &= [u_2\{1,1,1\}_{e_1} + u_3\{1,1,1\}_{e_3^*}, \{1,0,0\}_{e_1} + \{0,1,0\}_{e_3^*}] \\ &= u_2[\{1,1,1\}_{e_1}, \{0,1,0\}_{e_3^*}] - u_3[\{1,0,0\}_{e_1}, \{1,1,1\}_{e_3^*}] \\ &= (u_2 - u_3) e_1 \wedge e_3^*.\end{aligned}$$

We are now in a position to compute $\Phi_\chi(t, h)$.

Proof of Lemma 6.1. Set $\lambda = \det(h)/t$. From the computations above, if $n = \exp(X)$, one obtains

$$n^{-1}\widetilde{v_E} = \widetilde{v_E} + (u_2 - u_3)E_{13}.$$

Then

$$\begin{aligned}th^{-1}n^{-1}\widetilde{v_E} &= th^{-1}\widetilde{v_E} + t\det(h)^{-1}(u_2 - u_3)E_{13} \\ &= \lambda^{-1}\det(h)(h^{-1}\cdot\widetilde{v_E}) + \frac{u_2 - u_3}{\lambda}E_{13}.\end{aligned}$$

Now, $h^{-1} = \det(h)^{-1}\left(\begin{smallmatrix} d & -b \\ -c & a\end{smallmatrix}\right)$. We obtain

$$\begin{aligned}th^{-1}n^{-1}\widetilde{v_E} = &\left(\alpha_1 E_{12} + \frac{\beta_1}{3}v_1 + \frac{\gamma_1}{3}\delta_3 + \delta_1 E_{23}\right) + \frac{u_2 - u_3}{\lambda}E_{13} \\ &+ \lambda^{-1}\frac{1}{3}\left(d\{2,-1,-1\}_{e_1} - c\{-2,1,1\}_{e_3^*} - b\{1,-2,1\}_{e_1} + a\{-1,2,-1\}_{e_3^*}\right).\end{aligned}$$

One can now use this expression to verify the result. Indeed, rewriting this expression, one obtains

$$\begin{aligned}th^{-1}n^{-1}\widetilde{v_E} = &\,\alpha_1 E_{12} + \delta_1 E_{23} + \frac{u_2 - u_3}{\lambda}E_{13} \\ &+ \lambda^{-1}d\{1,0,0\}_{e_1} + \lambda^{-1}b\{0,1,0\}_{e_1} + \frac{1}{3}(\beta_1 - \lambda^{-1}d - \lambda^{-1}b)\{1,1,1\}_{e_1} \\ &+ \lambda^{-1}c\{1,0,0\}_{e_3^*} + \lambda^{-1}a\{0,1,0\}_{e_3^*} + \frac{1}{3}(\gamma_1 - \lambda^{-1}c - \lambda^{-1}a)\{1,1,1\}_{e_3^*}.\end{aligned}$$

Now, in order for the integral over $N^{0,E}\backslash N$ in (3.4) to be nonzero, one requires $\lambda \in \mathbb{Z}_p$. Moreover, the resulting integral is $|\lambda|$ times the characteristic function of the quantities:

- $\alpha_1, \delta_1 \in \mathbb{Z}_p$;
- $\lambda^{-1}a, \lambda^{-1}b, \lambda^{-1}c, \lambda^{-1}d \in \mathbb{Z}_p$;
- $\frac{1}{3}(\beta_1 - \lambda^{-1}d - \lambda^{-1}b)$ and $\frac{1}{3}(\gamma_1 - \lambda^{-1}c - \lambda^{-1}a)$ in $\mathbb{Z}_p$.

The lemma follows. □

6.2. **The local unramified computations.** For ease of notation, we now set $c(x) := c(T(x))$ for $x \in \mathrm{GL}_2(\mathbb{Q}_p)$. Recall the integral $I(s) = I_p(s)$ from (3.3). Using the exact same calculations as in [12, pg. 22], we notice

$$\begin{aligned}I(s+1) &= \frac{1}{1-z}\left(\sum_{[h]\in\text{cubic ring}} L(hv)|\det(h)|^{-2}z^{\mathrm{val}(\det(h))-c(h)}\left(1 - z^{c(h)+1}\right)\mathrm{char}(c(h) \geq 0)\right) \\ &= \frac{1}{1-z}\left(\sum_{[h]\in\text{cubic ring}} L(hv)|\det(h)|^{-2}P_h(z)\right),\end{aligned}$$

where $P_h(z) := z^{\mathrm{val}(\det(h))-c(h)}\left(1 - z^{c(h)+1}\right)\mathrm{char}(c(h) \geq 0)$.

To finish the evaluation of $I(s)$ in terms of L-functions, we must apply $M(\pi_p, s)$ to $I(s+1)$ (see [12, Section 5.4, 5.8]). The computations follow line-by-line just as in *loc cit.* To make clear that the proofs from [12] hold for $p = 2, 3$, we fill in various details that were omitted in that paper.

We begin by explaining the proof of [12, Lemma 5.10].

Proof of Lemma 5.10 of [12]. First we claim that f_0 factors into linear factors over an unramified field extension L of $\mathbb{Q}_p$. To see this, let T_0 be the cubic ring corresponding to f_0. Then by assumption, $T_0 \otimes \mathbb{Q}_p$ is an unramified extension of $\mathbb{Q}_p$. It follows that for some unramified field extension $L/\mathbb{Q}_p$, one has $T_0 \otimes L \approx L \times L \times L$. The binary cubic corresponding to the right-hand side is split, say $xy(x+y)$. The association between binary cubics and cubic rings is clearly compatible with base change, so it follows that f_0 factors over L.

Let $\mathcal{O}_L$ denote the ring of integers in L. By Gauss's Lemma, we conclude that f_0 factors into linear factors over $\mathcal{O}_L$, say $f_0 = L_1 \cdot L_2 \cdot L_3$. By using this factorization of f_0 and that p is a uniformizer in $\mathcal{O}_L$, the lemma follows without much difficulty.

For example, suppose that we are in the final case, so $f_0 \equiv \alpha\ell^3$ modulo p. Without loss of generality, assume $\ell = x$. Then by the factorization of f_0 given above, we can write $f_0(x, y) = \beta L_1' L_2' L_3'$ with $\beta \in \mathcal{O}_L^\times$ and $L_1' \equiv L_2' \equiv L_3' \equiv x$ modulo p. It follows that $\frac{1}{p} f_0(px, y)$ has content 2, showing that one of the $q+1$-sublattices has content 2.

The rest of the proof proceeds similarly. $\square$

We now explain some of the omitted aspects of [12] that occur after Lemma 5.10 of *loc cit.* We have

$$P_h(z) = z^{v-c}(1 - z^{c+1}) \operatorname{char}(c(h) \geq 0).$$

Recall from [12, Section 5.7] the function

$$M_h(z) = q^2 P_{hp}(z) + P_{hp^{-1}}(z) + (N(f_{max} \cdot h) - 1)P_h(z) + qP_{h*T(p)}(z) + P_{h*T(p^{-1})}(z).$$

The work after Lemma 5.10 of [12] is to evaluate $B_0(z)P_h(z) - z^2 M_h(z)$, where

$$B_0(z) = 1 + (p+1)z + pz^2 + (p^2+p)z^3 + p^2 z^4.$$

We first elaborate on this evaluation in case $c(h) \geq 2$, which is the case explained in *loc cit.* Let

$$g(z) = p^2 z^{v-c+1}(1 - z^{c+2}) + z^{v-c-1}(1 - z^c) + pz^{v-c}(1 - z^{c+1}).$$

Then, when $c \geq 2$, the first three terms in the above expression for $M_h(z)$ give $g(z)$. The point is that when h is changed, v and c change, as these depend on h. One has $v(hp) = v(h) + 2$ and $c(hp) = c(h) + 1$. Thus $P_{hp}(z) = z^{v-c+1}(1 - z^{c+2})$. Similarly, $P_{hp^{-1}}(z) = z^{v-c-1}(1 - z^c)$, because $v(hp^{-1}) = v(h) - 2$ and $c(hp^{-1}) = c(h) - 1$. Finally, because $c \geq 1$, we have $N = N(f_{max} \cdot h) = q + 1$, so $(N-1)P_h(z) = qz^{v-c}(1 - z^{c+1})$. Putting these computations together gives

$$q^2 P_{hp}(z) + P_{hp^{-1}}(z) + (N-1)P_h(z) = g(z).$$

The terms in $M_h(z)$ with the Hecke operators are computed using Lemma 5.10. For example, in the first case where f_0 is irreducible modulo p, $P_{hT(p)}(z) = (q+1)z^{v-c+2}(1 - z^c)$, because $v(hg_i) = v(h) + 1$ and $c(hg_i) = c(h) - 1$, using Lemma 5.10. Here g_i are the coset representatives for the Hecke operator $T(p)$. Similarly,

$$P_{hT(p^{-1})}(z) = (q+1)z^{v-c+1}(1 - z^{c-1}).$$

Combining the above expressions gives the expression for $M_h(z)$ at the bottom on page 27 of [12] [2].

[2] We remark also that there is a typo at the bottom of page 27 of *loc cit*: in the case $f_0 = \alpha\ell^3$, the term $z^{v-c+2}(1 - z^{c+2})$ should instead say $z^{v-c-2}(1 - z^{c+2})$

We now claim that the cases where $c = 1$ in fact do not need to be considered separately from those where $c \geq 2$. Indeed, this is because the terms that vanish because of the $\mathrm{char}(c(h) \geq 0)$ in case $c = 1$ all have a $(1 - z^{c-1})$ in them, and so vanish anyway.

We now explain a bit the calculation of $B_0(z)P_h(z) - z^2 M_h(z)$ in case $c = 0$. First note that, in the case $c = 0$,

$$N = N(f_{max} \cdot h) = \left\{ \begin{array}{lllll} 0 & \text{if} & c = 0 & \text{and} & f_0 \text{ is irreducible modulo } p \\ 1 & \text{if} & c = 0 & \text{and} & f_0 = \ell q \\ 3 & \text{if} & c = 0 & \text{and} & f_0 = \ell_1 \ell_2 \ell_3 \\ 2 & \text{if} & c = 0 & \text{and} & f_0 = \ell_1^2 \ell_2 \\ 1 & \text{if} & c = 0 & \text{and} & f_0 = \alpha \ell^3 \end{array} \right\}.$$

From this expression for N, one computes $M_h(z)$ in case $c = 0$ as follows.

(1) f_0 irreducible modulo p: In this case $h = 1$ necessarily and $M_h(z) = q^2 z(1-z^2) + (-1)(1-z)$. One has $P_h(z) = 1 - z$ and then $B_0(z)P_h(z) - z^2 M_h(z) = (1 + qz)(1 - z^3)$.

(2) $f_0 \equiv \ell q \pmod p$: One has $M_h(z) = q^2 z^{v+1}(1 - z^2) + z^v(1 - z) + qz^{v+1}(1 - z)$ and then $B_0(z)P_h(z) - z^2 M_h(z) = z^v(1 + qz)(1 - z^2)$.

(3) $f_0 \equiv \ell_1\ell_2\ell_3 \pmod p$: One has $M_h(z) = q^2 z^{v+1}(1 - z^2) + 2z^v(1 - z) + 3qz^{v+1}(1 - z)$ and then $B_0(z)P_h(z) - z^2 M_h(z) = z^v(1 + qz)(1 - z)^2(1 + 2z)$.

(4) $f_0 \equiv \ell_1^2\ell_2 \pmod p$: $M_h(z) = q^2 z^{v+1}(1-z^2) + z^v(1-z) + qz^{v+1}(1-z) + qz^v(1-z^2) + z^{v-1}(1-z)$ and then $B_0(z)P_h(z) - z^2 M_h(z) = z^v(1 + qz)(1 - z)(1 - z^2)$.

(5) $f_0 \equiv \alpha\ell^3 \pmod p$: $M_h(z) = q^2 z^{v+1}(1 - z^2) + qz^{v-1}(1 - z^3) + z^{v-2}(1 - z^2)$ and then $B_0(z)P_h(z) - z^2 M_h(z) = 0$.

Now, for $B_0(z)P_h(z) - z^2 M_h(z)$ what results is the same expression as on the top of page 28 in *loc cit*, except with $c = 0$. This completes our evaluation of $I(s)$.

7. The Eisenstein series

Recall from (3.1) the normalized Eisenstein series $E^*_\ell(g, s)$. The purpose of this section is to prove the following theorem.

Theorem 7.1. *The normalized Eisenstein series satisfies the functional equation*

$$E^*_\ell(g, s) = E^*_\ell(g, 5 - s).$$

This theorem is an immediate consequence of Langlands' functional equation, with the difficulty being the computation of the appropriate intertwining operator of the section $f_\ell(g, s)$.

Consider the diagonal maximal torus T' of G' consisting of the elements

$$t = \mathrm{diag}(t_1, t_2, t_3, t_4, t_4^{-1}, t_3^{-1}, t_2^{-1}, t_1^{-1}).$$

For $1 \leq j \leq 4$, let r'_j denote the characters of T' that takes the element t to t_j. We fix a maximal torus T of G that maps to T' under the map $G \to G'$, and write r_j for the restriction of r'_j to T. We label the Dynkin diagram of G by roots $\alpha_1 = r_1 - r_2, \alpha_2 = r_3 + r_4, \alpha_3 = r_3 - r_4, \alpha_4 = r_2 - r_3$. Then α_4 is the central vertex of the diagram.

We abuse notation and also denote by r_j the restriction to T of the character $t \mapsto |t_j|$ of T'. Then the inducing character for our Eisenstein series is $|\nu|^s = s(r_1 + r_2)$. This is in $\mathrm{Ind}_B^G(\delta_B^{1/2}\lambda_s)$ with $\delta_B^{1/2} = 3r_1 + 2r_2 + r_3$ and $\lambda_s = (s - 3)r_1 + (s - 2)r_2 - r_3$.

Let N be the unipotent radical of the Heisenberg parabolic, so that the roots in N are $r_1 - r_3, r_1 - r_4, r_1 + r_4, r_1 + r_3, r_2 - r_3, r_2 - r_4, r_2 + r_4, r_2 + r_3, r_1 + r_2$. The long intertwiner for N is $w = [412343214] = [412434214]$. Here the notation $[ijk]$ means that one performs a reflection in the roots i, j, k from right to left. To see that this expression of w as a product of simple reflections is correct, one checks that this w makes the roots in N negative, and it has length 9.

Denote by $M(w,s)f_\ell(g,s) = \int_{N(\mathbb{A})} f_\ell(w^{-1}ng,s)\,dn$. To prove Theorem 7.1, we use the following result.

Proposition 7.2. *One has*

$$M(w,s)f_\ell(g,s) = c_\ell(s)f_\ell(g,5-s),$$

where

$$c_\ell(s) = \frac{\Lambda(s-3)^2\Lambda(s-4)\Lambda(2s-5)}{\Lambda(s-1)^2\Lambda(s)\Lambda(2s-4)}\frac{\Gamma(s-2)\Gamma(s-3)\Gamma(s-2)\Gamma(s-1)}{\Gamma(s-\ell-3)\Gamma(s-\ell-2)\Gamma(s+\ell-1)\Gamma(s+\ell-2)}.$$

Indeed, Theorem 7.1 follows from Proposition 7.2, the Definition (3.1) and Langlands' functional equation by noting the equality

$$\frac{\Gamma(s-2)\Gamma(s-3)}{\Gamma(s-\ell-2)\Gamma(s-\ell-3)} = \frac{\Gamma(4-s+\ell)\Gamma(3-s+\ell)}{\Gamma(4-s)\Gamma(3-s)}.$$

For the rest of this section, we focus on proving Proposition 7.2.

To explain the proof of Proposition 7.2, we introduce some convenient notation. Let x, y be indeterminates, s be a complex parameter, and fix a positive even integer $\ell > 0$. Set $f_1 = x+y$ and $f_2 = x-y$. Note that the span of the polynomials $x^{2\ell-2j}y^{2j} + x^{2j}y^{2\ell-2j}$ for $0 \le j \le \ell/2$ is the same as the span of the polynomials $f_1^{2\ell-2k}f_2^{2k} + f_1^{2k}f_2^{2\ell-2k}$ for $0 \le k \le \ell/2$. Call this span V_{even}. We will define a few operators on the space V_{even}. For a non-negative integer $k \ge 0$ and $z \in \mathbb{C}$, let $(z)_k = z(z+1)(z+2)\cdots(z+k-1)$, which is the so-called Pochhammer symbol.

For a complex number s, define $[s;x,y]$ as the operator on V_{even} given by the map

$$x^{2\ell-2j}y^{2j} + x^{2j}y^{2\ell-2j} \mapsto \frac{\left(\frac{1-s}{2}\right)_{|\frac{\ell}{2}-j|}}{\left(\frac{1+s}{2}\right)_{|\frac{\ell}{2}-j|}}\left(x^{2\ell-2j}y^{2j} + x^{2j}y^{2\ell-2j}\right),$$

where $0 \le j \le \ell/2$.

Similarly, let $[s;f_1,f_2]$ be the operator on V_{even} that is given by

$$f_1^{2\ell-2j}f_2^{2j} + f_1^{2j}f_2^{2\ell-2j} \mapsto \frac{\left(\frac{1-s}{2}\right)_{|\frac{\ell}{2}-j|}}{\left(\frac{1+s}{2}\right)_{|\frac{\ell}{2}-j|}}\left(f_1^{2\ell-2j}f_2^{2j} + f_1^{2j}f_2^{2\ell-2j}\right)$$

for $0 \le j \le \ell/2$.

Proof of Proposition 7.2. We record how the simple reflections move the character λ_s around, and how it acts on the inducing section $f_\ell(g,s)$.

- $\lambda_s = (s-3)r_1 + (s-2)r_2 + (-1)r_3$
- $[4]$; $\frac{\Lambda(s-1)}{\Lambda(s)}[s-1, f_1f_2]$; $(s-3)r_1 + (-1)r_2 + (s-2)r_3$
- $[1]$; $\frac{\Lambda(s-2)}{\Lambda(s-1)}[s-2;x,y]$; $(-1)r_1 + (s-3)r_2 + (s-2)r_3$
- $[2]$; $\frac{\Lambda(s-2)}{\Lambda(s-1)}[s-2;x,y]$; $(-1)r_1 + (s-3)r_2 + (2-s)r_4$
- $[4]$; $\frac{\Lambda(s-3)}{\Lambda(s-2)}[s-3;f_1,f_2]$; $(-1)r_1 + (s-3)r_3 + (2-s)r_4$
- $[3]$; $\frac{\Lambda(2s-5)}{\Lambda(2s-4)}[2s-5,x,y]$; $(-1)r_1 + (2-s)r_3 + (s-3)r_4$
- $[4]$; $\frac{\Lambda(s-2)}{\Lambda(s-1)}[s-2,f_1,f_2]$; $(-1)r_1 + (2-s)r_2 + (s-3)r_4$
- $[2]$; $\frac{\Lambda(s-3)}{\Lambda(s-2)}[s-3,x,y]$; $(-1)r_1 + (2-s)r_2 + (3-s)r_3$
- $[1]$; $\frac{\Lambda(s-3)}{\Lambda(s-2)}[s-3,x,y]$; $(2-s)r_1 + (-1)r_2 + (3-s)r_3$
- $[4]$; $\frac{\Lambda(s-4)}{\Lambda(s-3)}[s-4,f_1,f_2]$; $(2-s)r_1 + (3-s)r_2 + (-1)r_3$.

The notation above has the following meaning, which we explain by an example: Set $\lambda'_s = (s-3)r_1 + (-1)r_2 + (s-2)r_3$. Then when one applies the intertwining operator associated to the reflection [4] to the inducing section $f_\ell(g,s)$, one obtains the unique K_f-spherical, K_∞-equivariant element of $\mathrm{Ind}(\delta_B^{1/2}\lambda'_s)$, whose value at $g=1$ is $\frac{\Lambda(s-1)}{\Lambda(s)}[s-1, f_1 f_2](x^\ell y^\ell)$. Denote the resulting inducing section by $f'_\ell(g,s)$, and set $\lambda''_s = (-1)r_1 + (s-3)r_2 + (s-2)r_3$. Then, when one applies the intertwining operator associated to the reflection [1] to $f'_\ell(g,s)$, one obtains the unique K_f-spherical, K_∞-equivariant element of $\mathrm{Ind}(\delta_B^{1/2}\lambda''_s)$, whose value at $g=1$ is $\frac{\Lambda(s-2)}{\Lambda(s-1)}[s-2;x,y]\frac{\Lambda(s-1)}{\Lambda(s)}[s-1, f_1 f_2](x^\ell y^\ell)$.

We defer a proof for the justification of the above list until the next subsection. Granted this, the $\Lambda(s)$-terms multiply to

$$\frac{\Lambda(s-3)^2\Lambda(s-4)\Lambda(2s-5)}{\Lambda(s-1)^2\Lambda(s)\Lambda(2s-4)}.$$

The other terms give a "polynomial intertwiner"

(7.1)

$$M_{poly}(s) = [s-4, f_1, f_2]\circ[s-3,x,y]^2\circ[s-2,f_1,f_2]\circ[2s-5,x,y]\circ[s-3,f_1,f_2]\circ[s-2,x,y]^2\circ[s-1,f_1,f_2].$$

The proposition now follows from Proposition 7.4 below. □

7.1. **The root intertwiners.** The purpose of this subsection is to fill in the omitted details in the proof of Proposition 7.2.

We require the following lemma.

Lemma 7.3. *Let B denote the upper-triangular Borel of SL_2. For $\theta \in \mathbb{R}$, set $k_\theta = \left(\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta)\end{smallmatrix}\right)$. Suppose $f_{\mathrm{SL}_2,j}(g,s)$ is the section in $\mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{SL}_2(\mathbb{R})}(\delta_B^{1/2}\delta_B^{s/2})$ satisfying $f_{\mathrm{SL}_2,j}(gk_\theta,s) = e^{ij\theta}f_{\mathrm{SL}_2,j}(g,s)$ for all $g\in\mathrm{SL}_2(\mathbb{R})$ and $k_\theta \in \mathrm{SO}(2)$ as above. Then*

$$\int_{\mathbb{R}} f_{\mathrm{SL}_2,j}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right)g, s\right)\,dx = i^j\frac{\Gamma_{\mathbb{C}}(s)}{\Gamma_{\mathbb{R}}(s-j+1)\Gamma_{\mathbb{R}}(s+j+1)}f_{\mathrm{SL}_2,j}(g,-s).$$

When j is even this is

$$\int_{\mathbb{R}} f_{\mathrm{SL}_2,j}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right)g, s\right)\,dx = \frac{\Gamma_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(s+1)}\frac{\left(\frac{1-s}{2}\right)_{|j/2|}}{\left(\frac{1+s}{2}\right)_{|j/2|}}f_{\mathrm{SL}_2,j}(g,-s).$$

Proof. The proof is standard. To give some details anyway, it suffices to consider the case $g=1$. Then

$$\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & -x(x^2+1)^{-1} \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} (x^2+1)^{-1/2} & \\ & (x^2+1)^{1/2}\end{smallmatrix}\right)\left(\begin{smallmatrix} x(x^2+1)^{-1/2} & -(x^2+1)^{-1/2} \\ (x^2+1)^{-1/2} & x(x^2+1)^{-1/2}\end{smallmatrix}\right),$$

from which one obtains

$$\begin{aligned}\int_{\mathbb{R}} f_{\mathrm{SL}_2,j}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right), s\right)\,dx &= \int_{\mathbb{R}} (x^2+1)^{-(s+1)/2}\left(\frac{x+i}{(x^2+1)^{1/2}}\right)^j\,dx \\ &= \int_{\mathbb{R}} (x+i)^{-(s-j+1)/2}(x-i)^{-(s+j+1)/2}\,dx.\end{aligned}$$

This last integral is evaluated in [10, Lemma 7.2.3, pg. 279]. One obtains

$$i^j 2\pi 2^{-s}\frac{\Gamma(s)}{\Gamma(\frac{s-j+1}{2})\Gamma(\frac{s+j+1}{2})} = i^j\frac{\Gamma_{\mathbb{C}}(s)}{\Gamma_{\mathbb{R}}(s-j+1)\Gamma_{\mathbb{R}}(s+j+1)}.$$

Note that when j is even,

$$\begin{aligned}\frac{\Gamma_{\mathbb{C}}(s)}{\Gamma_{\mathbb{R}}(s-j+1)\Gamma_{\mathbb{R}}(s+j+1)} &= \frac{\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)}{\Gamma_{\mathbb{R}}(s-j+1)\Gamma_{\mathbb{R}}(s+j+1)} \\ &= \frac{\Gamma_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(s+1)}\frac{\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(s+1)}{\Gamma_{\mathbb{R}}(s-j+1)\Gamma_{\mathbb{R}}(s+j+1)},\end{aligned}$$

and

$$\begin{aligned}\frac{\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(s+1)}{\Gamma_{\mathbb{R}}(s-j+1)\Gamma_{\mathbb{R}}(s+j+1)} &= \frac{\Gamma((s+1)/2)^2}{\Gamma((s-j+1)/2)\Gamma((s+j+1)/2)} \\ &= \frac{\left(\frac{s+1-|j|}{2}\right)_{|j/2|}}{\left(\frac{s+1}{2}\right)_{|j/2|}} \\ &= (-1)^{j/2}\frac{\left(\frac{1-s}{2}\right)_{|j/2|}}{\left(\frac{1+s}{2}\right)_{|j/2|}}\end{aligned}$$

The result then follows. □

To apply the lemma, we use the following calculations. The root $\mathfrak{sl}_2$'s give rise to the following elements in the Lie algebra of G, corresponding to the element $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ in $\mathfrak{sl}_2$:

- $\alpha_1 = r_1 - r_2 : b_1 \wedge b_{-2} + b_{-1} \wedge b_2 = u_1 \wedge u_2 - u_{-1} \wedge u_{-2} = -\frac{i}{2}(h_\ell^+ + h_\ell'^+ + h_\ell^- + h_\ell'^-)$
- $\alpha_2 = r_3 + r_4 : b_3 \wedge b_4 + b_{-3} \wedge b_{-4} = v_1 \wedge v_2 + v_{-1} \wedge v_{-2} = -\frac{i}{2}(h_\ell^+ - h_\ell'^+ - h_\ell^- + h_\ell'^-)$
- $\alpha_3 = r_3 - r_4 : b_3 \wedge b_{-4} + b_{-3} \wedge b_4 = v_1 \wedge v_2 - v_{-1} \wedge v_{-2} = -\frac{i}{2}(h_\ell^+ - h_\ell'^+ + h_\ell^- - h_\ell'^-)$
- $\alpha_4 = r_2 - r_3 : b_2 \wedge b_{-3} + b_{-2} \wedge b_3 = u_2 \wedge v_1 - u_{-2} \wedge v_{-1} = -\frac{i}{2}(-H^+ - H'^+ + H^- + H'^-)$ where $H^? = e_\ell^? + f_\ell^?$ and $? \in \{+, '+, -, '-\}$.

Combining the above calculations with Lemma 7.3, one arrives at the SL_2-intertwiners as stated in Proposition 7.2.

7.2. **The polynomial intertwiner.** Recall the operator $M_{poly}(s)$ from equation (7.1) in the proof of Proposition 7.2. The purpose of this subsection is to prove the following result, from which Proposition 7.2 follows.

Proposition 7.4. *One has $M_{poly}(s)x^\ell y^\ell = c_{poly,\ell}(s)x^\ell y^\ell$ with*

$$\begin{aligned}c_{poly,\ell}(s) &= \frac{(s-3)(s-4)^2(s-5)^2\cdots(s-\ell-2)^2(s-\ell-3)}{(s+\ell-2)(s+\ell-3)^2(s+\ell-4)^2\cdots(s-1)^2(s-2)} \\ &= \frac{\Gamma(s-2)\Gamma(s-3)\Gamma(s-2)\Gamma(s-1)}{\Gamma(s-\ell-3)\Gamma(s-\ell-2)\Gamma(s+\ell-1)\Gamma(s+\ell-2)}\end{aligned}$$

We require the following lemma.

Lemma 7.5. *Let $F_w(u,v) = (1-2u-2v+(u-v)^2)^{-w}$. Set*

$$p_{j,k}(w) = \frac{\Gamma(2w+j+k)\Gamma(w+j+k+1/2)\Gamma(w+1/2)}{\Gamma(2w)\Gamma(w+k+1/2)\Gamma(w+j+1/2)}.$$

Then

$$F_w(u,v) = (1-2u-2v+(u-v)^2)^{-w} = \sum_{j,k\geq 0} p_{j,k}(w)\frac{u^j v^k}{j!k!}.$$

Proof. First note that the $p_{j,k}(w)$ are polynomials in w. Now, if the polynomials $p_{j,k}(w)$ are indeed the Taylor coefficients of $F_w(u,v)$, then these polynomials must satisfy the expression

$$\begin{aligned} p_{j,k}(w-1) =& p_{j,k}(w) - 2jp_{j-1,k}(w) - 2kp_{j,k-1}(w) + j(j-1)p_{j-2,k}(w) \\ & -2jkp_{j-1,k-1}(w) + k(k-1)p_{j,k-2}(w). \end{aligned} \tag{7.2}$$

This relationship comes from multiplying out the identity

$$(1-2u-2v+u^2-2uv+v^2)\left(\sum_{j,k} p_{j,k}(w)\frac{u^j v^k}{j!k!}\right) = F_{w-1}(u,v) = \sum_{j,k} p_{j,k}(w-1)\frac{u^j v^k}{j!k!}.$$

Now, we claim two things:

(1) One can verify (7.2) directly;
(2) Combined with the fact that the $p_{j,k}(w)$ are polynomials, the identity (7.2) implies the lemma.

For the first claim, one uses the functional equation $s\Gamma(s) = \Gamma(s+1)$ of the Gamma function to relate the polynomials $p_{j,k}(w)$ to $p_{j,k}(w-1)$. For example, one can obtain

$$p_{j,k}(w) = \frac{(2w-2+j+k)(2w-1+j+k)(w+j+k-1/2)(w-1/2)}{(2w-2)(2w-1)(w+j-1/2)(w+k-1/2)} p_{j,k}(w-1).$$

One can obtain similar expressions relating $p_{j-1,k}(w), p_{j-1,k-1}(w), \ldots$ to $p_{j,k}(w-1)$. Then the identity (7.2) becomes an identity in rational functions of w, j, k, which one can verify directly.

For the second claim, note that the Taylor coefficients of $F_w(u,v)$ are necessarily polynomials in w. Thus to see that they are equal to $p_{j,k}(w)$, it suffices to check that they are equal at infinitely many integers. But the identity (7.2) allows one to induct, and thus verify the Taylor expansion of $F_w(u,v)$ for all *negative* integers w. This implies the lemma. □

Lemma 7.6. *Let*

$$c'(s) = \frac{(s-\ell)(s-\ell+2)\cdots(s-4)(s-2)}{(s+1)(s+3)\cdots(s+\ell-3)(s+\ell-1)}.$$

Then

(1) $D(s-1;x,y) \circ D(s;f_1,f_2)x^\ell y^\ell = \frac{1}{2^\ell}c'(s)f_1^\ell f_2^\ell$;
(2) $D(s-1;f_1,f_2) \circ D(s;x,y)f_1^\ell f_2^\ell = 2^\ell c'(s)x^\ell y^\ell$.

Proof. Part (2) follows from Part (1) by switching the roles of x, y with f_1, f_2. We now prove Part (1).

Set $w = \frac{1-s-\ell}{2}$. Then

$$(-1)^{\ell/2}4^\ell D(s;f_1,f_2)\frac{x^\ell y^\ell}{\ell!} = \frac{\Gamma(w)^2}{\Gamma(w+\ell/2)^2}\sum_j \frac{\Gamma(w+\ell-j)\Gamma(w+j)}{\Gamma(w)^2}\frac{f_1^{2\ell-2j}f_2^{2j}}{(\ell-j)!j!}.$$

Summing the right-hand side over ℓ, one obtains

$$\begin{aligned} \frac{\Gamma(w)^2}{\Gamma(w+\ell/2)^2}(1-f_1^2)^{-w}(1-f_2^2)^{-w} &= \frac{\Gamma(w)^2}{\Gamma(w+\ell/2)^2}(1-2(x^2+y^2)+(x^2-y^2)^2)^{-w} \\ &= \frac{\Gamma(w)^2}{\Gamma(w+\ell/2)^2}\sum_{j,k\geq 0} p_{j,k}(w)\frac{x^{2j}y^{2k}}{j!k!}. \end{aligned}$$

Note that $s-1=w+\frac{1}{2}$. Using Lemma 7.5 and applying $D(s-1,x,y)$ to the above expression gives

$$(-1)^{\ell/2}\frac{\Gamma(w)^2}{\Gamma(w+\ell/2)^2}\sum_{j,k}(-1)^k\frac{\Gamma(2w+j+k)\Gamma(w+j+k+1/2)\Gamma(w+1/2)}{\Gamma(2w)\Gamma(w+(\ell+1)/2)^2}\frac{x^{2j}y^{2k}}{j!k!}.$$

Thus $4^\ell D(s-1;x,y)D(s;f_1,f_2)\frac{x^\ell y^\ell}{\ell!}$ is the sum over ℓ of

$$\frac{\Gamma(w)^2\Gamma(2w+\ell)\Gamma(w+\ell+1/2)\Gamma(w+1/2)}{\Gamma(w+\ell/2)^2\Gamma(w+(\ell+1)/2)^2\Gamma(2w)}\frac{(x^2-y^2)^\ell}{\ell!}.$$

Rewriting in terms of s gives the lemma. Indeed, the product of Γ functions can be written as a product

$$\frac{\Gamma(2w+\ell)}{\Gamma(2w)}\cdot\frac{\Gamma(w+\ell+1/2)}{\Gamma(w+(\ell+1)/2)}\cdot\frac{\Gamma(w+1/2)}{\Gamma(w+(\ell+1)/2)}\cdot\frac{\Gamma(w)^2}{\Gamma(w+\ell/2)^2},$$

where each of these individual ratio of Γ functions is a rational function of w. For this rational function one obtains

$$2^\ell\frac{(2w+\ell+1)(2w+\ell+3)\cdots(2w+2\ell-1)}{(2w)(2w+2)(2w+4)\cdots(2w+\ell-2)}=2^\ell\frac{(s-\ell)(s-\ell+2)\cdots(s-6)(s-4)(s-2)}{(s+1)(s+3)\cdots(s+\ell-3)(s+\ell-1)}.$$

This latter product is $2^\ell c'(s)$. This completes the proof. □

Proof of Proposition 7.4. The proposition now follows easily from the factorization (7.1) of $M_{poly}(s)$ and Lemma 7.6. □

8. Archimedean zeta integral

In this section, we explicitly compute an archimedean integral that is part of the Rankin-Selberg integral. Below, we use the symbol $\doteq$ to denote equality up to a nonzero constant.

Recall from Section 3.3 that we define

$$I^*(s;\ell)=2^s\Gamma_{\mathbb{R}}(s-1)\Gamma_{\mathbb{C}}(s+\ell-1)\Gamma_{\mathbb{C}}(s+\ell-2)I(s;\ell), \tag{8.1}$$

where

$$I(s;\ell)=\int_{N^{0,E}(\mathbb{R})\backslash G_2(\mathbb{R})}\{f_\ell(\gamma_0 g,s),\mathcal{W}_\chi(g)\}_K\,dg.$$

Here $\mathcal{W}_\chi$ is the generalized Whittaker function. This means that $\mathcal{W}_\chi : G_2(\mathbf{R})\to Sym^{2\ell}(V_2)$ is a smooth function of moderate growth; it satisfies the condition

$$\mathcal{W}_\chi(ngk)=\chi(n)k^{-1}\cdot\mathcal{W}_\chi(g)\quad\text{ for all } n\in N(\mathbb{R}),\, k\in K \text{ and } g\in G_2(\mathbb{R}),$$

and we have $\mathcal{D}_\ell\mathcal{W}_\chi=0$ for the Schmid operator $\mathcal{D}_\ell$(see [12, p. 10]). Also, the braces $\{\,,\,\}_K$ denote the unique (up to scalar multiple) K-equivariant pairing on $Sym^{2\ell}(V_2)$.

Our goal is to prove the following theorem.

Theorem 8.1. *One has*

$$I^*(s;\ell)\doteq\Gamma_{\mathbb{R}}(s-1)\Gamma_{\mathbb{C}}(s+\ell-3)\Gamma_{\mathbb{C}}(s+\ell-2)\Gamma_{\mathbb{C}}(s+2\ell-3).$$

Note that by (8.1), it suffices to compute the integral $I(s;\ell)$, for which an expression was found in Section 6 of [12]. To state that result, we define the function

$$J'(s)=|q(v_E)|^{-s}\int_{V^*}|q(v)|^s e^{-|\langle v,r_0(i)\rangle|^2}\,\frac{dV}{|q(v)|}. \tag{8.2}$$

Here V^* is the $GL_2(\mathbb{R})$-orbit that consists of the binary cubics that split over $\mathbb{R}$ and dV denotes the Haar measure on V^*. Also, $v_E=(0,\frac{1}{3},\frac{1}{3},0)$ corresponds to the binary cubic x^2y+xy^2, and $r_0(i) = (1, -i, -1, i)$ corresponds to the binary cubic $(x - iy)^3$. The quartic form q and the symplectic pairing $\langle\, ,\, \rangle$ are as defined in Section 2.

Our first step in computing $I(s;\ell)$ is the following result.

Proposition 8.2. *One has*

$$I(s;\ell) \doteq \pi^{-s} \frac{\Gamma(s+2\ell-3)\Gamma(s+\ell-2)\Gamma((s+\ell-3)/2)^2}{\Gamma((s+\ell)/2)\Gamma(s+\ell-3)\Gamma((3s+3\ell-7)/2)} J'\left(\frac{s+\ell-2}{2}\right).$$

Proof. Let χ' denote the archimedean part of the character $\psi(\langle v_E, \overline{n}\rangle)$, so that $\chi'(n) = e^{2\pi i \langle v_E, \overline{n}\rangle}$. In the notation of [12] (compare with the second displayed equation on page 30 of [12]), one has

$$I(s;\ell) = \int_{\mathrm{GL}_2(\mathbb{R})} \int_{(N^{0,E}\backslash N)(\mathbb{R})} \frac{|\det(m)|^{-3} e^{2\pi i \langle v_E, \overline{n}\rangle}}{||x(n,m)||^{(s+\ell)}} \{pr_K(x(n,m))^\ell, \mathcal{W}_{\chi'}(m)\}_K \, dn \, dm.$$

The $\Gamma((s+\ell)/2)$ of *loc cit* has disappeared because $I(s;\ell)$ is defined in terms of the flat section $f_\ell(g;s)$ whereas $I(s,\Phi)$ in [12], that is

$$I(s,\Phi) = \int_{N^{0,E}(\mathbb{R})\backslash GL_1(\mathbb{R})\times G_2(\mathbb{R})} |t|^s \{\Phi(tg^{-1}\widetilde{v_E}), \mathcal{W}_\chi(g)\}_K \, dg,$$

is defined in terms of a section that takes the value $\Gamma((s+\ell)/2)$ at $g = 1$.

Now, by following the same argument as in page 30 of [12], one obtains

$$I(s;\ell) \doteq \int_{\mathrm{GL}_2(\mathbb{R})\times(N^{0,E}\backslash N)(\mathbb{R})} \frac{|\det(m)|^{s+\ell-2} e^{2\pi i\beta}}{(|\alpha|^2 + |\beta|^2)^{(s+\ell)/2}} \left(\sum_j \binom{\ell}{j} (i\beta)^{\ell-j} |\alpha|^j K_0^{(j)}(2\pi|\alpha|) \right) dn\, dm.$$

Using the change of variables $\beta \mapsto (2\pi)^{-1}\beta$ and $m \mapsto (2\pi 1_2)^{-1} m$ (note that α depends on m) one gets

$$I(s;\ell) \doteq (2\pi)^{-s} \int_{\mathrm{GL}_2(\mathbb{R})\times(N^{0,E}\backslash N)(\mathbb{R})} \frac{|\det(m)|^{s+\ell-2} e^{i\beta}}{(|\alpha|^2 + |\beta|^2)^{(s+\ell)/2}} \left(\sum_j \binom{\ell}{j} (i\beta)^{\ell-j} |\alpha|^j K_0^{(j)}(|\alpha|) \right) dn\, dm.$$

Consequently, $I(s;\ell) = (2\pi)^{-s}\Gamma((s+\ell)/2)^{-1} I(s,\Phi)$. The result now follows from the first part of Theorem 6.2 in [12].

We remark that the factor $|q(v_E)|^{-s}$ was mistakenly omitted in the first part of Theorem 6.2. It should first appear in the fifth displayed equation on page 32, as $|\det(g)|^2 = |q(v_E)|^{-1}|q(v)|$ and then be carried over to the expression for $I(s,\Phi)$ in the first part of Theorem 6.2.

□

As our second step, we now prove

Proposition 8.3. *Let J'(s) be as in* (8.2)*. One has*

$$J'(s) \doteq 2^{-6s}\Gamma(2s) \frac{\Gamma(3s - 1/2)}{\Gamma(s+1/2)^3}.$$

Proof. To compute $J'(s)$, it suffices to integrate over those elements of V^* whose coefficient of x^3 is nonzero. Such a binary cubic can be written as

$$t(x - r_1 y)(x - r_2 y)(x - r_3 y)$$

with $t, r_1, r_2, r_3 \in \mathbb{R}$. To compute the integral $J'(s)$, we make the variable change

- $a = t$
- $b = -t(r_1 + r_2 + r_3)$
- $c = t(r_1 r_2 + r_2 r_3 + r_3 r_1)$

- $d = -tr_1r_2r_3$.

The Jacobian of this transformation equals

$$\frac{\partial(a,b,c,d)}{\partial(t,r_1,r_2,r_3)} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ *_1 & t & t & t \\ *_2 & t(r_2+r_3) & t(r_1+r_3) & t(r_1+r_2) \\ *_3 & tr_2r_3 & tr_1r_3 & tr_1r_2 \end{vmatrix} = \pm t^3(r_1-r_2)(r_2-r_3)(r_3-r_1),$$

where $*_1, *_2, *_3$ denote some real numbers. Note that one has

$$q(x^2y+xy^2)^{-1}q(t(x-r_1y)(x-r_2y)(x-r_3y)) = t^4(r_1-r_2)^2(r_2-r_3)^2(r_3-r_1)^2.$$

By combining this with the change of variables, one obtains

$$\begin{aligned} J'(s) &= \int_{t,r_1,r_2,r_3} t^{4s-4}e^{-t^2(1+r_1^2)(1+r_2^2)(1+r_3^2)} \prod_{1\le i<j\le 3} |r_i-r_j|^{2s-2}\frac{\partial(a,b,c,d)}{\partial(t,r_1,r_2,r_3)}d(t,r_1,r_2,r_3) \\ &= \int_{t,r_1,r_2,r_3} t^{4s-1}e^{-t^2(1+r_1^2)(1+r_2^2)(1+r_3^2)} \prod_{1\le i<j\le 3} |r_i-r_j|^{2s-1}\, d(t,r_1,r_2,r_3) \\ &\doteq \Gamma(2s)\int_{r_1,r_2,r_3} |1+r_1^2|^{-2s}|1+r_2^2|^{-2s}|1+r_3^2|^{-2s} \prod_{1\le i<j\le 3} |r_i-r_j|^{2s-1}\, d(r_1,r_2,r_3) \end{aligned}$$

The integral on the last line is a special case of the Selberg integral. From (1.19) in [2] with $\alpha=\beta=2s$, $\gamma=s-1/2$ and $n=3$, it follows that

$$\begin{aligned} J'(s) &\doteq 2^{-6s}\Gamma(2s)\prod_{j=0}^{2}\frac{\Gamma(4s-1-(2+j)(s-1/2))\Gamma(1+(j+1)(s-1/2))}{\Gamma(2s-j(s-1/2))^2\Gamma(s+1/2)} \\ &= 2^{-6s}\Gamma(2s)\frac{\Gamma(3s-1/2)}{\Gamma(s+1/2)^3}. \end{aligned}$$

□

Proof of Theorem 8.1. It immediately follows from Proposition 8.3 that

$$J'\left(\frac{s+\ell-2}{2}\right) \doteq 2^{-3s-3\ell+6}\Gamma(s+\ell-2)\frac{\Gamma((3s+3\ell-7)/2)}{\Gamma((s+\ell-1)/2)^3}.$$

By combining this with Proposition 8.2, one obtains

$$I(s;\ell) \doteq (8\pi)^{-s}\frac{\Gamma(s+2\ell-3)\Gamma(s+\ell-2)^2\Gamma((s+\ell-3)/2)^2}{\Gamma(s+\ell-3)\Gamma((s+\ell-1)/2)^3\Gamma((s+\ell)/2)}.$$

Then by (8.1),

$$\begin{aligned} I^*(s;\ell) &= 2^s\Gamma_{\mathbb{R}}(s-1)\Gamma_{\mathbb{C}}(s+\ell-1)\Gamma_{\mathbb{C}}(s+\ell-2)I(s;\ell) \\ &\doteq (4\pi)^{-s}\Gamma_{\mathbb{R}}(s-1)\Gamma_{\mathbb{C}}(s+\ell-1)\Gamma_{\mathbb{C}}(s+\ell-2) \\ &\quad\times\frac{\Gamma(s+2\ell-3)\Gamma(s+\ell-2)}{\Gamma((s+\ell-1)/2)\Gamma((s+\ell)/2)}\frac{\Gamma(s+\ell-2)\Gamma((s+\ell-3)/2)^2}{\Gamma(s+\ell-3)\Gamma((s+\ell-1)/2)^2} \\ &\doteq \Gamma_{\mathbb{R}}(s-1)\Gamma_{\mathbb{C}}(s+\ell-1)\Gamma_{\mathbb{C}}(s+\ell-2) \\ &\quad\times\frac{\Gamma_{\mathbb{C}}(s+2\ell-3)\Gamma_{\mathbb{C}}(s+\ell-2)}{\Gamma_{\mathbb{R}}(s+\ell-1)\Gamma_{\mathbb{R}}(s+\ell)}\frac{\Gamma_{\mathbb{C}}(s+\ell-2)\Gamma_{\mathbb{R}}(s+\ell-3)^2}{\Gamma_{\mathbb{C}}(s+\ell-3)\Gamma_{\mathbb{R}}(s+\ell-1)^2} \\ &= \Gamma_{\mathbb{R}}(s-1)\Gamma_{\mathbb{C}}(s+\ell-3)\Gamma_{\mathbb{C}}(s+\ell-2)\Gamma_{\mathbb{C}}(s+2\ell-3) \end{aligned}$$

where we used the duplication formula. The theorem follows. □

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