THE GENERALIZED INJECTIVITY CONJECTURE

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Abstract. We prove a conjecture of Casselman and Shahidi stating that the unique irreducible generic subquotient of a standard module is necessarily a subrepresentation for a large class of connected quasi-split reductive groups, in particular for those which have a root system of classical type (or product of such groups). To do so, we prove and use the existence of strategic embeddings for irreducible generic discrete series representations, extending some results of Moeglin.

[1]

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1. Introduction

1.1. Let G be a quasi-split connected reductive group over a non-Archimedean local field F of characteristic zero. We assume we are given a standard parabolic subgroup P with Levi decomposition $P = MU$ as well as an irreducible, tempered, generic representation τ of M. Let now ν be an element in the dual of the real Lie algebra of the split component of M; we take it in the positive Weyl chamber. The induced representation $I_P^G(\tau, \nu) := I_P^G(\tau_\nu)$, called the standard module, has a unique irreducible quotient, $J(\tau_\nu)$, often named the Langlands quotient. Since the representation τ is generic (for a non-degenerate character of U, see the Section 2), i.e. has a Whittaker model,

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[1] 2010 *Mathematics Subject Classification* 11F70, 22E50

the standard module $I_P^G(\tau_\nu)$ is also generic. Further, by a result of Rodier [32] any generic induced module has a unique irreducible generic subquotient.

In their paper Casselman and Shahidi [11] conjectured that:

(A) $J(\tau_\nu)$ is generic if and only if $I_P^G(\tau_\nu)$ is irreducible.
(B) The unique irreducible generic subquotient of $I_P^G(\tau_\nu)$ is a subrepresentation.

These questions were originally formulated for real groups by Vogan [40]. Conjecture (B), was resolved in [11] provided the inducing data be cuspidal. Conjecture (A), known as the Standard Module Conjecture, was first proven for classical groups by Muić in [29], and was settled for quasi-split p-adic groups in [21] assuming the Tempered L Function Conjecture proven a few years later in [22].

The second conjecture, known as the Generalized Injectivity Conjecture was proved for classical groups $SO(2n+1)$, $Sp(2n)$, and $SO(2n)$ for P a maximal parabolic subgroup, by Hanzer in [16].

In the present work we prove the Generalized Injectivity Conjecture (Conjecture (B)) for a large class of quasi-split connected reductive groups provided the irreducible components of a certain root system (denoted Σ_σ) are of type A, B, C or D (see Theorem 1.1 below for a precise statement). Following the terminology of Borel-Wallach [4.10 in [6]], for a standard parabolic subgroup P, τ a tempered representation and $\eta \in (a_M^*)^+$, a positive Weyl chamber, (P, τ, η) is referred as Langlands data, and η is the Langlands parameter, see the Definition 3.1 in this manuscript.

We will study the unique irreducible generic subquotient of a standard module $I_P^G(\tau_\eta)$ and make *first* the following reductions:

- τ is discrete series representation of the standard Levi subgroup M
- P is a maximal parabolic subgroup.

  Then, η is written $s\tilde{\alpha}$, see the Subsection 1.8 for a definition of the latter.

Then, our approach has two layers: First we realized the generic discrete series τ as a subrepresentation of an induced module $I_{P_1\cap M}^M(\sigma_\nu)$ for a unitary generic cuspidal representation of M_1 (using Proposition 2.5 of [22]), and the parameter ν is dominant (i.e in some positive closed Weyl chamber) in a sense later made precise; Using induction in stages, we can therefore embed the standard module $I_P^G(\tau_{s\tilde{\alpha}})$ in $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$.

Let us denote $\nu + s\tilde{\alpha} := \lambda$. The unique generic subquotient of the standard module is also the unique generic subquotient in $I_{P_1}^G(\sigma_\lambda)$. By a result of Heiermann-Opdam [Proposition 2.5 of [22]], this generic subquotient appears as a subrepresentation of yet another induced representation $I_{P'}^G(\sigma'_{\lambda'})$ characterized by a parameter λ' in the closure of some positive Weyl chamber.

In an ideal scenario, λ and λ' are dominant with respect to P_1 (resp. P'), i.e. λ and λ' are in the closed positive Weyl chamber, and we may then build a bijective operator between those two induced representations using the dominance property of the Langlands parameters.

In case the parameter λ is not in the closure of the positive Weyl chamber, two alternatives procedures are considered: first, another strategic embedding of the irreducible generic subquotient in the representation induced from $\sigma''_{\lambda''}$ (relying on extended Moeglin's Lemmas) when the parameter λ'' (which depends on the form of λ) has a very specific aspect (this is Proposition 6.4); or (resp. and) showing the intertwining operator between $I_{P'}^G(\sigma'_{\lambda'})$ (resp. $I_{P_1}^G(\sigma''_{\lambda''})$) and $I_{P_1}^G(\sigma_\lambda)$ has non-generic kernel.

1.2. In order to study a larger framework than the one of classical groups studied in [16], we will use the notion of *residual points* of the μ function (the μ function is the main ingredient of the Plancherel density for p-adic groups (see the Definition 2.1 and Subsection 2.2).

Indeed, as briefly suggested in the previous point, the triple (P_1, σ, λ), introduced above, plays a pivotal role in all the arguments developed thereafter, and of particular importance, the parameter λ is related to the μ function in the following ways:

- When σ_λ is a residual point for the μ function (abusively one will say that λ is a residual point once the context is clear), the unique irreducible generic subquotient in the module induced from σ_λ is discrete series (A result of Heiermann in [18], see Proposition 2.2).
- Once the cuspidal representation σ is fixed, we attach to it the set Σ_σ, a root system in a subspace of $a^*_{M_1}$ defined using the μ function. More precisely, let α be a root in the set of reduced roots of A_{M_1} in Lie(G) and $(M_1)_\alpha$ be the centralizer of $(A_{M_1})_\alpha$ (the identity component of the kernel of α in A_{M_1}), we will consider the set

$$\Sigma_\sigma = \{\alpha \in \Sigma_{red}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\}$$

  it is a subset of $a^*_{M_1}$ which is a root system in a subspace of $a^*_{M_1}$ (cf [37] 3.5) and we suppose the irreducible components of Σ_σ are of type A, B, C or D. Let us denote W_σ the Weyl group of Σ_σ.

  This is where stands the particularity of our method, to deal with all possible standard modules, we needed an explicit description of this parameter λ lying in $a^*_{M_1}$. Thanks to Opdam's work in the context of affine Hecke algebras and Heiermann's one in the context of p-adic reductive groups such descriptive approach is made possible: Indeed, we have bijective correspondences between the following sets explained in Section 4:
  {dominant residual point}
  {Weighted Dynkin diagram(s)}

The notion of Weighted Dynkin diagram is established and recalled in the Appendix B.1.

We use this correspondence to express the coordinates of the dominant residual point and name this expression of the residual point a *residual segment* generalizing the classical notion of segments (of Bernstein-Zelevinsky). We associate to such a residual segment *set(s) of Jumps* (a notion connected to that of Jordan blocks elements in the classical groups setting of Moeglin-Tadić in [27]).

Further, the μ function is intrinsically related to the *intertwining operators* mentioned in the previous subsection, see the end of the Subsection 1.5.

1.3. Having defined the root system Σ_σ, let us present the main result of this paper:

Theorem 1.1 (Generalized Injectivity conjecture for quasi-split group)**.** *Let G be a quasi-split, connected group defined over a p-adic field F (of characteristic zero) such that its root system is of type A, B, C or D (or product of these). Let π_0 be the unique irreducible generic subquotient of the standard module $I_P^G(\tau_\nu)$, then π_0 embeds as* a subrepresentation *in the standard module $I_P^G(\tau_\nu)$.*

Theorem 1.2 (Generalized Injectivity conjecture for quasi-split group)**.** *Let G be a quasi-split, connected group defined over a p-adic field F (of characteristic zero). Let π_0 be the unique irreducible generic subquotient of the standard module $I_P^G(\tau_\nu)$, let σ be an irreducible, generic, cuspidal representation of M_1 such that a twist by an unramified real character of σ is in the cuspidal support of π_0.*

Suppose that all the irreducible components of Σ_σ are of type A, B, C or D, then, under certain conditions on the Weyl group of Σ_σ (explained in Section 6.1, in particular Corollary 6.1.1), π_0 embeds as a subrepresentation *in the standard module $I_P^G(\tau_\nu)$.*

Theorem 1.1 results from 1.2. The Theorem 1.2 is true when the root system of the group G contains components of type E, F provided Σ_σ is irreducible of type A. We do not know if an analogue of Corollary 6.1.1 hold for groups whose root systems are of type E or F. Further, in the exceptional groups of type E or F, many cases where the cuspidal support of π_0 is (P_0, σ) (generalized principal series) cannot be dealt with the methods proposed in this work, see Section 10 for details.

1.4. Let us briefly comment on the organisation of this manuscript, therefore giving a general overview of our results and the scheme of proof. In subsequent point (see 1.5), we will give details on the ingredients of proofs.

In Section 3, we formulate the problem in an as broad as possible context (any quasi-split reductive p-adic group G) and prove a few results on intertwining operators.

As M.Hanzer in [16], we distinguish two cases: the case of a generic discrete series subquotient, and the case of a non-discrete series generic subquotient. As stated in 1.2, the case of discrete series subquotient corresponds to σ_λ (in the cuspidal support of the generic discrete series) being a residual point.

As just stated in 1.2, our approach uses the bijection between Weyl group orbits of residual points and weighted Dynkin diagrams as studied in [30] and explained in the Appendix B.

Through this approach, we can explicit the Langlands parameters of subquotients of the representations $I_{P_1}^G(\sigma_\lambda)$ induced from the generic cuspidal support σ_λ and classify them using the order on parameters in $a_{M_1}^*$ as given in Chapter XI, Lemma 2.13 in [6]. In particular, the minimal element for this order (in a sense later made precise) characterizes the unique irreducible generic non-discrete series subquotient, see Theorem 5.2.

Although requiring to get acquainted with the notions of residual points, and then residual segments, our methods have two advantages.

The first is proving the Generalized Injectivity Conjecture for a large class of quasi-split reductive groups (provided a certain construction of the standard Levi subgroup M_1 and the irreducible components of Σ_σ to be of type A, B, C or D; we have verified those conditions when the root system of the quasi-split (hence reductive) group is of type A, B, C or D), and recovering the results of Hanzer through alternative proofs.

In particular, a key ingredient (which was not used by Hanzer in [16]) in our method is an embedding result of Heiermann-Opdam (Proposition 2.1).

The second is a self-contained and uniform (in the sense that cases of root systems of type B, C and D are all treated in the same proofs) treatment.

Although based on the ideas of Hanzer in [16], our approach includes a much larger class of quasi-split groups and some cases of exceptional groups.

We separate this work into two different problems. The first problem is determining the conditions on $\lambda \in a_{M_1}^*$ so that the unique generic subquotient of $I_{P_1}^G(\sigma_\lambda)$ with σ irreducible unitary generic cuspidal representation of a standard Levi M_1 is a subrepresentation. The results on this problem are presented in Theorem 6.1.

The second problem is to show that any standard module can be embedded in a module induced from cuspidal generic data, with $\lambda \in a_{M_1}^*$ satisfying one of the conditions mentioned in Theorem 6.1. This is done in the Section 7 and the following.

Regarding the first problem: in the Subsection 6.3 we present an embedding result for the unique irreducible generic discrete series subquotient of the generic standard module (see Proposition 6.4) relying on two extended Moeglin's Lemmas (see Lemmas 14 and 15) and the result of Heiermann-Opdam (see Proposition 2.1). This embedding and the use of standard intertwining operators with non-generic kernel allow us to prove the Theorem 6.1.

Once achieved the Theorem 6.1, it is rather straightforward to prove the Generalized Injectivity Conjecture for discrete series generic subquotient, first when P is a maximal parabolic subgroup and secondly for *any parabolic subgroup* in Section 7.1.

In Subsection 7.2, we continue with the case of a generic non-discrete series subquotient, and further conclude with the case of the standard module induced from a tempered representation τ in Corollary 7.2.1 and Corollary 9.3.

The proof of Theorem 1.2 is done in several steps. First, we prove it for the case of an irreducible generic discrete series subquotient assuming τ discrete series, and Σ_σ irreducible in Proposition 7.1.

We use this latter result for the case of a non-square integrable irreducible generic subquotient in Proposition 7.3; and also for the case of standard modules induced from non-maximal standard parabolic (Theorems 7.1 and 7.2). Then, the case of τ tempered follows (Corollary 7.2.1). The reader familiar with the work of Bernstein-Zelevinsky on GL_n (see [33] or [43]) may want to have a look at Section 8 where we treat the case of Σ_σ of type A to get a quicker overview on some tools used in this work. The case of Σ_σ reducible is done in Section 9 and relies on the Appendix C.

1.5. **A sketch of the argument.** Let us consider the following standard module $I_P^G(\tau_{s\tilde{\alpha}})$, τ is an irreducible generic discrete series representation of a standard maximal Levi subgroup M of G. Using the result of Heiermann-Opdam (Proposition 2.1), it can be embedded in $I_{P_1\cap M}^M(\sigma_\nu)$. The representation σ is unitary, the parameter ν is in $\overline{(a_{M_1}^{M\,*})^+}$ and σ_ν is a residual point for μ^M. The Proposition 4.3 will then translate this 'residual point for μ^M' condition into knowing the existence and type of root systems $\Sigma_\sigma^M \subseteq \Sigma^M(A_{M_1})$.

$$\Sigma_\sigma^M = \{\alpha \in \Sigma_{red}^M(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\}$$

The root system Σ_σ^M compares easily to Σ_σ since $M_1 \subseteq M$.

As we consider maximal standard Levi subgroups of G, $M = M_\Omega \supset M_1$, corresponding to subsets $\Omega = \Delta - \{\beta\} \subset \Delta$ where β is a non extremal simple root of the Dynkin diagram of G. The subset Ω is a union of two connected components, and Σ_σ^M is a direct sum of two irreducible components $\Sigma_{\sigma,1}^M \bigcup \Sigma_{\sigma,2}^M$ of type A and $\mathcal{T}$, whenever Σ_σ is of type $\mathcal{T}$.

Typically, if Σ_σ is irreducible and $\mathcal{T}$ denotes its type, let $\Delta_\sigma := \{\alpha_1, \ldots, \alpha_d\}$ be the basis of Σ_σ (following our choice of basis for the root system of G).

The type of Σ_σ (determining the ones of Σ_σ^M) depend on the reducibility point (which is necessarily in the set $\{0, 1/2, 1\}$ since σ is generic), of some induced representations:

That is, for each α in Σ_σ^M (resp. Σ_σ), the reducibility point Λ of $I_{P_1\cap(M_1)_\alpha}^{(M_1)_\alpha}(\sigma_\Lambda)$ (see Proposition 4.3 and Example 4.1 following this Proposition) determines the type of weighted Dynkin diagram to be considered to evaluate the coordinates of the parameter $\nu \in \overline{(a_{M_1}^M)^{*+}}$ (resp. $\lambda = \nu + s\tilde{\alpha} \in a_{M_1}^{G*}$) corresponding to the residual point σ_ν (σ_λ when relevant).

The Proposition 4.3 also gives conditions on the rank of the root system Σ_σ, for σ_λ to be a residual point.

1.5.1. *The orbit.* In fact, once a cuspidal representation σ of the Levi subgroup M_1 is fixed, we consider the Weyl group W_σ-orbit of the (possibly residual) point σ_λ: in this orbit there is a unique λ parameter which is dominant, i.e. in the closure of the positive Weyl chamber. This dominant parameter in the dual of the Lie algebra $a_{M_1}^*$ will be described using the bijection between weigthed Dynkin diagrams and dominant residual points. In the canonical basis of this vector space the parameter is written as a string of (half)-integers (The half-integers are precisely those numbers that are half of an odd integer. The notation (half)-integers means either half-integers or integers). The string of (half)-integers which depends on the weights of the Dynkin diagram. Such string of (half)-integers will be called *residual segments*, where the notion of *segments* stands in analogy with the notion introduced by Bernstein-Zelevinsky in [4].

1.5.2. *Explicitely.* In this work, we will first assume Σ_σ is irreducible and prove the result under this restriction, this is also the case into consideration in this introduction. The case of Σ_σ reducible is considered in Proposition 9.2.

Since σ_ν is in the cuspidal support of the generic discrete series τ, applying the condition on the rank mentioned in the second paragraph of 1.5 (see Proposition 4.3) we have: $rk(\Sigma_\sigma^M) = d_1 - 1 + d_2$ and write

$$\Sigma_\sigma^M := A_{d_1-1} \bigcup \mathcal{T}_{d_2}$$

such that ν corresponds to *residual segments* ν_A and $\nu_{\mathcal{T}}$. The coordinates of these two vectors (of respective length d_1 and d_2) are computed using the weights of Weighted Dynkin diagrams (see our definition of residual segments in Definition 4.2).

Further, we twist the discrete series τ with

$$s\tilde{\alpha} \in {a_M^*}^+$$

this twist is added on the linear part (i.e corresponding to A_{d_1-1}). Consequently, $\nu_{\mathcal{T}}$ is left unchanged and is thus $\lambda_{\mathcal{T}}$, whereas ν_A becomes $\lambda_A = \nu_A + s\tilde{\alpha}$.

In this very specific context, we can characterize the set of two residual segments

$$(\nu_A, \nu_{\mathcal{T}})$$

The first residual segment of type A_{d_1-1} is uniquely characterized by two (half)-integers a, b with $a > b$ and the residual segment of type $\mathcal{T}_{d_2}$ is uniquely characterized by a tuple $\underline{n}$.

We call each such triple $(a, b, \underline{n})$ a *cuspidal string* and call W_σ-cuspidal string the orbit of the Weyl group W_σ of this *cuspidal string* (see the definitions in Section 4.3).

An example of this construction consists in the representation of a standard Levi subgroup $GL_{k\times d_1} \times G(k')$ of a classical group $G(n)$ of rank n. It is a tensor product of a Steinberg representation (of $GL_{k\times d_1}$) with π an irreducible generic discrete series of a classical group of smaller rank, $G(k')$, $n = 2kd_1 + k'$, $d_1 = a - b + 1$:

$$St_{d_1}(\rho)|.|^{\frac{a+b}{2}} \otimes \pi$$

The irreducible generic discrete series π corresponds to a residual segment $(\underline{n})$.

If we obtain from the vector of coordinates of $(\lambda_A, \lambda_{\mathcal{T}})$ a residual segment of length $d = \mathrm{rk}(\Sigma_\sigma)$ and type $\mathcal{T}$, σ_λ is a residual point for μ^G and the induced representation $I_{P_1}^G(\sigma_\lambda)$ has a discrete series subquotient (as explained in Proposition 4.3); this is the case where the unique irreducible generic subquotient is discrete series (by Theorem 5.1).

It is now time to use appropriately standard intertwining operators. This is where the μ-function intervenes a second time since this function enters in the definition of *intertwining operators*: A key aspect of this work is an appropriate use of (standard) intertwining operators, more precisely the use of intertwining operators with non-generic kernel. Using the functoriality of induction, it is always possible to reduce the study of intertwining operators to *rank one* intertwining operators (i.e consider the well-understood intertwining operator $J_{s_{\alpha_i}P_1|P_1}$ between $I_{P_1\cap(M_1)_{\alpha_i}}^{M_1}(\sigma_\lambda)$ and $I_{\overline{P_1\cap(M_1)_{\alpha_i}}}^{M_1}(\sigma_\lambda)$); and in particular if σ is irreducible cuspidal (see Theorem 2.1). At the level of rank one intertwining operator (where $I_{P_1\cap(M_1)_{\alpha_i}}^{M_1}(\sigma_\lambda)$ is the direct sum of two non-isomorphic representations, see Theorem 2.1), determining the non-genericity of the kernel of the map $J_{s_{\alpha_i}P_1|P_1}$ reduces to a simple condition on the relevant coordinates (i.e the coordinates determined by α_i) of $\lambda \in a_{M_1}^*$.

The case of non-discrete series generic subquotient makes an easy example to illustrate the usefulness of standard intertwining operators.

The Weyl group W_σ fixes the irreducible unitary cuspidal representation σ and acts on the parameter λ in $a_{M_1}^*$. In the W_σ-orbit of $(a, b, \underline{n})$, we will find a *cuspidal string* $(a', b', \underline{n'})$ such that the unique irreducible generic subquotient (non-square integrable i.e referred as tempered or non-tempered), denoted $I_{P'}^G(\tau'_{\nu'})$ embeds in $I_{P_1}^G(\sigma_{(a',b')+(\underline{n'})}) := I_{P_1}^G(\sigma(a', b', \underline{n'}))$.

The parameter ν' corresponds to the minimal element for the order on parameters in $a_{M_1}^*$ given in Chapter XI, Lemma 2.13 in [6], and this minimality condition is used in the Appendix of the author's PhD thesis [14] to identify the form of the cuspidal string $(a', b', \underline{n'})$ in the W_σ-orbit of the cuspidal string $(a, b, \underline{n})$.

Intertwining operators with non-generic kernel (see Proposition 3.2) allow us to transfer generic irreducible pieces (such as $I_{P'}^G(\tau'_{\nu'})$) from $I_{P_1}^G(\sigma(a', b', \underline{n'})$ to $I_{P_1}^G(\sigma((a, b, \underline{n}))$.

Since the latter induced module also contains $I_P^G(\tau_{s\tilde{\alpha}})$, by multiplicity one the irreducible generic subquotient, we conclude that $I_P^G(\tau_{s\tilde{\alpha}})$ contains $I_{P'}^G(\tau'_{\nu'})$ as a subrepresentation.

1.6. Let us come back on the case of an irreducible discrete series generic subquotient.

It requires a more careful analysis of the properties of *residual segments*. As explained in Section 4, to a residual segment $(\underline{n})$, we associate a *set of Jumps* (a notion very similar to that of *Jordan blocks* from Moeglin-Tadić [27]); and then using extended Moeglin's Lemmas (see Lemmas 14 and 15), and the result of Heiermann-Opdam (Proposition 2.1), we prove an embedding result, Proposition 6.4 (equivalent to the Proposition 3.1 in [16] for classical groups) used to prove the generalized injectivity conjecture in this context.

1.7. The methods of proof developed thereafter will be illustrated under the following restriction: Let n be the rank of the group $G(n)$, and let assume the form of the Levi subgroup M_1 is isomorphic to $\prod_i \underbrace{GL_{k_i}}_{d_i \text{ times}} \times G(k_0)$ where the multisets $\{k_0; (k_1, \ldots, k_r)\}, n = k_0 + d_1k_1 + \ldots d_rk_r, k_0 \geq 0$, index the conjugacy classes of Levi subgroups of the group $G(n)$. This condition is satisfied for all classical groups and their variants (we borrow this expression from Moeglin [25]).

In this context, because of the restriction on the form of the Levi subgroup M_1, the generic representation σ_λ of M_1 which lies in the cuspidal support takes the form:

$$\rho|.|^a \otimes \rho|.|^{a-1} \ldots \otimes \rho|.|^b \otimes \underbrace{\sigma_2|.|^{\ell_2} \ldots \otimes \sigma_2|.|^{\ell_2}}_{n_{\ell_2} \text{ times}} \ldots \underbrace{\sigma_2|.|^0 \otimes \ldots \otimes \sigma_2|.|^0}_{n_{0,2} \text{ times}} \otimes \ldots \otimes$$
$$\underbrace{\sigma_r|.|^{\ell_r} \ldots \otimes \otimes \sigma_r|.|^{\ell_r}}_{n_{\ell_r} \text{ times}} \ldots \underbrace{\otimes \sigma_r|.|^0 \ldots \otimes \sigma_r|.|^0}_{n_{0,r} \text{ times}} \otimes \sigma_c$$

where σ_i $i = 2, \ldots, r$ (resp. ρ) are unitary cuspidal representations of GL_{k_i} (resp. GL_{k_1}) and σ_c a cuspidal representation of $G(k_0)$.

The tuple $(a, \ldots, b)$ is a decreasing sequence of (half)-integers corresponding to a residual segment of type A; whereas for each $i \geq 2$, the residual segment (of type B, C or D) is $(\underline{n_i}) := (0, \ldots, 0, n_{\ell_i}, \ldots, n_{1,i}, n_{0,i})$.

Since we are dealing with a generic cuspidal support, the reducibility point (0,1/2, or 1) of the induced representation of $G(k_0 + k_i)$: $I_{P_1}^{G(k_0+k_i)}(\rho|.|^s \otimes \sigma_c)$ explicitly determine the form of the parameters, as explained in Proposition 4.3 and the Example 4.1 following this Proposition.

Therefore, a corollary of our Theorem 1.1 is the following:

Corollary 1.2.1. *The generalized injectivity conjecture is true for all classical groups and their variants.*

An extended account of this work is given in the PhD thesis of the author ([14]), in particular the reader will find most of our results proved in the context of classical groups and their variants. Most of these results were already known by the work of Hanzer in [16], we recover them using similar tools but in a novel way; in particular we are relying on the result of Heiermann-Opdam (Proposition 2.1).

In the PhD thesis of the author, in the Appendix, we also illustrate our method of proof on GL_n and further prove the Generalized Injectivity Conjecture for its derived subgroup SL_n. More generally, for $G \subset \tilde{G}$, having the same derived subgroup, it is enough to prove the Generalized

Injectivity conjecture for $\tilde{G}$, then the result follows for G. In particular, we will prove the Generalized Injectivity Conjecture for odd and even Spin groups, since we prove it for odd and even GSpin.

Remark.
- The case of quasi-split non split is included in this treatment. Let $B = TU$ be a Borel subgroup of G. When choosing A_0 the maximal split subtorus of T, the root system $\Sigma(A_0, G)$ (set of roots of G with respect to A_0) relative to F may be non-reduced then it is of type BC. It is then sufficient to consider the reduced root system $\Sigma_{red}(A_0, G)$. Therefore the root systems considered in this work are of classical type (A, B, C, D) or exceptional types (E, F, G).
- The characteristic null hypothesis is used in particular in Proposition 4.3 through Shahidi's result on reducibility points of induced from generic cuspidal representations

1.8. **Exceptional groups.** In Section 10, we explain how our arguments apply to some standard modules of exceptional groups and which cases will certainly require a different type of argument. Our analysis do not allow us to conclude on the Generalized Injectivity for exceptional groups, except for G_2 which is treated there. Our work has led us to analyse extensively the set of projections (resp. restrictions) of roots of Σ to $a_\Theta = a_{M_1}/a_G$ (resp. A_{M_1}) where $M_1 = M_\Theta$ is a standard parabolic as presented in [15]. In the context of classical groups, the Corollary 6.1.1 (in Section 6.3) establishes a description of the set $W(M_1)$ relatively to W_σ which is crucially used in our proof of Generalized Injectivity. Understanding how $W(M_1)$ compares to W_σ in the context of exceptional groups could allow us to reach some conclusions in the missing cases. The analysis conducted in [15] may help in further analysing the subtleties specific to exceptional groups' cases.

From here, we will use the following notations:

Notations.
- *Standard module induced from a maximal parabolic subgroup:*
  Let $\Theta = \Delta - \{\alpha\}$ for α in Δ, and let $P = P_\Theta$ be a maximal parabolic subgroup of G. We denote ρ_P the half sum of positive roots in U,and for α the unique simple root for G which is not a root for M,
  $$\tilde{\alpha} = \frac{\rho_P}{\langle \rho_P, \alpha \rangle}$$
  (Rather than $\tilde{\alpha}$, in the split case, we could also take the fundamental weight corresponding to α).
  Since ν is in a_M^* (of dimension rank(G) - rank(M)= 1 since M is maximal), and should satisfy $\left\langle \nu, \check{\beta} \right\rangle > 0$ for all $\beta \in \Delta - \Theta = \{\alpha\}$, the standard module in this case is $I_P^G(\tau_{s\tilde{\alpha}})$ where $s \in \mathbb{R}$ such that $s > 0$, and τ is an irreducible tempered representation of M.
- For the sake of readability we sometimes denote $I_{P_1}^G(\sigma(\lambda)) := I_{P_1}^G(\sigma_\lambda)$ when the parameter λ is expressed in terms of residual segments.
- Let σ be an irreducible cuspidal representation of a Levi subgroup $M_1 \subset M$ in a standard parabolic subgroup P_1, and let λ be in $(a_{M_1}^*)$, we will denote $Z^M(P_1, \sigma, \lambda)$ the unique irreducible generic discrete series (resp. essentially square-integrable) in the standard module $I_{P_1 \cap M}^M(\sigma_\lambda)$.
  We will omit the index when the representation is a representation of G: $Z(P_1, \sigma, \lambda)$; often λ will be written explicitly with residual segments to emphasize the dependency on specific sequences of exponents.

Acknowledgements. This work is part of the author's PhD thesis under the supervision of Volker Heiermann, at Aix-Marseille University. The author has benefited from a grant of Agence Nationale de la Recherche with reference ANR-13-BS01-0012 FERPLAY. We are very grateful to Patrick Delorme for a careful reading and detailed comments on various part of this work. We also thank Dan Ciabotaru, Jean-Pierre Labesse, Omer Offen, François Rodier, Allan Silberger, and Marko Tadić for interesting suggestions and discussions.

2. Preliminaries

2.1. **Basic objects.** Throughout this paper we will let F be a non-Archimedean local field of characteristic 0. We will denote by G the group of F-rational points of a quasi-split connected reductive group defined over F. We fix a minimal parabolic subgroup P_0 (which is a Borel B since G is quasi-split) with Levi decomposition $P_0 = M_0U_0$ and A_0 a maximal split torus (over F) of M_0. P is said to be standard if it contains P_0.
More generally, if P rather contains A_0, it is said to be semi-standard. Then P contains a unique Levi subgroup M containing A_0, and M is said to be semi-standard.

For a semi-standard Levi subgroup M, we denote $\mathcal{P}(M)$ the set of parabolic subgroups P with Levi factor M.

We denote by A_M the maximal split torus in the center of M, $W = W^G$ the Weyl group of G defined with respect to A_0 (i.e. $N_G(A_0)/Z_G(A_0)$). The choice of P_0 determines an order in W, and we denote by w_0^G the longest element in W.

If Σ denote the set of roots of G with respect to A_0, the choice of P_0 also determines the set of positive roots (resp., negative roots, simple roots) which we denote by Σ^+ (resp., Σ^-, Δ).

To a subset $\Theta \subset \Delta$ we associate a standard parabolic subgroup $P_\Theta = P$ with Levi decomposition MU, and denote A_M the split component of M. We will write a_M^* for the dual of the real Lie-algebra a_M of A_M, $(a_M)^*_{\mathbb{C}}$ for its complexification and a_M^{*+} for the positive Weyl chamber in a_M^* defined with respect to P.
Further $\Sigma(A_M)$ denotes the set of roots of A_M in Lie(G). It is a subset of a_M^*. For any root $\alpha \in \Sigma(A_M)$, we can associate a coroot $\check{\alpha} \in a_M$. For $P \in \mathcal{P}(M)$, we denote $\Sigma(P)$ the subset of positive roots of A_M relative to P.

Let Rat(M) be the group of F-rational characters of M, we have:

$$a_M^* = \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{R} \text{ and } (a_M)^*_{\mathbb{C}} = a_M^* \otimes_{\mathbb{R}} \mathbb{C}$$

For $\chi \otimes r \in a_M^*$, $r \in \mathbb{R}$, , and λ in a_M, the pairing $a_M \times a_M^* \to \mathbb{R}$ is given by: $\langle \lambda, \chi \otimes r \rangle = \lambda(\chi).r$
Following [41] we define a map

$$H_M : M \to a_M = \mathrm{Hom}(\mathrm{Rat}(M), \mathbb{R})$$

such that

$$|\chi(m)|_F = q^{-\langle \chi, H_M(m) \rangle}$$

for every F-rational character χ in a_M^* of M, q being the cardinality of the residue field of F. Then H_P is the extension of this homomorphism to P, extended trivially along U.

We denote by $X(M)$ the group of unramified characters of M. This space consists of all continuous characters of M into $\mathbb{C}^*$ which are trivial on the distinguished subgroup $M^1 = \bigcap_{\chi \in \mathrm{Rat}(M)} \mathrm{Ker}|\chi(.)|$ of M. Its relation with $(a_M)^*_{\mathbb{C}}$ is given by the surjection

$$(a_M)^*_{\mathbb{C}} \to X(M)$$

which associates the character $\chi_\nu = q^{-\langle \nu, H_M(.) \rangle}$ to the element ν in $(a_M)^*_{\mathbb{C}}$. The kernel of this map if of the form $\frac{2\pi i}{log q}\Lambda$, for a certain lattice Λ of $(a_M)^*$. This surjection gives $X(M)$ the structure of a complex algebraic variety, where $X(M) \cong (\mathbb{C}^*)^d, d = \dim_{\mathbb{R}} a_M$. Thus there are notions of polynomial and rational functions on $X(M)$.

Let us assume that (σ, V) is an admissible complex representation of M. We adopt the convention that the isomorphism class of (σ, V) is denoted by σ. If χ_ν is in $X(G)$, then we write $(\sigma_\nu, V_{\chi_\nu})$ for the representation $\sigma \otimes \chi_\nu$ on the space V.

Let (σ, V) be an admissible representation of finite length of M, a Levi subgroup containing M_0 a minimal Levi subgroup, centralizer of the maximal split torus A_0. Let P and P' be in $\mathcal{P}(M)$. Consider the intertwining integral:

$$(J_{P'|P}(\sigma_\nu)f)(g) = \int_{U \cap U' \backslash U'} f(u'g)du' \quad f \in I_P^G(\sigma_\nu)$$

where U and U' denote the unipotent radical of P and P', respectively.

For ν in $X(M)$ with $\mathrm{Re}(\langle \nu, \check{\alpha} \rangle) > 0$ for all α in $\Sigma(P) \cap \Sigma(P')$ the defining integral of $J_{P'|P}(\sigma_\nu)$ converges absolutely. Moreover, $J_{P'|P}$ defined in this way on some open subset of $\mathcal{O} = \{\sigma_\nu | \nu \in X(M)\}$ becomes a rational function on $\mathcal{O}$ ([41] Theorem IV 1.1). Outsides its poles, this defines an element of

$$\mathrm{Hom}_G(I_P^G(V_\chi), I_{P'}^G(V_\chi))$$

Moreover, for any χ in $X(M)$, there exists an element v in $I_P^G(V_\chi)$ such that $J_{P'|P}(\sigma_\chi)v$ is not zero ([41], IV.1 (10))

In particular, for all ν in an open subset of a_M^*, and $\overline{P}$ the opposite parabolic subgroup to P, we have an intertwining operator

$$J_{\overline{P}|P}(\sigma_\nu) : I_P^G(\sigma_\nu) \to I_{\overline{P}}^G(\sigma_\nu)$$

and for ν in $(a_M^*)^+$ far away from the walls it is defined by the convergent integral:

$$(J_{\overline{P}|P}(\sigma_\nu)f)(g) = \int_{\overline{U}} f(ug)du$$

The intertwining operator is meromorphic in ν and the map $J_{\overline{P}|P}J_{P|\overline{P}}$ is a scalar. Its inverse equals the Harish-Chandra μ function up to a constant and will be denoted $\mu^G(\sigma_\nu)$.

Convention. By [34] Sections 3.3 and 1.4, we can fix a non-degenerate character ψ of U which, for every Levi subgroup M, is compatible with $w_0^G w_0^M$. We will still denote ψ the restriction of ψ to $M \cap U$. Every generic representation π of M becomes generic with respect to ψ after changing the splitting in U. Throughout this paper, generic means ψ-generic. When the groups are quasi-split and connected, by a theorem of Rodier, the standard ψ-generic modules have exactly one ψ-generic irreducible subquotient.

2.2. **The μ function.** Harish-Chandra's μ-function is the main ingredient of the Plancherel density for a p-adic reductive group G [41]. It assigns to every discrete series representation of a Levi subgroup a complex number and can be analytically extended to a meromorphic function on the space of essentially square-integrable representations of Levi subgroups.

Let $Q = NV$ be a parabolic subgroup of a connected reductive group G over F and σ an irreducible unitary cuspidal representation of N, then the Harish-Chandra's μ-function μ^G corresponding to G defines a meromorphic function $a_{N,\mathbb{C}}^* \to \mathbb{C}$, $\lambda \to \mu^G(\sigma_\lambda)$ (cf. [18], Proposition 4.1, [36], 1.6) which (in a certain context, see Proposition 4.1 in [18]) can be written:

$$\mu^G(\sigma_\lambda) = f(\lambda) \prod_{\alpha \in \Sigma(Q)} \frac{(1 - q^{\langle \check{\alpha}, \lambda \rangle})(1 - q^{-\langle \check{\alpha}, \lambda \rangle})}{(1 - q^{\epsilon_\alpha + \langle \check{\alpha}, \lambda \rangle})(1 - q^{\epsilon_\alpha - \langle \check{\alpha}, \lambda \rangle})}$$

where f is a meromorphic function without poles and zeroes on a_N^* and the ϵ_α are non-negative rational numbers such that $\epsilon_\alpha = \epsilon_{\alpha'}$ if α and α' are conjugate. We refer the reader to Sections IV.3 and V.2 of [41] for some further properties of the Harish-Chandra μ function.

Clearly the μ function denoted above μ^G can be defined with respect to any reductive group G, in particular we will use below the functions μ^M for a Levi subgroup M.

Let $P_1 = M_1 U_1$ be a standard parabolic subgroup. In [19] and [20], with the notations introduced in the Section 3.2.1, the following results are mentioned:

Theorem 2.1 (Harish-Chandra, see [20], 1.2). *Fix a root $\alpha \in \Sigma(P_1)$ and an irreducible cuspidal representation σ of M_1.*

a) If $\mu^{(M_1)_\alpha}(\sigma) = 0$ then there exists a unique (see Casselman's notes, 7.1 in [10]) non trivial element s_α in $W^{(M_1)_\alpha}(M_1)$ so that $s_\alpha(P_1 \cap (M_1)_\alpha) = \overline{P_1} \cap (M_1)_\alpha$ and $s_\alpha\sigma \cong \sigma$.

b) If there exists a unique non trivial element s_α in $W^{(M_1)_\alpha}(M_1)$ so $s_\alpha(P_1 \cap (M_1)_\alpha) = \overline{P_1} \cap (M_1)_\alpha$ and $s_\alpha\sigma \cong \sigma$. Then $\mu^{(M_1)_\alpha}(\sigma) \neq 0 \Leftrightarrow I^{(M_1)_\alpha}_{P_1\cap(M_1)_\alpha}(\sigma)$ is reducible.

If it is reducible, it is the direct sum of two non isomorphic representations.

Where the μ function's factor in this setting is:

$$\mu^{(M_1)_\beta}(\sigma_\lambda) = c_\beta(\lambda).\frac{(1-q^{\langle\check{\beta},\lambda\rangle})(1-q^{-\langle\check{\beta},\lambda\rangle})}{(1-q^{\epsilon_{\check{\beta}}+\langle\check{\beta},\lambda\rangle})(1-q^{\epsilon_{\check{\beta}}-\langle\check{\beta},\lambda\rangle})}$$

Lemma 1 (Lemma 1.8 in [20]). *Let $\alpha \in \Delta_\sigma$, $s = s_\alpha$ and assume $(M_1)_\alpha$ is a standard Levi subgroup of G. The operator $J_{sP_1|P_1}$ are meromorphic functions in σ_λ for σ unitary cuspidal representation and λ a parameter in $(a^{(M_1)_\alpha}_{M_1}*)$.*

The poles of $J_{sP_1|P_1}$ are precisely the zeroes of $\mu^{(M_1)_\alpha}$. Any pole has order one and its residu is bijective. Furthermore, $J_{P_1|sP_1}J_{sP_1|P_1}$ equals $(\mu^{(M_1)_\alpha})^{-1}$ up to a multiplicative constant.

Let us summarize the different cases:

- If $\mu^{(M_1)_\alpha}$ has a pole at σ_λ; then, the operators $J_{P_1|sP_1}$ and $J_{sP_1|P_1}$ (which are necessarily both non-zero) cannot be bijective. Indeed, at σ_λ their product is zero, if any was bijective, it would imply the other is zero.
- If $\mu^{(M_1)_\alpha}$ has a zero in σ_λ; it is Lemma 1 above.

Further by a general result concerning the μ function, it has one and only one pole on the positive real axis if and only if, for σ a unitary irreducible cuspidal representation, $\mu(\sigma) = 0$. Therefore for each $\alpha \in \Sigma_\sigma$, by definition, there will be one λ on the positive real axis such that $\mu^{(M_1)_\alpha}$ has a pole.

Example 2.1. Consider the group $G = GL_{2n}$ and one of its maximal Levi subgroups $M := GL_n \times GL_n$. Set $\sigma_s := \rho|\det|^s \otimes \rho|\det|^{-s}$ with ρ irreducible unitary cuspidal representation of GL_n. Then, $\mu(\rho\otimes\rho) = 0$ and it is well known that at $s = \pm 1/2$, $\mu(\sigma_s)$ has a pole and the operators $J_{P|\overline{P}}$ and $J_{\overline{P}|P}$ are not bijective.

2.3. **Some results on residual points.** Let Q be any parabolic subgroup of G, with Levi decomposition $Q = LU$. We recall that the parabolic rank of G (with respect to L) is $rk_{ss}(G) - rk_{ss}(L)$, where rk_{ss} stands for the semi-simple rank. The following definition will be useful:

Definition 2.1 (residual point). A point σ_ν for σ an irreducible unitary cuspidal representation of L is called a residual point for μ^G if

$$|\{\alpha \in \Sigma(Q) | \langle\check{\alpha},\nu\rangle = \pm\epsilon_\alpha\}| - 2|\{\alpha \in \Sigma(Q) | \langle\check{\alpha},\nu\rangle = 0\}| = \dim(a^*_L/a^*_G) = rk_{ss}(G) - rk_{ss}(L)$$

where ϵ_α appears in the Section 2.2.

Remark. Since the μ function depends only on a complex variable identified with $\sigma\otimes\chi_\lambda$, for $\lambda \in (a^G_L)^*$; once the unitary cuspidal representation σ is fixed we will freely talk about λ (rather than σ_λ) as a residual point.

The main result of Heiermann in [18] is the following:

Theorem 2.2 (Corollary 8.7 in [18]). *Let $Q = LU$ be a parabolic subgroup of G, σ a unitary cuspidal representation of L, and ν in a^*_L. For the induced representation $I^G_Q(\sigma_\nu)$ to have a discrete series subquotient, it is necessary and sufficient for σ_ν to be a residual point for μ^G and the restriction of σ_ν to A_G (the maximal split component in the center of G) to be a unitary character.*

We will also make a crucial use of the following result from [22]:

Proposition 2.1 (Proposition 2.5 in [22])**.** *Let π be an irreducible generic representation which is a discrete series of G. There exists a standard parabolic subgroup $Q = LU$ of G and a unitary generic cuspidal representation (σ, E) of L, with $\nu \in \overline{(a_L^*)^+}$ such that π is a subrepresentation of $I_Q^G(\sigma_\nu)$.*

We recall the Langlands' classification (see for instance [6] Theorem 2.11 or [23])

Theorem 2.3 (Langlands' classification)**.**
1. *Let $P = MU$ be a standard parabolic subgroup of G, τ (the equivalent class of) an irreducible tempered representation of M and $\nu \in a_M^{+*}$.Then the induced representation $I_P^G(\tau_\nu)$ has a unique irreducible quotient, the Langlands quotient denoted $J(P, \nu, \tau)$*
2. *Let π be an irreducible admissible representation of G. Then there exists a unique triple (P, ν, τ) as in (1) such that $\pi = J(P, \nu, \tau)$. We call this triple the Langlands data, and ν will be called the Langlands parameter of π.*

Theorem 2.4 (*Standard module conjecture* proved in [21] and [22])**.** *Let $\nu \in a_M^{*+}$, and τ be an irreducible tempered generic representation of M. Denote $J(\tau, \nu)$ the Langlands quotient of the induced representation $I_P^G(\tau_\nu)$. Then, the representation $J(\tau, \nu)$ is generic if and only if $I_P^G(\tau_\nu)$ is irreducible.*

3. Setting and first results on intertwining operators

3.1. **The setting.** Following [22], let us denote $a_{M_1}^{M*} = \mathbb{R}\Sigma^M \subset a_{M_1}^{G*}$, where Σ^M are the roots in Σ which are in M (with basis Δ^M) (see also [31] V.3.13).

With the setting and notations as given at the end of the introduction (see 1.8), we consider τ a generic discrete series of M. By the above proposition (Proposition 2.1) there exists a standard parabolic subgroup $P_1 = M_1U_1$ of G, and we could further assume $M_1 \subset M$, σ_ν a cuspidal representation of M_1, Levi subgroup of $M \cap P_1$ such that τ is a generic discrete series that appears as subrepresentation of $I_{M\cap P_1}^M(\sigma_\nu)$, with ν is in the closed positive Weyl chamber relative to M, $\overline{(a_{M_1}^{M*})^+}$. Moreover, σ_ν is a residual point for μ^M.

By transitivity of induction, we have:

$$I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_P^G(I_{M\cap P_1}^M(\sigma_\nu))_{s\tilde{\alpha}} = I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$$

where $s \in \mathbb{R}$ satisfies $s > 0$ and $\tilde{\alpha} = \langle \rho_P, \alpha \rangle^{-1} \rho_P$ (Rather than $\tilde{\alpha}$, we could also take the fundamental weight corresponding to α, but we will rather follow a convention of Shahidi [see [11]]).

Convention. The reader should note that our standard module $I_P^G(\tau_{s\tilde{\alpha}})$ is induced from an essentially square integrable representation $\tau_{s\tilde{\alpha}}$. The general case of a tempered representation τ will follow in the Corollary 7.2.1. Throughout this paper, we will adopt the following convention: τ will denote a discrete series representation, σ an (irreducible) cuspidal representation. Also following notations (as for instance in [16] or [27]), $\pi \leq \Pi$ means π is realised as a subquotient of Π, whereas $\pi \hookrightarrow \Pi$ is stronger, and means it embeds as a subrepresentation.

In the following sections we will study the generic subquotient of $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$ and consider the cases where either there exists a discrete series subquotient, or there isn't and therefore tempered or non-tempered generic (not square integrable) subquotients may occur.

Given a generic discrete series subquotient γ in $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$, using Proposition 2.1 above, it appears as a generic subrepresentation in some induced representation $I_{P'}^G(\sigma'_{\lambda'})$ for λ' in the closure of the positive Weyl chamber with respect to P', and σ' irreducible cuspidal generic.

The set-up is summarized in the following diagram:

$$\begin{tikzcd}
\gamma \leq & I_P^G(\tau_{s\tilde{\alpha}}) \arrow[r, hook] & I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}}) \\
\gamma \arrow[rr, hook] & & I_{P'}^G(\sigma'_{\lambda'}) \arrow[u, dotted]
\end{tikzcd}$$

We will investigate the existence of a bijective up-arrow on the right of this diagram.

3.2. Intertwining operators.

Lemma 2. *Let P_1 and Q be two parabolic subgroups of G having the same Levi subgroup M_1.*

Then there exist an isomorphism $r_{P_1|Q}$ between the two induced modules $I_Q^G(\sigma_\lambda)$ and $I_{P_1}^G(\sigma_\lambda)$ for any irreducible unitary cuspidal representation σ whenever λ is dominant for both P_1 and Q.

Proof. We first assume that Q and P_1 are adjacent (Two parabolic subgroups Q and P_1 are adjacent along α if $\Sigma(P_1) \cap -\Sigma(Q) = \{\alpha\}$). We denote β the common root of $\Sigma(\overline{Q})$ and $\Sigma(P_1)$. $\overline{Q}$ is the parabolic subgroup opposite to Q with Levi subgroup M_1.
We have

$$I_Q^G(\sigma_\lambda) = I_{Q_\beta}^G(I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda))$$

where $(M_1)_\beta$ is the centralizer of A_β (the identity component in the kernel of β) in G, a semi-standard Levi subgroup (confer section 1 in [41]), and the same inductive formula holds replacing Q by P_1.
Since λ is dominant for both Q and P_1, $\langle\lambda,\beta\rangle \geq 0$ (since β is a root in $\Sigma(P_1)$), but also $\langle\lambda,-\beta\rangle \geq 0$ since $-\beta$ is a root in $\Sigma(Q)$. Therefore $\left\langle\check{\beta},\lambda\right\rangle = 0$.
We have λ in $a_{M_1}^*$ which decomposes as

$$(a_{M_1}^{(M_1)_\beta})^* \oplus (a_{(M_1)_\beta})^*$$

and we write $\lambda = \mu \oplus \eta$. The dual of the Lie algebra, $(a_{M_1}^{(M_1)_\beta})^*$, is of dimension one (since M_1 is a maximal Levi subgroup in $(M_1)_\beta$) generated by $\check{\beta}$. If $\left\langle\check{\beta},\lambda\right\rangle = 0$, the projection of λ on $(a_{M_1}^{(M_1)_\beta})^*$ is also zero. That is $\left\langle\check{\beta},\mu\right\rangle = 0$ or χ_μ is unitary.

Therefore with σ unitary, and χ_μ a unitary character, the representations

$$I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\mu) \qquad \text{and} \qquad I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\mu)$$

are unitary. Since they trivially satisfy the conditions (i) of Theorem 2.9 in [4] (see also [31] VI.5.4) they have equivalent Jordan-Hölder composition series, and are therefore isomorphic (As unitary representations, having equivalent Jordan-Hölder composition series). Tensoring with χ_η preserves the isomorphism between

$$I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\mu) \qquad \text{and} \qquad I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\mu)$$

That is, there exist an isomorphism between $I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$ and $I_{P\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$. The induction of this isomorphism therefore gives an isomorphism between $I_Q^G(\sigma_\lambda)$ and $I_{P_1}^G(\sigma_\lambda)$ that we call $r_{P_1|Q}$.

If we further assume that Q and P_1 are not adjacent, but can be connected by a sequence of adjacent parabolic subgroups of G,

$$\{Q = Q_1, Q_2, Q_3, \ldots, Q_n = P_1\}$$

with

$$\Sigma(Q_i) \cap \Sigma(\overline{Q_{i+1}}) = \{\beta_i\}$$

We have the following set-up :

$$I_Q^G(\sigma_\lambda) \overset{r_{Q_2|Q}}{\rightarrow} I_{Q_2}^G(\sigma_\lambda) \overset{r_{Q_3|Q_2}}{\rightarrow} I_{Q_3}^G(\sigma_\lambda)\dots \overset{r_{Q_n|Q_{n-1}}}{\rightarrow} I_{P_1}^G(\sigma_\lambda)$$

Again, under the assumption that λ is dominant for P_1 and Q, we have $\langle \beta_i, \lambda\rangle \geq 0$ and $\langle -\beta_i, \lambda\rangle \geq 0$ for each β_i in $\Sigma(P_1)\cap\Sigma(\overline{Q})$, hence $\left\langle \check{\beta}_i, \lambda\right\rangle = 0$. Therefore there exists an isomorphism between $I_{Q_i}^G(\sigma_\lambda)$ and $I_{Q_{i+1}}^G(\sigma_\lambda)$ denoted $r_{Q_{i+1}|Q_i}$.
The composition of the isomorphisms $r_{Q_{i+1}|Q_i}$ will eventually give us the desired isomorphism between $I_Q^G(\sigma_\lambda)$ and $I_{P_1}^G(\sigma_\lambda)$.

Proposition 3.1. *Let $I_{P'}^G(\sigma'_{\lambda'})$ and $I_{P_1}^G(\sigma_\lambda)$ be two induced modules with σ (resp.σ') irreducible cuspidal representation of M_1 (resp M'), $\lambda \in a_{M_1}^*$, $\lambda' \in a_{M'}^*$, sharing a common subquotient, then:*

(1) There exists an element g in G such that ${}^gP' := gP'g^{-1}$ and P_1 have the same Levi subgroup.
(2) If λ and λ' are dominant for P_1 (resp. P'), there exists an isomorphism R_g between $I_{P'}^G(\sigma'_{\lambda'})$ and $I_{P_1}^G(\sigma_\lambda)$

Proof. First, since the representations $I_{P'}^G(\sigma'_{\lambda'})$ and $I_{P_1}^G(\sigma_\lambda)$ share a common subquotient by Theorem 2.9 in [4], there exists an element g in G such that $M_1 = gM'g^{-1}$, ${}^g\sigma'_{\lambda'} = \sigma_\lambda$ and $g\lambda' = \lambda$, where ${}^g\sigma(x) = \sigma(g^{-1}xg)$ for $x \in M_1$.
The last point follows from the equality ${}^g\chi_{\lambda'} = \chi_{g\lambda'}$.

For the second point, we first apply the map $t(g)$ between $I_{P'}^G(\sigma'_{\lambda'})$ and $I_{{}^gP'}^G({}^g\sigma'_{\lambda'})$ which is an isomorphism that sends f on $f(g^{-1}.)$

As λ' is dominant for P', $g\lambda' = \lambda$ is dominant for ${}^gP'$, and we can further apply the isomorphism defined in the previous lemma (Lemma 2): $r_{P_1|{}^gP'}(\sigma_\lambda)$ (Since P_1 and ${}^gP'$ have the same Levi subgroup: M_1), we will therefore have:

$$I_{P'}^G(\sigma'_{\lambda'}) \overset{t(g)}{\rightarrow} I_{{}^gP'}^G({}^g\sigma', g.\lambda') \overset{r_{P_1|{}^gP'}}{\rightarrow} I_{P_1}^G(\sigma_\lambda)$$

and R_g is the isomorphism given by the composition of $t(g)$ and $r_{P_1|{}^gP'}$.

3.2.1. *Intertwining operators with non-generic kernels.*

Definition 3.1. A set of Langlands data for G is a triple (P, τ, ν) with the following properties:

(1) $P = MU$ is a standard parabolic subgroup of G
(2) ν is in $(a_M^*)^+$
(3) τ is (the equivalence class of) an irreducible tempered representation of M.

Our objective is to embed an irreducible generic subquotient as a subrepresentation in a module induced from the data (P_1, σ, λ) knowing it embeds in one with Langlands' data (P', σ', λ'). Notice that (P_1, σ, λ) is not necessarily a Langlands data since, as explained in the beginning of Section 4, the parameter λ is not necessarily in the positive Weyl chamber $(a_{M_1}^*)^+$. If the intertwining operator between those two induced modules has non-generic kernel, the generic subrepresentation will necessarily appear in the image of the intertwining operator, and therefore will appear as a *subrepresentation* in the induced module with Langlands' data (P_1, σ, λ). We detail the conditions to obtain the non-genericity of the kernel of the intertwining operator.

Proposition 3.2. *Let P_1 and Q be two parabolic subgroups of G having the same Levi subgroup M_1.*

Consider the two induced modules $I_Q^G(\sigma_\lambda)$ and $I_{P_1}^G(\sigma_\lambda)$, and assume σ is an irreducible generic cuspidal representation and λ is dominant for P_1 and anti-dominant for Q. Then there exists an intertwining map from $I_Q^G(\sigma_\lambda)$ to $I_{P_1}^G(\sigma_\lambda)$ which has non-generic kernel.

Proof. We first assume that Q and P_1 are adjacent. We denote β the common root of $\Sigma(Q)$ and $\Sigma(\overline{P_1})$.

We have $I_Q^G(\sigma_\lambda) = I_{Q_\beta}^G(I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda))$ where $(M_1)_\beta$ is the centralizer of A_β (the identity component in the kernel of β) in G, a semi-standard Levi subgroup (confer Section 1 in [41]), and the same inductive formula holds replacing Q by P_1. Then, there are two cases: The case of $\left\langle \check{\beta}, \lambda \right\rangle = 0$ is Lemma 2. If $\left\langle \check{\beta}, \lambda \right\rangle > 0$, let us consider the intertwining operator defined in Section 2 between $I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$ and $I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$ and assume it is not an isomorphism. The representation σ being cuspidal, these modules are length two representations by the Corollary 7.1.2 of Casselman's [10]. Let S be the kernel of this intertwining map and the Langlands quotient $J(\sigma, P_1 \cap (M_1)_\beta, \lambda)$ its image. One has the exact sequences:

$$0 \to S \to I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda) \to J(\sigma, P_1 \cap (M_1)_\beta, \lambda) \to 0$$

$$0 \to J(\sigma, P_1 \cap (M_1)_\beta, \lambda) \to I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda) \to S \to 0$$

Further, the projection from

$$I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$$

to

$$I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)/J(\sigma, P_1 \cap (M_1)_\beta, \lambda) \cong S \subset I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$$

defines a map whose kernel, $J(\sigma, P_1 \cap (M_1)_\beta, \lambda)$, is not generic (by the main result of [21] which proves the Standard module Conjecture). In other words, we have the following exact sequence:

$$0 \to J(\sigma, P_1 \cap (M_1)_\beta, \lambda) \to I_{Q\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda) \xrightarrow{A} I_{P_1\cap(M_1)_\beta}^{(M_1)_\beta}(\sigma_\lambda)$$

Inducing from $(P_1)_\beta$ to G, one observes that the kernel of the induced map $(I_{(P_1)_\beta}^G(A))$ is the induction of the kernel $J(\sigma, P_1 \cap (M_1)_\beta, \lambda)$. Therefore the kernel of the induced map is non-generic (here, we use the fact that there exists an isomorphism between the Whittaker models of the inducing and the induced representations, using result of [32] and [12]).

Assume now that Q and P_1 are not adjacent, but can be connected by a sequence of adjacent parabolic subgroups of G,

$$\{Q = Q_1, Q_2, Q_3, \ldots, Q_n = P_1\}$$

with

$$\Sigma(Q_i) \cap \Sigma(\overline{Q_{i+1}}) = \{\beta_i\}$$

We have the following set-up :

$$I_Q^G(\sigma_\lambda) \overset{r_{Q_2|Q}}{\to} I_{Q_2}^G(\sigma_\lambda) \overset{r_{Q_3|Q_2}}{\to} I_{Q_3}^G(\sigma_\lambda) \ldots \overset{r_{Q_n|Q_{n-1}}}{\to} I_{P_1}^G(\sigma_\lambda)$$

Assume that certain maps $r_{Q_{i+1}|Q_i}$ have a kernel, by the same argument as above their kernels are non-generic and therefore the kernel of the composite map is non-generic. Indeed, we have the next Lemma 3.

Lemma 3. *The composition of operators with non-generic kernel has non-generic kernel.*

Proof. Consider first the composition of two operators, A and B as follows:

$$I_Q^G(\sigma_\lambda) \xrightarrow{A} I_{Q_2}^G(\sigma_\lambda) \xrightarrow{B} I_{P_1}^G(\sigma_\lambda)$$

Clearly, the kernel of the composite $(B \circ A)$ contains the kernel of A and the elements in the space of the representation $I_Q^G(\sigma_\lambda)$, x, such that $A(x)$ is in the kernel of B.

This means we have the following sequence of homomorphisms:

$$0 \to \ker(A) \to \ker(B\circ A) \xrightarrow{A} \ker(B)\cap \mathrm{Im}(A) \to 0$$

pull-back by A^{-1} of element in $\ker(B)$. The pull-back of a non-generic kernel yields a non-generic subspace in the pre-image. The fact that this sequence is exact is clear except for the surjectivity of the map $\ker(B\circ A) \xrightarrow{A} \ker(B)\cap \mathrm{Im}(A)$. But, if $y\in \ker(B)\cap \mathrm{Im}(A)$, then there exists x such that $A(x) = y$ and we have $B\circ A(x) = B(y) = 0$ since $y \in \ker(B)$.

If both $\ker(B)$ and $\ker(A)$ are non-generic, the kernel of $(B\circ A)$ is itself non-generic. Extending the reasoning to a sequence of rank one operators with non-generic kernels yields the result.

We have observed that the nature of intertwining operators rely on the dominance of the parameters λ and λ'. We now need a more explicit description of these parameters; to do so we will call on a result first presented in [30] in the Hecke algebra context (Theorem B.5 in Appendix B) and further developed in [19].

4. Description of residual points via Bala-Carter

With the notations of Section 3, we will study generic subquotient in induced modules $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$ and $I_{P'}^G(\sigma'_{\lambda'})$.

One needs to observe, following the construction of our setting in Section 3, that ν is in the closed positive Weyl chamber relative to M, $\overline{(a_{M_1}^{M*})^+}$, whereas $s\tilde{\alpha}$ is in the positive Weyl chamber $(a_M^*)^+$, therefore it is not expected that $\nu+s\tilde{\alpha}$ should be in the closure of the positive Weyl chamber $\overline{(a_{M_1}^*)^+}$.

In particular, let α be the only root in $\Sigma(A_0)$ which is not in Lie(M), we may have $\langle \nu, \check{\alpha}\rangle < 0$ and therefore for some roots $\beta \in \Sigma(A_{M_1})$, written as linear combination containing the simple root α, we may also have: $\left\langle \nu + s\tilde{\alpha}, \check{\beta}\right\rangle < 0$.

However, by the result presented in Appendix B, if $\nu + s\tilde{\alpha}$ is a residual point, it is in the Weyl group orbit of a dominant residual point (i.e. one whose expression can be directly deduced from a weighted Dynkin diagram). We therefore define:

Definition 4.1 (dominant residual point)**.** A residual point σ_λ for σ an irreducible cuspidal representation is dominant if λ is in the closed positive Weyl chamber $\overline{(a_M^*)^+}$.

Bala-Carter theory allows to describe explicitly the Weyl group orbit of a residual point. In the context of reductive p-adic groups studied in [19] (see in particular Proposition 6.2 in [19]), the fact that σ_λ lies in the cuspidal support of a discrete series can be translated somehow to the assertion that σ_λ corresponds to a distinguished nilpotent orbit in the dual of the Lie algebra ${}^L\mathfrak{g}$, and therefore by Proposition B.4 (see also B.5) in Appendix B to a weighted Dynkin diagram. Notice that Proposition B.4 requires: G to be a semi-simple adjoint group; a certain parameter k_α to equal one for any root α in Φ; further, it concerns only the case of unramified characters.

In the present work we treat the case of weighted Dynkin diagrams of type A, B, C, D. The key proposition is Proposition 4.3 below.

Our setting. Recall that in Section 3 we embedded the standard module as follows:

$$I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_P^G(I_{M\cap P_1}^M(\sigma_\nu))_{s\tilde{\alpha}} = I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$$

By hypothesis, σ_ν is a residual point for μ^M.

$\lambda = \nu + s\tilde{\alpha}$ is in $a_{M_1}^*$.

Describing explicitly the form of the parameter $\lambda \in a_{M_1}^*$ is essential for two reasons: first, to determine the nature (i.e discrete series, tempered, or non-tempered representations) of the irreducible generic subquotients in the induced module $I_{P_1}^G(\sigma_\lambda)$; secondly, to describe the intertwining operators and in particular the (non)-genericity of their kernels.

We will explain the following correspondences:

$$\text{(1)}\quad \{\text{dominant residual point}\} \leftrightarrow \{\text{Weighted Dynkin diagram}\} \leftrightarrow \{\text{residual segments}\} \leftrightarrow \{\text{Jumps of the residual segment}\}$$

The connection between residual points and roots systems involved for Weighted Dynkin Diagrams require a careful description of the involved participants:

The root system. Let us now recall that $W(M_1)$ the set of representatives in W of elements in the quotient group $\left\{w \in W | w^{-1}M_1 w = M_1\right\}/W^{M_1}$ of minimal length in their right classes modulo W^{M_1}.

Assume σ is a unitary cuspidal representation of a Levi subgroup M_1 in G, and let $W(\sigma, M_1)$ be the subgroup of $W(M_1)$ stabilizer of σ. The Weyl group of Σ_σ is W_σ, the subgroup of $W(M_1,\sigma)$ generated by the reflexions s_α.

Proposition 4.1 (3.5 in [37]). *The set $\Sigma_\sigma := \left\{\alpha \in \Sigma_{red}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\right\}$ is a root system.*

For $\alpha \in \Sigma_\sigma$, let s_α the unique element in $W^{(M_1)_\alpha}(M_1,\sigma)$ which conjugates $P_1 \cap M_\alpha$ and $\overline{P_1} \cap (M_1)_\alpha$. The Weyl group W_σ of Σ_σ identifies to the subgroup of $W(M_1,\sigma)$ generated by reflexions s_α, $\alpha \in \Sigma_\sigma$.

$\check{\alpha}$ the unique element in $a_{M_1}^{(M_1)_\alpha}$ which satisfies $\langle \check{\alpha}, \alpha\rangle = 2$.

Then $\Sigma_\sigma^\vee := \{\check{\alpha} | \alpha \in \Sigma_\sigma\}$ is the set of coroots of Σ_σ, the duality being that of a_{M_1} and $a_{M_1}^*$.

The set $\Sigma(P_1) \cap \Sigma_\sigma$ is the set of positive roots for a certain order on Σ_σ.

Remark. An equivalent proposition is proved in [20] (Proposition 1.3). There, the author considers $\mathcal{O}$ the set of equivalence classes of representations of the form $\sigma \otimes \chi$ where χ is an unramified character of M_1. He proves that the set $\Sigma_{\mathcal{O},\mu} := \left\{\alpha \in \Sigma_{\text{red}}(A_{M_1}) | \mu^{(M_1)_\alpha} \text{has a zero on } \mathcal{O}\right\}$ is a root system.

The Weyl group of G relative to a maximal split torus in M_1 acts on $\mathcal{O}$. The previous statement holds replacing W_σ by $W(M_1, \mathcal{O})$, the subgroup of $W(M_1)$ stabilizer of $\mathcal{O}$.

Lemma 4. *If σ is the trivial representation of $M_1 = M_0$ and λ is in the Weyl chamber a_0^*, the root system Σ_σ is the root system of the group G relative to A_0 (with length given by the choice of P_0).*

Proof. Recall that

$$\Sigma_\sigma := \left\{\alpha \in \Sigma_{\text{red}}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\right\}$$

is a root system.

We now apply this definition to the trivial representation. Clearly, for any $\alpha \in \Sigma(A_0)$, the trivial representation is fixed by any element in $W^{(M_0)_\alpha}(M_0)$, and therefore by s_α satisfying $s_\alpha(P_0 \cap (M_0)_\alpha) = \overline{P_0} \cap (M_0)_\alpha$.

It is well-known that the induced representation $I_{P_0 \cap (M_0)_\alpha}^{(M_0)_\alpha}(\mathbf{1})$ is irreducible; therefore using Harish-Chandra's Theorem (Theorem 2.1) above, $\mu^{(M_0)_\alpha}(\mathbf{1}) = 0$. Then

$$\left\{\alpha \in \Sigma_{\text{red}}(A_0) | \mu^{(M_0)_\alpha}(\mathbf{1}) = 0\right\} := \left\{\alpha \in \Sigma(A_0) | \mu^{(M_0)_\alpha}(\mathbf{1}) = 0\right\} = \{\alpha \in \Sigma(A_0)\}.$$

In general, the root system Σ_σ is the disjoint union of irreducible or empty components $\Sigma_{\sigma,i}$ for $i = 1, \ldots, r$. This will be detailed in the Subsection 4.4.2.

Proposition 4.2. *Let G be a quasi-split group whose root system Σ is of type A, B, C or D. Then the irreducible components of Σ_σ are of type A, B, C or D.*

Proof. See the main result of the article [15] recalled in the the Appendix C.

How the root system Σ_σ determines the Weighted Dynkin diagrams to be used in this work.

Proposition 4.3. *Assume G quasi-split over F. Let M_1 be a Levi subgroup of G and σ a generic irreducible unitary cuspidal representation of M_1. Put $\Sigma_\sigma = \{\alpha \in \Sigma_{red}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\}$. Let*

$$d = rk_{ss}(G) - rk_{ss}(M_1).$$

The set Σ_σ is a root system in a subspace of $a^*_{M_1}$ (cf. Silberger in [37] 3.5). Suppose that the irreducible components of Σ_σ are all of type A, B, C or D. Denote, for each irreducible component $\Sigma_{\sigma,i}$ of Σ_σ, by $a^{M^i*}_{M_1}$ the subspace of $a^{G*}_{M_1}$ generated by $\Sigma_{\sigma,i}$, by d_i its dimension and by $e_{i,1}, \ldots, e_{i,d_i}$ a basis of $a^{M^i*}_{M_1}$ (resp. of a vector space of dimension $d_i + 1$ containing $a^{M^i*}_{M_1}$ if $\Sigma_{\sigma,i}$ is of type A) so that the elements of the root system $\Sigma_{\sigma,i}$ are written in this basis as in Bourbaki [8].

For each i, there is a unique real number $t_i > 0$ such that, if $\alpha = \pm e_{i,j} \pm e_{i,j'}$ lies in $\Sigma_{\sigma,i}$, then $I^{(M_1)_\alpha}_{P_1\cap(M_1)_\alpha}(\sigma_{\frac{t_i}{2}(\pm e_{i,j} \pm e_{i,j'})})$ is reducible.

If $\Sigma_{\sigma,i}$ is of type B or C, then there is in addition a unique element $\epsilon_i \in \{1/2, 1\}$ such that $I^{(M_1)_{\alpha_{i,d_i}}}_{P_1\cap(M_1)_{\alpha_{i,d_i}}}(\sigma_{\epsilon_i t_i e_{i,d_i}})$ is reducible.

Let $\lambda = \sum_i \sum_{j=1}^{d_i} \lambda_{i,j} e_{i,j}$ be in $\overline{a^{G*+}_{M_1}}$ with $\lambda_{i,j}$ real numbers.

Then σ_λ is in the cuspidal support of a discrete series representation of G, if and only if the following two properties are satisfied

(i) $d = \sum_i d_i$;

(ii) For all i, $\frac{2}{t_i}(\lambda_{i,1}, \ldots, \lambda_{i,d_i})$ corresponds to the Dynkin diagram of a distinguished parabolic of a simple complex adjoint group of

- type D_{d_i} (resp. A_{d_i}) if $\Sigma_{\sigma,i}$ is of type D (resp. A);

otherwise:

- of type C_{d_i}, if $\epsilon_i = 1/2$;

- of type B_{d_i}, if $\epsilon_i = 1$.

Proof. As λ lies in $a^{G*}_{M_1}$, σ_λ lies in the cuspidal support of a discrete series representation of G, if and only if it is a residual point of Harish-Chandra's μ-function.

Denote $e^{\pm}_{i,j;i',j'}$ the rational character of A_{M_1} whose dual pairing with an element x of $a^G_{M_1}$ with coordinates

$$(x_{1,1}, \ldots, x_{1,d_1}, x_{2,1}, \ldots, x_{2,d_2}, \ldots, x_{r,1}, \ldots, x_{r,d_r})$$

in the dual basis equals $x_{i,j} x^{\pm 1}_{i',j'}$ and by $e^{\pm}_{i,j}$ the one whose dual pair equals $x^{\pm 1}_{i,j}$.

The μ-function decomposes as $\prod_{\alpha\in\Sigma(P)} \mu^{M_\alpha}$. By assumption, the function $\lambda \mapsto \mu^{M_\alpha}(\sigma_\lambda)$ won't have a pole or zero on $a^*_{M_1}$ except if $\alpha \in \Sigma_\sigma$. This means that

(i) α is of the form $e^-_{i,j;i,j'}$, $j < j'$;

(ii) α is of the form $e^+_{i,j;i,j'}$, $j < j'$, and $\Sigma_{\sigma,i}$ of type B, C or D;

(iii) α is of the form $e^+_{i,j}$ or $2e^+_{i,j}$ and $\Sigma_{\sigma,i}$ of respectively type B or C.

Let $(\lambda_{i,j})_{i,j}$ be a family of real numbers as in the statement of the proposition and put $\lambda = \sum_i \sum_{j=1}^{d_i} \lambda_{i,j} e_{i,j}$. It follows from Langlands-Shahidi theory (cf. the proof of Theorem 5.1 in [22]) that there is, for each i, a real number $t_i > 0$ and $\epsilon_i \in \{1/2, 1\}$, so that

If $\alpha = e^{\pm}_{i,j;i,j'} \in \Sigma_\sigma$, $j < j'$, then

$$\mu^{M_\alpha}(\sigma_\lambda) = c_\alpha(\sigma_{(\lambda_{i,j})_{i,j}}) \frac{(1 - q^{\lambda_{i,j}\pm\lambda_{i,j'}})(1 - q^{-\lambda_{i,j}\mp\lambda_{i,j'}})}{(1 - q^{t_i-\lambda_{i,j}\pm\lambda_{i,j'}})(1 - q^{t_i+\lambda_{i,j}\mp\lambda_{i,j'}})},$$

where $c_\alpha(\sigma_{(\lambda_{i,j})_{i,j}})$ denotes a rational function in $\sigma_{(\lambda_{i,j})_{i,j}}$, which is regular and nonzero for real $\lambda_{i,j}$.

If $\alpha = e_{i,j} \in \Sigma_\sigma$ or $\alpha = 2e_{i,j} \in \Sigma_\sigma$, then

$$\mu^{M_\alpha}(\sigma_{(\lambda_{i,j})_{i,j}}) = c_\alpha(\sigma_{(\lambda_{i,j})_{i,j}}) \frac{(1-q^{\lambda_{i,j}})(1-q^{-\lambda_{i,j}})}{(1-q^{\epsilon_i t_i - \lambda_{i,j}})(1-q^{\epsilon_i t_i + \lambda_{i,j}})}$$

with $\epsilon_i = 1, 1/2$.

Put $\kappa_i^+ = 0$ if $\Sigma_{\sigma,i}$ is of type A and put $\kappa_i = 0$ if $\Sigma_{\sigma,i}$ is of type A or D and otherwise $\kappa_i = \kappa_i^+ = 1$. As λ is in the closure of the positive Weyl chamber, it follows that, for σ_λ to be a residual point of Harish-Chandra's μ-function, it is necessary and sufficient, that for every i, one has

$$d_i = |\{(j,j')|j<j', \lambda_{i,j} - \lambda_{i,j'} = t_i\}| + \kappa_i^+|\{(j,j')|j<j', \lambda_{i,j} + \lambda_{i,j'} = t_i\}| + \kappa_i|\{j|\lambda_{i,j} = \epsilon_i t_i\}| \tag{2}$$

$$-2[|\{(j,j')|j<j', \lambda_{i,j} - \lambda_{i,j'} = 0\}| + \kappa_i^+|\{(j,j')|j<j', \lambda_{i,j} + \lambda_{i,j'} = 0\}| + \kappa_i|\{j|\lambda_{i,j} = 0\}|]. \tag{3}$$

If $\kappa_i = 0$ or $\epsilon_i = 1$, then this is the condition for $\frac{2}{t_i}(\lambda_{i,1}, \dots, \lambda_{i,d_i})$ defining a distinguished nilpotent element in the Lie algebra of an adjoint simple complex group of type A_{d_i}, D_{d_i} or B_{d_i} as in 5.7.5 in [9]. If $\epsilon_i = 1/2$, one sees that $\frac{2}{t_i}(\lambda_{i,1}, \dots, \lambda_{i,d_i})$ defines a distinguished nilpotent element in the Lie algebra of an adjoint simple complex group of type C_{d_i}.

In other words, $\frac{2}{t_i}(\lambda_{i,1}, \dots, \lambda_{i,d_i})$ corresponds to the Dynkin diagram of a distinguished parabolic subgroup of an adjoint simple complex group of type B_n, C_n or D_n, if $\kappa_i^+ = 1$ and $\kappa_i\epsilon_i$ is respectively 1, 1/2 or 0, and of type A_n if $\kappa_i = 0$.

Example 4.1 (See also Proposition 1.13 in [20] and the Appendix of the author's PhD thesis [14]**).** In the context of classical groups, let us spell out the Levi subgroups and cuspidal representations of these Levi considered in the previous proposition:

Let M_1 be a standard Levi subgroup of a classical group G and σ a generic irreducible unitary cuspidal representation of M_1.

Then, up to conjugation by an element of G, we can assume:

$$M_1 = \underbrace{GL_{k_1} \times \dots GL_{k_1}}_{d_1 \text{ times}} \times \underbrace{GL_{k_2} \times \dots \times GL_{k_2}}_{d_2 \text{ times}} \times \dots \times \underbrace{GL_{k_r} \times \dots \times GL_{k_r}}_{d_r \text{ times}} \times G(k)$$

where $G(k)$ is a semi-simple group of absolute rank k of the same type as G and

$$\sigma = \sigma_1 \otimes \dots \otimes \sigma_1 \otimes \sigma_2 \otimes \dots \otimes \sigma_2 \dots\dots \otimes \sigma_r \otimes \dots \otimes \sigma_r \otimes \sigma_c$$

Let us assume $k \neq 0$, and $\sigma_i \not\cong \sigma_j$ if $j \neq i$.

We identify A_{M_1} to $\mathbb{T} = \mathbb{G}_m^{d_1} \times \mathbb{G}_m^{d_2} \times \dots \times \mathbb{G}_m^{d_r}$ and denote $\alpha_{i,j}$ the rational character of A_{M_1} (identified with $\mathbb{T}$) which sends an element

$$x = (x_{1,1}, \dots, x_{1,d_1}, x_{2,1}, \dots, x_{2,d_2}, \dots, x_{r,1}, \dots, x_{r,d_r})$$

to $x_{i,j}x_{i,j+1}^{-1}$ if $j < d_i$ and to x_{i,d_i} if $j = d_i$.

Let $(s_{i,j})_{i,j}$ be a family of non-negative real numbers, $1 \leq i \leq r$, $1 \leq j \leq d_i$ and $s_{i,j} \geq s_{i,j+1}$ for i fixed. Then,

$$\sigma_1|\cdot|^{s_{1,1}} \otimes \dots \sigma_1|\cdot|^{s_{1,d_1}} \otimes \sigma_2|\cdot|^{s_{2,1}} \otimes \dots \sigma_2|\cdot|^{s_{2,d_2}} \otimes \dots \otimes \sigma_r|\cdot|^{s_{r,1}} \otimes \dots \sigma_r|\cdot|^{s_{r,d_r}} \otimes \sigma_c.$$

is in the cuspidal support of a discrete series representations of G, if and only if the following properties are satisfied:

(i) one has $\sigma_i \simeq \sigma_i^\vee$ for every i;

ii) denote by s_i the unique element in $\{0, 1/2, 1\}$ such that the representation of $G(k+k_i)$ parabolically induced from $\sigma_i|\cdot|^{s_i} \otimes \sigma_c$ is reducible (we use the result of Shahidi on reducibility points for generic cuspidal representations).

Then, for all i, $2(s_{i,1}, \dots, s_{i,d_i})$ corresponds to the Dynkin diagram of a distinguished parabolic subgroup of a simple complex adjoint group of

- type D_{d_i} if $s_i = 0$; then $\Sigma_{\sigma,i} = \left\{\alpha_{i,1}, \ldots, \alpha_{i,d_i-1}, \alpha_{i,d_i-1} + 2\alpha_{i,d_i}\right\}$
- type C_{d_i} if $s_i = 1/2$; then $\Sigma_{\sigma,i} = \left\{\alpha_{i,1}, \ldots, 2\alpha_{i,d_i}\right\}$
- type B_{d_i} if $s_i = 1$; then $\Sigma_{\sigma,i} = \left\{\alpha_{i,1}, \ldots, \alpha_{i,d_i-1}, \alpha_{i,d_i}\right\}$.

For $i \neq j$, since $\sigma_i \not\cong \sigma_j$, we have $\Sigma_{\sigma,i} \neq \Sigma_{\sigma,j}$.
Then M^i is isomorphic to

$$\underbrace{GL_{k_1} \times \ldots GL_{k_1}}_{d_1 \text{ times}} \times \underbrace{GL_{k_2} \times \ldots \times GL_{k_2}}_{d_2 \text{ times}} \times \ldots \times \ldots \times \underbrace{GL_{k_r} \times \ldots \times GL_{k_r}}_{d_r \text{ times}} \times G(k + d_i k_i)$$

4.1. **From weighted Dynkin diagrams to residual segments.** The Dynkin diagram of a distinguished parabolic subgroup mentioned in the Proposition 4.3 are also called *Weighted Dynkin diagrams*: a definition is given in Appendix B.1 and their forms are given in Appendix A.

Let a parameter $\nu \in a^*_{M_1}$ be written $(\nu_1, \nu_2, \ldots, \nu_n)$ in a basis $\{e_1, e_2, \ldots, e_n\}$ (resp. $\{e_1, e_2, \ldots, e_n, e_{n+1}\}$ for type A) (such that this basis is the canonical basis associated to the classical Lie algebra a^*_0, as in [8] when $M_1 = M_0$) and assume it is a dominant residual point. As it is dominant, observe that $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_n \geq 0$ (resp $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_n$ for type A). Further it corresponds by the previous Proposition (4.3) to a weighted Dynkin diagram of a certain type A, B, C or D (see also Bala-Carter theory presented in Appendix B).

Let us explain the following correspondence:

$$\{\text{Weighted Dynkin diagram}\} \leftrightarrow \{\text{residual segment}\} \tag{4}$$

First, let us explain the following assignment:
WDD $\rightarrow \nu$, where ν is the vector with coordinates $\langle \nu, \alpha_i \rangle$.

Let us start with a weighted Dynkin diagram of type A, B, C or D. The weights under roots α_i are 2 (respectively 0) which correspond to $\langle \nu, \alpha_i \rangle = 1$ (respectively 0). (see the weighted Dynkin diagrams given in Appendix A).

Notice that we abusively use α_i rather than $\check{\alpha}_i$ in the product expression, to be consistant with the notations in the weighted Dynkin diagrams.

Using the expressions of α_i in the canonical basis (for instance $\alpha_i = e_i - e_{i+1}, 2e_i$, or e_i), we compute the vector of coordinates $(\nu_1, \nu_2, \ldots, \nu_n)$ with integers or half-integers entries.

For instance, for $\alpha_i = e_i - e_{i+1}$, when $\langle \nu, \alpha_i \rangle = \left\langle \sum_{i=1}^n \nu_i e_i, \alpha_i \right\rangle = 1$, we get $\nu_i - \nu_{i+1} = 1$, whereas if $\langle \nu, \alpha_i \rangle = 0$ then $\nu_i - \nu_{i+1} = 0$.

Conversely, let us be given a vector of coordinates $(\nu_1, \nu_2, \ldots, \nu_n)$ with integers or half-integers entries and the type of root system (A, B, C or D). Using the relations ν_i and ν_{i+1} for any i, we deduce the weights under each root α_i and therefore obtain the weighted Dynkin diagram.

Definition 4.2 (residual segment)**.** The residual segment of type B, C, D associated to the dominant residual point $\nu := (\nu_1, \nu_2, \ldots, \nu_n) \in \overline{a^{*+}_{M_1}}$ (depending on a fixed irreducible cuspidal representation σ of M_1) is the expression in coordinates of this dominant residual point in a particular basis of $a^*_{M_1}$ (the basis such that the roots in the Weighted Dynkin diagram are canonically expressed as in [8]).

It is therefore a decreasing sequence of positive (half)-integers uniquely obtained from a Weighted Dynkin diagram by the aforementioned procedure.

It is uniquely characterized by:

- An infinite tuple $(\ldots, 0, n_{\ell+m}, \ldots, n_\ell, n_{\ell-1}, \ldots, n_0)$ or $(\ldots, 0, n_{\ell+m}, \ldots, n_\ell, n_{\ell-1}, \ldots, n_{1/2})$ where n_i is the number of times the integer or half-integer value i appears in the sequence.
- The greatest (half)-integer in the sequence, ℓ, such that $n_\ell = 1, n_{\ell-1} = 2$ if it exists.
- the greatest integer, m, such that, for any $i \in \{1, \ldots, m\}$, $n_{\ell+i} = 1$ and for any $i > m$, $n_{\ell+i} = 0$.

This residual segment uniquely determines the weighted Dynkin diagram of type B, C or D from which it originates.

Therefore the values obtained for the n_i's depend on the Weighted Dynkin diagram (see the Appendix A) one observes the following relations:

- Type B: $n_\ell = 1, n_{\ell-1} = 2, n_{i-1} = n_i + 1$ or $n_{i-1} = n_i$, $n_0 = \frac{n_1 - 1}{2}$ if n_1 is odd or $n_0 = \frac{n_1}{2}$ if n_1 is even. (The regular orbit where $n_i = 1$ for all $i \geq 1$ is a special case)
- Type C: $n_{i-1} = n_i + 1$ or $n_{i-1} = n_i$; $n_{1/2} = n_{3/2} + 1$, $n_\ell = 1, n_{\ell-1} = 2$ (The regular orbit where $n_i = 1$ for all $i \geq 1/2$ is a special case)
- Type D:
  (1) $n_i = 1$ for all $i \geq \ell$ and $n_0 = 1$, $n_i = 2$ for all $i \in \{2, \dots, \ell - 1\}$.
  (2) $n_{i-1} = n_i + 1$ or $n_{i-1} = n_i$, $n_0 \geq 2$, $n_0 = \left\{ \begin{array}{l} \frac{n_1}{2} \text{ if } n_1 \text{ is even} \\ \frac{n_1+1}{2} \text{ if } n_1 \text{ is odd} \end{array} \right\}$

It will be denoted $(\underline{n})$.

The residual segment of type A (we say *linear residual segment*, referring to the general *linear* group) is characterized with the same three objects, and also corresponds bijectively to a weighted Dynkin diagram of type A. Then it is a decreasing sequence of (not necessarily positive) reals and the infinite tuple given above is $(\dots, 0, 1, 1, 1, \dots, 1)$, i.e $n_i \leq 1$ for all i. It is symmetrical around zero.

We will also abusively say *linear residual segment* for the translated version of a residual segment of type A; i.e if it is not symmetrical around zero.

We usually do not write the commas to separate the (half)-integers in the sequence.

The use of the terminology "segments" is explained through the following example.

An example: Bernstein-Zelevinsky's segments. Consider the weighted Dynkin diagram of type A:

$$\overset{\alpha_1}{\underset{2}{\circ}} - \overset{\alpha_2}{\underset{2}{\circ}} - \cdots \; \cdots \; \cdots - \overset{\alpha_n}{\underset{2}{\circ}}$$

As $\langle \nu, \alpha_i \rangle = 1$ for all $i \iff \nu_i - \nu_{i+1} = 1$ for all i; the vector of coordinates is therefore a strictly decreasing sequence of real numbers :$(a, a-1, a-2, \dots, b)$.

The group GL_n is an example of reductive group whose root system is of type A.

We may now recall the notions of segments for GL_n as defined in [4], and following the treatment in [33]. We fix an irreducible cuspidal representation ρ, and denote $\rho(a) = \rho|\det|^a$. The representation $\rho_1 \times \rho_2$ denotes the parabolically induced representation from $\rho_1 \otimes \rho_2$.

Definition 4.3 (Segment, Linked segments)**.** [Bernstein-Zelevinsky; following [33]] Let $r|n$. A segment is an isomorphism class of irreducible cuspidal representations of a group GL_n, of the form $\mathcal{S} = \{\rho, \rho(1), \rho(2), \dots, \rho(r-1)\}$. We denote it $\mathcal{S} = [\rho, \rho(r-1)]$.

There is also a notion of intersection and union of two such segments explained in particular in [33]: the intersection of $\mathcal{S}_1$ and $\mathcal{S}_2$ is written $\mathcal{S}_1 \cap \mathcal{S}_2$, the union is written $\mathcal{S}_1 \cup \mathcal{S}_2$.

Let $\mathcal{S}_1 = [\rho_1, \rho_1']$, $\mathcal{S}_2 = [\rho_2, \rho_2']$ be two segments. We say $\mathcal{S}_1$ and $\mathcal{S}_2$ are linked if $\mathcal{S}_1 \nsubseteq \mathcal{S}_2, \mathcal{S}_2 \nsubseteq \mathcal{S}_1$ and $\mathcal{S}_1 \cup \mathcal{S}_2$ is a segment.

Once ρ is fixed, a segment is solely characterized by a string of (half)-integers, it seems therefore natural, in analogy with Bernstein-Zelevinsky's theory, to name any vector $(\nu_1, \dots, \nu_k)$ corresponding to a dominant residual point and therefore by Proposition 4.3 (see also B.4 and B.5) to a weighted Dynkin diagram: *a residual segment*.

If $\mathcal{S} = [\rho, \rho(r-1)]$ is a segment, the unique irreducible subrepresentation (resp. quotient) of $\rho \times \dots \times \rho(r-1)$ is denoted $Z(\mathcal{S})$ (resp. $L(\mathcal{S})$).

Further, it is well-known that $L(\mathcal{S})$ is the unique essentially square-integrable quotient in the induced module $\rho \times \dots \times \rho(r-1)$. For convenience, we rather use $Z(\mathcal{S})$ as essentially square-integrable subrepresentation in $\rho(r-1) \times \dots \times \rho$. Often, we denote it $Z(\rho, r-1, 0)$, and more generally $Z(\rho, a, b)$ for a and b any two real numbers such that $a - b \in \mathbf{Z}$. In the literature, the generalized Steinberg is also denoted $\mathrm{St}_k(\varrho)$, it is the canonical discrete series associated to the segment $[\varrho(\frac{k-1}{2}), \ldots, \varrho(\frac{1-k}{2})]$, for an irreducible cuspidal representation ϱ. Often, $\mathrm{St}_k(\mathbf{1})$ will simply be denoted St_k.

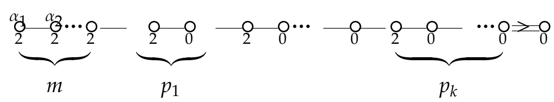
This is a general phenomenon, since by Theorem 2.2, for any quasi-split reductive group, we associate to any residual segment an essentially square- integrable (resp. discrete series) representation.

The well-known example of the Steinberg representation of GL_k is also characteristic since the Steinberg is the unique irreducible *generic* subquotient in the parabolically induced representation $\varrho(\frac{k-1}{2}) \times \ldots \times \varrho(\frac{1-k}{2})$.

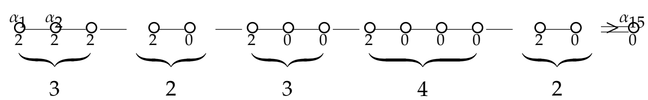
By Theorems 2.2 and 5.1, combined with Rodier's result, if the cuspidal support σ_λ, a residual point, is generic, then the induced representation is generic and the unique irreducible generic subquotient is essentially square integrable.

Therefore, the phenomenon presented here with the Steinberg subquotient, occurs more generally. When the generic representation σ_λ is a dominant residual point, the residual segment corresponding to λ characterizes the unique irreducible generic discrete series (resp. essentially square integrable) subquotient.

Example 4.2. Consider this example of type B:



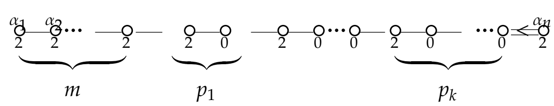
Consider B_{15} for instance, with $m = 3, p_1 = 2, p_2 = 3, p_3 = 4, p_4 = 2$:



We have $\langle \nu, \alpha_{15} \rangle = \langle \nu, 2e_{15} \rangle = 0$ and therefore $\nu_{15} = 0$.
$\langle \nu, \alpha_{14} \rangle = 0$ and therefore $\nu_{14} = \nu_{15} = 0$; $\langle \nu, \alpha_{13} \rangle = 1$, so $\nu_{13} - \nu_{14} = 1$.
Eventually the vector of coordinates corresponding to a dominant residual point, ν is

$$(\nu_1, \nu_2, \nu_3, \ldots, \nu_{13}, \nu_{14}, \nu_{15}) = (765433222111100)$$

Example 4.3. From the weigthed Dynkin diagram of type C_n:



We observe that $\langle \nu, \alpha_n \rangle = \langle \nu, 2e_n \rangle = 1$ and therefore $\nu_n = 1/2$ for some $i \leq n-1$, the weight under α_i is 2 and $\nu_i = 3/2$, etc. Residual segments of type C are therefore composed of half-integers.

4.2. **Set of Jumps associated to a residual segment.** In a following subsection (6.3), we will present certain embeddings of generic discrete series in parabolically induced modules. The proof of these embeddings necessitates to introduce the definition of the *set of Jumps* associated to a residual segment and therefore, transitively, to an irreducible generic discrete series.

These *Jumps* compose a finite set, *set of Jumps*, of (half)-integers a_i's, such that the set of integers $2a_i + 1$ is of a given parity. In the context of classical groups, the latter set (composed of elements of a given parity) coincides with the *Jordan block* defined in [27]. We will also use the notion of Jordan block in this subsection.

Let us recall our steps so far.

If we are given π_0, an irreducible generic discrete series of G, by Proposition 2.1 and Theorem 2.2, it embeds as a subrepresentation in $I_P^G(\sigma'_{\lambda'})$ for $\sigma'_{\lambda'}$ a dominant residual point. Further, by the results of [19] (see in particular Proposition 6.2), $\sigma'_{\lambda'}$ corresponds to a distinguished unipotent orbit and therefore a weigthed Dynkin diagram. Once $\Sigma_{\sigma'}$ is fixed (see the Subsection 4 or the introduction for the Definition of $\Sigma_{\sigma'}$), and assuming it is irreducible, the type of weighted Dynkin diagram is given. All details will be given in the next Section 4.3. By the previous argumentation (Subsection 4.1), we associate a residual segment $(\underline{n_{\pi_0}})$ to the irreducible generic discrete series π_0.

We illustrate these steps in the following example:

Example 4.4 (classical groups)**.** Let σ_λ be the cuspidal support of a generic discrete series π of a classical group (or its variants) $G(n)$, of rank n. First, assume $\sigma_\lambda := \rho|.|^a \otimes \dots \rho|.|^b \otimes \sigma_c$ where ρ is a unitary cuspidal representation of GL_k, and σ_c a generic cuspidal representation of $G(k'), k' < n$. Using Bala-Carter theory, since λ is a residual point, it is in the W_σ-orbit of a dominant residual point, which corresponds to a weighted Dynkin diagram of type B (resp.C, D) and further the above sequence of exponents $(a, \dots, b)$ is encoded $(\ell+m, \dots, \ell, \ell-1, \ell-1, \dots, 0) := (\underline{n})$ of type B (resp C, D). The type of weighted diagram only depends on the reducibility point of the induced representation of $G(k+k') : I^{G(k+k')}(\rho|.|^s \otimes \sigma_c)$ as explained in Proposition 4.3.

The bijective correspondence between Residual segments *and* set of Jumps*.* Let us start with the bijective map:

$(\underline{n}) \to$ set of Jumps of $(\underline{n})$

The length of a residual segment is the sum of the multiplicities: $n_{\ell+m} + n_{\ell+m-1} + \dots n_1 + n_0$.

We first write a length d residual segment $(\underline{n})$

$$((\ell+m), \dots, \underbrace{\ell}_{n_\ell \text{ times}}; \underbrace{\ell-1}_{n_{\ell-1} \text{ times}}, \dots, \underbrace{1}_{n_1 \text{ times}} \underbrace{0}_{n_0 \text{ times}})$$

as a length $2d+1$ (resp. $2d$) sequence of exponents (betokening an unramified character of the corresponding classical group, e.g. to B_d corresponds SO_{2d+1})

$$((\ell+m), \dots, \underbrace{\ell}_{n_\ell \text{ times}}; \underbrace{\ell-1}_{n_{\ell-1} \text{ times}}, \dots, \underbrace{1}_{n_1 \text{ times}} \underbrace{0}_{n_0 \text{ times}}, 0, \underbrace{0}_{n_0 \text{ times}} \underbrace{-1}_{n_1 \text{ times}} \dots \underbrace{-\ell}_{n_\ell \text{ times}}, \dots, -(\ell+m))$$

for type B_d only, we add the central zero

It is a decreasing sequence of $2d+1$ (for type B_d) or $2d$ (for type C_d, D_d) (half)-integers; from the previous Subsection (4.1), the reader has noticed that for $C_d, n_0 = 0$.

Then, we decompose this decreasing sequence as a multiset of $2n_0+1$ (resp. $2n_1$ for type D_d or $2n_{1/2}$ for type C_d) (it is the number of elements in the Jordan block) linear residual segments symmetrical around zero:

$$\{(a_1, a_1-1, \dots, 0, \dots, -a_1); (a_2, a_2-1, \dots, 0, \dots, -a_2); \dots$$
$$\dots; (a_{2n_0+1}, a_{2n_0+1}-1, \dots, 0, \dots, -a_{2n_0+1})\Big\}$$

(resp.

$$\{(a_1, a_1-1, \dots, 1/2, -1/2, \dots, -a_1); (a_2, a_2-1, \dots, 1/2, -1/2, \dots, -a_2); \dots$$
$$\dots; (a_{2n_{1/2}}, a_{2n_{1/2}}-1, \dots, 1/2, -1/2, \dots, -a_{2n_{1/2}})\Big\}$$

where a_1 is the largest (half)-integer in the above decreasing sequence, a_2 is the largest (half)-integer with multiplicity 2, and in general a_i is the largest (half)-integer with multiplicity i.

Definition 4.4 (set of Jumps)**.** The *set of Jumps* is the set:

$$\left\{a_1,\ldots,a_{2n_0+1}\right\}$$

(resp. $\left\{a_1,\ldots,a_{2n_{1/2}}\right\}$). As one notices, the terminology comes from the observation that multiplicities at each jump increases by one: $n_{a_{i+1}} = n_{a_i} + 1$.

Let us make a parallel for the reader familiar with Moeglin-Tadić terminology for classical groups [[27]] (see also Tadić's notes [38] and [39] for an introductory summary of these notions). In such context the Jordan block of the irreducible discrete series π associated to the residual segment $(\underline{n})$ (denoted Jord_π) is constituted of the integers:

$$\left\{2a_1+1, 2a_2+1;\ldots,2a_{2n_0+1}+1\right\}$$

(resp. $\left\{2a_1+1, 2a_2+1;\ldots,2a_{2n_{1/2}}+1\right\}$). This is not a complete characterization of a Jordan block: for a correct use of the definition of Jordan block, we should also fix a self-dual irreducible cuspidal representation ρ of a general linear group and an irreducible cuspidal representation σ_c of a smaller classical group.

We *abusively* use the terminology *Jordan block* to define one partition but such partition is only one of the constituents of the Jordan block as defined in [27].

Clearly the Jordan block is a set of distinct odd (resp even) integers. According to [27], the following condition should also be satisfied: $2d+1 = \sum_i (2a_i+1)$ for type B (resp. $2d = \sum_i(2a_i+1)$ for type C).

Moreover, we are now going to explain there is a canonical way to obtain for a given type (A, B, C, or D) and a fixed length d all distinguished nilpotent orbits, thus all Weighted Dynkin diagrams and therefore all residual segments of these given type and length.

This is given by Bala-Carter theory (see the Appendix B and in particular the Theorem B.4). First, one should partition the integer $2d+1$ (resp $2d$) into distinct odd (resp. even) integers (given $2d+1$, or $2d$ there is a finite number of such partitions). Each partition corresponds to a distinguished orbit and further to a dominant residual point, hence a residual segment.

In fact, each partition corresponds to a Jordan block of an irreducible discrete series π (whose associated residual segment is $(\underline{n_\pi})$). Let us detail the three cases (B, C and D).

Let us finally illustrate the following correspondence:

$$\mathrm{Jord}_\pi \to \text{set of Jumps } (\underline{n_\pi}) \to (\underline{n_\pi})$$

- In case of B_d, the set Jumps of (n_π) derives easily from the choice of *one* partition of $2d+1$ in distinct odd integers: $\mathrm{Jord}_\pi = \{2a_1+1, 2a_2+1,\ldots,2a_t+1\}$. Then Jumps of $(n_\pi) = \{a_1, a_2,\ldots,a_t\}$.

  Once this set of Jumps identified, one writes the corresponding symmetrical around zero linear segments $(a_i,\ldots,-a_i)$'s and by combining and reordering them, form a decreasing sequence of integers of length $2d+1$.

  This length $2d+1$ sequence is symmetrical around zero, with a length d sequence of positive elements, a central zero, and the symmetrical sequence of negative elements. The length d sequence of positive elements is the residual segment $(\underline{n})$.
- Again the case of C_d (by Theorem B.4 in Appendix B) $2d$ is partitioned into distinct even integers, each partition corresponds to a distinguished orbit and further to a dominant residual point, hence a residual segment.

  The correspondence is the following: to the Jordan block of a generic discrete series, π and its associated residual segment $\underline{n}_\pi$:

  $\mathrm{Jord}_\pi = \{2a_1+1, 2a_2+1,\ldots,2a_t+1\}$, for each a_i, one writes $(a_i, a_i-1,\ldots,1/2,-1/2,\ldots-a_i)$. One takes all elements in all these sequences, reorder them to get a $2d$ decreasing sequence of half-integers. The length d sequence of positive half-integers corresponds to residual segment $(\underline{n})$ of type C_d.

- In case of D_d, let $\mathrm{Jord}_\pi = \{2a_1+1, 2a_2+1, \ldots, 2a_t+1\}$ be the Jordan block of a generic discrete series, π; then write the corresponding linear segments $(a_i, \ldots, -a_i)$'s, with all these residual segments, form a decreasing sequence of integers of length $2d$. This length $2d$ sequence is symmetrical around zero. The length d sequence of positive elements in chosen to form the residual segment $(\underline{n})$.

Example 4.5 (B_{14})**.** Let us consider one partition of 2.14+1 into distinct odd integers: $\{11, 9, 5, 3, 1\}$.

For each odd integer in this partition, write it as $2a_i+1$ and write the corresponding linear residual segments $(a_i, \ldots, -a_i)$:

$$543210-1\ -2\ -3\ -4\ -5$$

$$43210-1\ -2\ -3\ -4$$

$$210-1\ -2$$

$$10-1$$

$$0$$

Re-assembling, we get

$$54433222111100; 0; 0\ 0\ -1\ -1\ -1\ -1\ -2\ -2\ -2\ -3\ -3\ -4\ -4\ -5$$

Then, the corresponding residual segment of length 14 (29=2.14+1) is: 54433222111100.

Example 4.6 (C_9)**.** Then $2d'_i$ is 18, and we decompose 18 into distinct even integers: 18; 14+4; 12+4+2; 16+2; 8+6+4, 12+6, 10+8. To each of these partitions corresponds the Weyl group orbit of a residual point and therefore a residual segment. The regular orbit (since the exponents of the associated residual segment form a regular character of the torus) correspond to 18. It is simply

$$(17/2, 15/2, 13/2, \ldots, 1/2)$$

The half-integer 17/2 is such that 2(17/2) + 1 =18.

Let us consider the third partition, 12+4+2, : 12= 2(11/2) + 1; 4= 2(3/2) + 1; 2 = 2(1/2) + 1. Each even integer gives a strictly decreasing sequence of half-integers (11/2,9/2,7/2,5/2,3/2,1/2); (3/2,1/2); (1/2). Finally, we reorder the nine half-integers obtained as a decreasing sequence :

$$(11/2, 9/2, 7/2, 5/2, 3/2, 3/2, 1/2, 1/2, 1/2)$$

The 6 other partitions correspond to:
$(15/2, 13/2, 11/2, 9/2, 7/2, 5/2, 3/2, 1/2, 1/2)$; $(11/2, 9/2, 7/2, 5/2, 5/2, 3/2, 3/2, 1/2, 1/2)$
$(13/2, 11/2, 9/2, 7/2, 5/2, 3/2, 3/2, 1/2, 1/2)$; $(9/2, 7/2, 7/2, 5/2, 5/2, 3/2, 3/2, 1/2, 1/2)$;
$(7/2, 5/2, 5/2, 3/2, 3/2, 3/2, 1/2, 1/2, 1/2)$

A few more examples of type B and D are treated in Appendix A.

Remark. Once given a residual segment, $(\underline{n})$, and its corresponding set of Jumps $a_1 > a_2 > \ldots > a_n$, one observes that for any i, $(a_i, \ldots, -a_{i+1})(\underline{n_i})$ is in W_σ-orbit of this residual segment, where $(a_i, \ldots, -a_{i+1})$ is a linear residual segment and $(\underline{n_i})$ a residual segment of the same type as $(\underline{n})$.

Therefore a set of asymmetrical linear segments $(a_i, \ldots, -a_{i+1})$ along with the smallest residual segment of a given type (e.g (100) for type B, resp. $(3/2, 1/2, 1/2)$ for type C) *or* a linear segments $(a_1, a_1-1, \ldots 0)$ (resp. $(a_1, a_1-1, \ldots 1/2)$ for type C) is in the W_σ-orbit of the residual segment $(\underline{n})$.

Clearly, a set of linear *symmetrical* segments cannot be in the W_σ-orbit of the residual segment $(\underline{n})$.

4.3. **Application of the theory of residual segments: reformulation of our setting.**

4.3.1. *Reformulation of our setting.* Let us come back to our setting (recalled at the beginning of the Subsection 4).

Let M_1 be a Levi subgroup of G and σ a generic irreducible unitary cuspidal representation of M_1. Put $\Sigma_\sigma = \{\alpha \in \Sigma_{red}(A_{M_1}) | \mu^{M_{1,\alpha}}(\sigma) = 0\}$ (resp. $\Sigma_\sigma^M = \{\alpha \in \Sigma_{red}^M(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\}$). The set Σ_σ is a root system in a subspace of $a_{M_1}^G *$ (resp. $(a_{M_1}^M)^*$)(cf. [37] 3.5).

Suppose that the irreducible components of Σ_σ are all of type A, B, C or D.

First assume Σ_σ is irreducible and let us denote $\mathcal{T}$ its type, and $\Delta_\sigma := \{\alpha_1, \dots, \alpha_d\}$ the basis of Σ_σ (following our choice of basis for the root system of G).

We will consider maximal standard Levi subgroups of G, $M \supset M_1$, corresponding to sets $\Delta - \{\underline{\alpha_k}\}$, for a simple root $\underline{\alpha_k} \in \Delta$ (here we use the notation $\underline{\alpha_k}$ to avoid confusion with the roots in $\overline{\Delta_\sigma}$). Since $M \supseteq M_1 = M_\Theta$, $\Theta \subset \Delta - \{\overline{\alpha_k}\}$, or in other words, if we denote α_k the projection of $\underline{\alpha_k}$ on the orthogonal of Θ in $a_{M_1}^*$ then $\alpha_k \in \Sigma_\Theta$ (see the Appendix C for precise definition and analysis of this set), and even more then $\alpha_k \in \Sigma_\sigma$.

If $\overline{\alpha_k}$ is not a extremal root of the Dynkin diagram of G, Σ^M decomposes in two disjoint components.

Remark. The careful reader has already noticed that it is possible that Σ^M breaks into *three* components rather than two: in the context Σ is of type D_n and $\underline{\alpha_k}$ in the above notation is the simple root $\alpha_{n-2} \in \Delta$. In this remark and in the Appendix C, we rather use the notation α_i to denote the simple roots in Σ; and $\overline{\alpha_i}$ their projections on the orthogonal to Θ.

By the calculations done in [15], to obtain any root system in Σ_Θ for Σ of type D_n, we need either α_{n-1} *and* α_n in Δ to be in Θ; or only one of them in Θ.

In case both of them are in Θ but α_{n-2} is not, we are reduced to the case of B_{n-2}. Then $\overline{\alpha_{n-2}} = e_{n-2}$ would be the last root in Σ_σ. Therefore if $M = M_{\Delta-\alpha_{n-2}}$, and therefore Σ_σ^M is irreducible; we treat the conjecture for this case in the Subsection 7.0.1.

In the case only one of them (without loss of generality α_{n-1}) is in Θ, the projection $\overline{\alpha_{n-2}} = e_{n-2} - \frac{e_n+e_{n-1}}{2}$ has squared norm equal to 3/2. This forbids this root to belong to Σ_Θ and therefore to be the root $\underline{\alpha_k}$ such that M is $M_{\Delta-\underline{\alpha_k}}$. Indeed as explained in the very beginning of Section 4.3, since $M_1 = M_\Theta \subseteq M$ the root $\underline{\alpha_k}$ which *is not a root in M* is not either a root in Θ.

Then, Σ_σ^M is a disjoint union of two irreducible components $\Sigma_{\sigma,1}^M \bigcup \Sigma_{\sigma,2}^M$ of type A and $\mathcal{T}$, one of which may be empty (if we remove extremal roots from the Dynkin diagram).

If we remove $\overline{\alpha_n}$, $\Sigma_{\sigma,2}^M$ is empty, and $\Sigma_{\sigma,1}^M$ is of type A, whereas if we remove $\overline{\alpha_1}$, $\Sigma_{\sigma,2}^M$ is of type $\mathcal{T}$ and $\Sigma_{\sigma,1}^M$ is empty.

Else we assume Σ_σ is not irreducible but a disjoint union of irreducible components or empty components $\Sigma_{\sigma,i}$ for $i = 1, \dots, r$ of type A, B, C or D: $\Sigma_\sigma = \bigcup_i \Sigma_{\sigma,i}$.

Then, the basis of Σ_σ is

$$\Delta_\sigma := \left\{\alpha_{1,1}, \dots, \alpha_{1,d_1}; \alpha_{2,1}, \dots, \alpha_{2,d_2}, \dots, \alpha_{i,1}, \dots, \alpha_{i,d_i}, \dots, \alpha_{r,1}, \dots, \alpha_{r,d_r}\right\}$$

Again, we will consider maximal standard Levi subgroup of G, $M \supset M_1$, corresponding to sets $\Delta - \{\overline{\alpha_k}\}$.

Then, for an index $j \in \{1, \dots, r\}$, $\Sigma_{\sigma,j}^M$ is a disjoint union of two irreducible components $\Sigma_{\sigma,j_1}^M \bigcup \Sigma_{\sigma,j_2}^M$ of type A and $\mathcal{T}$, one of which may be empty (if $\overline{\alpha_k}$ is an "extremal" root of the Dynkin diagram of G).

If we remove the last simple root, $\overline{\alpha_n}$, of the Dynkin diagram, Σ_{σ,j_2}^M is empty, and Σ_{σ,j_1}^M is of type A, whereas if we remove α_1, Σ_{σ,j_2}^M is of type $\mathcal{T}$ and Σ_{σ,j_1}^M is empty.

Therefore, it will be enough to prove our results and statements in the case of Σ_σ irreducible; since in case of reducibility, without loss of generality, we choose a component $\Sigma_{\sigma,j}$ and the same reasonings apply.

Now, in our setting (see the beginning of the Subsection 4), σ_ν is a residual point for μ^M. Recall Σ_σ is of rank $d = d_1 + d_2$. Therefore the residual point is in the cuspidal support of the generic discrete series τ if and only if (applying Proposition 4.3 above): $rk(\Sigma_\sigma^M) = d_1 - 1 + d_2$.

We write $\Sigma_\sigma^M := A_{d_1-1} \bigcup \mathcal{T}_{d_2}$ and ν corresponds to residual segments $(\nu_{1,1}, \ldots, \nu_{1,d_1})$ and $(\nu_{2,1}, \ldots, \nu_{2,d_2})$.

Let us assume that the representation σ_λ is in the cuspidal support of the essentially square integrable representation of M, $\tau_{s\tilde{\alpha}}$, where $\lambda = \nu + s\tilde{\alpha}$. We add the twist $s\tilde{\alpha}$ on the linear part (i.e corresponding to A_{d_1-1}), and therefore $(\nu_{2,1}, \ldots, \nu_{2,d_2})$ is left unchanged and is thus $(\lambda_{2,1}, \ldots, \lambda_{2,d_2})$, whereas $(\nu_{1,1}, \ldots, \nu_{1,d_1})$ becomes $(\lambda_{1,1}, \ldots, \lambda_{1,d_1})$.

Then, we need to obtain from $(\lambda_{1,1}, \ldots, \lambda_{1,d_1})(\lambda_{2,1}, \ldots, \lambda_{2,d_2})$ a residual segment of length d and type $\mathcal{T}$.

Indeed, it is the only option to insure σ_λ is a residual point (applying Proposition 4.3) for μ^G, in particular, since $d = d_1 + d_2$ (and therefore writing $\Sigma_\sigma = A_{d_1-1} \bigcup \mathcal{T}_{d_2}$ does not satisfy the requirement of Proposition 4.3).

4.3.2. *Cuspidal strings.* Assume we remove a non-extremal simple root of the Dynkin diagram, the parameter λ in the cuspidal support is therefore constituted of a couple of residual segments, one of which is a linear residual segment: $(a, \ldots, b)$, and the other is denoted $(\underline{n})$. It will be convenient to define the cuspidal support to be given by the tuple $(a, b, \underline{n})$ where $\underline{n}$ is a tuple $(\ldots, 0, n_{\ell+m}, \ldots, n_\ell, n_{\ell-1}, \ldots, n_1, n_0)$ characterization uniquely the residual segment. We define:

Definition 4.5 (cuspidal string)**.** Given two residual segments, strings of integers (or half-integers): $(a, \ldots, b)(\underline{n})$. The tuple $(a, b, \underline{n})$ where $\underline{n}$ is the $(\ell + m + 1)$-tuple

$$(n_{\ell+m}, \ldots, n_\ell, n_{\ell-1}, \ldots, n_1, n_0)$$

is named a cuspidal string.

Recall W_σ is the Weyl group of the root system Σ_σ.

Definition 4.6 (W_σ-cuspidal string)**.** Given a tuple $(a, b, \underline{n})$ where $\underline{n}$ is the $(\ell + m + 1)$-tuple $(n_{\ell+m}, \ldots, n_\ell, n_{\ell-1}, \ldots, n_1, n_0)$, the set of all three-tuples $(a', b', \underline{n'})$ where $\underline{n'}$ is a $(\ell' + m' + 1)$-tuple $(n'_{\ell'+m'}, \ldots, n'_{\ell'}, n'_{\ell'-1}, \ldots, n'_1, n'_0)$ in the W_σ orbit of $(a, b, \underline{n})$ is called W_σ-cuspidal string.

Remark. These definitions can be extended to include the case of t linear residual segments (i.e of type A) :$(a_1, \ldots, b_1)(a_2, \ldots, b_2)\ldots(a_t, \ldots, b_t)$ and a residual segment $(\underline{n})$ of type B, C or D, then the parameter in the cuspidal support will be denoted $(a_1, b_1; a_2, b_2; \ldots; a_t, b_t, \underline{n})$.

4.4. **Application to the case of classical groups.** We illustrate in the following subsection how these definitions naturally appear in the context of classical groups.

4.4.1. *Unramified principal series.* Let τ be a generic discrete series of $M = M_L \times M_c$, the maximal Levi subgroup in a classical group G, $M_L \subset P_L$ is a linear group and $M_c \subset P_c$ is a smaller classical group. It is a tensor product of an essentially square integrable representation of a linear group and an irreducible generic discrete series π of a smaller classical group of the same type as G.

$$\tau := St_{d_1}|.|^s \otimes \pi, \text{ with } s = \frac{a+b}{2}$$

Further, let us assume $(P_1, \sigma, \lambda) := (P_0, \mathbf{1}, \lambda)$. The twisted Steinberg is the unique subrepresentation in $I_{P_{0,L}}^{M_L}(a, \ldots, b)$, whereas $\pi \hookrightarrow I_{P_{0,c}}^{M_c}(\underline{n})$.

Therefore,

$$I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_c \times P_L}^G(I_{P_{0,L}}^{M_L}(a, \ldots, b) I_{P_{0,c}}^{M_c}(\underline{n})) \cong I_{P_0}^G((a, \ldots, b)(\underline{n}))$$

4.4.2. *The general case.* Assume τ is an irreducible generic essentially square integrable representation of a maximal Levi subgroup M of a classical group of rank $\sum_{i=1}^{r} d_i.\dim(\sigma_i) + k$. Then $\tau := St_{d_1}(\sigma_1)|.|^s \otimes \pi$, with $s = \frac{a+b}{2}$.

We study the cuspidal support of the generic (essentially) square integrable representations $St_{d_1}(\sigma_1)|.|^s$ and π.

By Proposition 2.1, $\pi \hookrightarrow I_{P_{1,c}}^{M_c}(\sigma_{\nu_c}^c)$ such that:

$$M_{1,c} = \underbrace{GL_{k_2} \times \ldots \times GL_{k_2}}_{d_2 \text{ times}} \times \ldots \times \underbrace{GL_{k_r} \times \ldots \times GL_{k_r}}_{d_r \text{ times}} \times G(k)$$

where $G(k)$ is a semi-simple group of absolute rank k of the same type as G.

We write the cuspidal representation $\sigma^c := \sigma_2 \otimes \ldots \sigma_2 \otimes \ldots \otimes \sigma_r \otimes \ldots \sigma_r \otimes \sigma_c$ of $M_{1,c}$ and assume the inertial classes of the representations of GL_{k_i}, σ_i, are mutually distinct and $\sigma_i \cong \sigma_i^\vee$ if $\sigma_i, \sigma_i^\vee$ are in the same inertial orbit.

The residual point ν_c is dominant: $\nu_c \in ((a_{M_1}^M)^*+$. Applying Proposition 4.3 below with ν_c and the root system Σ_σ^M, we have:

$$\nu_c := (\nu_2, \ldots, \nu_r)$$

where each ν_i for $i \in \{2, \ldots, r\}$ is a residual point, corresponding to a residual segment of type $B_{d_i}, C_{d_i}, D_{d_i}$.

Further,

$$\mathrm{St}_{d_1}(\sigma_1)|.|^s \hookrightarrow I_{P_{1,L}}^{M_L}(\sigma_1, \lambda_L) \cong I_{P_{1,L}}^{M_L}(\sigma_1|.|^a \otimes \sigma_1|.|^{a-1} \ldots \sigma_1|.|^b)$$

where λ_L is the residual segment of type A: $(a, a-1, \ldots, b)$, and M_L is the linear part of Levi subgroup M.

Such that eventually:

$$\sigma = \sigma_1 \otimes \sigma_1 \ldots \sigma_1 \otimes \sigma_2 \otimes \ldots \sigma_2 \otimes \ldots \otimes \sigma_r \otimes \ldots \sigma_r \otimes \sigma_c$$

And σ_λ can be rewritten:

$$\sigma_1|.|^a \otimes \sigma_1|.|^{a-1} \ldots \sigma_1|.|^b \otimes \underbrace{\sigma_2|.|^{\ell_2} \ldots \sigma_2|.|^{\ell_2}}_{n_{\ell_2} \text{ times}} \ldots \underbrace{\sigma_2|.|^0 \ldots \otimes \sigma_2|.|^0}_{n_{0,2} \text{ times}} \ldots \tag{5}$$
$$\underbrace{\sigma_r|.|^{\ell_r} \ldots \otimes \sigma_r|.|^{\ell_r}}_{n_{\ell_r} \text{ times}} \ldots \underbrace{\otimes \sigma_r|.|^0 \ldots \otimes \sigma_r|.|^0}_{n_{0,r} \text{ times}} \otimes \sigma_c$$

The character ν, representation of M_1, can be splitted in two parts ν_1 and $\underline{\nu} = (\nu_2, \ldots, \nu_r)$, residual points, giving the discrete series denoted $\mathrm{St}_{d_1}(\sigma_1)$ in $I_{P_{1,L}}^{M_L}(\sigma_1)$ and π in $I_{P_{1,c}}^{M_c}(\sigma_c, \underline{\nu})$. By a simple computation, it can be shown that the twist $s\tilde{\alpha}$ will be added on the 'linear part' of the representation and leaves the semi-simple part (classical part) invariant.

Namely ν is given by a vector $(\nu_1 = 0, \nu_2, \ldots, \nu_r)$ and we add the twist $s\tilde{\alpha}$ on the first element to get the vector: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ where each λ_i is a residual segment $(\underline{n_i})$ associated to the subsystem $\Sigma_{\sigma,i}$.

To use the bijection between W_σ orbits of residual points and weighted Dynkin diagrams, one needs to use a certain root system and its associated Weyl group. Then λ is a tuple of r residual segments of different types: $\left\{(\underline{n}_i)\right\}, i \in \{1, \ldots, r\}$. If the parameter λ is written as a r-tuple: $(\lambda_1, \ldots, \lambda_r)$, it is dominant if and only if each λ_i is dominant with respect to the subsystem $\Sigma_{\sigma,i}$.

We have not yet used the *genericity* property of the cuspidal support. This is where we use Proposition 4.3. The generic representation σ_c and the reducibility point of the representation induced from $\sigma_i|.|^s \otimes \sigma_c$ determine the type of the residual segment $(\underline{n}_i)$ obtained.

5. Characterization of the unique irreducible generic subquotient in the standard module

5.1. Let us first outline the results presented in this section.

Let us assume that the irreducible generic subquotient in the standard module is not discrete series. We characterize the Langlands parameter of this unique irreducible non-square integrable subquotient using an order on Langlands parameters given in Lemma 5 below: more precisely, in Theorem 5.2, we prove this unique irreducible generic subquotient is identified by its Langlands parameter being minimal for this order.

We then compare Langlands parameters in the Subsection 5.3, and along those results and Theorem 5.2, we will prove a lemma (Lemma 10) in the vein of Zelevinsky's Theorem at the end of this Section.

Finally, before entering the next section we need to come back on the depiction of the intertwining operators used in our context. This subsection 5.4 on intertwining operators also contains a lemma (Lemma 8) which is crucial in the proof of main Theorem 6.1 in the following Section.

5.2. **An order on Langlands' parameters.** Using Langlands' classification (see Theorem 2.3) and the *Standard module conjecture* (see Theorem 2.4), we can characterize the unique irreducible generic non-square integrable subquotient, denoted $I_{P'}^G(\tau'_{\nu'})$. In particular, on a given cuspidal support, we can characterize the form of the Langlands' parameter ν'. We introduce the necessary tools and results regarding this theory in this subsection.

To study subquotients in the standard module induced from a maximal parabolic subgroup P, $I_P^G(\tau_{s\tilde{\alpha}})$, we will use the following well-known lemma from [6]:

Let us recall their definition of the order:

Definition 5.1 (order). $\lambda_\mu \leq \lambda_\pi$ if $\lambda_\pi - \lambda_\mu = \sum_i x_i\alpha_i$ for simple roots α_i in a_0^* and $x_i \geq 0$.

Lemma 5 (Borel-Wallach, 2.13 in Chapter XI of [6]). *Let P, σ, λ_π be Langlands data. If (μ, H) is a constituent of $I_P^G(\sigma_{\lambda_\pi})$ the standard module, and if $\pi = J(P, \sigma, \lambda_\pi)$ is the Langlands quotient, then $\lambda_\mu \leq \lambda_\pi$, and equality occurs if and only if μ is $J(P, \sigma, \lambda_\pi)$.*

We will write this order on Langlands parameters:

$$\lambda_\mu {}_P\!\leq \lambda_\pi$$

Lemma 6. *Let $\nu = \sum_{i=1}^n a_i e_i$ in the canonical basis $\{e_i\}_i$ of $\mathbb{R}^n$. $0 {}_P\!\leq \nu$ if and only if $\sum_{i=1}^k a_i \geq 0$ for any k in non-D_n cases. In the case of D_n, one needs to specify $\sum_{i=1}^k a_i \geq 0$ for any $k \leq n-1$, $a_{n-1} \geq -a_n$ and $a_{n-1} \geq a_n$.*

Proof. From the expression $\nu = \sum_{i=1}^n a_i e_i$ in the canonical basis $\{e_i\}_i$ of $\mathbb{R}^n$, we can recover an expression of ν in the canonical basis of the Lie algebra a_0^*: $\nu = \sum_{i=1}^n x_i\alpha_i$. Let's explicit $\nu = \sum_i x_i\alpha_i$:

$$\nu = \sum_{i=1}^{n-1} x_i(e_i - e_{i+1}) + x_n\alpha_n =$$

$$x_1(e_1 - e_2) + x_2(e_2 - e_3) + x_3(e_3 - e_4) + \ldots + x_{n-1}(e_{n-1} - e_n) \begin{cases} & \text{for } A_{n-1} \\ +x_n(e_{n-1} + e_n) & \text{for } D_n \\ +x_n e_n & \text{for } B_n \\ +2x_n e_n & \text{for } C_n \end{cases}$$

Then,

$$\nu = \sum_{i=1}^n a_i e_i = x_1 e_1 + (x_2 - x_1)e_2 + (x_3 - x_2)e_3 + \ldots + \begin{cases} (x_{n-1} - x_{n-2})e_{n-1} - x_{n-1}e_n & \text{for } A_{n-1} \\ (x_{n-1} + x_n)e_{n-1} + (x_n - x_{n-1})e_n & \text{for } D_n \\ (x_{n-1} - x_{n-2})e_{n-1} + (x_n - x_{n-1})e_n & \text{for } B_n \\ (x_{n-1} - x_{n-2})e_{n-1} + (2x_n - x_{n-1})e_n & \text{for } C_n \end{cases}$$

$\nu = \sum_{i=1}^{n} x_i \alpha_i \geq 0 \Leftrightarrow x_i \geq 0\ \forall i$

From above $x_1 = a_1, x_2 - x_1 = a_2 \Leftrightarrow x_2 = a_1 + a_2, \ldots$ We have : $x_k = \sum_{i=1}^{k} a_i\ \forall k$ except for root system of type D_n, where for index $n-1$ and n, $2x_n = \sum_{i=1}^{n-1} a_i + a_n$ and $2x_{n-1} = \sum_{i=1}^{n-1} a_i - a_n$, and for C_n where $2x_n = \sum_{i=1}^{n} a_i$.

Notice that for A_{n-1}, $x_{n-1} = \sum_{i=1}^{n-1} a_i$ and $a_n = -x_{n-1}$ such that $\sum_{i=1}^{n} a_i = 0$.

Therefore $0 {}_P\!\leq \nu$ if and only if $\sum_{i=1}^{k} a_i \geq 0$ for any k in non-D_n cases. In the case of D_n, one needs to specify $\sum_{i=1}^{k} a_i \geq 0$ for any $k \leq n-1$, $\sum_{i=1}^{n-1} a_i \geq -a_n$ and $\sum_{i=1}^{n-1} a_i \geq a_n$.

Our next result, Theorem 5.2, will be used in the course of the proof of the Generalized Injectivity Conjecture for non-discrete series subquotients presented in the Sections 7 and 7.2. We use the notations of Section 3.

We will need the following theorem:

Theorem 5.1. *[Theorem 2.2 of [21]]*

Let $P = MU$ be a F-standard parabolic subgroup of G and σ an irreducible generic cuspidal representation of M. If the induced representation $I_P^G(\sigma)$ has a subquotient which lies in the discrete series of G (resp. is tempered) then the unique irreducible generic sub-quotient of $I_P^G(\sigma)$ lies in the discrete series of G (resp. is tempered).

Theorem 5.2. *Let $I_P^G(\tau_\nu)$ be a generic standard module and (P', τ', ν') the Langlands data of its unique irreducible generic subquotient.*

If (P'', τ'', ν'') is the Langlands data of any other irreducible subquotient, then $\nu' {}_P\!\leq \nu''$. The inequality is strict if the standard module $I_{P''}^G(\tau''_{\nu''})$ is generic.

In other words, ν' is the smallest Langlands parameter for the order (defined in Lemma 5) among the Langlands parameters of standard modules having (σ, λ) as cuspidal support.

Proof.

First using the result of Heiermann-Opdam (in [22]), we let $I_P^G(\tau_\nu)$ be embedded in $I_{P_1}^G(\sigma_{\nu_0+\nu})$ with cuspidal support $(\sigma, \lambda = \nu_0 + \nu)$.

Using Langlands' classification, we write $J(P', \tau', \nu')$ an irreducible generic subquotient of $I_P^G(\tau_\nu)$. Then the standard module conjecture claims that $J(P', \tau', \nu') \cong I_{P'}^G(\tau'_{\nu'})$.

The first case to consider is a *generic* standard module $I_{P''}^G(\tau''_{\nu'})$. From the unicity of the generic irreducible module with cuspidal support (σ, λ) (Rodier's Theorem), one sees that $J(P', \tau', \nu') \cong I_{P'}^G(\tau'_{\nu'}) \leq I_{P''}^G(\tau''_{\nu''})$.

Hence, $\nu' {}_{P''}\!< \nu''$.

Secondly, if the standard module $I_{P''}^G(\tau''_{\nu''})$ is any (non-generic) subquotient having (σ, λ) as cuspidal support, since this cuspidal support is generic one will see that one can replace τ'' by the generic tempered representation τ''_{gen} with same cuspidal support and conserve the Langlands parameter ν'' and we are back to the first case. This is explained in the next paragrapher. The lemma follows.

To replace the tempered representation τ'' of M'' the argument goes as follows: Since the representation σ in the cuspidal support of this representation is generic, by Theorem 5.1 the unique irreducible generic representation subquotient τ''_{gen} in the representation induced from this cuspidal support is tempered. As any representation in the cuspidal support of τ'' must lie in the cuspidal support of τ''_{gen}, any such representation must be conjugated to σ. That is there exists a Weyl group element $w \in W$ such that if

$$\tau'' \hookrightarrow I_{P_1 \cap M''}^{M''}(\sigma_{\nu_0})$$

then

$$\tau''_{\text{gen}} \hookrightarrow I_{P_1 \cap M''}^{M''}((w\sigma)_{w\nu_0})$$

Twisting by $\nu'' \in a^*_{M'}$ comes second. Therefore conjugation by this Weyl group element leaves invariant the Langlands parameter $\nu'' \in a^*_{M'}$, and $(\tau''_{\mathrm{gen}})_{\nu''}$ and $\tau''_{\nu''}$ share therefore the same cuspidal support.

5.3. **Linear residual segments.** Let $I_P^G(\tau_{s\tilde{\alpha}})$ be a standard module, we call the parameter $s\tilde{\alpha}$ *the Langlands parameter of the standard module*. We have seen that this Langlands parameter (the *twist*) depends only on the linear (not semi-simple) part of the cuspidal support, i.e the linear residual segment.

In this section and the following we use the notation $\mathcal{S}$ (see the Definition 4.3) to denote a *linear residual segment*, the underlying irreducible cuspidal representation ρ is implicit.

A simple computation gives that if a standard module $I_P^G(\tau_{s\tilde{\alpha}})$, where P is a maximal parabolic, embeds in $I_{P_1}^G(\sigma(a, b, \underline{n}))$ for a cuspidal string $(a, b, \underline{n})$, then $s = \frac{a+b}{2}$. The parameter $s\tilde{\alpha}$ is in $(a^*_M)^+$, but to use Lemma 5 we will need to consider it as an element of $a^*_{M_1}$.

Then, we say this Langlands parameter is *associated* to the linear residual segment $(a, \dots, b)$. In this subsection, we compare Langlands parameters associated to linear residual segments.

Lemma 7. *Let γ be a real number such that $a \geq \gamma \geq b$.*

Splitting a linear residual segment $(a, \dots, b)$ whose associated Langlands parameter is $\lambda = \frac{a+b}{2} \in a^*_M$ into two segments: $(a, \dots, \gamma+1)(\gamma, b)$ yields necessarily a larger Langlands parameter, λ' for the order given in Lemma 5.

Proof. We write $\lambda \in a^*_M$ as an element in $a^*_{M_1}$ to be able to use Lemma 5 (i.e the Lemma 5 also applies with $a^*_{M_1}$):

$$\lambda = (\underbrace{\frac{a+b}{2}, \dots, \frac{a+b}{2}}_{a-b+1 \text{ times}})$$

Similarly, we write λ'

$$\lambda' = (\underbrace{\frac{a+(\gamma+1)}{2}, \dots, \frac{a+(\gamma+1)}{2}}_{a-\gamma \text{ times}}, \underbrace{\frac{\gamma+b}{2}, \dots, \frac{\gamma+b}{2}}_{\gamma-b+1 \text{ times}})$$

$$\lambda' - \lambda = (\underbrace{\frac{(\gamma+1)-b}{2}, \dots, \frac{(\gamma+1)-b}{2}}_{a-\gamma \text{ times}}, \underbrace{\frac{\gamma-a}{2}, \dots, \frac{\gamma-a}{2}}_{\gamma-b+1 \text{ times}})$$

Therefore, $x_1 = \frac{(\gamma+1)-b}{2} > 0$. Since $x_k = \sum_{i=1}^k a_i$ as written in the proof of Lemma 6, one observes that $x_k > x_n$ for any $k < n = a-b+1$, and $x_n = \frac{(\gamma+1)-b}{2}(a-\gamma) + \frac{\gamma-a}{2}(\gamma-b+1) = (a-\gamma)(\frac{(\gamma+1)-b}{2} - \frac{-\gamma+b-1}{2}) = 0$. Hence $\lambda' \geq_P \lambda$ by Lemma 6.

Proposition 5.1. *Consider two linear (i.e of type A) residual segments, i.e stricly decreasing sequences of real numbers such that the difference between two consecutive reals is one: $\mathcal{S}_1 := (a_1, \dots, b_1); \mathcal{S}_2 := (a_2, \dots, b_2)$. Typically, one could think of decreasing sequences of consecutive integers or consecutive half-integers.*

Assume $a_1 > a_2 > b_1 > b_2$ so that they are linked in the terminology of Bernstein-Zelevinsky. Taking intersection and union yield two unlinked residual segments $\mathcal{S}_1 \cap \mathcal{S}_2 \subset \mathcal{S}_1 \cup \mathcal{S}_2$.

Denote $\lambda \in a^*_M$ the Langlands parameter $\lambda = (s_1, s_2)$ associated to $\mathcal{S}_1$ and $\mathcal{S}_2$, and expressed in the canonical basis associated to the Lie algebra a^*_0.

Denote $\lambda' \in a^*_M : \lambda' = (s'_1, s'_2)$ the one associated to the two unlinked segments $\mathcal{S}_1 \cap \mathcal{S}_2, \mathcal{S}_1 \cup \mathcal{S}_2$ ordered so that $s'_1 > s'_2$.

Then, $\lambda' {}_P\!\leq \lambda$.

Proof. Let $(a_1,\dots,b_1)(a_2,\dots,b_2)$ be two segments with $a_1 > a_2 > b_1 > b_2$ so that the two segments are linked. The associated Langlands parameter is:

$$\lambda = (\underbrace{\frac{a_1+b_1}{2},\dots,\frac{a_1+b_1}{2}}_{a_1-b_1+1 \text{ times}},\underbrace{\frac{a_2+b_2}{2},\dots,\frac{a_2+b_2}{2}}_{a_2-b_2+1 \text{ times}})$$

Then taking union and intersection of those two segments gives: $(a_1,\dots,b_2)(a_2,\dots,b_1)$ or $(a_2,\dots,b_1)(a_1,\dots,b_2)$ ordered so that $s'_1 > s'_2$.

The Langlands parameter will therefore be given by:

(1) If $\frac{a_1+b_2}{2} \geq \frac{a_2+b_1}{2}$:

$$\lambda' = (\underbrace{\frac{a_1+b_2}{2},\dots,\frac{a_1+b_2}{2}}_{a_1-b_2+1 \text{ times}},\underbrace{\frac{a_2+b_1}{2},\dots,\frac{a_2+b_1}{2}}_{a_2-b_1+1 \text{ times}})$$

(2) If $\frac{a_2+b_1}{2} > \frac{a_1+b_2}{2}$:

$$\lambda' = (\underbrace{\frac{a_2+b_1}{2},\dots,\frac{a_2+b_1}{2}}_{a_2-b_1+1 \text{ times}},\underbrace{\frac{a_1+b_2}{2},\dots,\frac{a_1+b_2}{2}}_{a_1-b_2+1 \text{ times}})$$

Then the difference $\lambda - \lambda'$ equals:

- In case (1)

$$(\underbrace{\frac{b_1-b_2}{2},\dots,\frac{b_1-b_2}{2}}_{a_1-b_1+1 \text{ times}},\underbrace{\frac{a_2-a_1}{2},\dots,\frac{a_2-a_1}{2}}_{b_1-b_2 \text{ times}},\underbrace{\frac{b_2-b_1}{2},\dots,\frac{b_2-b_1}{2}}_{a_2-b_1+1 \text{ times}},0,\dots,0)$$

  First, $x_1 = \frac{b_1-b_2}{2}$. Secondly, since $x_k = \sum_{i=1}^k a_i$ as written in the proof of Lemma 6, one observes that all subsequent x_k are greater or equal to x_n, for $n = a_1 - b_1 + 1 + a_2 - b_2 + 1$.

  And $x_n = \frac{b_1-b_2}{2}(a_1 - b_1 + 1) + \frac{a_2-a_1}{2}(b_1 - b_2) + \frac{b_2-b_1}{2}(a_2 - b_1 + 1) = \frac{b_1-b_2}{2}(a_1 - b_1 + 1 + a_2 - a_1 - (a_2 - b_1 + 1)) = 0$
- In case (2)

$$\lambda - \lambda' = (\underbrace{\frac{a_1-a_2}{2},\dots,\frac{a_1-a_2}{2}}_{a_2-b_1+1 \text{ times}},\underbrace{\frac{b_1-b_2}{2},\dots,\frac{b_1-b_2}{2}}_{a_1-a_2 \text{ times}},\underbrace{\frac{a_2-a_1}{2},\dots,\frac{a_2-a_1}{2}}_{a_2-b_2+1 \text{ times}})$$

  Here $x_1 = \frac{a_1-a_2}{2}$ $x_n = \frac{a_1-a_2}{2}(a_2 - b_1 + 1) + \frac{b_1-b_2}{2}(a_1 - a_2) + \frac{a_2-a_1}{2}(a_2 - b_2 + 1) = \frac{a_2-a_1}{2}(a_2 - b_1 + 1 + b_1 - b_2 - (a_2 - b_2 + 1)) = 0$.

Proposition 5.2. *The Langlands parameter λ', as defined in the previous Proposition 5.1, is the minimal Langlands parameter for the order given in Lemma 5 on this cuspidal support.*

Proof. Let us consider a decreasing sequence of real numbers such that the difference between two consecutive elements is one: $(a_1, a_1 - 1,\dots,a_2,\dots,b_1,\dots,b_2)$ with the following conditions: $a_1 > a_2 > b_1 > b_2$ and all real numbers between a_2 and b_1 are repeated twice. Let us call this sequence c.

We consider the set $\mathscr{S}$ of tuple of linear segments $\mathcal{S}_i = (a_i,\dots,b_i)$ (strictly decreasing sequence of reals) such that if $s_i = \frac{a_i+b_i}{2} \geq s_j = \frac{a_j+b_j}{2}$ then the linear segment $\mathcal{S}_i$ is placed on the left of $\mathcal{S}_j$, i.e. :

$$(\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_k) \in \mathscr{S} \Leftrightarrow s_1 \geq s_2 \dots \geq s_k$$

In this set $\mathscr{S}$, let us first consider the special case of a decreasing sequence $\delta \in \mathscr{S}$ where each segment is length one and $s_i = \mathcal{S}_i$.

Then the Langlands parameter is just δ:

$$\delta = (a_1, a_1 - 1, \ldots, a_2, a_2, \ldots, b_1, b_1, \ldots b_2)$$

Secondly, let us consider the case where all segments are mutually unlinked, then they have to be included in one another. The reader will readily notice that the only option is the following element in $\mathscr{S}$:

$$m := (a_1, \ldots, b_2)(a_2, \ldots, b_1)$$

Its Langlands parameter is:

$$\lambda' = (\underbrace{\frac{a_1 + b_2}{2}, \ldots, \frac{a_1 + b_2}{2}}_{a_1 - b_2 + 1 \text{ times}}, \underbrace{\frac{a_2 + b_1}{2}, \ldots, \frac{a_2 + b_1}{2}}_{a_2 - b_1 + 1 \text{ times}})$$

Let us show that $\delta \geq_P \lambda'$.

Clearly on the vector $\delta - \lambda'$: $x_1 = a_1 - \frac{a_1 + b_2}{2} > 0$, $x_k = \sum_{i=1}^k a_i$ and one observes that all subsequent x_k are greater or equal to x_n, and x_n is the sum of the elements (counted with multiplicities) in the vector δ minus $\frac{a_1+b_2}{2}(a_1 - b_2 + 1) + \frac{a_2+b_1}{2}(a_2 - b_1 + 1)$, therefore $x_n = 0$ as this sum ends up the same as in the proof of the previous proposition.

Let us show that m is the unique, irreducible element obtained in $\mathscr{S}$ when taking repeatedly intersection and union of any two segments in any element $s \in \mathcal{S}$.

Let us write an arbitrary $s \in \mathscr{S}$ as $(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_p)$, since we had a certain number of reals repeated twice in c, it is clear that some of the $\mathcal{S}_i$ are mutually linked.

For our purpose, we write the vector of lengths of the segments in s: $(k_1, k_2, \ldots, k_p)$.

Let us assume, without loss of generality, that $\mathcal{S}_1$ and $\mathcal{S}_2$ are linked. Taking intersection and union, we obtain two unlinked segments $\mathcal{S}'_1 = \mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{S}'_2 = \mathcal{S}_1 \cap \mathcal{S}_2$. If $k_1 \geq k_2$, then $k'_1 = k_1 + a$, and $k'_2 = k_2 - a$, i.e. the greatest length necessarily increases.

Therefore, the potential $\sum_i k_i^2$ is increasing, while the number of segments is non-increasing.

The process ends when we cannot take anymore intersection and union of linked segments, then the longest segment contains entirely the second longest, this is the element $m \in \mathscr{S}$ introduced above.

Since at each step (of taking intersection and union of two linked segments) the Langlands parameter $\lambda_{s'}$ of the element $s' \in \mathscr{S}$ is smaller than at the previous step (by Proposition 5.1), it is clear that λ' is the minimal element for the order on Langlands parameter.

Remark. Let us assume we fix the cuspidal representation σ and two segments $(\mathcal{S}_1, \mathcal{S}_2)$. As a result of this proposition, the standard module $I_{P'}^G(\tau'_{\lambda'})$ induced from the unique irreducible generic essentially square integrable representation $\tau'_{\lambda'}$ obtained when taking intersection and union $(\mathcal{S}_1 \cap \mathcal{S}_2)$ and $(\mathcal{S}_1 \cup \mathcal{S}_2)$(i.e. which embeds in $I_{P_1}^G(\sigma((\mathcal{S}_1 \cap \mathcal{S}_2); (\mathcal{S}_1 \cup \mathcal{S}_2)))$ is irreducible by Theorem 5.2.

5.4. **Intertwining operators.** In the following result, we play for the first time with cuspidal strings and intertwining operators. We fix a unitary irreducible cuspidal representation σ of M_1 and let $(a, b, \underline{n})$ and $(a', b', \underline{n'})$ be two elements in some W_σ-cuspidal string; i.e, there exists a Weyl group element $w \in W_\sigma$ such that $w(a, b, \underline{n}) = (a', b', \underline{n'})$.

For the sake of readability we sometimes denote $I_{P_1}^G(\sigma(\lambda)) := I_{P_1}^G(\sigma_\lambda)$ when the parameter λ is expressed in terms of residual segments. We would like to study intertwining operators between $I_{P_1}^G(\sigma(a, b, \underline{n}))$ and $I_{P_1}^G(\sigma(a', b', \underline{n'}))$. As explained in Section 3, Proposition 3.2, this operator can be decomposed in rank one operators. Let us recall how one can conclude on the non-genericity of their kernels in the two main cases.

Example 5.1 (Rank one intertwining operators with non-generic kernel)**.** Let us assume Σ_σ is irreducible of type A, B, C or D. We fix a unitary irreducible cuspidal representation σ and let $\alpha = e_i - e_{i+1}$ be a simple root in Σ_σ. The element s_α operates on λ in $(a^G_{M_1})^*$. In this first example, we illustrate the case where s_α acts as a coordinates' transposition on λ written in the standard basis $\{e_i\}_i$ of $(a^G_{M_1})^*$.

Let us focus on two adjacent elements in the residual segment corresponding to λ (at the coordinates λ_i and λ_{i+1}): $\{a, b\}$, let us consider the rank one operator which goes from $I^{(M_1)_\alpha}_{P_1 \cap (M_1)_\alpha}(\sigma_{\ldots\{a,b\}\ldots})$ to $I^{(M_1)_\alpha}_{\overline{P_1} \cap (M_1)_\alpha}(\sigma_{\ldots\{a,b\}\ldots})$.

By Proposition 3.2 it is an operator with non-generic kernel if and only if $a < b$; Indeed if we denote $\lambda := (\ldots, a, b, \ldots)$, then $\langle \check{\alpha}, \lambda \rangle = a - b < 0$ (The action of s_α on λ leaves fixed the other coordinates of λ that we simply denote by dots).

Since $\alpha \in \Sigma_\sigma$, by point (a) in Harish-Chandra's Theorem [Theorem 2.1], there is a unique non-trivial element s_α in $W^{(M_1)_\alpha}(M_1)$ such that $s_\alpha(P_1 \cap (M_1)_\alpha) = \overline{P_1} \cap (M_1)_\alpha$ and which operates as the transposition from (a, b) to (b, a).

The rank one operator from $I^{(M_1)_\alpha}_{\overline{P_1} \cap (M_1)_\alpha}(\sigma_{\ldots,a,b,\ldots})$ to $I^{(M_1)_\alpha}_{s_\alpha(\overline{P_1} \cap (M_1)_\alpha)}(s_\alpha(\sigma_{\ldots,a,b,\ldots})) := I^{(M_1)_\alpha}_{P_1 \cap (M_1)_\alpha}(\sigma_{\ldots,b,a,\ldots})$ is bijective.

Eventually we have shown that the composition of those two which goes from $I^{(M_1)_\alpha}_{P_1 \cap (M_1)_\alpha}(\sigma_{\ldots,a,b,\ldots})$ to $I^{(M_1)_\alpha}_{P_1 \cap (M_1)_\alpha}(\sigma_{\ldots,b,a,\ldots})$ has non-generic kernel.

If the Weyl group W_σ is isomorphic to $S_n \rtimes \{\pm 1\}$, the Weyl group element corresponding to $\{\pm 1\}$ is the sign change and we operate this sign change on the latest coordinate of λ (extreme right of the cuspidal string).

By the same argumentation as in the first example, for $a > 0$, the operator $I^{(M_1)_\alpha}_{P_1 \cap (M_1)_\alpha}(\sigma_{\ldots -a})$ to $I^{(M_1)_\alpha}_{P_1 \cap (M_1)_\alpha}(\sigma_{\ldots a})$ has non-generic kernel.

Example 5.2. Let G be a classical group of rank n. Let us take σ an irreducible unitary generic cuspidal representation of M_1, a standard Levi subgroup of G. Let us assume Σ_σ is irreducible of type B, and take $\lambda := (s_1, s_2, \ldots, s_m)$ in $a^*_{M_1}$, ρ an irreducible unitary cuspidal representation of GL_k, and σ_c an irreducible unitary cuspidal representation of $G(k')$ $k' < n$. Then σ_λ is:

$$\sigma_\lambda := \rho|.|^{s_1} \otimes \rho|.|^{s_2} \otimes \ldots \otimes \rho|.|^{s_m} \otimes \sigma_c$$

The element s_{α_i} operates as follows:

$$s_{\alpha_i}(\rho|.|^{s_1} \otimes \ldots \rho|.|^{s_i} \otimes \rho|.|^{s_{i+1}} \ldots \otimes \rho|.|^{s_m} \otimes \sigma_c) = \rho|.|^{s_1} \otimes \ldots \otimes \rho|.|^{s_{i+1}} \otimes \rho|.|^{s_i} \otimes \ldots \otimes \rho|.|^{s_m} \otimes \sigma_c$$

Indeed, for such α_i (which is in Σ_σ), one checks that property (a) in Theorem 2.1 holds: $s_{\alpha_i}(\sigma) \cong \sigma$. This is verified for any $i \in \{1, \ldots, n\}$. The intertwining operator usually considered in this manuscript is induced by functoriality from the application $\sigma_\lambda \to s_{\alpha_i}(\sigma_\lambda)$.

Lemma 8. *Let $b' \leq \ell + m, b \leq a$. Fix a unitary irreducible cuspidal representation σ of a maximal Levi subgroup in a quasi-split reductive group G, and two cuspidal strings $(a, b, \underline{n})$ and $(a, b', \underline{n'})$ in a W_σ-cuspidal string (notice that the right end of these are equals with value a). If $b' \geq b$, the intertwining operator between $I^G_{P_1}(\sigma(a, b, \underline{n}))$ and $I^G_{P_1}(\sigma(a, b', \underline{n'}))$ has non-generic kernel.*

Proof.

In this proof, to detail the operations on cuspidal strings more explicitly we write the residual segments of type B, C, D defined in Definition 4.2 as:

$$((\ell + m)(\ell + m - 1) \ldots ((\ell + 1)\ell^{n_\ell}(\ell - 1)^{n_{\ell-1}}(\ell - 2)^{n_{\ell-2}} \ldots 2^{n_2}1^{n_1}0^{n_0})$$

where n_i denote the number of times the (half)-integer i is repeated. We present the arguments for integers, the proof for half-integers follows the same argumentation.

First, assume $b \geq 0$, and consider changes on the cuspidal strings

$$(a, \dots, b', b'-1, \dots b)((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0})$$

consisting in permuting successively all elements in $\{b, \dots, b'-1\}$ with their right hand neighbor, as soon as this right hand neighbor is larger. We incorporate all elements starting with b until $b'-1$ from the left into the right hand residual segment. The rank one intertwining operators associated to those permutations have non-generic kernel (see Example 5.1); hence the intertwining operator from $I_{P_1}^G(\sigma(a, b, \underline{n}))$ to $I_{P_1}^G(\sigma(a, b', \underline{n'}))$ as composition of those rank one operators has non-generic kernel.

Assume now $b < 0$ and write $b = -\gamma$. Let us show that there exists an intertwining operator with non-generic kernel from the module induced from $I_{P_1}^G(\sigma(a, -\gamma, \underline{n}))$ to the one induced from $I_{P_1}^G(\sigma(a, b', \underline{n'}))$.

The decomposition in rank one operators has the following two steps (the details on the first step are given in the next paragraph):

(1) (a) If $b' \geq 1 > b$ From the cuspidal string

$$(a, \dots, \gamma, \gamma-1, \dots, -\gamma)((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0})$$

to

$$(a, \dots, \gamma, \gamma-1, \dots, 1)((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0+1}, -1, \dots, -\gamma)$$

and then to

$$(a, \dots, \gamma, \gamma-1, \dots, 1)((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b+1}\dots 2^{n_2+1}1^{n_1+1}0^{n_0+1})$$

(b) If $0 \geq b' \geq b$ From the cuspidal string

$$(a, \dots, \gamma, \gamma-1, \dots, -\gamma)((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0})$$

to

$$(a, \dots, \gamma, \gamma-1, \dots, b')((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0+1}, b'-1, \dots, -\gamma)$$

and then to

$$(a, \dots, \gamma, \gamma-1, \dots, b')((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b+1}\dots 2^{n_2+1}1^{n_1+1}0^{n_0+1})$$

(2) In case (a), from $(a, \dots, 1)(\underline{n''})$ to $(a, \dots, b')(\underline{n'})$ by the same arguments as in the case $b \geq 0$ treated in the first paragraph of this proof.

We detail the operations in step 1:

(i) Starting with $-\gamma$, all negative elements in $\{0, \dots, -\gamma\}$ are successively sent to the extreme right of the second residual segment $(\underline{n})$. At each step, the rank one intertwining operator between (a, p) and (p, a) where p is a negative integer (or half-integer) and $a > p$ has non-generic kernel.

(ii) We use rank one operators of the second type (sign chance of the extreme right element of the cuspidal string). Since they intertwine cuspidal strings where the last element changes from negative to positive, they have non-generic kernels. Then, the positive element is moved up left. The right-hand residual segment goes from

$$((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0+1}, -1, \dots, -\gamma)$$

to

$$((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0+1}, -1, \dots, \gamma)$$

and then to

$$((\ell+m)\dots \ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0+1}, \gamma, -1, \dots, -(\gamma-1))$$

Once changed to positive, permuting successively elements from right to left, one can reorganize the residual segment $(\ell+m)\dots\ell^{n_\ell}(\ell-1)^{n_{\ell-1}}\dots b^{n_b}\dots 2^{n_2}1^{n_1}0^{n_0+1},\gamma,\dots 1)$ such as it is a decreasing sequence of (half)-integers. Again intertwining operators following these changes on the cuspidal string have non-generic kernels.

Example 5.3. Consider the cuspidal string (543210-1)(43 322 211 1 0) and the dominant residual point in its W_σ-cuspidal string: (54 433 3222 21111 10 0). To the Weyl group element $w \in W_\sigma$ associate an intertwining operator from the module induced with string (534210-1)(43 322 211 1 0) to the one induced with cuspidal-string (54 433 3222 21111 10 0) which has non-generic kernel.

Indeed one will decompose it into transpositions s_α such as (-1,4) to (4,-1) and similarly for any $4 > i \geq 0$: $(-1,i)$ to $(i,-1)$.

This process will result in (543210)(43 322 211 1 0 -1). Then one will change the -1 to 1, and by the above the associated rank-one operator also has non-generic kernel.

Then notice that the '4', '3' and '2' in the middle of the sequence can be moved to the left with a sequence of rank one operators with non- generic kernel such as :$(0,4)\to(4,0);\dots;(3,4)\to(4,3)$.

Lemma 9. *Let $(\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_t)$ be an ordered sequence of t linear segments and let us denote $\mathcal{S}_i=(a_i,\dots,b_i)$, for any i in $\{1,\dots,t\}$. This sequence is ordered so that for any i in $\{1,\dots,t\}$, $s_i=\frac{a_i+b_i}{2}\geq s_{i+1}=\frac{a_{i+1}+b_{i+1}}{2}$. Let us assume that for some indices in $\{1,\dots,t\}$ the linear residual segments are linked.*

Let us denote $(\mathcal{S}'_1,\mathcal{S}'_2,\dots,\mathcal{S}'_t)$ the ordered sequence corresponding to the end of the procedure of taking union and intersection of linked linear residual segments. This sequence is composed of at most t unlinked residual segments $\mathcal{S}'_i=(a'_i,\dots,b'_i), i\in\{1,\dots,t\}$.

Taking repeatedly intersection and union yields smaller Langlands parameters for the order defined in Lemma 5; and we denote the smallest element for this order, $\underline{s}'$. It corresponds to the sequence $(\mathcal{S}'_1,\mathcal{S}'_2,\dots,\mathcal{S}'_t)$ as explained in Lemma 5.2.

Then there exists an intertwining operator with non-generic kernel from the induced module $I_{P_1}^G(\sigma((\mathcal{S}'_1,\mathcal{S}'_2,\dots,\mathcal{S}'_t;\underline{n}))$ to $I_{P_1}^G(\sigma((\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_t;\underline{n}))$.

Proof. Let us first consider the case $t=2$.

Consider two linear (i.e of type A) residual segments, i.e stricly decreasing sequences of either consecutive integers or consecutive half-integers $\mathcal{S}_1 := (a_1,\dots,b_1);\mathcal{S}_2 := (a_2,\dots,b_2)$.

Assume $a_1 > a_2 > b_1 > b_2$ so that they are linked in the terminology of Bernstein-Zelevinsky. Taking intersection and union yield two unlinked linear residual segments $\mathcal{S}_1\cap\mathcal{S}_2\subset\mathcal{S}_1\cup\mathcal{S}_2$: $(a_1,\dots,b_2)(a_2,\dots,b_1)$ or $(a_2,\dots,b_1)(a_1,\dots,b_2)$ ordered so that $s'_1 > s'_2$.

As in the proof of Lemma 8, because $a_2 > b_2$ and also $b_1 > b_2$ there exists an intertwining operator with non-generic kernel from the module induced with cuspidal support $(a_1,\dots,b_2)(a_2,\dots,b_1)$ to the one induced with cuspidal support $(a_1,\dots,b_1)(a_2,\dots,b_2)$.

This intertwining operator is a composition of rank one intertwining operators associated to permutations which have non-generic kernel (see Example 5.1); as composition of those rank one operators, it has non-generic kernel.

Similarly, because $a_1 > a_2$, there exists an intertwining operator with non-generic kernel from the module induced with cuspidal support $(a_2,\dots,b_1)(a_1,\dots,b_2)$ to the one induced with cuspidal support $(a_1,\dots,b_1)(a_2,\dots,b_2)$.

Let us now assume the result of this lemma true for t linear residual segments.

Consequently, there exists an intertwining operator with non-generic kernel from $I_{P_1}^G(\sigma((\mathcal{S}'_1,\mathcal{S}'_2,\dots,\mathcal{S}'_t,\mathcal{S}_{t+1}),\underline{n})$ to $I_{P_1}^G(\sigma((\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_t,\mathcal{S}_{t+1}),\underline{n})$. In this case $\mathcal{S}_{t+1}$ and $\mathcal{S}'_t$ may be linked and taking union and intersection of them yields $\mathcal{S}'_{t+1}$ and $\mathcal{S}''_t$ and the existence of an intertwining operator with non-generic kernel from $I_{P_1}^G(\sigma((\mathcal{S}'_1,\mathcal{S}'_2,\dots,\mathcal{S}''_t,\mathcal{S}'_{t+1}),\underline{n})$ to $I_{P_1}^G(\sigma((\mathcal{S}'_1,\mathcal{S}'_2,\dots,\mathcal{S}'_t,\mathcal{S}_{t+1}),\underline{n})$. The latter argument is repeated if $\mathcal{S}''_t$ and $\mathcal{S}'_{t-1}$ are linked, and so on. Eventually there exists an intertwining operator with non-generic kernel from $I_{P_1}^G(\sigma((\mathcal{S}^*_1,\mathcal{S}^*_2,\dots,\mathcal{S}^*_t,\mathcal{S}^*_{t+1};\underline{n}))$ to $I_{P_1}^G(\sigma((\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_t,\mathcal{S}_{t+1};\underline{n}))$, where

$(\mathcal{S}_1^*, \mathcal{S}_2^*, \ldots, \mathcal{S}_t^*, \mathcal{S}_{t+1}^*)$ is the sequence of $t+1$ unlinked segments obtained at the end of the procedure of taking intersection and union.

5.5. **A Lemma in the vein of Zelevinsky's Theorem.** Recall this fundamental result of Zelevinsky, for the general linear group, which was also presented as Theorem 5 in [33]. We use the notation introduced in Definition 4.3.

Proposition 5.3 (Zelevinsky, [43], Theorem 9.7)**.** *If any two segments, $\mathcal{S}_i, \mathcal{S}_j$, j, i in $\{1, \ldots, n\}$ of the linear group are not linked, we have the irreducibility of $Z(\mathcal{S}_1) \times Z(\mathcal{S}_2) \times \ldots \times Z(\mathcal{S}_n)$ and conversely if $Z(\mathcal{S}_1) \times Z(\mathcal{S}_2) \times \ldots \times Z(\mathcal{S}_n)$ is irreducible, then all segments are mutually unlinked.*

Here, we prove a similar statement in the context of any quasi-split reductive group of type A.

Lemma 10. *Let τ be an irreducible generic discrete series of a standard Levi subgroup M in a quasi-split reductive group G. Let σ be an irreducible unitary generic cuspidal representation of a standard Levi subgroup M_1 in the cuspidal support of τ. Let us assume Σ_σ is irreducible of rank $d = rk_{ss}(G) - rk_{ss}(M_1)$ and type A.*

Let $\underline{s} = (s_1, s_2, \ldots, s_t) \in a_{M_1}^*$ be ordered such that $s_1 \geq s_2 \geq \ldots \geq s_t$ with $s_i = \frac{a_i + b_i}{2}$, for two real numbers $a_i \geq b_i$.

Then $I_P^G(\tau_{\underline{s}})$ is a generic standard module embedded in $I_{P_1}^G(\sigma_\lambda)$ and λ is composed of t residual segments $\{(a_i, \ldots, b_i), i = 1, \ldots, t\}$ of type A_{n_i}.

Let us assume that the t segments are mutually unlinked. Then λ is not a residual point and therefore the unique irreducible generic subquotient of the generic module $I_{P_1}^G(\sigma_\lambda)$, is not a discrete series. This irreducible generic subquotient is $I_P^G(\tau_{\underline{s}})$. In other words, the generic standard module $I_P^G(\tau_{\underline{s}})$ is irreducible. Further, for any reordering $\underline{s'}$ of the tuple $\underline{s}$, which corresponds to an element $w \in W$ such that $w\underline{s} = \underline{s'}$ and discrete series τ' of M' such that $w\tau = \tau'$, $wM = M'$. $I_{P'}^G(\tau'_{\underline{s'}})$ is isomorphic to $I_P^G(\tau_{\underline{s}})$.

Proof.

By the result of Heiermann-Opdam (Proposition 2.1), there exists a standard parabolic subgroup P_1, a unitary cuspidal representation σ, a parameter $\nu \in \overline{(a_{M_1}^M *)^+}$ such that the generic discrete series τ embeds in $I_{M_1 \cap M}^M(\sigma_\nu)$. By Heiermann's Theorem (see Theorem 2.2), ν is a residual point so it is composed of residual segments of type A_{n_i}. Then twisting by $\underline{s}$ and inducing to G, we obtain:

$$I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda) \text{ where } \lambda = (a_i, \ldots, b_i)_{i=1}^t$$

Let π be the unique irreducible generic subquotient of the generic standard module $I_P^G(\tau_{\underline{s}})$. Then using Langlands' classification and the standard module conjecture $\pi = J(P', \tau', \nu') \cong I_{P'}^G(\tau'_{\nu'})$.

Assume τ' is discrete series. We apply again the result of Heiermann-Opdam to this generic discrete series to embed $I_{P'}^G(\tau'_{\nu'})$ in $I_{P'_1}^G(\sigma'_{\lambda'})$.

As any representation in the cuspidal support of $\tau_{\underline{s}}$ must lie in the cuspidal support of π, any such representation much be conjugated to $\sigma'_{\lambda'}$, therefore λ' is in the Weyl group orbit of λ. Let us consider this Weyl group orbit under the assumption that the t segments $\{(a_i, \ldots, b_i), i = 1, \ldots, t\}$ are unlinked.

Whether the union of any two segments in $\{(a_i, \ldots, b_i), i = 1, \ldots, t\}$ is not a segment, or the segments are mutually included in one another, it is clear there are no option to take intersections and unions to obtain new linear residual segments. Further, starting with λ, to generate new elements in its W_σ-orbit, one can split the segments $\{(a_i, \ldots, b_i), i = 1, \ldots, t\}$. By Lemma 7, this procedure yields necessarily larger Langlands parameters. Therefore there is no option to reorganize them to obtain residual segments (a'_j, b'_j) of type $A_{n'_j}$ such that $n'_j \neq n_i$ for some $i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, s\}$, for some r such that $\sum_{j=1}^r n'_j = \sum_{i=1}^t n_i$.

The second option is to permute the order of the segments $\{(a_i, \dots, b_i), i = 1, \dots, t\}$ to obtain any other parameter λ' in the Weyl group orbit of λ. From this λ', one clearly obtains the parameter $\nu' := \underline{s'}$ as a simple permutation of the tuple $\underline{s}$.

On the Langlands parameter $\underline{s}$, which is the unique among the (ν')'s described in the previous paragraph in the Langlands situation (we consider all standard modules $I_{P'}^G(\tau'_{\nu'})$), we can use Theorem 5.2 to conclude that the generic standard module $I_P^G(\tau_{\underline{s}})$ for $\nu = \underline{s}$ is irreducible.

Now, we want to show $I_{P'}^G(\tau'_{\underline{s'}})$ is isomorphic to $I_P^G(\tau_{\underline{s}})$.

Looking at the cuspidal support, it is clear that there exists a Weyl group element in $W(M, M')$ sending σ_λ to $\sigma'_{\lambda'}$, and therefore $\tau_{\underline{s}}$ to the Langlands data $(w\tau)_{w\underline{s}} := \tau'_{\underline{s'}}$.

Consider first the case of a maximal parabolic subgroup P in G. Set $\underline{s} = (s_1, s_2)$, $\underline{s'} = (s_2, s_1)$ and τ' is a generic discrete series representation. We apply the map $t(w)$ between $I_P^G(\tau_{\underline{s}})$ and $I_{wP}^G((w\tau)_{w\underline{s}})$ which is an isomorphism. By definition, the parabolic wP has Levi M'. Then, by Lemma 5.4 [3] (see also the Remark 2.10 in [4]) since the Levi subgroups and inducing representations are the same, the Jordan-Hölder composition series of $I_{wP}^G(\tau'_{\underline{s'}})$ and $I_{P'}^G(\tau'_{\underline{s'}})$ are the same, and since $I_P^G(\tau_{\underline{s}})$ is irreducible, they are isomorphic and irreducible.

Secondly, consider the case when the two parabolic subgroups P and P', with Levi subgroup M and M', are connected by a sequence of adjacent parabolic subgroups of G.

Using Theorem 5.2 with any Levi subgroup in G, in particular a Levi subgroup M_α (containing M as a maximal Levi subgroup) shows that the representation $I_{P\cap M_\alpha}^{M_\alpha}(\tau_{\underline{s}})$ is irreducible.

Then, we are in the context of the above paragraph and $I_{s_\alpha(\overline{P}\cap M_\alpha)}^{M_\alpha}((s_\alpha\tau)_{s_\alpha\underline{s}})$ (the image of the composite of the map $J_{\overline{P}\cap M_\alpha|P\cap M_\alpha}$ with the map $t(s_\alpha)$) is irreducible, and isomorphic to $I_{P\cap M_\alpha}^{M_\alpha}(\tau_{\underline{s}})$.

Let us denote Q the parabolic subgroup adjacent to P along α. Induction from M_α to G yields that $I_Q^G(s_\alpha\tau)_{s_\alpha\underline{s}})$ is isomorphic to $I_P^G(\tau_{\underline{s}})$. Writing the Weyl group element w in $W(M, M')$ such that $wM = M'$ as a product of elementary symmetries s_{α_i}, and applying a sequence of intertwining maps as above yields the isomorphism between $I_P^G(\tau_{\underline{s}})$ and $I_{P'}^G(\tau'_{\underline{s'}})$.

Example 5.4 (See [4], 2.6)**.** Let $W^G = N_G(A_0)/Z_G(A_0)$ for a maximal split torus A_0 in G. Let M and N be standard Levi subgroups of G. We set $W(M, N) = \left\{w \in W^G | w(M) = N\right\}$; it is clear that $W^N.W(M, N).W^M = W(M, N)$.

The subgroups M and N are associated (the notation $M \sim N$) if $W(M, N) \neq \emptyset$.

Any element $w \in W(M, N)$ determines the functor $w : AlgM \to AlgN$; and representation $\rho \in AlgM, \rho' \in AlgN$ are called associated if $\rho' \cong w\rho$ for $w \in W(M, N)$ (the notation $\rho \sim \rho'$).

Let $G = G_n = GL_n, \alpha = (n_1, \dots, n_r)$ and $\beta = (n'_1, \dots, n'_s)$ be partitions of n. To each partition $\alpha = (n_1, \dots, n_r)$ corresponds the standard Levi subgroup $G_\alpha = G_{n_1} \times G_{n_2} \times \dots \times G_{n_r}$. Set $M = G_\alpha, N = G_\beta$. Then the condition $M \sim N$ means $r = s$ and the family $(n_1, \dots, n_r)$ is a permutation of $(n'_1, \dots, n'_s)$. Such permutation corresponds to elements of $W(M, N)/W^N$. Let $\rho_i \in IrrG_{n_i}, \rho'_i \in IrrG_{n'_i}, \rho = \otimes\rho_i \in IrrM$, and $\rho' = \otimes\rho'_i \in IrrN$, then $\rho \sim \rho'$ iff the set $(\rho_1, \dots, \rho_r)$ and $(\rho'_1, \dots, \rho'_r)$ are equal up to permutation.

6. Conditions on the parameter λ so that the unique irreducible generic subquotient of $I_{P_1}^G(\sigma_\lambda)$ is a subrepresentation

The goal of this section is to present specific forms of the parameter $\lambda \in a_{M_1}^*$ such that the unique irreducible generic subquotient of $I_{P_1}^G(\sigma_\lambda)$ with σ irreducible unitary generic cuspidal representation of any standard Levi M_1 is a subrepresentation. There is an obvious choice of parameter satisfying this condition as it is proven in the following Lemma:

Lemma 11. *Let σ be an irreducible generic cuspidal representation of M_1 and σ_λ be a dominant residual point and consider the generic induced module $I_{P_1}^G(\sigma_\lambda)$. Its unique irreducible generic square-integrable subquotient is a subrepresentation.*

Proof. From Theorem 2.2, since λ is a residual point, $I_{P_1}^G(\sigma_\lambda)$ has a discrete series subquotient. From Rodier's Theorem, it also has a unique irreducible generic subquotient, denote it γ.

From Theorem 5.1, this unique irreducible generic subquotient is discrete series. Consider this unique generic discrete series subquotient, by Proposition 2.1, there exists a parabolic subgroup P' such that $\gamma \hookrightarrow I_{P'}^G(\sigma'_{\lambda'})$, and λ' dominant for P'. Then the lemma follows from Proposition 3.1 in Section 3.

Our main result in this section is the following theorem.

Theorem 6.1. *Let us consider $I_{P_1}^G(\sigma_\lambda)$ with σ irreducible unitary generic cuspidal representation of a standard Levi M_1, and $\lambda \in a_{M_1}^*$ such that (M_1, σ) satisfies the conditions (CS) (see Definition 6.1). Let W_σ be the Weyl group of the root system Σ_σ. The unique irreducible generic subquotient of $I_{P_1}^G(\sigma_\lambda)$ is necessarily a subrepresentation if the parameter λ is one of the following:*

(1) If λ is a residual point:
   - *(a) λ is a dominant residual point.*
   - *(b) λ is a residual point of the form $(a, a_-)(\underline{n})$ with (a, a_-) two consecutive jumps in the Jumps set associated to the dominant residual point in its W_σ-orbit.*
   - *(c) λ is a residual point of the form $(a, b)(\underline{n})$ such that the dominant residual point in its W_σ-orbit has associated Jumps set containing (a, a_-) as two consecutive jumps and $b > a_-$.*

(2) If λ is not a residual point
   - *(a) λ is of the form $(a', b')(\underline{n'})$ such that the Langlands' parameter $\nu' = \frac{a'+b'}{2}$ is minimal for the order on Langlands parameter (see Subsection 5.2)*
   - *(b) If λ is of the form $(a, b)(\underline{n})$ with $a = a', b' < b$ in the W_σ-orbit of a parameter as in (2).a).*

The proof of this theorem given in Subsection 6.4, relies on Moeglin's extended lemmas and an embedding result (6.4).

6.1. **On some conditions on the standard Levi M_1 and some relationships between $W(M_1)$ and W_σ.** Let G be a quasi-split reductive group over F (resp. a product of such groups) whose root system Σ is of type A, B, C or D, π_0 is an irreducible generic discrete series of G whose cuspidal support contains the representation σ_λ of a standard Levi subgroup M_1, where $\lambda \in a_{M_1}^*$ and σ is an irreducible unitary cuspidal generic representation.

Let
$$d = rk_{ss}(G) - rk_{ss}(M_1) = \dim a_{M_1} - \dim a_G$$
Let us denote $M_1 = M_\Theta$. Then $\Delta - \Theta$ contains d simple roots.

Let us denote $\Delta(P_1)$ the set of non-trivial restrictions (or projections) to A_{M_1} (resp. to $a_{M_1}^G$) of simple roots in Δ such that elements in $\Sigma(P_1)$ (roots which are positive for P_1) are linear combinations of simple roots in $\Delta(P_1)$.

Let us denote $\Delta(P_1) = \{\alpha_1, \dots, \alpha_{d-1}, \beta_d\}$ and $\underline{\alpha}_i$ the simple root in Δ which projects onto α_i in $\Delta(P_1)$.

As (M_1, σ_λ) is the cuspidal support of an irreducible discrete series, as explained in the Proposition 4.3, the set Σ_σ is a root system of rank d in $\Sigma(A_{M_1})$ and its basis, when we set $\Sigma(P_1) \cap \Sigma_\sigma$ as the set of positive roots for Σ_σ, is Δ_σ.

Proposition 6.1. *With the context of the previous paragraphs. Let Σ_σ be irreducible. If $\Delta(P_1) = \{\alpha_1, \dots, \alpha_{d-1}, \beta_d\}$ then $\Delta_\sigma = \{\alpha_1, \dots, \alpha_{d-1}, \alpha_d\}$, where α_d can be different from β_d if Σ_σ is of type B, C, D.*

Proof. This is a result of the case-by-case analysis conducted in the independent paper [15], where Δ_Θ denotes there the $\Delta(P_1)$ considered in this Proposition. From its definition Σ_σ is a subsystem in Σ_Θ. If Σ_Θ contains a root system of type BC_d, it is clear that the last root, denoted α_d, of this system (which is either the short of long root depending on the chosen reduced system) can be different from β_d if Σ_σ is of type D_d.

We have not included the root β_d in Δ_σ because (as opposed to the context of classical groups) it is possible that there exists σ an irreducible cuspidal representation such that $s_{\beta_d}\sigma \not\cong \sigma$.

A typical example of the above Proposition (6.1) is when Σ if of type B, C and Σ_σ is of type D, then it occurs that $\Delta(P_1)$ contains $\beta_d = e_d$ or $\beta_d = 2e_d$ whereas Δ_σ contains $\alpha_d = e_{d-1} + e_d$.

This proposition allows us to use our results on intertwining operators with non-generic kernel (see Proposition 3.2, and Example 5.1).

In the context of Harish-Chandra's Theorem 2.1, the element denoted s_α corresponds to the element $\widetilde{w_0}^{(M_1)_\alpha}\widetilde{w_0^{M_1}}$ as defined in Chapter 1 in [34].

Let us describe it:

Let P be a standard parabolic, $P = MN$. Let $\Theta \subset \Delta$, $M = M_\Theta$. In [34], Shahidi defines $\widetilde{w_0}$ as the element in $W(A_0, G)$ which sends Θ to a subset of Δ but every other root $\beta \in \Delta - \Theta$ to a negative root.

If $\widetilde{w_0^G}$, $\widetilde{w_0^M}$ are the longest elements in the Weyl groups of A_0 in G and M, respectively, then $\widetilde{w_0} = \widetilde{w_0^G}\widetilde{w_0^M}$.

The length of this element in W is the difference of the lengths of each element in this composition. Therefore, if a representative of this element in G normalizes M, since it is of minimal length in its class in the quotient $\left\{w \in W | w^{-1}Mw = M\right\}/W^M$, this representative belongs to $W(M)$.

When P is maximal and self-associate (meaning $\widetilde{w_0}(\Theta) = \Theta$) then if α is the simple root of A_M in Lie(N), $\widetilde{w_0}(\alpha) = -\alpha$.

In this case $w_0 N w_0^{-1} = N^-$, the opposite of N for w_0 a representative of $\widetilde{w_0}$ in G.

Remark. Applying the previous paragraph to the context of $P_1 \cap (M_1)_\beta$ and $(M_1)_\beta$, we first show that $\widetilde{w_0}^{(M_1)_\beta}\widetilde{w_0^{M_1}}(\Theta) = \Theta$. Then, one notices that $\widetilde{w_0}^{(M_1)_\beta}\widetilde{w_0^{M_1}}$ sends β to $-\beta$.

In analogy with the notations of Theorem 2.1, let us denote $\widetilde{w_0}^{(M_1)_\beta}\widetilde{w_0^{M_1}} = s_\beta$, we have: $s_\beta(P_1 \cap (M_1)_\beta) = \overline{P_1} \cap (M_1)_\beta$, then $s_\beta\lambda = \lambda$ if λ is in $\overline{(a_{M_1}^G*)}^+$ and is a residual point of type D.

By definition, if $\alpha \in \Sigma_\sigma$, by Harish-Chandra's Theorem 2.1, $s_\alpha(P_1 \cap (M_1)_\alpha) = \overline{P_1} \cap (M_1)_\alpha$ and $s_\alpha.M_1 = M_1$, and this means that s_α is a representative in G of a Weyl group element sending Θ on Θ.

Corollary 6.1.1. *Let σ be an irreducible cuspidal representation of a standard Levi subgroup M_1 and let us assume that Σ_σ is irreducible of rank $d = rk_{ss}(G) - rk_{ss}(M_1)$ and type A, B, C or D, then:*

(1) For any α in $\Delta(P_1)$, $s_\alpha \in W(M_1)$.
(2) $W(M_1) = W_\sigma \cup \left\{s_{\beta_d}W_\sigma\right\}$.
(3) Let σ' (resp. σ) be an irreducible cuspidal representation of a standard Levi subgroup M_1' (resp. standard Levi subgroup M_1). Let us assume they are the cuspidal support of the same irreducible discrete series. Then $M_1' = M_1$.

Proof.

Point (1):

Let us assume Θ has the form given in Appendix C, Theorem C.1, that is a disjoint union of irreducible components: $\bigcup_{i=1}^n \Theta_i$. Then, let us show that for any α in $\Delta(P_1)$, $s_\alpha \in W(M_1)$.

By definition, s_α is a representative in G of the element $\widetilde{w_0^{(M_1)_\alpha}}\widetilde{w_0^{(M_1)}}$.

Let us first assume that α_i is the restriction of the simple root connecting Θ_i and Θ_{i+1}, both of type A, in the Dynkin diagram of G.

Then
$$\Delta^{(M_1)_{\alpha_i}} = \Theta_i \cup \{\alpha_i\} \cup \Theta_{i+1} \bigcup_{j\neq i,i+1} \Theta_j$$

The element $\widetilde{w_0^{M_1}}$ operates on each component as the longest Weyl group element for that component: it sends $\alpha_k \in \Theta_i$ to $-\alpha_{\ell_i+1-k}$ if ℓ_i is the length of the connected component Θ_i.

In a second time, $\widetilde{w_0^{(M_1)_{\alpha_i}}}$ operates on $\Theta_i \cup \{\alpha_i\} \cup \Theta_{i+1}$ in a similar fashion, and trivially on each component in $\bigcup_{j\neq i,i+1} \Theta_j$.

Secondly, let us assume that β is the restriction of the simple root connecting Θ_{n-1} of type A and Θ_n of type B, C or D in the Dynkin diagram of G.

$\widetilde{w_0^{(M_1)}}(\Theta_{n-1}) = \Theta_{n-1}$ (since this element simply permutes and multiply by (-1) the simple roots in Θ_{n-1}), while $\widetilde{w_0^{(M_1)}}(\Theta_n) = -\Theta_n$. Further, $\widetilde{w_0^{(M_1)_\beta}}$ acts as (-1) on all the simple roots in $\Theta_{n-1} \cup \Theta_n$.

Eventually, $\widetilde{w_0^{(M_1)_\beta}}\widetilde{w_0^{(M_1)}}$ fixes Θ_n pointwise and sends each root in Θ_{n-1} to another root in Θ_{n-1}. It also fixes pointwise $\bigcup_{j\neq n-1,n}^n \Theta_j$.

Therefore, for any α in $\Delta(P_1)$, $\widetilde{w_0^{(M_1)_\alpha}}\widetilde{w_0^{(M_1)}}(\Theta) = \Theta$, hence $s_\alpha \in \left\{w \in W | w^{-1}M_1 w = M_1\right\}$.

Furthermore, since the length of this element is the difference of the lengths of each element in this composition, it is clear that s_α is of minimal length in its class in the quotient $\left\{w \in W | w^{-1}M_1 w = M_1\right\} / W^{M_1}$, hence this element is in $W(M_1)$.

Point (2)

Any element in $W(M_1)$ is a representative of minimal length in its class in the quotient $\left\{w \in W | w^{-1}M_1 w = M_1\right\} / W^{M_1}$. The $s_\alpha = \widetilde{w_0^{(M_1)_\alpha}}\widetilde{w_0^{(M_1)}}$ described above where the elements $\alpha \in \Delta(P_1)$ are a set of generators of $W(M_1)$. Recall from Proposition 6.1 that $\Delta(P_1) = \{\alpha_1, \ldots, \alpha_{d-1}, \beta_d\}$ and $\Delta_\sigma = \{\alpha_1, \ldots, \alpha_{d-1}, \alpha_d\}$, where α_d can be different from β_d if Σ_σ is of type B, C, D. Therefore $W(M_1) = W_\sigma \cup \left\{s_{\beta_d} W_\sigma\right\}$.

We also recall that in the context of Σ_σ of type D_d and $\Sigma(P_1)$ of type B_d or C_d : $s_{\alpha_d} = s_{\alpha_{d-1}} s_{\beta_d} s_{\alpha_{d-1}} s_{\beta_d}$.

Point (3)

Let us denote $M_1' = M_{\Theta'}$, and $M_1 = M_\Theta$ and assume that Θ and Θ' are written as $\bigcup_{i=1}^n \Theta_i$, where, for any $i \in \{1, \ldots, n-1\}$, Θ_i is an irreducible component of type A.

Since the cuspidal data are the support of the same irreducible discrete series, by Theorem 2.9 in [4], there exists $w \in W^G$ such that $M_1' = w.M_1$, $\sigma' = w.\sigma$. Since M_1' is isomorphic to M_1, Θ' is isomorphic to Θ.

Therefore applying the observations made in the first part of the proof of this Proposition to M_1 and M_1', we observe Θ and Θ' share the same constraints: their components of type A are all of the same cardinal and the interval between any two of these consecutive components is of length one. Also, since Θ' is isomorphic to Θ, its last component Θ_m' is of the same type as Θ_m. Therefore $\Theta' = \Theta$.

Hence $M_1 = M_1'$.

Remark. This implies that if $P_1 = M_1 U_1$ and $P_1' = M_1' U_1'$ are both standard parabolic subgroups such that their Levi subgroups satisfy the conditions of the previous Proposition, they are actually equal.

6.2. **A few preliminary results for the proof of Moeglin's extended lemmas.** Let us recall Casselman's square-integrability criterion as stated in [41] whose proof can be found in ([10],(4.4.6)). Let $\Delta(P)$ be a set of simple roots, then ${}^+a_P^{G*}$,resp. ${}^+\overline{a}_P^{G*}$, denote the set of χ in a_M^* of the following form: $\chi = \sum_{\alpha\in\Delta(P)} x_\alpha \alpha$ with $x_\alpha > 0$,resp $x_\alpha \geq 0$. Further, denote π_P the Jacquet module of π with respect to P, and $\mathcal{E}xp$ the set of exponents of π as defined in Section I.3 in [41].

Proposition 6.2 (Proposition III.1.1 in [41]). *The following conditions are equivalent:*

(1) π is square-integrable;
(2) for any semi-standard parabolic subgroup $P = MU$ of G, and for any χ in $\mathcal{E}xp(\pi_P)$, $Re(\chi) \in {}^{+}a_P^G{*}$
(3) for any standard parabolic subgroup $P = MU$ of G, proper and maximal, and for any χ in $\mathcal{E}xp(\pi_P)$, $Re(\chi) \in {}^{+}a_P^G{*}$.

In the following two lemmas we will apply the previous Proposition as follows:

Proposition 6.3. *Let π_0 embed in $I_{P_1}^G(\sigma_\lambda)$. Let us write the parameter λ as a vector in the basis $\{e_i\}_{i\geq 0}$ (the basis of $a_{M_1}^*$ as chosen in the Definition 4.2, for instance) as $((x, y) + \underline{\lambda})$ for a linear residual segment (x, y), and assume $\sum_{k\in[x,y]} k \leq 0$. Then π_0 is not square-integrable.*

Proof. Indeed, if
$$\pi_0 \hookrightarrow I_{P_1}^G(\sigma((x, y) + \underline{\lambda}))$$
by Frobenius reciprocity, the character χ_λ appears as exponent of the Jacquet module of π_0 with respect to P_1. Let us write λ as
$$\sum_i x_i(e_i - e_{i+1}) + \underline{\lambda} = \sum_i y_i e_i + \underline{\lambda}$$
it is clear that, for any integer j, $x_j = \sum_{i=1}^{j} y_j$, and notice there is an index j' such that $x_{j'} = \sum_{k\in[x,y]} k$. Therefore, using the hypothesis of the Proposition, $x_{j'} = \sum_{k\in[x,y]} k \leq 0$. But then χ_λ does not satisfy the requirement of Proposition 6.2 since $x_{j'}$ is negative.

We will also use the following well-known result:

Theorem 6.2 ([31], Theorem VII.2.6). *Let (π, V) be a admissible irreducible representation of G. Then (π, V) is tempered if and only if there exists a standard parabolic subgroup of G, $P = MN$, and a square integrable irreducible representation (σ, E) of M such that (π, V) is a subrepresentation of $I_P^G(\sigma)$.*

Le us repeat here Lemma 1:

Lemma 12 (Lemma 1.8 in [20]). *Let $\alpha \in \Delta_\sigma$, $s = s_\alpha$ and assume $(M_1)_\alpha$ is a standard Levi subgroup of G. The operator $J_{sP_1|P_1}$ are meromorphic functions in σ_λ for σ unitary cuspidal representation and λ a parameter in $(a_{M_1}^{(M_1)_\alpha}{*})$.*

The poles of $J_{sP_1|P_1}$ are precisely the zeroes of $\mu^{(M_1)_\alpha}$. Any pole has order one and its residue is bijective. Furthermore, $J_{P_1|sP_1}J_{sP_1|P_1}$ equals $(\mu^{(M_1)_\alpha})^{-1}$ up to a multiplicative constant.

Further by a general result concerning the μ function, it has one and only one pole on the positive real axis if and only if, for fixed σ unitary irreducible cuspidal representation, $\mu(\sigma) = 0$ (This is clear from the explicit formula given by Silberger [36]).

Therefore for each $\alpha \in \Sigma_\sigma$, by definition, there will be one λ on the positive real axis such that $\mu^{(M_1)_\alpha}$ has a pole.

Lemma 13. *Let $\beta \in \Delta(P_1)$, and assume $\beta \notin \Delta_\sigma$, then the elementary intertwining operator associated to $s_\beta \in W$ is bijective at σ_λ $\forall \lambda \in a_{M_1}^*$.*

Proof. Set $s = s_\beta$ for $\beta \in \Delta(P_1)$, and $\beta \notin \Delta_\sigma$. Recall we have $J_{P_1|sP_1}J_{sP_1|P_1}$ equals $(\mu^{(M_1)_\beta})^{-1}$ up to a multiplicative constant.

Recall $\mathcal{O}$ denotes the set of equivalence classes of representations of the form $\sigma \otimes \chi$ where χ is an unramified character of M_1.

The operator $\mu^{(M_1)_\beta} J_{P_1|s_\beta P_1}$ is regular at each unitary representation in $\mathcal{O}$ (see [41], V.2.3), $J_{s_\beta P_1|P_1}$ is itself regular on $\mathcal{O}$, since this operator is polynomial on $X^{nr}(G)$.

By the general result mentioned after Lemma 1, the function $\mu^{(M_1)_\beta}$ has a pole at σ_λ for λ on the positive real axis, if $\mu^{(M_1)_\beta}(\sigma) = 0$. Therefore, by definition, since $\beta \notin \Delta_\sigma$, there is no pole at σ_λ.

Further, since the regular operators $J_{P_1|sP_1}$ and $J_{sP_1|P_1}$ are non-zero at any point, if $\mu^{(M_1)_\beta}$ does not have a pole at σ_λ, these operators $J_{P_1|sP_1}$ and $J_{sP_1|P_1}$ are bijective.

A consequence of this lemma is that for any root $\beta \in \Sigma(P_1)$ which admits a reduced decomposition without elements in Δ_σ, the intertwining operators associated to s_β are everywhere bijective.

6.3. **Extended Moeglin's Lemmas.** In this section and the following the core of our argumentation relies on the form of the parameters λ; changes on the form of these parameters are induced by actions of Weyl group elements (see for instance Example 5.2). In fact, the Weyl group operates on σ_λ and any Weyl group element decomposes in elementary symmetries s_{α_i} for $\alpha_i \in \Delta$. This kind of decomposition is explained in details in I.1.8 of the book [42]. If α_i is in Δ_σ, by Harish-Chandra's Theorem (Theorem 2.1), $s_{\alpha_i}\sigma \cong \sigma$; however recall that for $\beta_d \in \Delta(P_1)$ (see Proposition 6.1), we may not have $s_{\beta_d}\sigma \cong \sigma$.

The three next lemmas, inspired by Remark 3.2 page 154 and Lemma 5.1 in Moeglin [26] are used in our main embedding Proposition 6.4 (of the irreducible generic discrete series) result.

Recall that in general $P_{\Theta'}$ is the parabolic subgroup associated to the subset $\Theta' \subset \Delta$, and $M_{\Theta'}$ contains all the roots in Θ'. Recall that we denote $\underline{\alpha}_i$ the simple root in Δ which restricts to α_i in $\Delta(P_1)$.

Definition 6.1. Let (M_1, σ) be the generic cuspidal support of an irreducible generic discrete series.

Let us denote $M_1 = M_\Theta$. Let us assume that $\Theta = \bigcup_{i=1}^n \Theta_i$, where, for any $i \in \{1, \dots, n-1\}$, Θ_i is an irreducible component of type A.

We say this cuspidal support satisfies the conditions *(CS)* (given in Proposition 6.1 and Corollary 6.1.1) if:

- Σ_σ is irreducible of rank d.
- If $\Delta(P_1) = \{\alpha_1, \dots, \alpha_{d-1}, \beta_d\}$ then $\Delta_\sigma = \{\alpha_1, \dots, \alpha_{d-1}, \alpha_d\}$, where α_d can be different from β_d if Σ_σ is of type B, C, D.
- For any $i \in \{1, \dots, n-1\}$, Θ_i has fixed cardinal. Furthermore, the interval between any two disjoint consecutive components Θ_i, Θ_{i+1} is of length one.

Lemma 14. *Let π_0 be a generic discrete series of a quasi-split reductive group G (of type A, B, C or D) whose cuspidal support (M_1, σ_λ) satisfies the condition* (CS) *(see the Definition 6.1).*

Let

$$x, y \in \mathbb{R}, k - 1 = x - y \in \mathbb{N}$$

This defines the integer k.

Let us denote

$$M' = M_{\Delta - \{\underline{\alpha}_1, \dots, \underline{\alpha}_{k-1}, \underline{\alpha}_k\}}$$

Let us assume there exists $w_{M'} \in W^{M'}(M_1)$, and an irreducible generic representation τ which is the unique generic subquotient of $I_{P_1 \cap M'}^{M'}(\sigma_{\lambda_1^{M'}})$ such that

$$\pi_0 \hookrightarrow I_{P'}^G(\tau_{(x,y)}) \hookrightarrow I_{P_1}^G((w_{M'}\sigma)_{(x,y)+\lambda_1^{M'}});\ \ \lambda_1^{M'} \in a_{M_1}^{M'} \tag{6}$$

Let us assume y is minimal for this property.

Then τ is square integrable.

Proof.

Let us first remark that in Equation 6 the parameter in $a_{M_1}^*$ is decomposed as

$$\underbrace{(x, y)}_{\text{combination of } \alpha_1, \dots, \alpha_{k-1}} \quad + \quad \underbrace{\lambda_1^{M'}}_{\text{combination of } \alpha_{k+1}, \dots, \beta_d}$$

Let us denote τ the generic irreducible subquotient in $I_{P_1\cap M'}^{M'}(\sigma_{\lambda_1^{M'}})$, and let us show that τ is square integrable.

Assume on the contrary that τ is not square-integrable.

Then τ is tempered (but not square integrable) or non-tempered. Langlands' classification [Theorem 2.3] insures us that τ is a Langlands quotient $J(P'_L, \tau', \nu')$ for a parabolic subgroup $P'_L \supseteq P_1$ of M' or equivalently a subrepresentation in $I_{P'_L}^{M'}(\tau'_{\nu'})$, $\nu' \in \overline{((a_{M'_L}^{M'})^*)^-}$ (Equivalently $\nu'_{P'_L} \leq 0$, the inequality is strict in the non-tempered case).

This is equivalent to claim there exists an irreducible generic cuspidal representation σ', (half)-integers ℓ, m with $\ell - m + 1 \in \mathbb{N}$ and $m \leq 0$ such that:

$$\tau \hookrightarrow I_{P'_L}^{M'}(\tau'_{\nu'}) \hookrightarrow I_{P_1\cap M'}^{M'}(\sigma'((\ell, m) + \lambda_2{}^{M'})) \tag{7}$$

$$\sum_{k\in[\ell,m]} k \leq 0 \tag{*}$$

We have extracted the linear segment (ℓ, m) out of the segment $\lambda_1^{M'}$ and named $\lambda_2^{M'}$ what is left.

Let us justify Equation (*): The parameter ν' reads

$$(\underbrace{\ldots, \frac{\ell+m}{2}, \ldots}_{\ell-m+1 \text{ times}}, 0, \ldots, 0)$$

$$\nu'_{P'_L} \leq 0 \Leftrightarrow \frac{\ell+m}{2} \leq 0 \Leftrightarrow m \leq -\ell \Leftrightarrow \sum_{k\in[\ell,m]} k \leq 0$$

From Equation (7)

$$\pi_0 \hookrightarrow I_{P'}^G(\tau_{(x,y)}) \hookrightarrow I_{P_1}^G(\sigma'((x,y) + (\ell, m) + \lambda_2^{M'})) \tag{8}$$

Since π_0 also embeds as a subrepresentation in $I_{P_1}^G(\sigma_\lambda)$, by Theorem 2.9 in [4] (see also [31] VI.5.4) there exists a Weyl group element w in W^G such that $w.M_1 = M_1$, $w.\sigma' = \sigma$ and $w((x,y)+(\ell,m)+\lambda_2^{M'}) = \lambda$. This means we can take w in $W(M_1)$.

But we can be more precise on this Weyl group element: from Equation (7) and the hypothesis in the statement of the Lemma, we see we can take it in $W^{M'}(M_1)$ and it leaves the leftmost part of the cuspidal support, $\sigma_{(x,y)}$, invariant, this element therefore depends on x and y. We denote this element $w_{M'}$.

Let

$$M'' = M_{\Delta - \left\{\underline{\alpha}_q, \ldots, \underline{\beta}_d\right\}}$$

where $q = x - y + 1 + \ell - m + 1$.

Now, let us consider two cases. First, let us assume $m \geq y$. If the two linear segments are unlinked and the generic subquotient in $I_{P_1\cap M''}^{M''}(\sigma'((x,y) + (\ell, m)))$ is irreducible, applying Lemma 10, we can interchange them in the above Equation (8) and we reach a contradiction to the Casselman Square Integrability criterion applied to the discrete series π_0 (considering its Jacquet module with respect to P_1, see Proposition 6.3 using $\sum_{k\in[\ell,m]} k \leq 0$).

By Proposition 5.2 and Remark 5.3, if the two linear segments are linked the irreducible generic subquotient $\tau_{L,gen}$ of

$$I_{P_1\cap M''}^{M''}((w_{M'}\sigma)((x,y) + (\ell, m)))$$

embeds in

$$I_{P_1\cap M''}^{M''}((w.w_{M'}\sigma)((\ell, y) + (x, m)))$$

(for some Weyl group element $w \in W^{M''}(M_1)$, such that $w.w_{M'}\sigma \cong w_{M'}\sigma$).

By Lemma 9 there exists an intertwining operator with non generic kernel sending $\tau_{L,gen}$ to $I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$. Then by unicity of the generic piece in $I^G_{P_1}((w_{M'}\sigma)((x,y)+(\ell,m)+\lambda^{M'}_2))$, π_0 embeds in $I^G_{P''}((\tau_{L,gen})_{\lambda^{M'}_2})$.

Therefore, inducing to G, we have

$$\pi_0 \hookrightarrow I^G_{P''}((\tau_{L,gen})_{\lambda^{M'}_2}) \hookrightarrow I^G_{P'''}(I^{M'''}_{P_1\cap M''}((w_{M'}\sigma)((\ell,y)+(x,m)+\lambda^{M'}_2))$$

but then since $\sum_{k\in[\ell,y]} k \leq 0$ (since $m \geq y$), we reach a contradiction to the Casselman Square Integrability criterion applied to the discrete series π_0 (considering its Jacquet module with respect to P_1).

Secondly, let us assume $m < y$. The induced representation

$$I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$$

is reducible only if $\ell \in]x, y-1]$. Then using Proposition 5.2 and Remark 5.3, we know that the irreducible generic subquotient $\tau_{L,gen}$ of

$$I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$$

should embed in

$$I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((x,m)+(\ell,y)))$$

(or only in $I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((x,m)))$ if $\ell = y-1$).

Applying Lemma 10, we also know that it embeds in $I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((\ell,y)+(x,m)))$ (we can interchange the order of the two unlinked segments (ℓ,y) and (x,m)). Then, using Lemma 9 and unicity of the generic irreducible piece as above, we embed π_0 in $I^G_{P''}((\tau_{L,gen})_{\lambda^{M'}_2}) \hookrightarrow I^G_{P_1}((w_{M'}\sigma)((x,y)+(\ell,m)+\lambda^{M'}_2))$.

But π_0 does not embed in $I^G_{P_1}((w_{M'}\sigma)((x,m)+(\ell,y)+\lambda^{M'}_2)))$ since y is minimal for such (embedding) property.

Therefore, $\tau_{L,gen}$ rather embeds in the quotient $I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((\ell,m)+(x,y)))$ of $I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((x,y)+(\ell,m)))$.

Then π_0 embed in

$$I^G_{P''}((\tau_{L,gen})_{\lambda^{M'}_2}) \hookrightarrow I^G_{P''}(I^{M''}_{P_1\cap M''}((w_{M'}\sigma)((\ell,m)+(x,y))))_{\lambda^{M'}_2} = I^G_{P_1}((w_{M'}\sigma)((\ell,m)+(x,y)+\lambda^{M'}_2))$$

Since $\sum_{k\in[\ell,m]} k \leq 0$, using Proposition 6.3, we reach a contradiction.

Lemma 15. *Let π_0 be a generic discrete series of G whose cuspidal support satisfies the conditions* CS *(see the Definition 6.1). Let a, a_- be two consecutive jumps in the set of Jumps of π_0.*

Let us assume there exists an irreducible representation π' of a standard Levi $M' = M_{\Delta-\{\underline{\alpha}_1,\ldots,\underline{\alpha}_{a-a_-}\}}$ such that

$$\pi_0 \hookrightarrow I^G_{P'}(\pi'_{(a,a_-+1)}) \hookrightarrow I^G_{P_1}(\sigma_{(a,a_-+1)+\lambda}). \tag{9}$$

Then there exists a generic discrete series π of $M'' = M_{\Delta-\{\underline{\alpha_{a+a_-+1}}\}}$ such that:

π_0 embeds in $I^G_{P''}((\pi)_{s\widetilde{\alpha}_{a+a_-}}) \hookrightarrow I^G_{P_1}(\sigma((a,-a_-)+(\underline{n})))$ with $s = \frac{a-a_-}{2}$ and $(\underline{n})$ a residual segment.

We split the proof in two steps:

Step A. We first need to show that π' is necessarily tempered following the argumentation given in [26].

Assume on the contrary that π' is not tempered. Langlands' classification [Theorem 2.3] insures us that π' is a subrepresentation in $I_{P_L}^{M'}(\tau_\nu)$, for a parabolic standard subgroup $P_L \supseteq P_1$ and

$$\nu \in ((a_L^{M'})^*)^-$$

This is equivalent to claim there exists x, y with $x - y + 1 \in \mathbb{N}$, and $y \leq 0$, a Levi subgroup

$$L = M_{\Delta - \{\underline{\alpha}_1, \ldots, \underline{\alpha}_{a-a_-}\} \cup \{\underline{\alpha}_{x-y}\}}$$

a unitary cuspidal representation $w_{M'}\sigma$ in the $W(M_1)^{M'}$ group orbit of σ, and the element $\lambda \in (a_{M_1}^{M'})^*$ decomposes as $(x, y) + \lambda_1^{M'}$ such that:

$$\pi' = I_{P_L}^{M'}(\tau_\nu) \hookrightarrow I_{P_1 \cap M'}^{M'}((w_{M'}\sigma)((x, y) + \lambda_1^{M'}))$$

$$(*) \qquad \sum_{k \in [x,y]} k < 0$$

The first equality in the first equation is due to the Standard module conjecture since π' is generic. The second equation (*) results from the following sequences of equivalences: $\nu <_{P_L} 0 \Leftrightarrow \frac{x+y}{2} < 0 \Leftrightarrow y < -x \Leftrightarrow \sum_{k \in [x,y]} k < 0$.

The element $w_{M'}$ in $W(M_1)^{M'}$ leaves the leftmost part, $\sigma_{(a,a_-+1)}$, invariant.

Then from Equation (9) and inducing to G:

$$\pi_0 \hookrightarrow I_{P_1}^G((w_{M'}\sigma)((a, a_- + 1) + (x, y) + \lambda_1^{M'}))$$

We can change $(a, a_- + 1)(x, y)$ to $(x, y)(a, a_- + 1)$ if and only if the two segments $(a, \ldots, a_- + 1)$ and $(x, \ldots, y)$ are unlinked (see the Lemma 10). As $y \leq 0$, this condition is equivalent to $x \notin]a, a_-]$.

If we can change, since $\sum_{k \in [x,y]} k < 0$, we get by Proposition 6.3 a contradiction to the square integrability of π_0.

Assume therefore we cannot change, then the two segments are linked by Proposition 5.1. Let $M''' = M_{\Delta - \{\underline{\alpha}_q, \ldots, \underline{\beta}_d\}}$ where $q = a - a_- + x - y + 1$.

The induced representation

$$I_{P_1 \cap M'''}^{M'''}((w_{M'}\sigma)((a, \ldots, a_- + 1) + (x, \ldots, y)))$$

has a generic submodule which is:

$$Z^{M'''}(P_1, w_L.w_{M'}\sigma, (a, \ldots, y)(x, \ldots, a_- + 1))$$

(for some Weyl group element w_L such that $w_L.w_{M'}\sigma \cong w_{M'}\sigma$)

We twist these by the character $\lambda_1^{M'}$ central for M'''.

and therefore, by unicity of the irreducible generic piece:

$\pi_0 \hookrightarrow I_{P'''}^G(Z^{M'''}(P_1, w_{M'}\sigma, (a, \ldots, y)(x, \ldots, a_- + 1))_{\lambda_1^{M'}})$

$\hookrightarrow I_{P'''}^G(I_{P_1 \cap M'''}^{M'''}((w_{M'}\sigma)((a, \ldots, a_- + 1) + (x, \ldots, y)))_{\lambda_1^{M'}}) = I_{P_1}^G((w_{M'}\sigma)((a, \ldots, y) + (x, \ldots, a_- + 1) + \lambda_1^{M'})$

Let $Q' = L'U'$, we rewrite this as:

$$\pi_0 \hookrightarrow I_{Q'}^G(Z^{L'}(P_1, w'_L.w_{M'}\sigma, (a, \ldots, y)(\lambda_2^{M'}))) \hookrightarrow I_{P_1}^G((w'_L.w_{M'}\sigma)((a, \ldots, y) + \lambda_2^{M'}))$$
$$:= I_{P_1}^G((w_{M'}\sigma)((a, \ldots, y) + \lambda_2^{M'}))$$

for some Weyl group element w'_L such that $w'_L.w_{M'}\sigma \cong w_{M'}\sigma$.

Further, we have $y < -a_-$ since y is negative, $x \geq a_-$ and $\sum_{k \in [x,y]} k < 0$. In this context, the above Lemma 14 claims there exists $y' \leq y$:

$$\pi_0 \hookrightarrow I_{P_1}^G((w_{M'}\sigma)((a, \ldots, y') + \lambda_3^{M'}))$$

And then the unique irreducible generic subquotient π'_0 of $I_{P_1 \cap N'}^{N'}(\sigma_{\lambda_3^{M'}})$ is square-integrable, or equivalently $\sigma_{\lambda_3^{M'}}$ is a residual point for $\mu^{N'}$ (The type is given by $\Sigma_\sigma^{N'}$). Further, $\sigma_{(a,\ldots,y')+\lambda_3^{M'}}$ is a residual point for μ^G (type given by Σ_σ), corresponding to the generic discrete series π_0.

Then the *set of Jumps* of the residual segment associated to π_0 contains the *set of Jumps* of the residual segment associated to π'_0 and two more elements a and $-y'$ but then $a > -y' > a_-$ and this contradicts the fact that a and a_- are two consecutive jumps.

We have shown that π' is necessarily tempered.

Step B. Let $(\underline{n}_{\pi_0})$ be the residual segment canonically associated to a generic discrete series π_0. Let us now denote a_{i+1} the greatest integer smaller than a_i in the *set of Jumps* of $(\underline{n}_{\pi_0})$. Therefore, the half-integers, a_i and a_{i+1} satisfy the conditions of this lemma.

As the representation π' is tempered, by Theorem 6.2, there exists a standard parabolic subgroup $P_\#$ of M' and a discrete series τ' such that $\pi' \hookrightarrow I_{P_\#}^{M'}(\tau')$.

Again, as an irreducible generic discrete series representation of a non necessarily maximal Levi subgroup, using the result of Heiermann-Opdam (Proposition 2.1), there exists an irreducible cuspidal representation σ' and a standard parabolic $P_{1,\#}$ of $M_\#$ such that τ' embeds in $I_{P_{1,\#}}^{M_\#}(\sigma'((\frac{a-a_--1}{2}, -\frac{a-a_--1}{2}) + \bigoplus_j (a_j, -a_j) + (\underline{n}_{\pi''_0})))$, where $(\underline{n}_{\pi''_0})$ is a residual segment corresponding to an irreducible generic discrete series π''_0 and $(\frac{a-a_--1}{2}, -\frac{a-a_--1}{2})$ along with $(a_j, -a_j)$'s are linear residual segments for (half)-integers a_j.

Clearly, the point $(\frac{a-a_--1}{2}, \ldots, -\frac{a-a_--1}{2}) + \bigoplus_j (a_j, -a_j) + (\underline{n}_{\pi''_0})$ is in $\overline{a_{M_1}^{M_{\#}*}}^+$.

Then

$$\pi' \hookrightarrow I_{P_{1,\#}U_\#}^{M'}(\sigma'((\frac{a-a_--1}{2}, \ldots, -\frac{a-a_--1}{2}) + \bigoplus_j (a_j, \ldots, -a_j) + (\underline{n}_{\pi''_0}))) \tag{10}$$

Since $P_{1,\#}U_\#$ is standard in P' which is standard in G, there exists a standard parabolic subgroup P'_1 in G, such that, when inducing Equation 10, we obtain:

$$\pi_0 \hookrightarrow I_{P'}^G(\pi'_{(a,\ldots,a_-+1)}) \hookrightarrow I_{P'_1}^G(\sigma'_{(a,\ldots,a_-+1)+\bigoplus_j (a_j,\ldots,-a_j)+(\underline{n}_{\pi''_0})}) \tag{11}$$

Let us denote $(a, \ldots, a_- + 1) + \bigoplus_j (a_j, \ldots, -a_j) + (\underline{n}_{\pi''_0}) := \lambda'$.

Since π_0 also embeds as a subrepresentation in $I_{P_1}^G(\sigma_{(a,\ldots,a_-+1)+\lambda})$, by Theorem 2.9 in [4] (see also [31] VI.5.4) there exists a Weyl group element w in W^G such that $w.M_1 = M'_1, w.\sigma = \sigma'$ and $w((a, a_- + 1) + \lambda) = \lambda'$.

Since Σ_σ is irreducible and M'_1 is standard, we have by Point (3) in Corollary 6.1.1 that $M'_1 = M_1$, and we can take w in $W(M_1)$. Further since P_1 and P'_1 are standard parabolic subgroups of G, and Σ_σ is irreducible they are actually equal (see Remark 6.1).

Now, by Point (2) in Corollary 6.1.1 any element in $W(M_1)$ is either in W_σ or decomposes in elementary symmetries in W_σ and $s_{\beta_d} W_\sigma$ and :

$$\sigma' = w\sigma = \begin{cases} \sigma \text{ if } w \in W_\sigma \\ \text{Else } s_{\beta_d}\sigma \end{cases}$$

Let us assume we are in the context where $\sigma' = s_{\beta_d}\sigma \not\cong \sigma$. As explained in the first part of Section 6 (see Proposition 6.1), this happens if Σ_σ is of type D.

Let us apply the bijective operator (see Lemma 13) from $I^{(M_1)_{\beta_d}}_{P_1\cap(M_1)_{\beta_d}}((s_{\beta_d}\sigma)_{\lambda'})$ to $I^{(M_1)_{\beta_d}}_{\overline{P_1}\cap(M_1)_{\beta_d}}((s_{\beta_d}\sigma)_{\lambda'})$ and then the bijective map $t(s_{\beta_d})$ (the definition of the map $t(g)$ has been given in the proof of Proposition 3.1) to $I^{(M_1)_{\beta_d}}_{s_{\beta_d}(\overline{P_1}\cap(M_1)_{\beta_d})}(\sigma_{s_{\beta_d}\lambda'}) = I^{(M_1)_{\beta_d}}_{P_1\cap(M_1)_{\beta_d}}(\sigma_{s_{\beta_d}\lambda'})$.

As explained in Remark 6.1, $s_{\beta_d}\lambda' = \lambda'$ since λ' is a residual point of type D.

Therefore, we have a bijective map from $I^{(M_1)_{\beta_d}}_{P_1\cap(M_1)_{\beta_d}}((s_{\beta_d}\sigma)_{\lambda'})$ to $I^{(M_1)_{\beta_d}}_{P_1\cap(M_1)_{\beta_d}}(\sigma_{\lambda'})$.

The induction of this bijective map gives a bijective map from $I^G_{P_1}(\sigma'_{(a,\dots,a_-+1)+\bigoplus_j(a_j,\dots,-a_j)+(\underline{n}_{\pi_0''})})$ to $I^G_{P_1}(\sigma_{(a,\dots,a_-+1)+\bigoplus_j(a_j,\dots,-a_j)+(\underline{n}_{\pi_0''})})$.

Therefore we may write Equation (11) as:

$$\pi_0 \hookrightarrow I^G_{P'}(\pi'_{(a,\dots,a_-+1)}) \hookrightarrow I^G_{P_1}(\sigma_{(a,\dots,a_-+1)+\bigoplus_j(a_j,\dots,-a_j)+(\underline{n}_{\pi_0''})}) \tag{12}$$

Let us set $a = a_i$, $a_- = a_{i+1}$ for a_i, a_{i+1} two consecutive elements in the set of Jumps of $(\underline{n}_{\pi_0})$.

Therefore, $(a_i,\dots,\dots,a_{i+1}+1)\bigoplus_j(a_j,\dots,-a_j) + (\underline{n}_{\pi_0''})$ is in the Weyl group orbit of the residual segment associated to π_0: $(\underline{n}_{\pi_0})$.

Let us show that $(a_i,\dots,a_{i+1}+1)(a_{i+1},\dots,-a_{i+1})(\underline{n}^i)$ is in the W_σ-orbit of $(\underline{n}_{\pi_0})$.

One notices that in the tuple $\underline{n}_{\pi_0}$ of the residual segment $(\underline{n}_{\pi_0})$ the following relations are satisfied:

$$n_{a_i} = n_{a_{i+1}} - 1 \tag{13}$$

$$n_i = n_{i-1} - 1 \text{ or } n_i = n_{i-1},\ \forall i > 0 \tag{14}$$

Therefore, when we withdraw $(a_i,\dots,a_{i+1}+1)$ from this residual segment, we obtain a segment $(\underline{n'})$ which cannot be a residual segment since $n'_{a_{i+1}} = n'_{a_{i+1}+1} + 2$ for $i \neq 1$; or if $i = 1$, $n'_{a_2} = 2$ but a_2 is now the greatest element in the set of Jumps associated to the segment $(\underline{n'})$, so we should have $n'_{a_2} = 1$.

Therefore, to obtain a residual point (residual segment $(\underline{n}_{\pi_0''})$), we need to remove twice a_{i+1}.

Then, for any $0 < j < a_{i+1}$, if we remove twice j, $n'_j = n_j - 2$ and, for all i, the relations $n'_j = n'_{j-1} - 1$ or $n'_j = n'_{j-1}$ are still satisfied. As we also remove one zero, we have for $j = 0$, $n'_0 = n_0 - 1$ which is compatible with removing twice $j = 1$.

The residual segment left, thus obtained, will be denoted $(\underline{n}^i)$. We have shown that $(a_i,\dots,a_{i+1}+1)(a_{i+1},\dots,-a_{i+1})(\underline{n}^i)$ is in the W_σ-orbit of $(\underline{n}_{\pi_0})$.

Since $(\underline{n}^i)$ is a residual segment, from the conditions detailed in Equations 13 and 14 (see also Remark 4.2 in Section 4.2) no symmetrical linear residual segment $(a_k, -a_k)$ can be extracted from $(\underline{n}^i)$ to obtain another residual segment $(\underline{n}_{\pi_0''})$ such that $(a_i,\dots,a_{i+1}+1)(a_{i+1},\dots,-a_{i+1})(a_k,-a_k)(\underline{n}_{\pi_0''})$ is in the W_σ-orbit of $(\underline{n}_{\pi_0})$.

So $(\underline{n}_{\pi_0''}) = (\underline{n}^i)$ and

$$\pi'_{(a,a_-+1)} \hookrightarrow I^{M'}_{P_1}(\sigma((a_i, a_{i+1}+1) + (a_{i+1}, -a_{i+1}) + (\underline{n}^i)))$$

Eventually, using induction in stages Equation (10) rewrites:

$$\pi_0 \hookrightarrow I^G_{P_1}(\sigma((a_i, a_{i+1}+1) + (a_{i+1}, -a_{i+1}) + (\underline{n}^i)) = \Theta$$

and since the two segments $(a_i, \ldots, a_{i+1}+1)$ and $(a_{i+1}, \ldots, -a_{i+1})$ are linked, we can take their union and deduce there exists an irreducible generic essentially square integrable representation π_{a_i} of a Levi subgroup M^{a_i} in P^{a_i} which once induced embeds as a subrepresentation in Θ and therefore by multiplicity one of the irreducible generic piece, π_0, we have:

$$\pi_0 \hookrightarrow I_{P^{a_i}}^G(\pi_{a_i}) \hookrightarrow I_{P_1}^G(\sigma((a_i, -a_{i+1}) + (\underline{n}^i)))$$

Proposition 6.4. *Let $(\underline{n}_{\pi_0})$ be a residual segment associated to an irreducible generic discrete series π_0 of G whose cuspidal support satisfies the conditions* CS *(see the Definition 6.1).*

Let $a_1 > a_2 > \ldots > a_n$ be Jumps *of this residual segment. Let $P_1 = M_1U_1$ be a standard parabolic subgroup, σ be a unitary irreducible cuspidal representation of M_1 such that $\pi_0 \hookrightarrow I_{P_1}^G(\sigma(\underline{n}_{\pi_0}))$.*

For any i, there exists a standard parabolic subgroup $P^{a_i} \supset P_1$ with Levi subgroup M^{a_i}, residual segment $(\underline{n}^i)$ and an irreducible generic essentially square-integrable representation $\pi_{a_i} = Z^{M^{a_i}}(P_1, \sigma, (a_i, -a_{i+1})(\underline{n}^i))$ such that π_0 embeds as a subrepresentation in

$$I_{P^{a_i}}^G(\pi_{a_i}) \hookrightarrow I_{P_1}^G(\sigma((a_i, -a_{i+1}) + (\underline{n}^i)))$$

Proof. By the result of Heiermann-Opdam [Proposition 2.1] and Lemma 11, to any residual segment $(\underline{n}_{\pi_0})$ we associate the unique irreducible generic discrete series subquotient in $I_{P_1}^G(\sigma(\underline{n}_{\pi_0}))$.

Then as explained in the Subsection 4.2 this residual segment defines uniquely *Jumps* : $a_1 > a_2 > \ldots > a_n$.

Start with the two elements $a_1 = \ell + m$ and $a_2 = \ell - 1$ and consider the following induced representation:

$$\begin{aligned} (15) \quad & I_{P_1}^G(\sigma((\ell+m, a_2+1=\ell)(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 0^{n_0})) \\ & \qquad = I_P^G(I_{P_1\cap M}^M(\sigma((\ell+m, a_2=\ell)(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 0^{n_0})) \end{aligned}$$

Let us denote $\nu := (\ell+m, a_2+1=\ell)(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 0^{n_0}$.

The induced representation $I_{P_1\cap M}^M(\sigma((\ell+m, a_2+1=\ell)(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 0^{n_0})) := I_{P_1\cap M}^M(\sigma_\nu)$ is a generic induced module.

The form of ν implies σ_ν is not necessarily a residual point for μ^M. Indeed, the first linear residual segment $(\ell+m, a_2+1=\ell)$ is certainly a residual segment (of type A), but the second not necessarily.

Let π be the unique irreducible generic subquotient of $I_{P_1\cap M}^M(\sigma_\nu)$ (which exists by Rodier's Theorem). We have: $\pi \leq I_{P_1\cap M}^M(\sigma_\nu)$ and $I_P^G(\pi) \leq I_P^G(I_{P_1\cap M}^M(\sigma_\nu)) := I_{P_1}^G(\sigma_\lambda)$.

Assume $I_P^G(\pi)$ has an irreducible generic subquotient π_0' different from π_0, then π_0' and π_0 would be two generic irreducible subquotients in $I_{P_1}^G(\sigma_\lambda)$ contradicting Rodier's theorem. Hence $\pi_0 \leq I_P^G(\pi)$.

Further, since π_0 embeds as a subrepresentation in

$$I_P^G(I_{P_1\cap M}^M(\sigma((\ell+m, a_2+1=\ell)+(\ell-1)^{n_{\ell-1}}(\ell-2)^{n_{\ell-2}}\ldots 0^{n_0})) := I_{P_1}^G(\sigma_\lambda)$$

it also has to embed as a subrepresentation in $I_P^G(\pi)$.

Therefore applying Lemma 15, we conclude there exists a residual segment $(\underline{n}^1)$ an essentially square integrable representation π_{a_1} such that π_0 embeds as a subrepresentation in

$$I_{P^{a_1}}^G(\pi_{a_1}) \hookrightarrow I_{P_1}^G((\sigma((a_1, -a_2) + (\underline{n}^1)))$$

Let us consider now the elements $a_2 = \ell - 1$ and a_3. As in the proof of Lemma 10, since the linear residual segments $(a_1, \ell-1)$ and $(\ell-1)$ are unlinked, we apply a composite map from the induced representation $I_{P_1\cap M'}^{M'}(\sigma((a_1, \ell-1)+(\ell-1)+\ldots 0^{n_0}))$ to $I_{P_1\cap M'}^{M'}(\sigma((\ell-1)+(a_1, \ell-1))+\ldots+0^{n_0}))$. We can interchange the two segments and as in the proof of Lemma 10, applying this intertwining map and inducing to G preserves the unique irreducible generic subrepresentation of $I_{P_1}^G(\sigma_\lambda)$.

We repeat this argument with
$I^{M''}_{P_1\cap M''}(\sigma((\ell-1)+(a_1,\ell-2)+(\ell-2)+\dots 0^{n_0}))$ and $I^{M''}_{P_1\cap M''}(\sigma(\ell-1)+(\ell-2)+(a_1,\ell-2)+\dots+0^{n_0}))$
and further repeat it with all exponents til a_3+1.

Eventually, the unique irreducible subrepresentation π_0 appears as a subrepresentation in $I^G_{P_1}(\sigma((a_2,a_3+1)+(a_1,a_3+1)+(\ell-2)^{n_{\ell-2}-2}\dots(a_3+1)^{n_{a_3+1}-2}\dots 1^{n_1}0^{n_0})$.

$$\pi_0 \hookrightarrow I^G_{P'^{a_2}}(I^{M'^{a_2}}_{P_1\cap M'^{a_2}}(\sigma(a_2,a_3+1)+(a_1,a_3+1)+(\ell-2)^{n_{\ell-2}-2}\dots(a_3+1)^{n_{a_3+1}-2}\dots 1^{n_1}0^{n_0}))$$
$$:= I^G_{P'^2}(I^{M'^{a_2}}_{P_1\cap M'^{a_2}}((w\sigma)_{w\nu}))$$

where $w\in W_\sigma$.

Let π be the unique irreducible generic subquotient of $I^{M'^2}_{P_1\cap M'^2}(\sigma_{w\nu})$ (which exists by Rodier's Theorem). We have: $\pi\leq I^{M'^{a_2}}_{P_1\cap M'^{a_2}}(\sigma_{w\nu})$ and

$$I^G_{P'^2}(\pi)\leq I^G_{P'^{a_2}}(I^{M'^{a_2}}_{P_1\cap M'^{a_2}}(\sigma_{w\nu})) := I^G_{P_1}(\sigma_{w\lambda})$$

Assume $I^G_{P'^{a_2}}(\pi)$ has an irreducible generic subquotient π_0' different from π_0, then π_0' and π_0 would be two generic irreducible subquotients in $I^G_{P_1}((w\sigma)_{w\lambda})$ contradicting Rodier's theorem. Hence $\pi_0\leq I^G_{P'^{a_2}}(\pi)$.

Further, since π_0 embeds as a subrepresentation in

$$I^G_{P'^2}(I^{M'^{a_2}}_{P_1\cap M'^{a_2}}(\sigma((a_2,a_3+1)+(a_1,a_3+1)+(\ell-2)^{n_{\ell-2}-2}\dots(a_3+1)^{n_{a_3+1}-2}\dots 1^{n_1}0^{n_0}) := I^G_{P_1}(\sigma_{w\lambda})$$

it also embeds as a subrepresentation in $I^G_{P'^2}(\pi)$.

Therefore applying Lemma 15, we conclude there exists a residual segment $(\underline{n}^2)$ and an essentially square- integrable representation $\pi_{a_2} = Z^{M^2}(P_1\cap M^{a_2},\sigma,(a_2,-a_3)(\underline{n}^2))$ such that π_0 embeds as a subrepresentation in $I^G_{P^{a_2}}(\pi_{a_2})\hookrightarrow I^G_{P_1}(\sigma((a_2,-a_3)+(\underline{n}^2))$.

Similarly, for any two consecutive elements in the *set of Jumps*, a_i and a_{i+1}, the same argumentation (i.e first embedding π_0 as a subrepresentation in $I^G_{P'^{a_i}}(\pi)$ using intertwining operators, and conclude with Lemma 15) yields the embedding:

$$\pi_0\hookrightarrow I^G_{P^{a_i}}(\pi_{a_i})\hookrightarrow I^G_{P^{a_i}}(I^{P^{a_i}}_{P_1\cap M^i}(\sigma((a_i,-a_{i+1})+(\underline{n}^i)))$$

for an irreducible generic essentially square-integrable representation

$$\pi_{a_i} = Z^{M^{a_i}}(P_1\cap M^{a_i},\sigma,(a_i,-a_{i+1})(\underline{n}^i))$$

of the Levi subgroup M^{a_i}.

6.4. **Proof of the Theorem 6.1. Proof.**

- (1)a) is the result of Lemma 11.
- (1)b) is the result of Proposition 6.4.
- (1)c)
  Let us denote π_0 the unique irreducible subquotient in $I^G_{P_1}(\sigma_{(a,b)\underline{n}})$. By Proposition 2.1, there exists a parabolic subgroup P' such that π_0 embeds as a subrepresentation in the induced module $I^G_{P'}(\sigma'_{\lambda'})$, for $\sigma'_{\lambda'}$ a dominant residual point for P'. Let $(w\sigma)_{w\lambda}$ be the dominant (for P_1) residual point in the W_σ-orbit of σ_λ, then (using Theorem 2.9 in [4] or Theorem VI.5.4 in [31]) π_0 is the unique irreducible generic subquotient in $I^G_{P_1}((w\sigma)_{w\lambda})$, and Proposition 3.1 gives us that these two ($I^G_{P'}(\sigma'_{\lambda'})$ and $I^G_{P_1}((w\sigma)_{w\lambda})$) are isomorphic.

  The point $(w\sigma)_{w\lambda}$ is a dominant residual point with respect to $P_1: w\lambda\in\overline{a^*_{M_1}}^+$ and there is a unique element in the orbit of the Weyl group W_σ of a residual point which is dominant and is explicitly given by a residual segment using the correspondence of the Subsection 2.5.1. We denote $w\lambda := (\underline{n}_{\pi_0})$ this residual segment. Since $w \in W_\sigma$, $(w\sigma)_{w\lambda} \cong \sigma_{w\lambda}$. Hence

$$\pi_0 \hookrightarrow I_{P_1}^G(\sigma(\underline{n}_{\pi_0}))$$

Since $a > b$, and $(\underline{n_{\pi_0}})$ is a residual segment, it is clear that a is a jump. [Indeed, if you extract a linear residual segment $(a, \ldots, b)$ such that $a > b$ from $(\underline{n}_{\pi_0})$ such that what remains is a residual segment, then $a = a$ has to be in the *set of Jumps* of the residual segment $(\underline{n}_{\pi_0})$ as defined in the Subsection 4.2]. Let us denote a_- the greatest integer smaller than a in the *set of Jumps*. Therefore, the (half)-integers, a and a_- satisfy the conditions of Proposition 6.4. We will further show in the next paragraph that $b \geq -a_-$. Let $P_\flat = P_{\Delta - \{\alpha_{a+a_-+1}\}}$ be a maximal parabolic subgroup, with Levi subgroup $M_\flat$, which contains P_1.

Let $\pi_a = Z^{M_\flat}(P_1, \sigma, w_{a_-}\lambda)$, for $w_{a_-} \in W_\sigma$ be the generic essentially square integrable representation with cuspidal support $(\sigma((a, -a_-)(\underline{n_{-a_-}}))$ associated to the residual segment $((a, -a_-) + (\underline{n_{-a_-}}))$ (in the W_σ-orbit of $(\underline{n}_{\pi_0})$).

It is some discrete series twisted by the Langlands parameter $s_{-a_-}\widetilde{\alpha_{a+a_-+1}}$ with $s_{-a_-} = \frac{a-a_-}{2}$. By the Proposition 6.4 we can write

$$\pi_0 \hookrightarrow I_{P_\flat}^G(\pi_a) \hookrightarrow I_{P_1}^G(\sigma((a, -a_-)(\underline{n_{-a_-}}))) \tag{16}$$

Here, we need to justify that given a, for any b we have: $b \geq -a_-$.

Consider again the residual segment $(\underline{n}_{\pi_0})$, and observe that by definition the sequence $(a, \ldots, -a_-)$ is the longest linear segment with greatest (half)-integer a that one can withdraw from $(\underline{n}_{\pi_0})$ such that the remaining segment $(\underline{n_{-a_-}})$ is a residual segment of the same type and $(a, \ldots, -a_-)(\underline{n_{-a_-}})$ is in the Weyl group orbit of $(\underline{n}_{\pi_0})$.

Further, this is true for any couple (a, a_-) of elements in the *set of Jumps* associated to the residual segment $(\underline{n}_{\pi_0})$. It is therefore clear that given a and a_- such that $s_{-a_-} = \frac{a-a_-}{2} > 0$ is the smallest positive (half)-integers as possible, we have $s_b = \frac{a+b}{2} \geq s_{-a_-} = \frac{a-a_-}{2}$ and b is necessarily greater or equal to $-a_-$.

Once this embedding given, using Lemma 8, there exists an intertwining operator with non-generic kernel from the induced module $I_{P_1}^G(\sigma((a, -a_-)(\underline{n_{-a_-}})))$ given in Equation (16) to any other induced module from the cuspidal support $\sigma(a, b, \underline{n_b})$ with $b \geq -a_-$.

Therefore

$$\pi_0 \hookrightarrow I_{P_1}^G(\sigma(a, b, \underline{n_b})) = I_{P_1}^G(\sigma_\lambda)$$

- (2)a)
  Since λ is not a residual point, the generic subquotient is non-discrete series. By Langlands' classification, Theorem 2.3, and the Standard module conjecture, it has the form $J_{P'}^G(\tau'_{\nu'}) \cong I_{P'}^G(\tau'_{\nu'})$. By Theorem 5.2, ν' corresponds to the minimal Langlands parameter (this notion was introduced in the Theorem 2.3) for a given cuspidal support.

  For an explicit description of the parameter ν, given the cuspidal string $(a, b, \underline{n})$, the reader is encouraged to read the analysis conducted in the Appendix of the author's thesis manuscript [14].

  The representation τ' (for e.g $St_q|.|^{\nu'} \otimes \pi'$ in the context of classical groups, for a given integer q) corresponds to a cuspidal string $(a', b', \underline{n'})$, and cuspidal representation σ', that is:

$$I_{P'}^G(\tau'_{\nu'}) \hookrightarrow I_{P'_1}^G(\sigma'(a', b', \underline{n'}))$$

  By the Theorem 2.9 in [4], we know the cuspidal data $(P_1, \sigma, (a', b', \underline{n'}))$ and $(P'_1, \sigma', \lambda' := (a', b', \underline{n'}))$ are conjugated by an element $w \in W^G$.

By Corollary 6.1.1 and since P_1 and P'_1 are standard parabolic subgroups (see Remark 6.1), we have $P_1 = P'_1$, $w \in W(M_1)$. Any element in $W(M_1)$ decomposes in elementary symmetries with elements in W_σ and $s_{\beta_d} W_\sigma$:

$$\sigma' = w\sigma = \begin{cases} \sigma \text{ if } w \in W_\sigma \\ \text{Else } s_{\beta_d}\sigma \end{cases}$$

Let us assume we are in the context where $\sigma' = s_{\beta_d}\sigma \ncong \sigma$. As explained in the first part of Section 6.3, this happens if Σ_σ is of type D.

Let us apply the bijective operator (see Lemma 13) from $I_{P_1\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(s_{\beta_d}\sigma)_{\lambda'})$ to $I_{\overline{P_1\cap(M_1)_{\beta_d}}}^{(M_1)_{\beta_d}}((s_{\beta_d}\sigma)_{\lambda'})$ and then the bijective map (the definition of the map $t(g)$ has been given in the proof of 3.1) $t(s_{\beta_d})$ to $I_{s_{\beta_d}(\overline{P_1}\cap(M_1)_{\beta_d})}^{(M_1)_{\beta_d}}(\sigma_{s_{\beta_d}\lambda'}) = I_{P_1\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(\sigma_{s_{\beta_d}\lambda'})$.

As explained in Remark 6.1, $s_{\beta_d}\lambda' = \lambda'$ since λ' is a residual point of type D.

Therefore, we have a bijective map from $I_{P_1\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(s_{\beta_d}\sigma)_{\lambda'})$ to $I_{P_1\cap(M_1)_{\beta_d}}^{(M_1)_{\beta_d}}(\sigma_{\lambda'})$.

The induction of this bijective map gives a bijective map from $I_{P'_1}^G(\sigma'(a', b', \underline{n'}))$ to $I_{P'_1}^G(\sigma(a', b', \underline{n'}))$.

- (2)b) Assume now that we consider a tempered or non-tempered subquotient in $I_{P_1}^G(\sigma(a, b, \underline{n}))$. We first apply the argumentation developed in the previous point (2)a) to embed it in $I_{P'_1}^G(\sigma(a', b', \underline{n'}))$. Then it is enough to understand how one passes from the cuspidal string $(a', b', \underline{n'})$ to $(a, b, \underline{n})$ to understand the strategy for embedding the unique irreducible generic subquotient as a subrepresentation $I_{P_1}^G(\sigma(a, b, \underline{n}))$.

  Starting from $(a, b, \underline{n})$, to minimize the Langlands parameter ν', we usually remove elements at the end of the first segment (i.e. the segment $(a, \ldots, b)$) to insert them on the second residual segment, or we enlarge the first segment on the right. This means either $a' < a$, or $b' < b$, or both.

  If $a' = a$, and $b' < b$, in particular if $b' < 0$, we have a non-generic kernel operator between $I_{P_1}^G(\sigma(a', b', \underline{n'}))$ and $I_{P_1}^G(\sigma(a, b, \underline{n}))$ as proved in Lemma 8.

6.5. **An order on the cuspidal strings in a W_σ-orbit.** It is possible to describe the set of points in the W_σ-orbit of a dominant residual point λ_D as follows.
Let us define a set of points $\mathcal{L}$ in the W_σ-orbit of a dominant residual point λ_D such that they are written as : $(a, b)(\underline{n})$ with at most one linear residual segment (a, b) satisfying the condition $a > b$. Then a is a Jump as explained in the proof of Theorem 6.1, point 1)c).

Let us attach a positive integer $C(1, \lambda) = \#\left\{\beta \in \Sigma_\sigma^+ | \left\langle \lambda, \check{\beta} \right\rangle < 0 \right\}$ to any of these points.
By definition, $C(1, \lambda_D) = 0$.
What are the points λ in $\mathcal{L}$ such that the function $C(1, \lambda)$ is maximal?

Lemma 16. *The function $C(1, .)$ on $\mathcal{L}$ is maximal for the points which are the form $(a, -a_-)(\underline{n})$ for (a, a_-) any two consecutive elements in the Jumps sets associated to λ_D.*

Proof. Let us choose a point in $\mathcal{L}$; since it is a point in $\mathcal{L}$, it uniquely determines a jump a (as its left end). For any fixed a, we show that the function $C(1, \lambda_a)$ is maximal for $\lambda_{a,-a_-} = (a, -a_-)(\underline{n})$. Let $\mathcal{L}_a$ denote the set of points in $\mathcal{L}$ such that the linear residual segment (if it exists) *has left end a*. The union of the $\mathcal{L}_a$ where a runs over the set of Jumps is $\mathcal{L}$.
Let us choose a point $\lambda_{a,b} = (a, b)(\underline{n})_b$ in $\mathcal{L}_a$ and denote L_b the length of the residual segment $(\underline{n})_b$. Recall also that $(\underline{n})_b = (\ell, \ldots b^{n_b} \ldots 0^{n_0})$

- Case $a > 0 > b$

  Consider λ_b and λ_{b+1}.

Let us consider first those roots which are of the forms $e_i - e_j$, $i > j$:

On $\lambda_{\mathfrak{b}}$ the number of these roots which have non-positive scalar product is: $(-b) \times L_{\mathfrak{b}} + (L_{\mathfrak{b}} - n_0) + (L_{\mathfrak{b}} - (n_0 + n_1)) + (L_{\mathfrak{b}} - (n_0 + n_1 + n_2 + \ldots + n_{\mathfrak{b}})) + C_{\mathfrak{b}+1}$ where $C_{\mathfrak{b}+1}$ is some constant depending on the multiplies n_i for $i \geq (\mathfrak{b} + 1)$.

Secondly, let us consider the roots of the forms $e_i + e_j$, $i > j$; on $\lambda_{\mathfrak{b}}$ the number of these roots which have non-positive scalar product is:

$$L_{\mathfrak{b}}-(n_{\mathfrak{b}}+n_{b+1}+n_\ell)+L_{\mathfrak{b}}-(n_{b-1}+n_{\mathfrak{b}}+n_{b+1}+\ldots+n_\ell)+L_{\mathfrak{b}}-(n_{b-2}+n_{b-1}+n_{\mathfrak{b}}+n_{b+1}+\ldots+n_\ell)+\ldots+\mathfrak{b}+\mathfrak{b}-1+\mathfrak{b}-2+\ldots+1$$

Finally, one should also take into account the roots of type $e_i, 2e_i$ or $e_i + e_d$ if d is the dimension of Σ_σ and of type B, C or D. There are b such roots in our context.

(17)
$$C(1,\lambda_{\mathfrak{b}+1}) = (-\mathfrak{b}-1)\times(L_{\mathfrak{b}}+1)+(L_{\mathfrak{b}}+1-n_0)+(L_{\mathfrak{b}}+1-(n_0+n_1))+\ldots+(L_{\mathfrak{b}}+1-(n_0+n_1+\ldots+n_{\mathfrak{b}}))+C_{b+1}$$
$$+L_{\mathfrak{b}}+1-(n_{\mathfrak{b}-1}+n_{\mathfrak{b}}+n_{\mathfrak{b}+1}+\ldots+n_\ell)+L_{\mathfrak{b}}-(n_{b-2}+n_{b-1}+n_{\mathfrak{b}}+n_{\mathfrak{b}+1}+\ldots+n_\ell)+\ldots+\mathfrak{b}-1+\mathfrak{b}-2+\ldots+1+\mathfrak{b}-1$$
$$C(1,\lambda_{\mathfrak{b}}) - C(1,\lambda_{\mathfrak{b}+1}) = L_{\mathfrak{b}} - (n_{\mathfrak{b}} + n_{b+1} + n_\ell) + \mathfrak{b} + \mathfrak{b} - (-L_{\mathfrak{b}} - \mathfrak{b} - 1 + \mathfrak{b} - 1 + \mathfrak{b} + \mathfrak{b} - 1)$$
$$C(1,\lambda_{\mathfrak{b}}) - C(1,\lambda_{\mathfrak{b}+1}) = 2L_{\mathfrak{b}} - (n_{\mathfrak{b}} + n_{\mathfrak{b}+1} + n_\ell) + 3$$

Therefore
$$C(1,\lambda_b) > C(1,\lambda_{\mathfrak{b}+1})$$

- Case $a > \mathfrak{b} > 0$

  Consider $\lambda_{\mathfrak{b}}$ and $\lambda_{\mathfrak{b}+1}$. The number $C(1,\lambda_{\mathfrak{b}})$ and $C(1,\lambda_{\mathfrak{b}+1})$ differ by $L_{\mathfrak{b}} - (n_0 + n_1 + \ldots + n_{\mathfrak{b}})$. As this number is clearly positive, we have: $C(1,\lambda_{\mathfrak{b}}) > C(1,\lambda_{\mathfrak{b}+1})$.

This shows that $C(1,.)$ decreases as the length of the linear residual segment (a,b) decreases.

Furthermore, from the definition of residual segment (Definition 4.2) and the observations made on cuspidal lines, the sequence $(a,\ldots,-a_-)$ is the *longest* linear segment with greatest (half)-integer a that one can withdraw from λ_D such that the remaining segment $(\underline{n_{-a_-}})$ is a residual segment of the same type and $(a,\ldots,-a_-)(\underline{n_{-a_-}})$ is in the W_σ-orbit of λ_D.
Therefore $C(1,\lambda_{a,-a_-})$ is maximal on the set $\mathcal{L}_a$.

As a consequence of this Lemma, we will denote the points of maximal $C(1,.)$, λ_{a_i} for any a_i in the jumps set of λ_D.

The elementary symmetries associated to roots in Σ_σ permute the (half)-integers appearing in the cuspidal line $(a,\mathfrak{b})(\underline{n})$.
We illustrate the set $\mathcal{L}$ with a picture.
Let us assume any two points in the W_σ-orbit are connected by a vertex if they share the same parameter a and/or the intertwining operator associated to the sequence of elementary symmetries connecting the two points has non-generic kernel. Any point in $\mathcal{L}$ is on a vertex joining the points of maximal $C(1,.)$ to λ_D. We obtain the following picture.

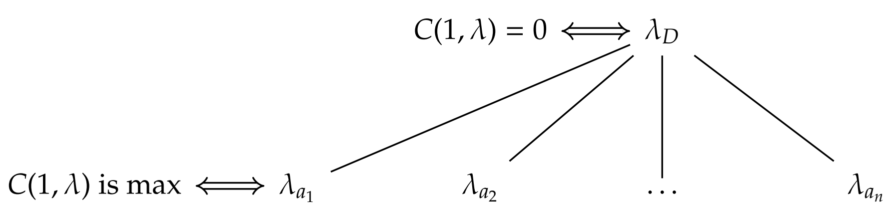


Figure 1. The set $\mathcal{L}$

Then the proof of the Theorem 6.1 could be thought about in this way: Relying on the extended Moeglin's Lemmas we obtain the embedding of the unique irreducible generic subquotient for a

set of parameters $\{\lambda_{a_i}\}_i$. Those parameters are indexed by the *jumps* a_i in a (finite) set of Jumps associated to the dominant residual point λ_D (they are in the W_σ-orbit of λ_D).
Once this key embedding given, for each jump a, we use intertwining operators with non-generic kernel to send the unique irreducible generic subrepresentation which lies in $I_{P_1}^G(\sigma_{\lambda_a}) = I_{P_1}^G(\sigma((a, -a_-)(\underline{n}))$ to $I_{P_1}^G(\sigma((a, b)(\underline{n'}))$, for any $b > -a_-$ where $(\underline{n'})$ is a residual segment of the same type as $(\underline{n})$.

7. Proof of the Generalized Injectivity Conjecture for Discrete Series Subquotients

Before entering the proof the Conjecture for Discrete Series Subquotients, let us mention two aside results.

First,in order to use Theorem 2.2, let us first prove the following lemma:

Lemma 17. *Under the assumption that μ^G has a pole in $s\tilde{\alpha}$ (assumption 1) for τ and μ^M has a pole in ν (for σ) of maximal order, for $\nu \in a^*_{M_1}$, $\sigma_{\nu+s\tilde{\alpha}}$ is a residual point.*

Proof. We will use the multiplicativity formula for the μ function (see Section IV 3 in [41], or the earlier result (Theorem 1) in [35]) :

$$\mu^G(\tau_{s\tilde{\alpha}}) = \frac{\mu^G}{\mu^M}(\sigma_{s\tilde{\alpha}+\nu})$$

We first notice that if μ^M has a pole in ν (for σ) of maximal order, for $\nu \in a^*_{M_1}$, μ^M also has a pole of maximal order in $\nu + s\tilde{\alpha}$ (Since $s\tilde{\alpha}$ is in a^*_M, we twist by a character of A_M which leaves the function μ^M unchanged). Under the assumption 1, the order of the pole in $\nu + s\tilde{\alpha}$ of the right side of the equation is:

$$\text{ord}(\text{pole for } \mu^G \text{ in } \nu + s\tilde{\alpha}) - (rk_{ss}(M) - rk_{ss}(M_1)) \geq 1$$

Since M is maximal we have: $(rk_{ss}(G) - rk_{ss}(M)) = \dim(A_M) - \dim(A_G) = 1$, then

$$\begin{aligned}(rk_{ss}(M) - rk_{ss}(M_1)) + 1 &= (rk_{ss}(M) - rk_{ss}(M_1)) + (rk_{ss}(G) - rk_{ss}(M)) \\ &= (rk_{ss}(G) - rk_{ss}(M_1))\end{aligned}$$

Hence ord(pole of μ^G in $\nu + s\tilde{\alpha}) \geq (rk_{ss}(G) - rk_{ss}(M_1))$, and the lemma follows.

The element $\nu + s\tilde{\alpha}$ being a residual point (a pole of maximal order for μ^G) for σ, by Theorem 2.2 we have a discrete series subquotient in $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$.

Further, consider the following classical lemma (see for instance [44]):

Lemma 18. *Take τ a tempered representation of M, and ν_0 in the positive Weyl chamber. If ν_0 is a pole for μ^G then $I_P^G(\tau_{\nu_0})$ is reducible.*

This lemma results from the fact that when τ is tempered and ν_0 in the positive Weyl chamber, $J_{\overline{P}|P}(\tau, .)$ is holomorphic at ν_0. If the μ function has a pole at ν_0 then $J_{\overline{P}|P}J_{P|\overline{P}}(\tau, .)$ is the zero operator at ν_0. The image of $J_{P|\overline{P}}(\tau, .)$ would then be in the kernel of $J_{\overline{P}|P}(\tau, .)$, a subspace of $I_P^G(\tau_{\nu_0})$ which is null if $I_P^G(\tau_{\nu_0})$ is irreducible. This would imply $J_{P|\overline{P}}$ is a zero operator which is not possible. So $I_P^G(\tau_{\nu_0})$ must be reducible.

Under the hypothesis of Lemma 17, the module $I_P^G(\tau_{s\tilde{\alpha}})$ has a generic discrete series subquotient. We aim to prove in this section that this generic subquotient is a subrepresentation.

We present here the proof of the generalized injectivity conjecture in the case of a standard module induced from a maximal parabolic $P = MU$. Then, the roots in Lie(M) are all the roots in Δ but α. We first present the proof in case α is not an extremal root in the Dynkin diagram of G, and secondly when it is an extremal root.

The context is the one of the previous Subsection: G is a quasi-split reductive group, of type A, B, C or D and Σ_σ is irreducible.

Proposition 7.1. *Let π_0 be an irreducible generic representation of a quasi-split reductive group G of type A, B, C or D which embeds as a subquotient in the standard module $I_P^G(\tau_{s\tilde{\alpha}})$, with $P = MU$ a maximal parabolic subgroup and τ discrete series of M.*

Let σ_ν be in the cuspidal support of the generic discrete series representation τ of the maximal Levi subgroup M and we take $s\tilde{\alpha}$ in $\overline{(a_M^*)^+}$, such that $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$ and denote $\lambda = \nu + s\tilde{\alpha}$ in $\overline{a_{M_1}^{M}}^{+*}$.

Let us assume that the cuspidal support of τ satisfies the conditions CS *(see the Definition 6.1).*

Let us assume that α is not an extremal simple root on the Dynkin diagram of Σ.

Let us assume σ_λ is a residual point for μ^G. This is equivalent to say that the induced representation $I_{P_1}^G(\sigma_\lambda)$ has a discrete series subquotient. Then, this unique irreducible generic subquotient, π_0, which is discrete series embeds as a submodule in $I_{P_1}^G(\sigma_\lambda)$ and therefore in the standard module $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$.

Proof.
First, notice that if $s = 0$, the induced module $I_P^G(\tau_{s\tilde{\alpha}})$ is unitary hence any irreducible subquotient is a subrepresentation; in the rest of the proof we can therefore assume $s\tilde{\alpha}$ in $(a_M^*)^+$.

Let us denote π_0 the irreducible generic discrete series representation which appears as subquotient in a standard module $I_P^G(\tau_{s\tilde{\alpha}})$ induced from a maximal parabolic subgroup P of G. We are in the context of the Subsection 4.3, and therefore we can write $\lambda := (a, \dots, b)(\underline{n})$, for some (half)-integers $a > b$, and residual segment $(\underline{n})$. In this context, as we denote $s\tilde{\alpha}$ the Langlands parameter twisting the discrete series τ, then $s = s_b = \frac{a+b}{2}$.

Notice that since σ_λ is in the W_σ-orbit of a dominant residual point whose parameter corresponds to a residual segment of type B, C or D, a and b are not only reals but *(half)-integers*. The conditions of application of Theorem 6.1 1)b) or 1)c) are satisfied and therefore the unique irreducible generic subquotient in $I_{P_1}^G(\sigma_\lambda)$ is a subrepresentation. By multiplicity one, it will also embed as a subrepresentation in the standard module $I_P^G(Z^M(P_1, \sigma, \lambda))$.

Remark. From the Theorem 6.1 and the argumentation given in the proof of the previous Proposition, it is easy to deduce that if π_0 appears as a submodule in the standard module

$$I_{P_\flat}^G(Z^{M_\flat}(P_1, \sigma, w_{a_-}\lambda))$$

with Langlands parameter $s_{a_-}\widetilde{\alpha_{a+a_-+1}}$, it also appears as a submodule in any standard module $I_P^G(Z^M(P_1, \sigma, (a, b, \underline{n_b}))$ with Langlands' parameter $s_b\tilde{\alpha} \geq s_{-a_-}\widetilde{\alpha_{a+a_-+1}}$ for the order defined in Lemma 5 as soon as $Z^M(P_1, \sigma, (a, b, \underline{n_b}))$ has equivalent cuspidal support.

7.0.1. *The case of Σ_σ^M irreducible.*

Proposition 7.2. *Let π_0 be an irreducible generic discrete series of G with cuspidal support (M_1, σ) and let us assume Σ_σ is irreducible. Let M be a standard maximal Levi subgroup such that Σ_σ^M is irreducible.*

Then, π_0 embeds as a subrepresentation in the standard module $I_P^G(\tau_{s\tilde{\alpha}})$, where τ is an irreducible generic discrete series of M.

Proof. Assume Σ_σ is irreducible of rank d, let $\Delta_\sigma := \{\alpha_1, \dots, \alpha_d\}$ be the basis of Σ_σ (following our choice of basis for the root system of G) and let us denote $\mathcal{T}$ its type.

We consider maximal standard Levi subgroups of G, $M \supset M_1$, such that the root system Σ_σ^M is irreducible. Typically if $M = M_{\Delta - \{\underline{\beta_d}\}}$.

Now, in our setting, σ_ν is a residual point for μ^M. It is in the cuspidal support of the generic discrete series τ if and only if (applying Proposition 4.3): $rk(\Sigma_\sigma^M) = d - 1$.

Let us denote $(\nu_2, \dots, \nu_d)$ the residual segment corresponding to the irreducible generic discrete series τ of M.

If $(\nu_2, \dots, \nu_d)$ is a residual segment of type A to obtain a residual segment $(\nu_1, \nu_2, \dots, \nu_d)$ of rank d and type:

- D: we need $\nu_d = 0$ and $\nu_1 = \nu_2 + 1$
- B: we need $\nu_d = 1$ and $\nu_1 = \nu_2 + 1$
- C: we need $\nu_d = 1/2$ and $\nu_1 = \nu_2 + 1$

If $(\nu_2, \dots, \nu_d)$ is a residual segment of type $\mathcal{T}$ (B, C, D) we need $\nu_1 = \nu_2 + 1$ to obtain a residual segment of type $\mathcal{T}$ and rank d.

In all these cases, the twist $s\tilde{\alpha}$ corresponds on the cuspidal support to add one element on the left to the residual segment $(\nu_2, \dots, \nu_d)$; then the segment $(\nu_1, \nu_2, \dots, \nu_d) := (\lambda_1, \lambda_2, \dots, \lambda_d)$ is a residual segment:

$$\pi_0 \leq I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$$

This is equivalent to say σ_λ is a *dominant* residual point and therefore, by Lemma 11, π_0 embeds as a subrepresentation in $I_{P_1}^G(\sigma_\lambda)$ and therefore in $I_P^G(\tau_{s\tilde{\alpha}})$ by multiplicity one of the generic piece in the standard module.

7.1. **Non necessarily maximal parabolic subgroups.** In the course of the main theorem in this section, we will need the following result:

Lemma 19. *Let $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_t$ be t unlinked linear segments with $\mathcal{S}_i = (a_i, \dots, b_i)$ for any i. If*

$$(a_1, \dots, b_1)(a_2, \dots, b_2) \dots (a_t, \dots, b_t)(\underline{n})$$

is a residual segment $(\underline{n'})$; then at least one segment $(a_i, \dots, b_i)$ merges with $(\underline{n})$ to form a residual segment $(\underline{n''})$.

Proof.

Consider the case of t unlinked segments, with at least one disjoint from the others, we aim to prove that this segment can be inserted into $(\underline{n})$ independently of the others to obtain a residual segment. For each such (disjoint from the others) segment $(a_i, \dots, b_i)$, inserted, the following conditions are satisfied:

$$\begin{cases} n'_{a_i+1} = n_{a_i+1} = n'_{a_i} - 1 = n_{a_i} + 1 - 1 \\ n'_{b_i} = n_{b_i} + 1 = n_{b_i-1} - 1 + 1 = n_{b_i-1} = n'_{b_i-1} \end{cases} \tag{18}$$

The relations $n'_{a_i+1} = n_{a_i+1}$ and $n'_{b_i-1} = n_{b_i-1}$ come from the fact that the elements $(a_i + 1)$ and $(b_i - 1)$ cannot belong to any other segment unlinked to $(a_i, \dots, b_i)$.

If for any i those conditions are satisfied $(\underline{n'})$ is a residual segment, by hypothesis.

Now, let us choose a segment which does not contain zero: (a_j, b_j). Since by the Equation (18) $n_{a_j+1} = n_{a_j}$ and $n_{b_j} = n_{b_j-1} - 1$, adding only $(a_j, \dots, b_j)$ yields equations as (18) and therefore a residual segment.

If this segment contains zero and is disjoint from the others, then adding all segments or just this one yields the same results on the numbers of zeroes and ones: $n'_0 = n''_0$, $n'_1 = n''_1$, therefore there is no additional constraint under these circumstances.

Secondly, let us consider the case of a chain of inclusions, that, without loss of generality, we denote $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \dots \supset \mathcal{S}_t$. Starting from $(\underline{n'})$, observe that adding the t linear residual segments yields the following conditions:

$$n'_{a_i+1} = n_{a_i+1} + i - 1 = n'_{a_i} - 1 = n_{a_i} + i - 1$$

$$n'_{b_i} = n_{b_i} + i = n_{b_i-1} - 1 + i = n'_{b_i-1}$$

Then, for any i, we clearly observe $n_{a_i+1} = n_{a_i}$; and $n_{b_i} = n_{b_i-1} - 1$. Assume we only add the segment $(a_1, \dots, b_1)$, then we observe $n''_{a_1+1} = n''_{a_1} - 1$ and $n''_{b_1} = n''_{b_1-1}$, satisfying the conditions for $(\underline{n''})$ to be a residual segment.

Assume $\mathcal{S}_t$ contains zero, then any $\mathcal{S}_i$ also. Assume there is an obstruction at zero to form a residual segment when adding $t - 1$ segments. If adding only $t - 1$ zeroes does not form a residual segment, but t zeroes do, we had $n'_0 = \frac{n_1}{2}$. Then $n_0 + t = \frac{n_1}{2} + t = \frac{n_1+2t}{2}$ (the option $n'_1 = n_1 + 2t + 1$ is immediately excluded since there is at most two '1' per segment $\mathcal{S}_i$).

We need to add $2t$ times '1'. Then we need at least $2t-1$ times '2' and $2t-2$ times '3'..etc. Since, $n'_1 = n_1 + 2t$ all $\mathcal{S}_i$'s will contain (10 -1). There is no obstruction at zero while adding solely $\mathcal{S}_1$ (i.e $n_0 + 1 = \frac{n_1+2}{2}$) and since $\mathcal{S}_1 \supset \mathcal{S}_2 \ldots \supset \mathcal{S}_t$ and $\mathcal{S}_1$ needs to contain $a_1 \geq \ell + m$, $\mathcal{S}_1$ can merge with $(\underline{n})$ to form a residual segment.

Finally, it would be possible to observe the case of a residual segment $\mathcal{S}_1$ containing $\mathcal{S}_2$ and $\mathcal{S}_3$ with $\mathcal{S}_2$ and $\mathcal{S}_3$ disjoint (or two-or more- disjoint chains of inclusions). Again, we have:

$$n'_{a_1+1} = n_{a_1+1} = n'_{a_1} - 1 = n_{a_1} + 1 - 1$$

Assume we only add the segment $(a_1, \ldots, b_1)$, then we observe $n''_{a_1+1} = n''_{a_1} - 1$ and $n''_{b_1} = n''_{b_1-1}$, satisfying the conditions for $(\underline{n''})$ to be a residual segment.

Remark. We show in this remark that if $s_i = \frac{a_i+b_i}{2} = s_j = \frac{a_j+b_j}{2}$, the linear segments $(a_i, \ldots, b_i)$ with $a_i > b_i$ and (a_j, b_j) with $a_j > b_j$ are such that one of them is included in the other (therefore unlinked).

If the length of the segments are the same, they are equal; without loss of generality let us consider the following case of different lengths:

$$a_i - b_i + 1 > a_j - b_j + 1 \tag{19}$$

Since $\frac{a_i+b_i}{2} = \frac{a_j+b_j}{2}$, $a_i + b_i = a_j + b_j$ and from Equation (19) $a_i - a_j > b_i - b_j$ replacing b_i by $a_j + b_j - a_i$, and further a_i by $a_j + b_j - b_i$, we obtain:

$$a_i - a_j > a_j + b_j - a_i - b_j \Leftrightarrow a_i > a_j$$
$$a_j + b_j - b_i - a_j > b_i - b_j \Leftrightarrow b_j > b_i$$

Therefore

$$a_i > a_j > b_j > b_i$$

Therefore, the content of the proofs of the next Theorem (7.1), when considering the case of equal parameters $s_i = s_j$, remain the same.

Theorem 7.1. *Let π_0 be an irreducible generic representation discrete series of a quasi-split reductive group G. Let us assume σ_ν is in the cuspidal support of a generic discrete series representation τ of a standard Levi subgroup M of G. Let us assume that the cuspidal support of τ satisfies the conditions (CS) (see the Definition 6.1). Let us take $\underline{s}$ in $\overline{(a^*_M)^+}$, such that $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_{\nu+\underline{s}})$ and denote $\lambda = \nu + \underline{s}$ in $\overline{a_{M_1}^{M}}^{+*}$. Let us assume σ_λ is a residual point for μ^G.*

Then, the unique irreducible generic square-integrable subquotient, π_0, in the standard module $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$ is a subrepresentation.

Proof.

Let us assume that Σ_σ^M is a disjoint union of t subsystems of type A and a subsystem of type $\mathcal{T}$.

Let $\underline{s} = (s_1, s_2, \ldots, s_t)$ be ordered such that $s_1 \geq s_2 \geq \ldots \geq s_t \geq 0$ with $s_i = \frac{a_i+b_i}{2}$, for two (half)-integers $a_i \geq b_i$.

Using the depiction of residual points in Subsection 4.3, we write the residual point

$$\sigma(\bigoplus_{i=1}^{t}(a_i, \ldots, b_i)(\underline{n}))$$

where λ reads $\bigoplus_{i=1}^{t}(a_i, \ldots, b_i)(\underline{n})$.

Let us denote the linear residual segments $(a_i, \ldots, b_i) := \mathcal{S}_i$ and assume that for some indices $i, j \in \{1, \ldots, t\}$, the segments $\mathcal{S}_i, \mathcal{S}_j$ are linked.

By Lemma 9, there exists an intertwining operator with non-generic kernel from $I_{P_1}^G(\sigma((\mathcal{S}'_1,\mathcal{S}'_2,\ldots,\mathcal{S}'_t;\underline{n})$ to $I_{P_1}^G(\sigma((\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_t;\underline{n})$. Therefore, if we prove the unique irreducible discrete series subquotient appears as subrepresentation in $I_{P_1}^G(\sigma((\mathcal{S}'_1,\mathcal{S}'_2,\ldots,\mathcal{S}'_t;\underline{n})))$, it will consequently appears as subrepresentation in $I_{P_1}^G(\sigma((\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_t;\underline{n})))$. This means we are reduced to the case of the cuspidal support σ_λ being constituted of t *unlinked* segments.

Further, notice that by the above remark [7.1] when $s_i = s_j$, the segments $\mathcal{S}_i$, and $\mathcal{S}_j$ are *unlinked*. This allows us to treat the case $s_1 = s_2 = \ldots = s_t > 0$ and $s_1 > s_2 = \ldots = s_t = 0$.

So let us assume all linear segments $(a_i,\ldots,b_i)$ are unlinked.

We prove the theorem by induction on the number t of linear residual segments.

First, $t = 0$, let $P_0 = G$, and π be the generic irreducible square integrable representation corresponding to the dominant residual point $\sigma_\lambda := \sigma(\underline{n}_{\pi_0})$.

$$I_{P_0}^G(\pi) \hookrightarrow I_{P_1}^G(\sigma((\underline{n}_{\pi_0}))$$

By Lemma 11, λ being in the closure of the positive Weyl chamber, the unique irreducible generic discrete series subquotient is necessarily a subrepresentation.

The proof of the step from $t = 0$ to $t = 1$ is Proposition 7.1.

Assume the result true for any standard module $I_{P'_{\Theta_{\leq t}}}^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma(\bigoplus_{i=1}^t(a_i,\ldots,b_i)(\underline{n})))$ with t or less than t linear residual segments, where $P'_{\Theta_{\leq t}}$ is any standard parabolic subgroup whose Levi subgroup is obtained by removing t or less than t simple (non-extremal) roots from Δ.

We consider now π_0 the unique irreducible generic discrete series subquotient in

$$I_{P_{\Theta_{t+1}}}^G(\tau'_{\underline{s'}}) \hookrightarrow I_{P_1}^G(\sigma(\bigoplus_{i=1}^t(a_i,\ldots,b_i)(a_{t+1},\ldots,b_{t+1})(\underline{n'})))$$

To distinguish with the case of a discrete series τ of P_{Θ_t}, we denote τ' the irreducible generic discrete series and s' in $a_{M_{\Theta_{t+1}}}{}^{*+}$.

Using Lemma 19, we know there is at least one linear segment with index $j \in [1, t+1]$ such that $(a_j,\ldots,b_j)$ can be inserted in $(\underline{n'})$ to form a residual segment. Without loss of generality, let us choose this index to be $t+1$ (else we use bijective intertwining operators on the unlinked segments to set $(a_j,\ldots,b_j)$ in the last position).

Then, there exists a Weyl group element w such that $w((a_{t+1},\ldots,b_{t+1})(\underline{n'})) = (\underline{n})$ for a residual segment $(\underline{n})$.

Let $M_1 = M_\Theta$ with $\Theta = \bigcup_{i=1}^s\Theta_i$ for some $s > t$ and $M' = M_{\Theta'}$ where $\Theta' = \bigcup_{i=1}^{s-2}\Theta_i\cup\Theta_t\cup\left\{\underline{\alpha}_t\right\}\cup\Theta_{t+1}$, if we assume (by convention) that the root $\underline{\alpha}_t$ connects the two connected components Θ_t and Θ_{t+1}.

Since $M'\cap P$ is a maximal parabolic subgroup in M', we can apply the result of Proposition 7.1 to π' the unique irreducible discrete series subquotient in $I_{P_1\cap M'}^{M'}(\sigma(a_{t+1},b_{t+1})(\underline{n'}))$.

Notice that $\Sigma^{M'}$ is a reducible root system, and therefore so is $\Sigma_\sigma^{M'}$; it is because we choose an irreducible component of $\Sigma^{M'}$ that we can apply the result of Proposition 7.1.

It appears as a subrepresentation in $I_{P_1\cap M'}^{M'}(\sigma(\underline{n}))$.

Then, since the parameter $\bigoplus_{i=1}^t(a_i,\ldots,b_i)$ corresponds to a central character χ for M', we have:

$$I_{P'}^G(\pi'_\chi) \hookrightarrow I_{P'}^G(I_{P_1\cap M'}^{M'}(\sigma(\underline{n}))_{\bigoplus_{i=1}^t(a_i,\ldots,b_i)}) \cong I_{P_1}^G(\sigma(\bigoplus_{i=1}^t(a_i,\ldots,b_i)(\underline{n})))$$

By Proposition 7.1, the subquotient π' appears as a subrepresentation in $I_{P_1\cap M'}^{M'}(\sigma(a_{t+1},\ldots,b_{t+1})(\underline{n'}))$ and therefore in the standard module embedded in $I_{P_1\cap M'}^{M'}(\sigma(a_{t+1},\ldots,b_{t+1})(\underline{n'}))$ by multiplicity one of the irreducible generic piece.

Since the parameter $\bigoplus_{i=1}^{t}(a_i,\ldots,b_i)$ correspond to a central character for M', we have:

$$I_{P'}^G(\pi'_\chi) \hookrightarrow I_{P'}^G(I_{P_1\cap M'}^{M'}(\sigma(a_{t+1},b_{t+1})(\underline{n'}))_{\bigoplus_{i=1}^{t}(a_i,\ldots,b_i)}) \cong I_{P_1}^G(\sigma(\bigoplus_{i=1}^{t}(a_i,\ldots,b_i)(a_{t+1},\ldots,b_{t+1})(\underline{n'})))$$

We have therefore two options:
Either $I_{P'}^G(\pi'_\chi)$ is irreducible and then it is the unique irreducible generic subrepresentation in

$$I_{P'}^G(I_{P_1\cap M'}^{M'}(\sigma(\bigoplus_{i=1}^{t}(a_i,\ldots,b_i)(a_{t+1},\ldots,b_{t+1})(\underline{n'})))$$

$$= I_{P_1}^G(\sigma(\bigoplus_{i=1}^{t}(a_i,\ldots,b_i)(a_{t+1},\ldots,b_{t+1})(\underline{n'})))$$

and by multiplicity one in $I_{P_{\Theta_{t+1}}}^G(\tau'_{\underline{s'}})$.

Either it is reducible, but then its unique irreducible generic subquotient is also the unique irreducible generic subquotient in $I_{P_1}^G(\sigma(\bigoplus_{i=1}^{t}(a_i,\ldots,b_i)(\underline{n})))$.

Then, by induction hypothesis, it embeds as a subrepresentation in $I_{P_1}^G(\sigma(\bigoplus_{i=1}^{t}(a_i,\ldots,b_i)(\underline{n})))$; and by multiplicity one of the generic piece, also in $I_{P'}^G(\pi'_\chi)$.

Hence it embeds in $I_{P_1}^G(\sigma(\bigoplus_{i=1}^{t}(a_i,\ldots,b_i)(a_{t+1},\ldots,b_{t+1})(\underline{n'})))$, and therefore in $I_{P_{\Theta_{t+1}}}^G(\tau'_{s'})$ concluding this induction argument, and the proof.

7.2. **Proof of the Generalized Injectivity Conjecture for Non-Discrete Series Subquotients.** We could have $I_P^G(\tau_{s\tilde{\alpha}})$ reducible without having hypothesis 1 in Lemma 18 satisfied, that is without having $s\tilde{\alpha}$ a pole of the μ function for τ; i.e the converse of the Lemma 18 doesn't necessarily hold.

It is only in this case that a non-tempered or tempered (but not square-integrable) generic subquotient may occur in $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$.

Proposition 7.3. *Let σ_ν be in the cuspidal support of a generic discrete series representation τ of a maximal Levi subgroup M of a quasi-split reductive group G. Let us take $s\tilde{\alpha}$ in $\overline{(a_M^*)^+}$, such that $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$ and denote $\lambda = \nu + s\tilde{\alpha}$ in $\overline{a_{M_1}^{M}}^{+*}$.*

Let us assume that the cuspidal support of τ satisfies the conditions CS *(see the Definition 6.1).*

Let us assume σ_λ is not a residual point for μ^G, and therefore the unique irreducible generic subquotient in $I_P^G(\tau_{s\tilde{\alpha}})$ is essentially tempered or non-tempered.

Then, this unique irreducible generic subquotient embeds as a submodule in $I_{P_1}^G(\sigma_\lambda)$ and therefore in the standard module $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$.

Proof.
First, notice that if $s = 0$ the induced module $I_P^G(\tau_{s\tilde{\alpha}})$ is unitary hence any irreducible subquotient is a subrepresentation, in the rest of the proof we can therefore assume $s\tilde{\alpha}$ in $(a_M^*)^+$.

Let us denote π_0 the irreducible generic tempered or non-tempered representation which appears as subquotient in a standard module $I_P^G(\tau_{s\tilde{\alpha}})$ induced from a maximal parabolic subgroup P of G.

We are in the context of the Subsection 4.3, and therefore we can write $\lambda := (a,\ldots,b) + (\underline{n})$, for some $a > b$, and residual segment $(\underline{n})$. Here, we assume σ_λ is not a residual point. Then $I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_{P_1}^G(\sigma(a,b,\underline{n}))$ has a unique irreducible generic subquotient which is tempered or non-tempered.

Following the proof of the Theorem 6.1 2)a) and b), we can write this unique irreducible generic subquotient $I_{P'}^G(\tau'_{\nu'})$, either it is embeds in an induced module which satisfies the conditions 2)a) or 2)b) of the Theorem 6.1 and then we can conclude by multiplicity one the unique generic subquotient. This is the context of existence of an intertwining operator with non-generic kernel between the induced module with cuspidal strings $(a', b', \underline{n'})$ and $(a, b, \underline{n})$.

Otherwise, one observes that passing from $(a', b', \underline{n'})$ to $(a, b, \underline{n})$ require certain elements γ, with $a \geq \gamma > a'$, to move up, i.e. from right to left. This means using rank one operators which change $(\gamma + n, \gamma)$ to $(\gamma, \gamma + n)$ for integers $n \geq 1$, those rank one operators may clearly have generic kernel.

In this context, we will rather use the results of Proposition 7.1.

Consider again $I_{P'}^G(\tau'_{\nu'})$ embedded in $I_{P_1}^G(\sigma(a', b', \underline{n'}))$. Let us denote π' the unique irreducible generic discrete series subquotient corresponding to the dominant residual point $\sigma((\underline{n'}))$:

Let $M'' = M_{\Delta - \{\underline{\alpha_1}, \ldots, \underline{\alpha_{a-b+1}}\}}$ be a standard Levi subgroup, we have:

$$\pi' \hookrightarrow I_{P_1 \cap M''}^{M''}((\sigma((\underline{n'})))$$

Since the character corresponding to the linear residual segment $(a', \ldots, b')$ is central for M'', we write:

$$\pi'_{(a', \ldots, b')} \hookrightarrow I_{P_1 \cap M''}^{M''}(\sigma((a', \ldots, b') + (\underline{n'}))) \cong I_{P_1 \cap M''}^{M''}(\sigma(\underline{n'}))_{(a', \ldots, b')}$$

Since $\tau'_{\nu'}$ is irreducible (and generic), we also have $\tau'_{\nu'} \hookrightarrow I_{P_1 \cap M'}^{M'}(\sigma((a', \ldots, b') + (\underline{n'})))$ we know:

$$\tau'_{\nu'} \hookrightarrow I_{P''}^{M'}(\pi'_{(a', \ldots, b')}) \hookrightarrow I_{P_1 \cap M'}^{M'}(\sigma((a', \ldots, b') + (\underline{n'}))) \tag{20}$$

By the generalized injectivity conjecture for square-integrable subquotient (Proposition 7.1), any standard module embedded in $I_{P_1 \cap M''}^{M''}(\sigma((\underline{n'})))$ has π' as subrepresentation. We may therefore embed π' as subrepresentation in

$$I_{P_1 \cap M''}^{M''}((w_\flat \sigma)((a^\flat, b^\flat, \underline{n}^\flat)))$$

with $w_\flat \sigma \cong \sigma$, and therefore inducing Equation 20 to G

$$I_{P'}^G(\tau'_{\nu'}) \hookrightarrow I_{P_1}^G((w_\flat \sigma)((a', \ldots, b') + (a^\flat, b^\flat)(\underline{n}^\flat))$$

The sequence $(a^\flat, b^\flat, \underline{n}^\flat)$ is chosen appropriately to have an intertwinning operator with non-generic kernel from $I_{P_1}^G(\sigma((a', \ldots, b') + (a^\flat, b^\flat, \underline{n}^\flat))$ to $I_{P_1}^G(\sigma(a, b, \underline{n}))$.

The unique irreducible generic subrepresentation $I_{P'}^G(\tau'_{\nu'})$ in $I_{P_1}^G(\sigma(a, b, \underline{n}))$ cannot appear in the kernel and therefore appears in the image of this operator. It therefore appears as a subrepresentation in $I_{P_1}^G(\sigma(a, b, \underline{n}))$ and by multiplicity one of the generic piece in $I_{P_1}^G(\sigma(a, b, \underline{n}))$, it also appears as subrepresentation in the standard module $I_P^G(\tau_{s\tilde{\alpha}})$.

Theorem 7.2. *Let σ_ν be in the cuspidal support of a generic discrete series representation τ of a standard Levi subgroup M of a quasi-split reductive group.*

Let us take $\underline{s}$ in $\overline{(a_M^*)^+}$, such that $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_{\nu+\underline{s}})$ and denote $\lambda = \nu + \underline{s}$ in $\overline{a_{M_1}^{M}}^{+*}$. Let us assume that σ_λ is not a residual point for μ^G and that the unique irreducible generic subquotient satisfies the conditions CS *(see the Definition 6.1).*

Then, the unique irreducible generic in $I_P^G(\tau_{\underline{s}})$ (which is essentially tempered or non-tempered) embeds as a subrepresentation in $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$.

Proof.

First, notice that, by the Remark 7.1, when $s_i = s_j$ the segments $\mathcal{S}_i$, and $\mathcal{S}_j$ are *unlinked*.

Using the argument given in Subsection 4.3, we write σ_λ as $\sigma(\bigoplus_{i=1}^t (a_i, \ldots, b_i)(\underline{n}))$, where λ reads $\bigoplus_{i=1}^t (a_i, \ldots, b_i)(\underline{n})$.

The proof goes along the same inductive line than in the proof of Proposition 7.1.

The case of $t = 1$ is Proposition 7.3. That is, given a cuspidal support (P_1, σ_λ), for any standard module induced from a maximal parabolic subgroup P: $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_\lambda)$, the unique irreducible generic subquotient is a subrepresentation. We use an induction argument on the number t of linear residual segments obtained when removing t simple roots to define the Levi subgroup $M \subset P$. Considering that an essentially tempered or non-tempered irreducible generic subquotient in a standard module with t linear residual segments $I_{P_{\Theta_t}}^G(\tau_{\underline{s}})$ is necessarily a subrepresentation; one uses the same arguments than in the proof of Theorem 7.1 to conclude that a tempered or non-tempered irreducible generic subquotient in a standard module with $t+1$ linear residual segments $I_{P_{\Theta_{t+1}}}^G(\tau'_{\underline{s}'})$ is a subrepresentation, therefore proving the theorem.

Eventually, we now consider the generic subquotients of $I_P^G(\gamma_{s\tilde{\alpha}})$ when γ is a generic irreducible tempered representation.

Corollary 7.2.1 (Standard modules)**.** *Let G be a quasi-split reductive group of type A, B, C or D and let us assume Σ_σ is irreducible.*

The unique irreducible generic subquotient of $I_P^G(\gamma_{\underline{s}})$ when γ is a generic irreducible tempered representation of a standard Levi M is a subrepresentation.

Proof. Let $P = MU$.

By Theorem 6.2, as a tempered representation of M, γ appears as a subrepresentation of $I_{P_3\cap M}^M(\tau)$ for some discrete series τ and standard parabolic $P_3 = M_3U$ of G; τ is generic irreducible representation of the Levi subgroup M_3, therefore

$$I_P^G(\gamma_{\underline{s}}) \hookrightarrow I_P^G(I_{M\cap P_3}^M(\tau))_{\underline{s}} \cong I_{P_3}^G(\tau_{\underline{s}})$$

where P_3 is not necessarily a maximal parabolic subgroup of G. Since $\underline{s}$ is in $(a_M^*)^+$, $\underline{s}$ is in $\overline{(a_{M_3}^*)^+}$. Let us write this parameter $\bar{\underline{s}}$ when it is in $\overline{(a_{M_3}^*)^+}$.

The unique irreducible generic subquotients of $I_P^G(\gamma_{\underline{s}})$ are the unique irreducible generic subquotients of $I_{P_3}^G(\tau_{\bar{\underline{s}}})$, where $\bar{\underline{s}}$ is in $\overline{(a_{M_3}^*)^+}$. Since P_3 is not a maximal parabolic subgroup of G, we may now use Theorems 7.1 and 7.2 with $\bar{\underline{s}}$ in $\overline{(a_{M_3}^*)^+}$ to conclude that these unique irreducible generic subquotients, whether square-integrable or not, are subrepresentations.

8. Generalized Injectivity conjecture for Σ_σ of type A

Theorem 8.1. *Let G be a quasi-split reductive group of type A, B, C or D. Let P be a standard parabolic subgroup $P = MU$ of G.*

Let us consider $I_P^G(\tau_{\underline{s}})$ with τ an irreducible discrete series of M, $\underline{s} \in (a_M^*)^+$. Let σ be a unitary cuspidal representation of M_1 in the cuspidal support of τ and assume Σ_σ (defined with respect to G) is of type A and irreducible of rank $d = rk_{ss}(G) - rk_{ss}(M_1)$.

A typical example is when G is of type A.

Then, the unique irreducible generic subquotient of $I_P^G(\tau_{\underline{s}})$ is a subrepresentation.

Proof.

As proven in Appendix C, when G is of type A, if Σ_σ is irreducible of rank d, it is necessarily of type A.

Let τ be a discrete series of M a standard Levi subgroup of G. By the result of Heiermann-Opdam [Proposition 2.1], there exists a standard parabolic subgroup P_1 such that $\tau \hookrightarrow I_{P_1}^M(\sigma_\nu)$ with ν is in the closed positive Weyl chamber relative to M, $\overline{(a_{M_1}^{M*})^+}$. We consider the unique irreducible generic subquotient in the standard module $I_P^G(\tau_{\underline{s}}) \hookrightarrow I_{P_1}^G(\sigma_{\nu+\underline{s}})$. We denote $\lambda = \nu + \underline{s}$.

Let us first consider the case of M being a maximal Levi subgroup in G. M is obtained by removing a (non-extremal) root from Δ, and therefore obtain two subsystems of roots of type A_{i-1} and A_{d-i} in Σ_σ^M.

The character ν is constituted of two residual segments of type A_{i-1} and A_{d-i} : (ν_1, ν_2) and when we twist by $s\tilde{\alpha}$ we obtain the Langlands parameter $\lambda = (\nu_1 + s_1, \nu_2 + s_2)$ where $s_1 = \frac{a_1+b_1}{2} > s_2 = \frac{a_2+b_2}{2}$ and write $\lambda = (a_1, b_1; a_2, b_2)$ for two residual segments (a_j, b_j) of type A.

Assume σ_λ is a residual point, that is the sequence $(a_1, b_1; a_2, b_2)$ shall be a strictly decreasing sequence of real numbers (corresponding to a segment of type A_d).

This means $b_1 = a_2 + 1$, and therefore σ_λ is already in a dominant position with respect to P_1. So λ is a dominant residual point and therefore by Lemma 11 the unique irreducible generic discrete series subquotient embeds as a subrepresentation in $I_{P_1}^G(\sigma_\lambda)$ and consequently (by unicity of the generic irreducible piece) in the standard module $I_P^G(\tau_{\underline{s}})$.

Else, σ_λ is not a residual point.

In this case, either the two segments $(a_1, \ldots, b_1); (a_2, \ldots, b_2)$ are unlinked and therefore, by Lemma 10, the standard module $I_P^G(\tau_{\underline{s}})$ is irreducible; or the two segments are linked and the sequence $(a_1, b_1; a_2, b_2)$ can be reorganized in only one way to obtain two residual segments of type $A_{i'}$ and $A_{n-i'}$ which is to take intersection and union of $(a_1, \ldots, b_1)$ and $(a_2, \ldots, b_2)$. We obtain two unlinked segments $(a_2, \ldots, b_1) \subset (a_1, \ldots, b_2)$ [see the Definition 4.3] and by Lemma 5.1 the Langlands parameter (this notion was introduced in the Theorem 2.3) $\underline{s'} := (s'_1, s'_2)$ is smaller than (s_1, s_2).

By Proposition 5.1, the parameter $\underline{s'}$ is the minimal element for the order defined in Lemma 5. Let M' be the maximal Levi subgroup which corresponds to removing the root from the A_d Dynkin diagram to obtain $A_{i'-1}$ and $A_{d-i'}$. Let τ' be the discrete series subrepresentation of $I_{P_1 \cap M'}^{M'}(\sigma((\frac{a_2-b_1}{2}, \ldots, -\frac{a_2-b_1}{2})(\frac{a_1-b_2}{2}, \ldots, -\frac{a_1-b_2}{2})))$.

Since $\underline{s'}$ is the minimal element for the order defined in Lemma 5, by Theorem 5.2, the module $I_{P'}^G(\tau'_{\underline{s'}})$ is the unique irreducible generic subquotient of $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$.

Further, notice that in Lemma 10, $\underline{s'}$ can be ordered as one wishes, or said differently we order the residual segments $(a_2, \ldots, b_1) \subset (a_1, \ldots, b_2)$ as one wishes. Then we can embed as a subrepresentation $I_{P'}^G(\tau'_{\underline{s'}})$ in $I_{P_1}^G(\sigma(a_2, b_1, a_1, b_2))$.

We now consider the intertwining operator from $I_{P_1}^G(\sigma(a_2, b_1, a_1, b_2))$ to $I_{P_1}^G(\sigma((a_1, b_1, a_2, b_2))$. Since $a_1 \geq a_2$, we can use Lemma 8 to conclude that it has non-generic kernel. Therefore $I_{P'}^G(\tau'_{\underline{s'}})$ embeds as a subrepresentations in $I_{P_1}^G(\sigma(a_1, b_1, a_2, b_2))$ and therefore in $I_P^G(\tau_{\underline{s}})$ by unicity of the generic piece in the induced representation $I_{P_1}^G(\sigma((a_1, b_1, a_2, b_2))$.

Secondly, consider the case of a non-necessarily maximal standard Levi subgroup, then we have t subsystems of type A.

If $\nu + \underline{s} = \lambda := (\bigotimes(a_i, \ldots, b_i)_{i=1}^t)$ is a residual point, it shall be a decreasing sequence of real numbers, therefore in dominant position, and we can immediately conclude by Lemma 11 and the unique irreducible generic discrete series subquotient embeds as a subrepresentation in $I_{P_1}^G(\sigma_\lambda)$ and therefore in $I_P^G(\tau_{\underline{s}})$ by unicity of the generic piece.

Else, λ is not a residual point, and therefore the unique irreducible generic subquotient reads $I_{P'}^G(\tau'_{\underline{s'}})$ where $\underline{s'}$ is the smallest Langlands parameter with respect to the order defined in Lemma 5 on Langlands parameters. If all the linear residual segments $\{(a_i, \ldots, b_i)\}_{i=1}^t$ are unlinked, by Lemma 10, the standard module $I_P^G(\tau_{\underline{s}})$ is irreducible.

Otherwise, let us assume that for some indices i, j in $\{1, \ldots, t\}$, the two linear segments $(a_i, \ldots, b_i)$ and $(a_j, \ldots, b_j)$ are linked.

By Proposition 5.1, $\underline{s'} = (s'_1, s'_2, \dots, s'_t) < \underline{s}$ if it is obtained by taking repeatly intersection and union of all two linked linear segments (at each step taking intersection and union of two segments and leaving the other segments unchanged gives a smaller Langlands parameter by Proposition 5.1).

Let us denote $w\lambda = (\bigotimes(a'_i, \dots, b'_i)_{i=1}^t)$ (for some Weyl group element w in W_σ) the parameter obtained by taking repeatedly intersection and union of all two linked linear segments.

Further from Lemma 10, $\underline{s'}$ can be ordered as one wishes, or said differently how we order the t unlinked residual segments (a'_i, b'_i)'s in the parameter $w\lambda = ((a'_i, \dots, b'_i))_{i=1}^t$ does not matter. Now, the irreducible generic discrete series τ' is (by the result of Heiermann-Opdam, Proposition 2.1) a subrepresentation in $I_{P_1 \cap M'}^{M'}(\sigma(\oplus_{i=1}^t(\frac{a'_i - b'_i}{2}, -\frac{a'_i - b'_i}{2})))$.

Then, $I_{P'}^G(\tau'_{\underline{s'}})$ embeds as a subrepresentation in $I_{P_1}^G((w\sigma)_{w\lambda}) = I_{P_1}^G(\sigma_{w\lambda})$. The last equality because $w\sigma \cong \sigma$.

Further, there will exist a certain order on the unlinked residual segments $(a'_i, \dots, b'_i)$ allowing the existence of an intertwining operator with non-generic kernel from $I_{P_1}^G(\sigma_{w\lambda})$ to $I_{P_1}^G(\sigma_\lambda)$ using repeatedly Lemma 8. Therefore the generic module $I_{P'}^G(\tau'_{\underline{s'}})$ appears as a subrepresentation in $I_{P_1}^G(\sigma_\lambda)$ and therefore in $I_P^G(\tau_{\underline{s}})$.

9. The case Σ_σ reducible

Let us recall that the set Σ_σ is a root system in a subspace of $a_{M_1}^*$ (cf. [37] 3.5) and we assume that the irreducible components of Σ_σ are all of type A, B, C or D. In Proposition 4.3, we have denoted for each irreducible component $\Sigma_{\sigma,i}$ of Σ_σ, by $a_{M_1}^{M_i*}$ the subspace of $a_{M_1}^{G*}$ generated by $\Sigma_{\sigma,i}$, by d_i its dimension and by $e_{i,1}, \dots, e_{i,d_i}$ a basis of $a_{M_1}^{M_i*}$ (resp. of a vector space of dimension $d_i + 1$ containing $a_{M_1}^{M_i*}$ if $\Sigma_{\sigma,i}$ is of type A) so that the elements of the root system $\Sigma_{\sigma,i}$ are written in this basis as in Bourbaki, [8].

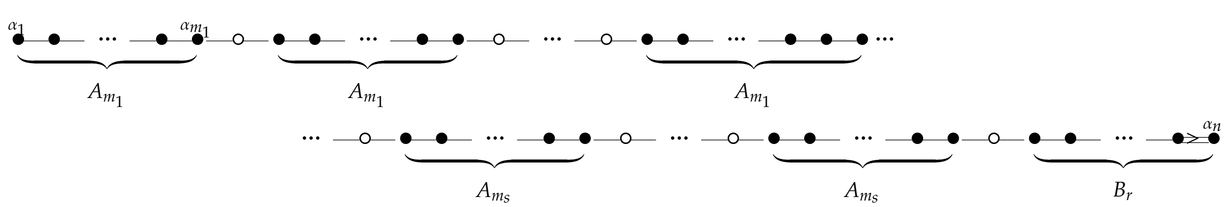
The following result is analogous to Proposition 1.10 in [20]. Recall $\mathcal{O}$ denotes the set of equivalence classes of representations of the form $\sigma \otimes \chi$ where χ is an unramified character of M_1.

Proposition 9.1. *Let $P'_1 = M_1U'_1$, and $P_1 = M_1U_1$. If the intersection of $\Sigma(P_1) \cap \Sigma(\overline{P'_1})$ with Σ_σ is empty, the operator $J_{P'_1|P_1}$ is well defined and bijective on $\mathcal{O}$.*

Proof. The operator $J_{P'_1|P_1}$ is decomposed in elementary operators which come from intertwining operators relative to $(M_1)_\alpha$ with $\alpha \notin \Sigma_\sigma$, so it is enough to consider the case where P_1 is a maximal parabolic subgroup of G and $P'_1 = \overline{P_1}$. Then, if $\alpha \notin \Sigma_\sigma$ and by the same reasoning than in the previous Lemma 13, the operator $J_{P'_1|P_1}$ is well defined and bijective at any point on $\mathcal{O}$.

Let G be a quasi-split reductive group over F, π_0 is an irreducible generic representation whose cuspidal support contains the representation σ_λ of a standard Levi subgroup M_1, $\lambda \in a_{M_1}^*$ and σ an irreducible unitary cuspidal generic representation.

In this subsection, we consider the case of a *reducible* root system Σ_σ. As explained in Appendix C, this case occurs in particular when Σ_Θ (see the notations in Appendix C) is reducible, and then Θ has connected components of type A of different lengths. An example is the following Dynkin diagram for Θ:



Let us assume Θ is a disjoint union of components of type A_{m_i} $i = 1 \dots s$; and $m_i \neq m_{i+1}$ for any i, where each component of type A_{m_i} appears d_i times. Set $m_i = k_i - 1$.

Let us denote $\Delta^i_{M_1} = \left\{\alpha_{i,1}, \dots, \alpha_{i,d_i}\right\}$ the non-trivial restrictions of roots in Σ, generating the set $a_{M_1}^{M^i}*$. Similar to the case of Σ_σ irreducible, we may have $\Delta_{\sigma_i} = \left\{\alpha_{i,1}, \dots, \beta_{i,d_i}\right\}$ where β_{i,d_i} can be different from α_{i,d_i} in the case of type B, C or D. For any $i \neq s$, the pre-image of the root α_{i,d_i} is *not* simple.

Indeed, for instance, in the above Dynkin diagram, the first root 'removed' is $e_{k_1} - e_{k_1+1}$, the second is $e_{2k_1} - e_{2k_1+1}$;...etc; they are simple roots and their restrictions to A_{M_1} are roots of $\Delta^1_{M_1}$ (the generating set of $a_{M_1}^{M^1}*$) ; the last root to consider is $e_{k_1d_1} - e_{n-r+1}$ which restricts to $e_{k_1d_1}$; then the preimage of $e_{k_1d_1}$ is not simple.

However, since $e_{n-r} - e_{n-r+1}$ restricts to e_{n-r}; the pre-image of α_{s,d_s} is simple.

The Levi subgroup M^i is defined such that $\Delta^{M^i} = \Delta^{M_1} \cup \left\{\underline{\alpha_{i,1}}, \dots, \underline{\alpha_{i,d_i}}\right\}$ where $\Delta^i_{M_1} = \left\{\alpha_{i,1}, \dots, \alpha_{i,d_i}\right\}$.

It is a *standard* Levi subgroup for $i = s$. This is quite an important remark since most of our results in the previous sections were conditional on having *standard* parabolic subgroups.

Furthermore, since $\Sigma_{\sigma,i}$ generates $a_{M_1}^{M^i}*$ and is of rank d_i, the semi-simple rank of M^i is $d_i + rk_{ss}(M_1)$. Since $\Sigma_{\sigma,i}$ is irreducible, an equivalent of Proposition 6.1 is satisfied for M^i.

Proposition 9.2. *Let π_0 be an irreducible generic representation of a quasi-split reductive group G, and assume it is the unique irreducible generic subquotient in the standard module $I_P^G(\tau_{s\tilde{\alpha}})$, where M is a maximal Levi subgroup (and α is not an extremal simple root on the Dynkin diagram of Σ) of G and τ is an irreducible generic discrete series of M. Let us assume Σ_σ is reducible.*

Then π_0 is a subrepresentation in the standard module $I_P^G(\tau_{s\tilde{\alpha}})$.

Proof.

Let us repeat the initial context:

The representation τ is an irreducible generic discrete series of a maximal Levi subgroup $M = M_\Theta$ such that $I_P^G(\tau_{s\tilde{\alpha}})$ is a standard module. By Heiermann-Opdam's result, $\tau \hookrightarrow I^M_{P_1\cap M}(\sigma_\nu)$, for $\nu \in (a_{M_1}^{M\,*})^+$. Then, ν is a residual point for μ^M.

Let us write $\Sigma_\sigma^M = \bigcup_{i=1}^{r+1} \Sigma^M_{\sigma,i}$, then the residual point condition is dim $((a^M_{M_1})^*) = rk(\Sigma^M_\sigma) = \sum_{i=1}^{r+1} d_i^M$, where d_i^M is the dimension of $(a_{M_1}^{M^i})^*$ generated by $\Sigma^M_{\sigma,i}$. The residual point ν decomposes in $r+1$ disjoint residual segments: $\nu = (\nu_1, \dots, \nu_{r+1}) := (\underline{n_1}, \underline{n_2}, \dots, \underline{n_{r+1}})$.

Since Σ^M decomposes into two disjoint irreducible components, one of them being of type A, the restrictions of simple roots of this irreducible component of type A in Δ^M generates an irreducible component of Σ_σ of type A, let us denote this A component $\Sigma^M_{\sigma,I}$ for $I \in \{1, \dots, r+1\}$, $d_I = б - \gamma$, and denote $\nu_I + s\tilde{\alpha} := (б, \dots, \gamma)$ the twisted residual segment of type A.

Let us further assume that there is one index j such that there exists a residual segment $(\underline{n'_j})$ of length $б - \gamma + 1 + d_j$ and type $\mathcal{T}$ (B, C or D) in the W_σ-orbit of $(б, \gamma)(\underline{n_j})$ where the residual segment $(\underline{n_j})$ is of the same type as $\mathcal{T}$.

Since all intertwining operators corresponding to rank one operators associated to s_β for $\beta \notin \Delta_\sigma$ are bijective (see Lemma 13), all intertwining operators interchanging any two residual segments $(\underline{n_k})$ and $(\underline{n_{k'}})$ are bijective. Therefore, we can interchange the positions of all residual segments (or said differently interchange the order of the irreducible components for $i = 1, \dots, r+1$) and therefore set $(б, \dots, \gamma)(\underline{n_j})$ in the last position, i.e we set $I = r, j = r+1$. This flexibility is quite powerful since it allows us to circumvent the difficulty arising with M^i not being standard for any $i \neq r$.

When adding the root α to Θ (when inducing from M to G), we form from the disjoint union $\Sigma^M_{\sigma,r} \bigcup \Sigma^M_{\sigma,r+1}$ the irreducible root system that we denote $\Sigma_{\sigma,r}$.

The Levi subgroup M^r is the smallest standard Levi subgroup of G containing M_1, the simple root α and the set of simple roots whose restrictions to A_{M_1} lie in $\Delta^r_{M_1}$. It is a group of semi-simple rank $d_r + rk_{ss}(M_1)$.

We may therefore apply the results of the previous subsections with Σ_σ irreducible to this context: Let us assume first the unique irreducible generic subquotient π is discrete series.

From the result of Heiermann-Opdam, we have:

$$\pi \hookrightarrow I^{M^r}_{P_1 \cap M^r}(\sigma(\underline{n'_r}))$$

where the residual segment $(\underline{n'_r})$ is the dominant residual segment in the W_σ-orbit of $(\mathit{б}, \gamma, \underline{n_r})$.

The unramified character χ corresponding to the remaining residual segments (n_k)'s, $k \neq r, r+1$ is a central character of M^r (since it's expression in the $a^*_{M_1}$ is orthogonal to all the roots in Δ^{M^r}). Then:

$$\pi_\chi \hookrightarrow I^{M^r}_{P_1 \cap M^r}(\sigma(\underline{n'_r}))_{\bigoplus_{j \neq r, r+1}(\underline{n_j})}$$

As a result:

$$\pi_0 \hookrightarrow I^G_{P^r}(\pi_\chi) \hookrightarrow I^G_{P_1}(\sigma(\bigoplus_{j \neq r}(\underline{n_j}) + (\underline{n'_r}))) \tag{21}$$

In Equation (21), we claim π_0 embeds first in $I^G_{P_1}(\sigma(\bigoplus_{j \neq r} \underline{n_j}) + (\underline{n'_r})))$ by the Heiermann-Opdam embedding result (since the residual segment $\bigoplus_{j \neq r}(\underline{n_j}) + (\underline{n'_r})$ corresponds to a parameter in $\overline{(a^*_{M_1})^+}$), therefore it should embed in $I^G_{P^r}(\pi_\chi)$ by multiplicity one of the irreducible generic piece.

Applying our conclusion in the case of irreducible root system (in Proposition 7.1) to $\Sigma_{\sigma,r}$, we embed π in the induced module $I^{M^r}_{P_1 \cap M^r}(\sigma(\mathit{б}, \gamma, \underline{n_r}))$ as a subrepresentation (and therefore in a standard module $I^{M^r}_{P \cap M^r}(\tau_{\frac{\mathit{б}+\gamma}{2}})$ embedded in $I^{M^r}_{P_1 \cap M^r}(\sigma(\mathit{б}, \gamma, \underline{n_r}))$).

$$\pi_\chi \hookrightarrow I^{M^r}_{P_1 \cap M^r}(\sigma(\mathit{б}, \gamma, \underline{n_r}))_{\bigoplus_{j \neq r, r+1}(\underline{n_j})} \cong I^{M^r}_{P_1 \cap M^r}(\sigma(\mathit{б}, \gamma, \underline{n_r}) + \bigoplus_{j \neq r, r+1}(\underline{n_j}))$$

Therefore:

$$\pi_0 \hookrightarrow I^G_{P^r}(\pi_\chi) \hookrightarrow I^G_{P_1}(\sigma(\bigoplus_{j \neq r}(\underline{n_j}) + (\mathit{б}, \gamma, \underline{n_r}))$$

In case π is non-(essentially) square integrable, i.e tempered or non-tempered, and embeds in $I^{M^r}_{P_1 \cap M^r}((\sigma(\mathit{б}', \gamma', \underline{n'_r}))$ (see the construction in the Section 6.4, 2)a)), we had shown in Proposition 7.3 there existed an intertwining operator with non-generic kernel sending π in $I^{M^r}_{P_1 \cap M^r}(\sigma(\mathit{б}, \gamma, \underline{n_r}))$.

Since the other remaining residual segments $(\underline{n'_k})$'s, $k \neq r, r+1$ do not contribute when minimizing the Langlands parameter ν', the unique irreducible generic subquotient in

$$I^G_{P_1}(\sigma(\bigoplus_{k \neq r}(\underline{n_k}) + (\mathit{б}, \gamma, \underline{n_r})))$$

embeds in

$$I^G_{P_1}(\sigma(\bigoplus_{k \neq r}(\underline{n_k}) + (\mathit{б}', \gamma', \underline{n'_r})))$$

and we can use the inducting of the previously defined (at the level of M^r) intertwining operator to send this generic subquotient as a subrepresentation in $I^G_{P_1}(\sigma(\bigoplus_{k \neq r}(\underline{n_k}) + (\mathit{б}, \gamma, \underline{n_r})))$. We conclude the argument as usual: by multiplicity one, the generic piece also embeds as a subrepresentation in the standard module.

Proposition 9.3. *Let π_0 be an irreducible generic representation and assume it is the unique irreducible generic subquotient in the standard module $I_P^G(\tau_s)$, where the set of simple roots in M (Δ^M) is the set of simple roots Δ minus t simple roots, $s = (s_1, \ldots, s_t)$ such that $s_1 \geq s_2 \geq \ldots \geq s_t$ and τ is an irreducible generic discrete series.*

Then it is a subrepresentation.

Proof.

The representation τ is an irreducible generic discrete series of a non-maximal Levi subgroup M such that $I_P^G(\tau_s)$ is a standard module. By Heiermann-Opdam's result, $\tau \hookrightarrow I_{P_1 \cap M}^M(\sigma_\nu)$, for $\nu \in \overline{(a_{M_1}^{M\,*})^+}$. Then, ν is a residual point for μ^M.

Let us denote $M = M_\Theta$. Then $\Theta = \bigcup_{i=1}^{t+1} \Theta_i$ where Θ_i, for $i \in \{1, \ldots, t\}$ is of type A.

Since M_1 is a standard Levi subgroup of G contained in M, we can write $\Sigma_\sigma^M = \bigcup_{i=1}^{t+r} \Sigma_{\sigma,i}^M$, then the residual point condition is dim $((a_{M_1}^M)^*) = rk(\Sigma_\sigma^M) = \sum_{i=1}^{r+t} d_i^M$, where d_i^M is the dimension of $(a_{M_1}^{M^i})^*$ generated by $\Sigma_{\sigma,i}^M$. The residual point ν decomposes in t linear residual segments along with r residual segments: $\nu = (\nu_1, \ldots, \nu_{r+t}) := (\underline{n_1}, \underline{n_2}, \ldots, \underline{n_{r+t}})$.

Adding the twist $s = (s_1, \ldots, s_t)$, we obtain a parameter λ in $(a_{M_1}^G)^*$ composed of t twisted linear residual segments $\{(a_i, \ldots, b_i)\}_{i=1}^t$ and r residual segments $(\underline{n_1}, \underline{n_2}, \ldots, \underline{n_r})$.

Let us first assume that λ is a residual point.

This means all linear residual segments can be incorporated in the r residual segments of type $\mathcal{T}$ to form residual segments $\left\{(\underline{n'_j})\right\}_{j=1}^r$ of type $\mathcal{T}$ and length d_i such that $\sum_i d_i = d$ where d is $rk_{ss}(G) - rk_{ss}(M_1) = \dim a_{M_1} - \dim a_G$. It is also possible that, as twisted linear residual segments they are already in a form as in Proposition 7.2. In that case, the linear residual segment need not be incorporated in any residual segment of type $\mathcal{T}$.

Furthermore, as in the proof of Theorem 7.1, we can reduce our study to the case of unlinked residual linear segments.

By Heiermann-Opdam's Proposition (2.1):

$$\pi_0 \hookrightarrow I_{P_1}^G(\sigma(\bigoplus_j \underline{n'_j}))$$

Let us consider the last irreducible component $\Sigma_{\sigma,r}$ of Σ_σ and the residual segment $(\underline{n'_r})$ associated to it.

Let us assume this irreducible subsystem is obtained from some subsystems $\Sigma_{\sigma,i}^M$ of type A denoted $A_q, \ldots, A_s$ and one of type $\mathcal{T}$ when inducing from M to G

$$\left\{A_q, \ldots, A_s\right\} \leftrightarrow \{\mathcal{T}\} \tag{22}$$

$$\left\{(b_{r,q}, \ldots, \gamma_{r,q}), \ldots, (b_{r,s}, \ldots, \gamma_{r,s})\right\} \leftrightarrow \left\{(\underline{n_r})\right\} \tag{23}$$

The Levi subgroup M^r is the smallest standard Levi subgroup of G containing M_1, s simple roots (among the t simple roots in $\Delta - \Theta$) and the set of roots whose restrictions to A_{M_1} lie in $\Delta_{M_1}^r$. It is a group of semi-simple rank $d_r + rk_{ss}(M_1)$.

We may therefore apply the results of the previous subsections with Σ_σ irreducible to this context: the unique irreducible generic discrete series, π, in the induced module $I_{P_1 \cap M^r}^{M^r}(\sigma(\bigoplus_{j=q}^s (b_{r,j}, \gamma_{r,j}) + (\underline{n_r}))$ is a subrepresentation.

As in the proof of the previous Proposition 9.2, since π also embeds in $I_{P_1\cap M^r}^{M^r}(\sigma(\underline{n_r'}))$, when we add the twist by the central character corresponding to $\bigoplus_{k\neq r}(n_k')$, we obtain:

$$\pi_0 \hookrightarrow I_P^G(\pi_\chi) \hookrightarrow I_{P^r}^G(I_{P_1\cap M^r}^{M^r}(\sigma(\bigoplus_{j=k}^{s}(\mathcal{b}_{r,j},\ldots,\gamma_{r,j}) + (\underline{n_r}))_{\bigoplus_{k\neq r,r+1}(\underline{n_k'})}))$$

In case π is non-tempered, and embeds (as a subrepresentation) in $I_{P_1\cap M^r}^{M^r}((\sigma(\mathcal{b}',\gamma',\underline{n_r''}))$, we had shown in Proposition 7.3 there existed an intertwining operator with non-generic kernel sending π in $I_{P_1\cap M^r}(\sigma(\mathcal{b},\gamma,\underline{n_r}))$.

Since the other remaining residual segments (n_k')'s, $k\neq r$ do not contribute when minimizing the Langlands parameter ν', the unique irreducible generic subquotient in

$$I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n_k'}) + (\mathcal{b},\gamma,\underline{n_r})))$$

embeds in

$$I_{P^r}^G(\sigma(\bigoplus_{k\neq r}(\underline{n_k'}) + (\mathcal{b}',\gamma',\underline{n_r'})))$$

and we can use the inducting of the previously defined intertwining operator to send this generic subquotient as a subrepresentation in $I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n_k'}) + (\mathcal{b},\gamma,\underline{n_r})))$.

Then

$$\pi_0 \hookrightarrow I_{P^r}^G(\pi_\chi) \hookrightarrow I_{P_1}^G(\sigma(\bigoplus_{k\neq r}(\underline{n_k'}) + \bigoplus_{j=q}^{s}(\mathcal{b}_{r,j},\gamma_{r,j}) + (\underline{n_r})))$$

We conclude the argument as usual: by multiplicity one, the generic piece also embeds as a subrepresentation in the standard module.

Using bijective intertwining operators, we now reorganize this cuspidal support so as to put the linear residual segments $\bigoplus_{j=q}^{s}(\mathcal{b}_{r,j},\gamma_{r,j})$ on the left-most part and $\Sigma_{\sigma,r-1}$ in the right-most part. The residual segment $(\underline{n_{r-1}'})$ is (possibly) again formed of some linear residual segments $(\mathcal{b}_i,\gamma_i)$ and the residual segment $(\underline{n_{r-1}})$. We argue just as above. Since the linear residual segments are unlinked, we can reorganize them so as to insure $s_1\geq s_2\geq \ldots s_t$.

Eventually repeating this procedure,

$$\pi_0 \hookrightarrow I_{P_1}^G(\sigma(\bigoplus_{i=1}^{t}(\mathcal{b}_i,\gamma_i) + \bigoplus_{j=1}^{r}(\underline{n_j})))$$

Further, by multiplicity one, the generic piece also embeds as a subrepresentation in the standard module.

Corollary 9.0.1. *Let π_0 be an irreducible generic representation of G and assume it is the unique irreducible generic subquotient in the standard module $I_P^G(\gamma_{\underline{s}})$, where M is a standard Levi subgroup of G. Let us assume Σ_σ is reducible.*

Then it is a subrepresentation.

Proof. Let $P = MU$. We argue as in the Corollary 7.2.1: using the Theorem 6.2, the tempered representation of M, γ, appears as a subrepresentation of $I_{P_3\cap M}^M(\tau)$ for some discrete series τ and standard parabolic subgroup $P_3 = M_3U$ of G; τ is a generic irreducible representation of the standard Levi subgroup M_3, therefore

$$I_P^G(\gamma_{\underline{s}}) \hookrightarrow I_P^G(I_{M\cap P_3}^M(\tau))_{\underline{s}} \cong I_{P_3}^G(\tau_{\underline{s}})$$

where P_3 is not necessarily a maximal parabolic subgroup of G.

Since $\underline{s}$ is in $(a_M^*)^+$, $\underline{s}$ is in $\overline{(a_{M_3}^*)^+}$. Let us write this parameter $\bar{\underline{s}}$ when it is in $\overline{(a_{M_3}^*)^+}$.

The unique irreducible generic subquotients of $I_P^G(\gamma_{\underline{s}})$ are the unique irreducible generic subquotients of $I_{P_3}^G(\tau_{\underline{\bar{s}}})$, where $\underline{\bar{s}}$ is in $\overline{(a_{M_3}^*)^+}$. Since P_3 is not a maximal parabolic subgroup of G, we use the result of the previous Proposition 9.3.

10. Exceptional groups

The arguments developed in the context of reductive groups whose roots systems are of classical type may apply in the context of exceptional groups provided the set $W(M_1)$ is equal to the Weyl group W_σ or differ by one element as in the Corollary 6.1.1. However, this hypothesis shall not be necessarily satisfied, as the E_8 Example 5.3.3 in [37] illustrates: in this example, where W_σ, the Weyl group of Σ_σ, is of type D_8, it shall be rather different from $W(A_0)$.

In an auxiliary work [15], we have observed that in most cases where a root system of rank $d = \dim(a_{M_1}^*/a_G^*)$ occurs in Σ_Θ, it is of type A or D; or of very small rank (such as in F_4).

Further, the main result of [15] (Theorem 2) is that only classical root systems occur in Σ_Θ; except when G is of type E_8 and $\Theta = \{\alpha_8\}$.

This latter case along with the case of $\Theta = \emptyset$ (in the context of exceptional groups), $\Sigma_\Theta = \Sigma$, $M_1 = P_0 = B$ and σ a generic irreducible representation of P_0 (in particular the case of trivial representation σ) shall be treated in an independent work since the combinatorial arguments given in this work shall not apply as easily.

Furthermore, it might be necessary for the case E_8 and $\Theta = \{\alpha_8\}$ to obtain a result analogous to the Proposition 4.3 *which includes the exceptional root systems*; it would allow to use the weighted Dynkin diagrams (of exceptional type) to express the coordinates of residual points.

(1) Let us assume Σ_Θ contains Σ_σ of type A and the basis of Δ_Θ contains at least two projections of simple roots in Δ: α and β. Let us assume that the standard module is $I_P^G(\tau_{s\tilde{\alpha}})$ such that τ is a discrete series of M and $\Delta_M = \Delta - \{\alpha\}$. The proof of Theorem 8.1 carries over this context if the Levi M' given there is such that $\Delta_{M'} = \Delta - \{\beta\}$ and one should pay attention to the choice of (order of simple roots in the) basis Δ_Θ to insure that the parameter ν' for the root system $\Sigma_\sigma^{M'}$ splits into two residual segments appropriately (hence also an appropriate choice of M determining Σ_σ^M). Let us simply recall that from the Lemmas 10 and 5.1, we know that if there is an embedding of the irreducible generic subquotient $I_{P'}^G(\tau'_{\underline{s}'})$ into $I_{P'_1}^G(\sigma'_{\lambda'})$, the parameter λ' is in the W_σ-orbit of λ, hence $M'_1 = w.M_1 = M_1$ and $\sigma' = w.\sigma = \sigma$ since $w \in W_\sigma$.
(2) Under the assumption that $W(M_1)$ equals W_σ or $W(M_1) = W_\sigma \cup \left\{s_{\beta_d} W_\sigma\right\}$ (see Corollary 6.1.1) and $s\beta_d\lambda = \lambda$, the cases where Σ_σ is irreducible of type D_d in Σ_Θ can be dealt with the methods proposed in this work.

It follows:

Proposition 10.1. *Let G be a quasi-split reductive group of exceptional type, Σ its root system, and Δ a basis of Σ. Let P be a standard parabolic subgroup $P = MU$ of G.*

Let us consider $I_P^G(\tau_{\underline{s}})$ with τ an irreducible discrete series of M, $\underline{s} \in (a_M^*)^+$. Let σ be a unitary cuspidal representation of M_1 in the cuspidal support of τ and assume Σ_σ (defined with respect to G) is of type A and irreducible of rank $d = rk_{ss}(G) - rk_{ss}(M_1)$. Further assume that Δ_σ contains at least two restrictions of simple roots in Δ.

Then, the unique irreducible generic subquotient of $I_P^G(\tau_{\underline{s}})$ is a subrepresentation.

10.1. Generalized Injectivity in G_2.

Theorem 10.1. *Let G be of type G_2.*
Let π_0 be the unique irreducible generic subquotient of a standard module $I_P^G(\tau_s)$, then it is a subrepresentation.

We follow the parametrization of the root system of G_2 as in Muić [28]: α is the short root and β the long root. We have $M_\alpha \cong GL_2$, $M_\beta \cong GL_2$. Without loss of generality, let us assume τ is a discrete series representation of $M = M_\alpha$, the reasoning is the same for M_β. As τ is a discrete series for GL_2, $\tau = St_2$.

$$\tau \hookrightarrow I_B^{M_\alpha}(|.|^{1/2}|.|^{-1/2})$$

We twist τ with $s\tilde{\alpha}$

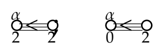
$$\tau_{s\tilde{\alpha}} \hookrightarrow I_B^{M_\alpha}(|.|^{1/2}|.|^{-1/2}) \otimes |.|^s$$
$$I_P^G(\tau_{s\tilde{\alpha}}) \hookrightarrow I_B^G(|.|^{s+1/2}|.|^{s-1/2})$$

Conjecturally for two values of s (since there are only two weighted Dynkin diagrams conjecturally in bijection with dominant residual points) we obtain a dominant residual point of type G_2. Since they are dominant residual points, the unique generic subquotient in $I_B^G(|.|^{s+1/2}|.|^{s-1/2})$ is a subrepresentation, and therefore appears as subrepresentation in $I_P^G(\tau_{s\tilde{\alpha}})$.

If the value of s is such that $(s+1/2, s-1/2)$ is not a dominant residual point. The set up considered is that of $St_2 \hookrightarrow I_B^M(|.|^{1/2}|.|^{-1/2})$ twisted by $|.|^s$ so that it embeds in $I_B^M(|.|^{s+1/2}|.|^{s-1/2})$. Since $s > 0$, $I_P^G(St_2|.|^s) \hookrightarrow I_B^G(|.|^{s+1/2}|.|^{s-1/2})$. Using the result of Casselman-Shahidi (generalized injectivity conjecture for cuspidal inducing data) it is clear that the generic irreducible subquotient in $I_B^G(|.|^{s+1/2}|.|^{s-1/2})$ embeds as a subrepresentation.

10.1.1. *The case of a non-discrete series induced representation.* We now consider the general case of a standard module, with τ a tempered representation of $M \cong GL_2$. As an irreducible tempered representation of GL_2, $\tau \cong I_B^{GL_2}(\mathbf{1} \otimes \mathbf{1})$. Then the standard module is $I_P^G(\tau_s) \cong I_P^G(I_B^{GL_2}(\mathbf{1} \otimes \mathbf{1}) \otimes |.|^s) \cong I_B^G((\mathbf{1} \otimes \mathbf{1}) \otimes |.|^s) = I_B^G(|.|^s|.|^s)$. Since $I_B^G(\mathbf{1} \otimes \mathbf{1})$ is unitary, its unique generic subquotient is a subrepresentation; the twist by $|.|^s$ leaves it a subrepresentation in $I_B^G(|.|^s|.|^s)$.

10.1.2. *Residual segments.* As a aside, we compute the residual segments of type G_2 here. The weighted Dynkin diagrams for G_2 are:



Let $\lambda = (\lambda_1, \lambda_2)$ means that $\lambda = \lambda_1(2\alpha + \beta) + \lambda_2(\alpha + \beta)$. On the other hand, it is known that

$$< 2\alpha + \beta, \alpha^\vee >= 1 \ , < \alpha + \beta, \alpha^\vee >= -1 \ , < 2\alpha + \beta, \beta^\vee >= 0 \ , < \alpha + \beta, \beta^\vee >= 1 \tag{24}$$

From the first weighted Dynkin diagram above, the parameter λ satisfies:

$$< \lambda, \alpha^\vee >= 1, < \lambda, \beta^\vee >= 1$$

From the above relations 24, one should be able to compute that the residual segment is $\lambda = (2, 1)$.

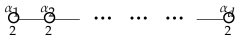
In the second weighted Dynkin diagram, the parameter λ satisfies:

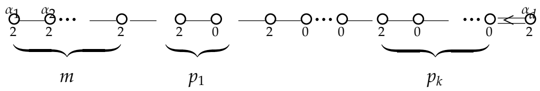
$$< \lambda, \alpha^\vee >= 0, \ < \lambda, \beta^\vee >= 1$$

And using the above relations 24, we conclude that the residual segment is (1,1).

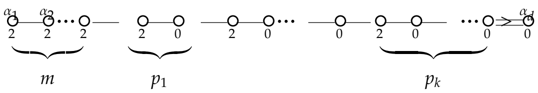
Appendix A. Weighted Dynkin diagrams

The diagrams presented here are also presented in Carter's book [9], page 175.

A_d. 

C_d. 

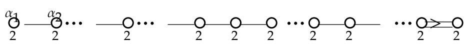
with $m + p_1 + \dots p_k + 1 = d$, $p_1 = 2$, $p_{i+1} = p_i$ or $p_i + 1$ for each i. ($k = 0$, $m = l - 1$ is a special case)

B_d. 

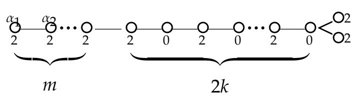
with $m + p_1 + \dots p_k = d$, $p_1 = 2$, $p_{i+1} = p_i$ or $p_i + 1$ for $i = 1, 2, \dots, k-2$ and

$$p_k = \begin{cases} \frac{p_{k-1}}{2} \text{ if } p_{k-1} \text{ is even} \\ \frac{p_{k-1}-1}{2} \text{ if } p_{k-1} \text{ is odd} \end{cases}$$

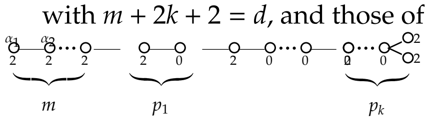
In addition the diagram:



is distinguished.

D_d. 

with $m + 2k + 2 = d$, and those of the form



with $m + p_1 + \dots p_k = l$, $p_1 = 2$, $p_{i+1} = p_i$ or $p_i + 1$ for $i = 1, 2, \dots, k-2$ and

$$p_k = \begin{cases} \frac{p_{k-1}}{2} \text{ if } p_{k-1} \text{ is even} \\ \frac{p_{k-1}+1}{2} \text{ if } p_{k-1} \text{ is odd} \end{cases}$$

A.1. Examples of *Set of Jumps* and residual segments.

Example A.1 (B_9)**.** Let $d_i' = 9$. Then $2d_i' + 1$ is 19, and we decompose 19 into distinct odd integers: 19; 11+7+1 ; 13+5+1; 15+3+1. So they are four different weighted Dynkin diagrams for B_9. The integers a_i's are respectively $\{9\}$; $\{5, 3\}$; $\{6, 2\}$; $\{7, 1\}$.

Example A.2 (D_9)**.** Then $2d_i'$ is 18, and we decompose 18 into distinct odd integers: 1 + 17; 15+3; 11+7; To each of these partitions correspond the Weyl group orbit of a residual point and therefore a residual segment. The regular orbit (since the exponents of the associated residual segment form a regular character of the torus) correspond to 1+17. It is simply $(8, 7, \dots 1, 0)$.

The other residual segments are: (765432110);(654322110); (543322110); (4 32 211 100) and the corresponding Jordan blocks are $\{15, 3\}$; $\{13, 5\}$; $\{11, 7\}$; $\{9, 5, 3, 1\}$.

Appendix B. Bala-Carter theory

In this section, we discuss unipotent conjugacy classes in a connected reductive complex algebraic group. The discussion can be reduced to the case in which G is semi-simple since the natural homomorphism from G to G/Z_G induces a bijection between unipotent conjugacy classes of G and those of G/Z_G (Proposition 5.1.1 in [9]).

Using a further bijection between unipotent conjugacy classes of G and nilpotent $Ad(G)$-orbits on the Lie algebra $\mathfrak{g}$ (A theorem of Springer- Steinberg, see [7]), we will explain the classification of the latter.

So let G be a semi-simple adjoint group over $\mathbb{C}$, and $\mathfrak{g}$ its Lie algebra over $\mathbb{C}$. It is well-known that if $\mathfrak{g}$ is semi-simple then a Cartan subalgebra $\mathfrak{t}$ is commutative, and $\mathfrak{g}$ is completely reducible under $\mathfrak{t}$, acting by the adjoint representation[see [5]]. We can consider $\Phi_0 = \Phi(\mathfrak{t};\mathfrak{g})$ the roots of $\mathfrak{t}$ in $\mathfrak{g}$, Φ_0^+ the corresponding set of positive roots, and $\Delta \subset \Phi_0$ a set of simple roots.

There is a decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \overline{\mathfrak{n}}$, where $\overline{\mathfrak{n}}$ is the nilpotent radical of the Borel subalgebra opposite to $\mathfrak{b}$.

Let $\mathcal{N} = \mathcal{N}_{\mathfrak{g}}$ be the cone of nilpotent elements in $\mathfrak{g}$. This cone is the disjoint union of a finite number of G- orbits. In the 1950's different parametrizations of the set of nilpotent G-orbits in $\mathfrak{g}$, $G\backslash\mathcal{N}$ were proposed: partition-type classifications and weighted Dynkin diagrams, we will discuss the second.

B.1. **Weigthed Dynkin diagrams.** Let $\mathcal{O}$ be a nilpotent orbit in $G\backslash\mathcal{N}$ and let $x \in \mathcal{O}$ be a representative element. A theorem of Jacobson- Morozov extends x to a standard ($\mathfrak{sl}_2$) triple $\{x, h, y\} \in \mathfrak{g}$, where h can be chosen to lie in the fundamental dominant Weyl chamber :

$$\{h' \in \mathfrak{g} | \mathrm{Re}(\alpha(h')) \geq 0, \forall \alpha \in \Delta \text{ and whenever } \mathrm{Re}(\alpha(h')) = 0, \mathrm{Im}(\alpha(h')) \geq 0\}$$

Theorem B.1 (Kostant,[24])**.** *Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$. A nilpotent orbit $\mathcal{O}$ is completely determined by the values $[\alpha_1(h), \alpha_2(h), \dots, \alpha_n(h)]$.*

For every simple root α in Δ, we have $\langle \alpha, h\rangle \in \{0, 1, 2\}$ (see section 3.5 in [13]).

If we label every node of the Dynkin diagram of $\mathfrak{g}$ with the eigenvalues $\alpha(h) = \langle \alpha, h\rangle$ of h on the corresponding simple root space $\mathfrak{g}_\alpha$, then all labels are 0,1 or 2. We call such a labeled Dynkin diagram, **a weighted Dynkin diagram**.

B.2. **The Bala-Carter classification.** The drawback of partition-type classifications was that they only apply to classical Lie algebras whereas a "good" parametrization of nilpotent orbits should be applicable to any semisimple Lie algebra. In two seminal papers ([1], [2]), appearing in 1976, Bala and Carter achieved such parametrization.

The key notion used by Bala and Carter was the notion of distinguished nilpotent element. It is an element that is not contained in any proper Levi subalgebra. Alternatively, a nilpotent element $n \in \mathfrak{g}$ is called distinguished if it does not commute with any non-zero semi-simple element of $\mathfrak{g}$. Or also, a nilpotent element X (resp. orbit $\mathcal{O}_X$) is distinguished if the only Levi subalgebra containing X (resp. meeting $\mathcal{O}_X$) is $\mathfrak{g}$ itself.

By focusing on the special properties of the orbits of distinguished elements in Levi subalgebras they could eventually parametrize all nilpotent orbits in $\mathfrak{g}$.

We now need to introduce the definition of distinguished parabolic subgroup and distinguished parabolic subalgebra.

Definition B.1 (distinguished parabolic subgroup)**.** Let P_J be a standard parabolic subgroup of G a group of adjoint type, with Levi decomposition $P_J = N_J L_J$. The Levi subgroup L_J decomposes as $L_J' Z(L_J)$ where L_J' is semisimple and $Z(L_J)$ is a torus.

The parabolic subgroup P_J is defined to be distinguished provided
$\dim L_J = \dim N_J/N_J'$.
For a subset $J \subseteq \Delta$, one defines a function $\eta_J : \Phi_0 \to 2\mathbb{Z}$ which equals 0 on any root in Δ_J and 2 for any root in $\Delta - \Delta_J$, then $N_J' = \prod_{\eta_J(\alpha)>2} N_\alpha$, N_α is the root subgroup corresponding to the root α. See Section 5.8 in [9].

Definition B.2 (distinguished parabolic subalgebra)**.** A parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$ is called distinguished if $\dim \mathfrak{l} = \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$, in which $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is a Levi decomposition of $\mathfrak{p}$, with Levi part $\mathfrak{l}$.

The main theorem is the following:

Theorem B.2 (5.9.5 in [9]). *Let G be a simple algebraic group of adjoint type over F. Suppose the characteristic p of F is either zero, or $p > 3(h-1)$ where h is the Coxeter number of G. Let $\mathfrak{g}$ be the Lie algebra of G. Then:*

(1) There is a bijective map between the G-orbits of distinguished nilpotent elements of $\mathfrak{g}$ and the conjugacy classes of the distinguished parabolic subgroups of G. The G-orbit corresponding to a given parabolic subgroup P contains the dense orbit of P acting on the Lie algebra of its unipotent radical.

(2) There is a bijective map between the G-orbits of nilpotent elements of $\mathfrak{g}$ and the G-classes of pairs $(L, P_{L'})$ where L is a Levi subgroup of G and $P_{L'}$ a distinguished parabolic subgroup of the semi-simple part L' of L. The G-orbit corresponding to a given pair $(L, P_{L'})$ contains the dense orbit of $P_{L'}$ acting on the Lie algebra of its unipotent radical.

In term of Lie algebras, we have the following one-to-one correspondences:

$$\begin{Bmatrix}\text{Distinguished nilpo-}\\ \text{tent } \mathrm{Ad}(G)\text{-orbits of}\\ \mathfrak{g}\end{Bmatrix} \leftrightarrow \begin{Bmatrix}G \text{ conjugacy classes of dis-}\\ \text{tinguished parabolic subalge-}\\ \text{bras of } \mathfrak{g}\end{Bmatrix} \tag{25}$$

$$\{\text{Nilpotent } \mathrm{Ad}(G)\text{-orbits of } \mathfrak{g}\} \leftrightarrow \begin{Bmatrix}G \text{ conjugacy classes of pairs}\\ (\mathfrak{p}, \mathfrak{m}) \text{ of } \mathfrak{g}\end{Bmatrix} \tag{26}$$

in which $\mathfrak{m}$ is a Levi factor, $\mathfrak{p} \subseteq \mathfrak{m}'$ is a distinguished parabolic subalgebra of the semi-simple part of $\mathfrak{m}$.

We sketch the ideas behind these correspondences.

As above, given a non-zero nilpotent element in $\mathfrak{g}$, let $\{e, h, f\}$ denote the standards basis of the $\mathfrak{sl}_2$ Lie algebra. The Jacobson-Morozov Lie algebra homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ satisfies $\phi(e) = n \in \mathfrak{n}$ and $\phi(h) = \gamma$ is in the dominant chamber of $\mathfrak{t}$.

The adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$ yields a grading $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}(i)$ in which

$$\mathfrak{g}(i) = \{x \in \mathfrak{g} | ad(\gamma)(x) = ix\}\,;[\mathfrak{g}(i), \mathfrak{g}(j)] \subseteq \mathfrak{g}(i+j)$$

and $n \in \mathfrak{g}(2)$. Further, set

$$\begin{cases}\mathfrak{p} = \mathfrak{p}(\gamma) = \oplus_{i\geq 0}\mathfrak{g}(i)\\ \mathfrak{u} = \oplus_{i>0}\mathfrak{g}(i)\\ \mathfrak{l} = \mathfrak{g}(0)\end{cases} \tag{27}$$

The Lie subalgebra $\mathfrak{p}$ contains $\mathfrak{b}$, and is thus a parabolic subalgebra whose Levi decomposition is $\mathfrak{p} = \mathfrak{u} \oplus \mathfrak{l}$.

On the other hand, starting with a subset $J \subseteq \Delta$, and denoting $\mathfrak{p}_J$ the standard parabolic subalgebra, one defines a function $\eta_J : \Phi_0 \to \mathbb{Z}$, defined on roots of Δ as twice the indicator function of J and extended linearly to all roots.

We obtain a grading: $\mathfrak{g} = \oplus_{i\geq 0}\mathfrak{g}_J(i)$ by declaring $\mathfrak{g}_J(0) = \mathfrak{t} \oplus \sum_{\eta_J(\alpha)=0} \mathfrak{g}_\alpha$ and otherwise $\mathfrak{g}_J(i) = \sum_{\eta_J(\alpha)=i} \mathfrak{g}_\alpha$. Then, $\mathfrak{p}_J = \oplus_{i\geq 0}\mathfrak{g}_J(i)$ and its nilpotent radical is $\mathfrak{n}_J = \oplus_{i>0}\mathfrak{g}_J(i)$.

To summarize, to the standard triple containing n one attaches a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$.

If $\dim \mathfrak{g}(1) = 0$, then we call n (resp. O_n) an even nilpotent element (even nilpotent orbit, respectively).

Proposition B.1 (Corollary 3.8.8 in [13]). *A weighted Dynkin diagram has labels 0 or 2 if and only if it corresponds to an even nilpotent orbit (i.e, if* $\dim \mathfrak{g}(1) = 0$*)*

Proposition B.2. *The standard parabolic subalgebra $\mathfrak{p}_J$ is distinguished if and only if* $\dim \mathfrak{g}_J(0) = \dim \mathfrak{g}_J(2)$*. In this case, if n is any element in the unique open orbit of the parabolic subgroup P_J on its nilpotent radical $\mathfrak{n}_J$, then the parabolic subalgebra associated to n as in (27) equals $\mathfrak{p}_J$.*

A distinguished nilpotent element also satisfies the following:

Proposition B.3. *A nilpotent element $n \in \mathfrak{g}$ is distinguished if and only if* $\dim \mathfrak{g}(0) = \dim \mathfrak{g}(2)$*. Moreover, if $n \in \mathfrak{g}$ is distinguished, then* $\dim \mathfrak{g}(1) = 0$.

Theorem B.3 (Theorem 8.2.3 in [13]). *Any distinguished orbit in $\mathfrak{g}$ is even.*

Theorem B.4 (Theorem 8.2.14 in [13]).

(1) If $\mathfrak{g}$ is of type A, then the only distinguished orbit is principal.

(2) If $\mathfrak{g}$ is of type B, C or D, then an orbit is distinguished if and only if its partition has no repeated parts. Thus the partition of a distinguished orbit in types B, D has only odd parts, each occurring once, while the partition of a distinguished orbit in type C has only even parts, each occurring once.

We can now write the correspondences:
Pick a distinguished element n. By Proposition B.3, $\mathfrak{p}$ is a standard parabolic subalgebra $\mathfrak{p}_J$ for $J = \{\alpha \in \Delta | \mathfrak{g}_\alpha \subseteq \mathfrak{g}(2)\}$ which is distinguished by Proposition B.2, and we obtain the map inducing the first bijective correspondence: $n \to \mathfrak{p}$

By Proposition B.1, since we are given this distinguished parabolic algebra $\mathfrak{p}$, $\gamma = \phi(h)$ is an even Weighted Dynkin Diagram for the semi-simple Lie algebra $\mathfrak{g}$.

For the second, one can choose a minimal Levi subalgebra $\mathfrak{m}$ containing n (cf Prop 5.9.3 in [9]) which modulo conjugation, can be assumed to be a Levi factor of a parabolic subalgebra containing $\mathfrak{b}$. By minimality of $\mathfrak{m}$, it follows that $n \in \mathfrak{m}' = [\mathfrak{m}, \mathfrak{m}]$ is a distinguished nilpotent element in $\mathfrak{m}'$, and then by Proposition B.2, there is a distinguished parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{m}'$ corresponding to n. One can construct a map induced by $n \to (\mathfrak{m}, \mathfrak{p})$. On the other direction, one associates to a conjugacy class of the pair $(\mathfrak{m}, \mathfrak{p})$ the orbit $\mathrm{Ad}(G)n$ in which $n \in \mathfrak{n}_\mathfrak{p}$ is any element in the unique dense adjoint orbit of P on $\mathfrak{n}_\mathfrak{p}$, with the latter being the nilradical of $\mathfrak{p}$ and P the parabolic subgroup of G associated to $\mathfrak{p}$.

B.3. **Distinguished Nilpotent orbits and residual points.** The connection with the notion of residual point is now made accessible.
Let G be a Chevalley (semi-simple) group and $T \subseteq B$ a maximal split torus and a Borel subgroup. We have a root datum $\mathcal{R}(G, B, T)$. By reversing the role of $X^*(T)$ and $X_*(T)$, we obtain a new root datum $\mathcal{R}^\vee = (X_*(T), \Delta, X^*(T), \Delta^\vee)$. Let $({}^LG, {}^LB, {}^LT)$ be the triple with root datum $\mathcal{R}^\vee$. The L-group ${}^LG$ is the dual group, with maximal torus ${}^LT$, and Borel subgroup ${}^LB$. Denote the respective Lie algebra ${}^L\mathfrak{g}$, ${}^L\mathfrak{t}$ and ${}^L\mathfrak{b}$. Let $(V^*, \langle , \rangle)$ be a finite dimensional Euclidean space containing and spanned by the root system: $\Delta \subseteq V^*$, the canonical pairing between V and V^* is denoted by $\langle , \rangle$. We fix an inner product on V by transport of structure from $(V^*, \langle , \rangle)$ via the canonical isomorphism $V^* \to V$ associated with $\langle , \rangle$. Thus this map becomes an isometry, and for each $\alpha \in \Delta$, the coroot $\check{\alpha} \in V$ is given as the image of $2\langle \alpha, \alpha \rangle^{-1} \alpha \in V^*$.

To this data we associate the Weyl group W_0 generated by the reflexions s_α ($s_\alpha(x) = x - \langle x, \check{\alpha} \rangle \alpha$ and $s_\alpha(y) = y - \langle \alpha, y \rangle \check{\alpha}$) over the hyperplanes $H_\alpha \subseteq V^*$ consisting of elements $x \in V^*$ which are orthogonal to $\check{\alpha}$ with respect to $\langle , \rangle$.

Let us make a remark before stating the correspondence result related to our use in this manuscript:

Remark. The bijective correspondence (below) is originally formulated for residual subspaces. Let k be the "coupling parameter" as defined in [17]. An affine subspace $L \subseteq V$ is called residual if, for a root system Φ (in a root datum)

$$\#\{\alpha \in \Phi | \langle \alpha, L \rangle = k\} = \#\{\alpha \in \Phi | \langle \alpha, L \rangle = 0\} + \mathrm{codim} L$$

(If $\mathcal{R}$ is semi-simple, there exist residual subspace which are singletons $\{\lambda\} \subseteq V$, the residual points).

For example, when the parameter k (called "coupling parameter" in [17]) equals 1, the Weyl vector $\rho = \frac{1}{2} \sum_{\alpha \in \Phi} \alpha$ is a residual point, since the above equation is verified. More generally, for any $k = (k_\alpha)_{\alpha \in \Phi}$, the vector $\rho(k) = \frac{1}{2} \sum_{\alpha \in \Phi} k_\alpha \alpha$ is a residual point.

Then the bijective correspondence is given between the set of nilpotent orbits in the Langlands dual Lie algebra ${}^L\mathfrak{g}$ and the set of W_0- orbits of residual subspaces.

We mention the following result partially related to Proposition 4.3. The bijective correspondence concerns only unramified characters and we fix the parameter $k_\alpha = 1$ for all $\alpha \in \Phi_0$.

Proposition B.4. *There is a bijective correspondence $\mathcal{O}_{W_0\lambda(\mathcal{O})} \leftrightarrow W_0\lambda(\mathcal{O})$ between the set of distinguished nilpotent orbits in the Langlands dual Lie algebra ${}^L\mathfrak{g}$ and the set of W_0-orbits of residual points.*

Proof. This particular bijection is a specific case of the larger bijective correspondence given between the set of nilpotent orbits in the Langlands dual Lie algebra ${}^L\mathfrak{g}$ and the set of W_0-orbits of residual subspaces. It is discussed in details in [[30], Appendices A and B], but also in [[19], Proposition 6.2].

Let $({}^L\mathfrak{m}, {}^L\mathfrak{p})$ be a representative of a class, for which ${}^L\mathfrak{m} = {}^L\mathfrak{g}$ and ${}^L\mathfrak{p} \subseteq {}^L\mathfrak{g}$ is a standard distinguished parabolic subalgebra. We have a corresponding distinguished nilpotent orbit $\mathcal{O}$. With Proposition B.2, the data ${}^L\mathfrak{p}$ is equivalent to the assignment of an even weighted Dynkin diagram: $2\lambda(\mathcal{O})$.

Since we have $\dim \mathfrak{g}(0) = \dim \mathfrak{h} + \#\{\alpha \in \Phi | \langle \check{\alpha}, 2\lambda(\mathcal{O})\rangle = 0\}$ and

$$\dim \mathfrak{g}(2) = \#\{\alpha \in \Phi | \langle \check{\alpha}, 2\lambda(\mathcal{O})\rangle = 2\}$$

The assignment of an even weighted Dynkin diagram implies $\dim \mathfrak{g}(0) = \dim \mathfrak{g}(2)$ and this equality sets $\lambda(\mathcal{O})$ as a residual point.

The definition of $\lambda(\mathcal{O})$ depends on the choice of positive roots and Borel subgroup ${}^LB$. A different choice yields a different element on the same W_0-orbit.

For the sake of completeness, we quote the proposition as given in [[30], Appendices A and B]:

Proposition B.5 (Proposition 8.1 in [30]). *(i) If r is a residual point with polar decomposition $r = sc = s\exp(\gamma) \in T_u T_{rs}$ and γ is dominant, then the centralizer $C_{\mathfrak{g}}(s)$ of s in $\mathfrak{g} := Lie(G)$ is a semi-simple subalgebra of $\mathfrak{g}$ of rank equal to rank($\mathfrak{g}$), and γ/k is the weighted Dynkin diagrams (confer page 175 of [9]) of a distinguished nilpotent class of $C_{\mathfrak{g}}(s)$.*

(ii) Conversely, let $s \in T_u$ be such that the centralizer algebra $C_{\mathfrak{g}}(s)$ is semisimple and let $e \in C_{\mathfrak{g}}(s)$ be a distinguished nilpotent element. If h denotes the weighted Dynkin diagram of e then $r = sc$ with $c := \exp(kh)$ is a residual point.

(iii) The above maps define a 1 - 1 correspondence between W_0-orbits of residual points on the one hand, and conjugacy classes of pairs (s, e) with $s \in G$ semisimple such that $C_{\mathfrak{g}}(s)$ is semisimple, and e a distinguished nilpotent element in $C_{\mathfrak{g}}(s)$.

(iv) Likewise there is a $1-1$ correspondence between W_0-orbits of residual points and conjugacy classes of pairs (s, u) with $C_G(s)$ semisimple and u a distinguished nilpotent element of $C_G(s)^0$.

Appendix C. Projections of roots systems

Let us first follow the notations of the book of [31], Chapter V. We will also use the notations of the Section 2. Let $X^*(G)$ denote the group of rational characters of G; it dual is $X_*(G)$. Let A_M be the split component in M and A_0 the maximal split component in M_0. We denote $a_0 = X_*(A_0) \otimes_{\mathbf{Z}} \mathbf{R}$ and $a_0^* = X^*(A_0) \otimes_{\mathbf{Z}} \mathbf{R}$.

The duality between $X^*(A_0)$ and $X_*(A_0)$ extends to a duality (canonical pairing) between the vector spaces a_0 and a_0^*. We have the following diagram (see the Chapter V of [31]):

$$\begin{array}{ccc} a_M^* = X^*(M) \otimes_{\mathbf{Z}} \mathbf{R} & \longrightarrow & X^*(A_M) \otimes_{\mathbf{Z}} \mathbf{R} = a_M^* \\ \uparrow & & \downarrow \\ a_G^* = X^*(G) \otimes_{\mathbf{Z}} \mathbf{R} & \longrightarrow & X^*(A_G) \otimes_{\mathbf{Z}} \mathbf{R} = a_G^* \end{array}$$

The horizontal arrows are isomorphisms. If we denote a_M^G the kernel of the vertical arrow on the right, we obtain:

$$a_M^* = a_G^* \oplus (a_M^G)^*$$

And in the dual:

$$a_M = a_G \oplus (a_M^G)$$

Let M be a standard Levi subgroup of G such that the set of simple roots in Lie(M) is $\Delta_M = \Theta$. Let us therefore denote $a_M = a_\Theta$.

Because of the existence of the scalar product (sustaining the duality), the restriction map from $(a_0^G)^*$ to $(a_\Theta^G)^*$ is a *projection* map from (a_0^G) to (a_Θ^G). With the notations of the Section 6, the roots in $\Delta(P_1)$ generating $(a_{M_1})^*$ are non -trivial restrictions of roots in $\Delta \setminus \Delta^{M_1}$ (Recall that in the notations of [42], I.1.6, Δ^{M_1} are the roots of Δ which are in M_1), and (a_{M_1}) is generated by the projection of roots in $\Delta^\vee \setminus \Delta^{M_1\,\vee}$.

In this Appendix, we will rather consider projections of roots.

Let a be a real euclidean vector space of finite dimension and Σ a root system in a with a basis Δ. Let $\Theta \subset \Delta$, to avoid trivial cases we assume Θ is a proper subset of Δ, i.e. that Θ is neither empty nor equal to Σ. Let us consider the projection of Σ on a_Θ and we denote Σ_Θ the set of all non-trivial projections of roots in Σ. Our context is that of $a = a_0^G := a_0/a_G$ quotient of the Lie algebra of the maximal split torus A_0 by the Lie algebra of the center of G. We consider Σ as root system of G, an order, and a basis Δ. Let M be a standard Levi subgroup of G such that the set of simple roots in Lie(M) is $\Delta^M = \Theta$. Then $a_\Theta = a_M/a_G$. We don't consider the trivial case where $M = M_0$ and $M = G$. Let us denote d the dimension of a_Θ, i.e the cardinal of $\Delta - \Theta$.

Let us also denote Δ_Θ the set of projections of the simple roots in $\Delta - \Theta$ on a_Θ. In general Σ_Θ is not a root system, however let us observe:

Lemma 20. *The elements in Σ_Θ are, in a unique way, linear combination with entire coefficients all with the same sign of the elements in Δ_Θ.*

We would like to determine the conditions under which Σ_Θ contains a root system (for a subspace of a_Θ) and what are the types of root system appearing. We will classify the subsystems of rank d appearing when they exist. Of course, there are always subsystems of rank 1 and as Θ is assumed to be non-empty there is no need to discuss the case where Σ is of rank 2 (in particular G_2). We will therefore consider the root systems Σ of rank $n \geq 3$ and $d \leq n-2$. Let us remark that we will find irreducible non reduced root systems: they are the BC_d which contain three subsystems of rank d: B_d, C_d and D_d.

Theorem C.1. *Let Σ be an irreducible root system of classical type (i.e of type A, B, C or D). The subsystems in Σ_Θ are necessarily of classical type. In addition, if the irreducible (connected) components of Θ of type A are all of the same length, the interval between each of them of length one, then Σ_Θ contains an irreducible root system of rank d (non necessarily reduced).*

We will use the following remark (see the Chapter VI in [8], in particular Equation (10) in VI.3 and Proposition 12 in VI.4). Let α and β be two non-orthogonal elements of a root system. Set

$$C = \left(\frac{1}{\cos(\alpha,\beta)}\right)^2 \qquad \text{and} \qquad R = \frac{||\alpha||^2}{||\beta||^2} .$$

The only possible values for C (the inverse of the square of the cosinus of the angle between two roots) are 4, 2 and $\frac{4}{3}$ whereas assuming the length of α larger or equal to the one of β, the quotient of the length is respectively 1, 2 or 3. Thus, if $||\alpha|| \geq ||\beta||$

$$\frac{C}{R} \in \{2^2, 1, (2/3)^2\} \qquad \text{and} \qquad CR = 4 .$$

We will therefore compute the quotient of length and the angles of the non-trivial projections of roots in Σ, in particular those of elements in $\Delta - \Theta$.

C.1. **The case of reducible Σ_Θ.** We have seen that in order to obtain a projected root system irreducible and of maximal rank, we had to impose several constraints. Let us explain once more some of them. Let us first consider two components A_m and A_q of Θ, let e_r and e_s be the vectors in the basis vectors of smallest index such $\Xi_r = \{e_r, \ldots, e_{r+m}\}$ corresponds to A_m and Ξ_s to A_q. Let us assume two simple consecutive roots α_{k-1} and α_k are outside of Θ and $k = r+m+1 = s-1$. Then $\Xi_k = \{e_k\}$. Let us consider the projections of α_{k-1} and α_k: Since e_k is orthogonal to all roots in Θ, $\overline{e_k} = e_k$.
Therefore:

$$||\overline{\alpha_{k-1}}||^2 = ||\overline{e_{k-1}} - \overline{e_k}||^2 = \frac{1}{m+1} + 1 .$$

$$||\overline{\alpha_k}||^2 = ||\overline{e_k} - \overline{e_{k+1}}||^2 = 1 + \frac{1}{q+1} .$$

$$(<\overline{\alpha_{k-1}},\overline{\alpha_k}>)^2 = 1\ .$$

Then

$$C = \left(\frac{1}{\cos(\overline{\alpha_{k-1}},\overline{\alpha_k})}\right)^2 = (\frac{1}{m+1}+1)(1+\frac{1}{q+1})$$

and if we assume $||\overline{\alpha_{k-1}}|| \geq ||\overline{\alpha_k}||$ i.e $m \geq q$, we have:

$$R = \frac{||\overline{\alpha_{k-1}}||^2}{||\overline{\alpha_k}||^2} = \frac{\frac{1}{m+1}+1}{\left(1+\frac{1}{q+1}\right)}$$

If α_k and α_{k-1} were to be part of a root system, we would need

$$\frac{C}{R} = \left(1+\frac{1}{m+1}\right)^2 \in \{2^2, 1, (2/3)^2\} \qquad \text{and} \qquad CR = \frac{1}{\left(1+\frac{1}{q+1}\right)}^2 = 4\ .$$

This implies $m = 0$ and $\left(1+\frac{1}{q+1}\right) = 1/4$ a contradiction. This illustrates the fact that in the main theorem (Theorem C.1) the intervals between the irreducible connected components of Θ need to be of length one, and *at most one*.

In general, the contrapositive of the main theorem of [15] (recalled as Theorem C.1 above) is the following:

Theorem C.2. *Let Σ be an irreducible root system of type B, C or D. If the irreducible (connected) components of Θ of type A are all* not *of the same length, the interval between each of them of length one, then Σ_Θ contains a reducible root system of rank d (non necessarily reduced); $\Sigma_\Theta = \bigoplus_i \Sigma_{\Theta,i}$ and if d_i is the rank of the irreducible i-th component, then $\sum_i d_i = d$.*

The number of irreducible components ($\Sigma_{\Theta,i}$) is as many as there are changes of length *plus one. That is, if there are d_1 components of type A_{m_1}, followed by d_2 components of type A_{m_2},etcetera until d_s components of type A_{m_s}, such that $m_i \neq m_{i+1}$ for any i, and one last component of type B or C or D, they are $s-1$ changes in the length (m_i) and therefore s irreducible connected components in Σ_Θ. The set Σ_Θ is composed of irreducible components of type A and possibly one component of type B, C or D.*

Proof. We have explained the condition on the interval being of at most length one in the paragraph preceding the statement of the theorem. We do not repeat here the methods of proof for the case of Σ_Θ irreducible which apply here: in particular the treatment of the case $e_n \notin \Theta$, the reduction to this case's argumentation when $e_n \in \Theta$, and the argumentation showing that the components of type A of Θ should be of the same length to obtain a root system in the projection.

We consider the case of root system of type B, C, D.

Let then assume that we have d_1 components of type A_{m_1} in Θ, by the argumentation given in [15], we obtain a root system of type BC_{d_1}. Let us assume that these d_1 components of type A_{m_1} are followed by d_2 components of type A_{m_2}, $m_2 \neq m_1$. Let us denote e_{1,d_1} the vector associated to the last component of type A_{m_1} and $e_{2,1}$ the vector associated to the first component of type A_{m_2}.

The projection $\overline{e_{1,d_1}} - \overline{e_{2,1}} = \frac{e_{(d_1-1)m_1+1}+e_{(d_1-1)m_1+2}+\ldots+e_{(d_1-1)m_1+m_1}}{m_1+1} - \frac{e_{d_1m_1+1}+e_{d_1m_1+2}+\ldots+e_{d_1m_1+m_2}}{m_2+1}$ of $e_{1,d_1} - e_{2,1}$ cannot be a root in Σ_Θ (it would contradict the conditions of validity of the value C and R when calculated with respect to the last root of the previously considered BC_{d_1}).

However, the projections of the roots corresponding to the intervals between any two of the d_2 components of type A_{m_2} (say of $\overline{e_s} - \overline{e_t}$) along with all roots of the form $\pm e_s$ or $\pm e_t$ (resp. $\pm 2e_s$ or $\pm 2e_t$) form a root system of type BC_{d_2}. Some specificities, such as root system of type C appearing in the projection for certain cases under Σ of type C or D carry over here (see [15]).

The key mechanism assuring that the sum of the d_i equals d is the observation that one need *three* consecutive components of type A_q of a given length q (followed by components of length $m \neq q$) to obtain in the projection a BC_3 *(hence of rank three!)* whereas one would obtain only a A_2 type of root system. This means that even if the root connecting the A_q to A_m is not a root in the projection, i.e "we are missing a simple root"; we get a simple root of type $\overline{e_i}$ or $\overline{2e_i}$.

One may notice that another possibility would be to obtain a reducible root system such as $A_1 \times A_1 \times \ldots \times A_1$. This case is not excluded but it would not be possible to find such a system of *maximal rank*.

Indeed, by the formulas written for the case of Σ of type A in [15], we had:

$$\left(< \overline{\alpha}, \overline{\beta} >\right)^2 = \frac{1}{(p+1)^2} .$$

This excludes the possibility of α and β being orthogonal.

Therefore for two consecutive roots in the projection (projections of simple roots), it is not possible to obtain a system of type $A_1 \times A_1$.

If there is a sequence of connected consecutive components of Θ of type A that we index by an integer i (in increasing order) and length q_i with $q_i \neq q_{i+1}$ for any i, let us denote $\overline{\alpha_i} = \overline{e_r} - \overline{e_s}$ where $e_r \in A_{q_i}$ and $e_s \in A_{q_{i+1}}$.

Further, let us denote $\overline{\alpha_{i+2}} = \overline{e_t} - \overline{e_z}$ where $e_t \in A_{q_{i+2}}$ and $e_z \in A_{q_{i+3}}$. The orthogonal roots $\overline{\alpha_i}$ and $\overline{\alpha_{i+2}}$ form a root system of type $A_1 \times A_1$.

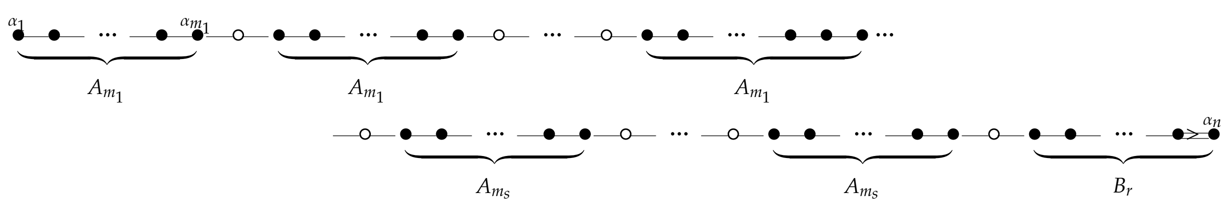
The root $\overline{\alpha_{i+1}} = \overline{e_s} - \overline{e_t}$ does not contribute to this subsystem.

Therefore, the maximal number of A_1 factor such that the reducible root system $A_1 \times A_1$ appear in Σ_Θ is $d/2$.

By a similar reasoning, it would be possible to obtain a reducible system of type $A_2 \times A_2 \times \ldots \times A_2$ if Θ is composed of a succession of connected components of type A such that the three first ones are of length m, the three next ones of length $q \neq m$ ..etc. Then the projection of the root connecting A_m and A_q would not contribute to this subsystem. Again, this would never give any reducible system of *maximal rank d*.

Because to any change of length of the A components, the corresponding root (connecting the two components of different length) cannot appear as a (simple) root in the projection, we are missing a root (of the set $\Delta - \Theta$ of size d) at any change of length. In the case Σ is of type A, this 'missing' root is not replaced by any short or long root (e_i or $2e_i$), therefore it is impossible to obtain a basis of root system in the projection. In other words, there does not exist any reducible root system of maximal rank in the projection Σ_Θ of Σ of type A.

Let us illustrate one case of the previous theorem with a Dynkin diagram of Σ of type B:



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