PROJECTION OF ROOT SYSTEMS

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Abstract

Let a be a real euclidean vector space of finite dimension and Σ a root system in a with a basis Δ. Let $\Theta \subset \Delta$ and $M = M_\Theta$ be a standard Levi of a reductive group G such that $a_\Theta = a_M/a_G$. Let us denote d the dimension of a_Θ, i.e the cardinal of $\Delta - \Theta$ and Σ_Θ the set of all non-trivial projections of roots in Σ. We obtain conditions on Θ such that Σ_Θ contains a root system of rank d.

1 Introduction

Let a be a real euclidean vector space of finite dimension and Σ a root system in a with a basis Δ. Let $\Theta \subset \Delta$, to avoid trivial cases we assume Θ is a proper subset of Δ, i.e. that Θ is neither empty nor equal to Σ.

We consider $\mathbf{G}$ a quasi-split reductive group over a local field F, and $\mathbf{T}$ a maximal torus of $\mathbf{G}$. As usual, the not-bold notation G denotes the F-points of $\mathbf{G}$.

In this article, we fix $a = a_0^G := a_0/a_G$ quotient of the Lie algebra of the maximal F-split torus in a maximal torus $\mathbf{T}$ by the Lie algebra of the maximal split torus A_M in the center of $\mathbf{G}$. We denote Σ the root system of G and Δ a basis of Σ. Let M be a standard Levi subgroup of G such that the set of simple roots in Lie(M) is $\Delta^M = \Theta$. Then $a_\Theta = a_M/a_G$.

Let us consider the projection of Σ on a_Θ (projection orthogonal to Θ) and we denote Σ_Θ the set of all non-trivial projections of roots in Σ. We do not consider the trivial case where $M = M_0$ and $M = G$. Let us denote d the dimension of a_Θ, i.e the cardinal of $\Delta - \Theta$.

Let us also denote Δ_Θ the set of projections of the simple roots in $\Delta - \Theta$ on a_Θ.

In this article, we determine the conditions under which Σ_Θ contains a root system (for a subspace of a_Θ) and what are the types of root system appearing. We will classify the subsystems of rank d appearing when they exist. We then say they are of *maximal rank*. Our main results are :

Theorem 1.1. *Let Σ be an irreducible root system of classical type (i.e of type A, B, C or D). The subsystems in Σ_Θ are necessarily of classical type. In addition, if the irreducible (connected) components of Θ of type A are all of the same length, and the interval between each of them of length one, then Σ_Θ contains an irreducible root system of rank d (non necessarily reduced).*

Theorem 1.2. *Let Σ be an irreducible root system of exceptional type (i.e of type E, F_4 or G_2). If Σ_Θ contains an irreducible system of rank d it is necessarily of classical type, except in the case of the orthogonal projection to any single root of E_8 where E_7 appears in the projection.*

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1.1 Motivation

This question emerges in an attempt to better understand the result of Silberger in [8]. In Section 3.5 of his work, he claims that

$$\Sigma_\sigma = \{\alpha \in \Sigma_{red}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\}$$

is a subset of $a^*_{M_1}$ which is a root system in a subspace of $a^*_{M_1}$. Here σ is a discrete series representation of a semi-standard Levi subgroup M_1 of a reductive group G and $\mu^{(M_1)_\alpha}$ is one factor in the product formula of the μ function (see also [9], Lemma V.2.1). Let us recall that the μ function is the main ingredient in the Plancherel measure, the unique Borel measure on the set of irreducible tempered representations of G which was defined by Harish-Chandra to formulate the Plancherel formula for p-adic groups.

Considering the restrictive definition of a root system, it is not immediately clear that the restrictions (resp. projections) of roots to A_{M_1} constitute a root system of rank $|\Delta - \Theta|$; and in general it is not the case. The reader will find many instances of non-existence of maximal rank root system in Σ_Θ in this paper. The goal of this paper is to make precise the conditions on the semi-standard Levi M_1 (i.e on Θ) such that $\Sigma(A_{M_1}) := \Sigma_\Theta$ contains a root system of maximal rank.

Silberger's result applied in the case where σ is unitary cuspidal, along with the results obtained in this work are fundamentally used in our work on the Generalized Injectivity Conjecture, a conjecture formulated by Casselman and Shahidi in [2].

Theorem (Generalized Injectivity for quasi-split group, [3]). Let G be a quasi-split, connected group defined over a p-adic field F (of characteristic zero). Let π_0 be the unique irreducible generic subquotient of the standard module $I_P^G(\tau_\nu)$, let σ be an irreducible, generic, cuspidal representation of M_1 such that a twist by an unramified real character of σ is in the cuspidal support of π_0.

Suppose that all the irreducible components of Σ_σ are of type A, B, C or D, then, under certain conditions on the Weyl group of Σ_σ, π_0 embeds as *a subrepresentation* in the standard module $I_P^G(\tau_\nu)$.

The condition of *maximal rank* of Σ_σ is also crucial to the existence of a discrete series subquotient in the induced module $I_{P_1}^G(\sigma_\lambda)$ whenever $\lambda \in a^*_{M_1}$ is known as a residual point, as studied in [4]. Heiermann's approach to the infinitesimal character of an irreducible discrete series requires the notion of residual point which itself requires the rank of Σ_σ to be maximal (see Definition 2.1 in [3]), see also Section 3.8 in [8]. The conditions we have obtained on the form of the Levi M_1 in order to obtain a maximal rank root system have already been implicitly used in the literature, see for instance Proposition 1.13 in [5].

Further, understanding which root systems (in particular classical or not) appear in the projections of exceptional root systems helped us circumscribe the limits of our work on the Generalized Injectivity conjecture. One key to understand the limits of our work lies in subtleties involving $W(M_1)$ (the set of representatives in W (Weyl group of Σ) of elements in the quotient group $\left\{w \in W | w^{-1} M_1 w = M_1\right\} / W^{M_1}$ of minimal length in their right classes modulo W^{M_1}) and W_σ (Weyl group of Σ_σ). Identifying the potential Σ_σ of maximal rank in the projections is the first step in doing so.

1.2 Method

Of course, there are always subsystems of rank 1 and as Θ is assumed to be non-empty there is no need to discuss the case where Σ is of rank 2 (in particular G_2). We will therefore consider the root systems Σ of rank $n \geq 3$ and $d \leq n-2$. Let us remark that we will find irreducible non reduced root systems : they are the BC_d which contain three subsystems of rank d : B_d, C_d and D_d.

We will use the following remark (see [1, Equation (10) in VI.3, Proposition 12 in VI.4, Chapter VI]). Let α and β be two non-orthogonal distinct elements of a root system. Set

$$C = \left(\frac{1}{\cos(\alpha,\beta)}\right)^2 \qquad \text{and} \qquad R = \frac{||\alpha||^2}{||\beta||^2} \ .$$

The only possible values for C (the inverse of the square of the cosinus of the angle between two roots) are 4, 2 and 4/3 whereas assuming the length of α larger or equal to the one of β, the quotient of the length is respectively 1, 2 or 3. Thus, if $||\alpha|| \geq ||\beta||$

$$\frac{C}{R} \in \{2^2, 1, (2/3)^2\} \qquad \text{and} \qquad CR = 4 \ .$$

We will therefore compute the quotient of lengths and the angles of the non-trivial projections of roots in Σ, in particular those of elements in $\Delta - \Theta$.

In general Σ_Θ is not a root system, however let us observe :

LEMMA 1.3. *The elements in Σ_Θ are, in a unique way, linear combination with entire coefficients all with the same sign of the elements in Δ_Θ.*

From Theorem 3 (page 156) or Corollary 3 (page 162) in Chapter 6, §1, Sections 6 and 7 in [1] ; we know any root in Σ can be written in a unique way as linear combination with entire coefficients all with the same sign of the elements in the basis Δ. Then the statement in Lemma 1.3 follows since the projection orthogonal to any subset $\Theta \subset \Delta$ (i.e projection onto $W^\perp$, if W is the vector space generated by Θ) is a linear application.

Acknowledgements

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2 Classical root systems

In this section, we prove Theorem 1.1 *via* a case-by-case analysis.

2.1 The case where Σ is of type A_n

Let us consider a_0 to be of dimension $n+1$ and with orthonormal basis $e_1, e_2, \ldots, e_{n+1}$. Let us denote Ξ this ordered basis, i.e the ordered set of the e_i. The elements of Σ are the $e_i - e_j$ with $i \neq j$; they generate a subspace a of dimension n and Δ is the set of simple roots $\alpha_i = e_i - e_{i+1}$. Let us denote $\overline{e_i}$ the projection of e_i on a_Θ. The Dynkin diagram of Θ is a union of irreducible (or connected) components of type A. Therefore, the data of Θ corresponds to a partition of the ordered set Ξ in a disjoint (ordered) union of ordered parts that we index by the smallest index appearing in the indices of the basis vectors associated :

$$\Xi = \Xi_1 \cup \cdots \cup \Xi_l \ .$$

The correspondence is defined as follows, the part :

$$\Xi_r = \{e_r, \cdots, e_{r+m}\}$$

is associated to the empty subset if $m = 0$ and to the subset of simple roots

$$\{\alpha_r, \cdots, \alpha_{r+m-1}\} \qquad \text{if } m \geq 1 \ .$$

Let us consider an element e_i in the basis Ξ of a_0. Let r be the smallest integer j such that $\overline{e_j} = \overline{e_r}$, and let $r+m$ be the largest. We will have $\overline{e_k} = \overline{e_i}$ for any k such that $r \leq k \leq r+m$. If $m = 0$, it is clear. Observe that if $m = 0$, the two simple consecutive roots α_{i-1} and α_i where e_i appears are outside Θ. Now, let $m \geq 0$, the root $e_r - e_{r+m}$ has a trivial projection on a_Θ and therefore by Lemma 1.3 all the simple roots that occur in the expression of this root shall be in Θ. As a result, the roots $\alpha_k = e_k - e_{k+1}$ belong to Θ for any k such that $r \leq k \leq r+m-1$ and we have :

$$\overline{e_k} = \frac{e_r + e_{r+1} + \cdots + e_{r+m}}{m+1}$$

for all k such that $r \leq k \leq r+m$. Indeed, this expression of $\overline{e_k}$ is then orthogonal to all the roots $\alpha_k = e_k - e_{k+1}$ for any k such that $r \leq k \leq r+m-1$.

Such a chain of simple roots is a connected component of length m of the Dynkin diagram associated to Θ. We have observed that such a connected component is empty when e_r is orthogonal to all the elements in Θ in which case $m = 0$ i.e the two consecutive simple roots α_{r-1} and α_r are outside Θ. If e_r is associated to a length m connected component of Θ and therefore belongs to an ordered part of cardinal $m+1$ of Ξ, the square of the length of $\overline{e_r}$ is :

$$||\overline{e_r}||^2 = \frac{1}{m+1} \ .$$

Let us consider three vectors e_r, e_s and e_t whose projections $\overline{e_r}, \overline{e_s}$ and $\overline{e_t}$ are distinct and are associated to three components of Θ of type A_m, A_p and A_q. Let $\alpha = e_i - e_j$ a root whose projection

$$\overline{\alpha} = \pm(\overline{e_r} - \overline{e_s}) \ .$$

$$||\overline{\alpha}||^2 = \frac{1}{m+1} + \frac{1}{p+1} \ .$$

Let us consider a root $\beta = e_k - e_l$ whose projection is

$$\overline{\beta} = \pm(\overline{e_s} - \overline{e_t})$$

we obtain

$$||\overline{\beta}||^2 = \frac{1}{p+1} + \frac{1}{q+1}$$

and the square of the scalar product of $\overline{\alpha}$ and $\overline{\beta}$ is

$$\left(< \overline{\alpha}, \overline{\beta} >\right)^2 = \frac{1}{(p+1)^2} \ .$$

Thus we have :

$$C = \left(\frac{1}{\cos(\overline{\alpha}, \overline{\beta})}\right)^2 = \left(1 + \frac{p+1}{m+1}\right)\left(1 + \frac{p+1}{q+1}\right),$$

and if we assume $||\overline{\beta}|| \geq ||\overline{\alpha}||$ i.e $q \geq m$, we have :

$$R = \frac{||\overline{\alpha}||^2}{||\overline{\beta}||^2} = \frac{\left(1 + \frac{p+1}{m+1}\right)}{\left(1 + \frac{p+1}{q+1}\right)}.$$

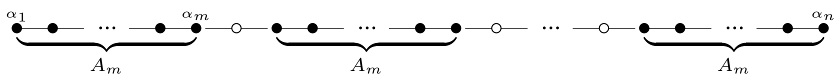
Then

$$\frac{C}{R} = \left(1 + \frac{p+1}{q+1}\right)^2 \in \{2^2, 1, (2/3)^2\} \qquad \text{and} \qquad CR = \left(1 + \frac{p+1}{m+1}\right)^2 = 4 \; .$$

The only possible case is $C/R = 4$ and thus $R = 1$ and $C = 4$. This implies $m = p = q$ and $\{\overline{\alpha}, \overline{\beta}\}$ generate a root system of type A_2 : $\pm(\overline{e_r} - \overline{e_s})$, $\pm(\overline{e_s} - \overline{e_t})$ and $\pm(\overline{e_r} - \overline{e_t})$.

LEMMA 2.1. *If Σ is of type A_n the only irreducible subsystems appearing in Σ_Θ are of type A. To have a root system of rank the dimension d of a_Θ it is necessary if $d > 1$, that the Dynkin diagram of Θ be a disjoint union of $d+1$ connected components of type A_m with $m \geq 0$, the intervals between each such component being of length one :*

$$n + 1 = (m+1)(d+1)$$



This corresponds to a partition of the ordered basis Ξ in an union of $d+1$ ordered parts of cardinal $m+1$:

$$\Xi = \Xi_1 \cup \cdots \cup \Xi_{d+1}$$

where

$$\Xi_r = \{e_{(r-1)(m+1)+1} \cdots e_{r(m+1)}\} \; .$$

In this case Σ_Θ is of type A_d.

PROOF. An irreducible subsystem is necessarily generated by the projections of roots of the form $\overline{\alpha} = \overline{e_i} - \overline{e_j}$ where the vectors $\overline{e_*}$ are all of the same length ; when we order these vectors following the $d+1$ indices, we obtain a basis of a subspace b_0 of a_0 containing a subspace b of codimension one in which the $\overline{e_i} - \overline{e_j}$ generate a system of type A. The rest of the corollary follows easily. $\square$

2.2 The case where Σ is of type B_n

In this case, the basis of a is constituted of the e_i for $i \in \{1, \cdots, n\}$ and the elements in Σ are the $\pm e_i$ and the $\pm e_i \pm e_j$ and Δ is formed of the $\alpha_i = e_i - e_{i+1}$ for $i \leq n-1$ and of $\alpha_n = e_n$. The set Θ is an union of irreducible components which are all of type A except for at most one which is of type B_k.

We distinguish two cases according to whether e_n belongs to Θ or not, i.e according to whether one of the components is of type B or not (case $k = 0$).

Case 1 ($k = 0$) : $e_n \notin \Theta$. In this case Θ is an union of components of type A. As in the previous case, let us consider three vectors e_r, e_s and e_t whose non-trivial projections $\overline{e_r}$, $\overline{e_s}$ ans $\overline{e_t}$ are distinct and associated to three components Θ of type A_m, A_p and A_q. Let us consider the roots of the form $\alpha = \pm e_i \pm e_j$ and $\beta = \pm e_k \pm e_l$ and let us suppose their projections write

$$\overline{\alpha} = \pm(\overline{e_r} \pm \overline{e_s}) \qquad \text{and} \qquad \overline{\beta} = \pm(\overline{e_s} \pm \overline{e_t}) \; .$$

The projections are non-trivial, non-collinear, and non-orthogonal. The computations done in the previous subsection show that this family of vectors form a root system if and only if $m = p = q$. We also have in the projection of Σ the vectors of the form :

$$\overline{\gamma} = \pm \overline{e_v} \qquad \text{for } v \in \{r, s, t\}$$

Thus a system of type B_3. Furthermore, $m \geq 1$, we also have in the projection of Σ, vectors of the form :

$$\overline{\delta} = \pm 2\overline{e_v} \qquad \text{for } v \in \{r, s, t\}$$

and in the end we obtain a root system of type BC_3.

Let us consider now two roots $\alpha = \pm e_i \pm e_j$ and $\delta = \pm e_k$ whose projections write $\overline{\alpha} = \pm(\overline{e_r} \pm \overline{e_s})$ and $\overline{\delta} = \pm \overline{e_s}$. We observe that

$$||\overline{\alpha}||^2 = \frac{1}{m+1} + \frac{1}{p+1} \qquad \text{and} \qquad ||\overline{\delta}||^2 = \frac{1}{p+1} \ .$$

Further $||\overline{\alpha}|| > ||\overline{\delta}||$ and we have :

$$\left(< \overline{\alpha}, \overline{\delta} >\right)^2 = \frac{1}{(p+1)^2} \ .$$

Therefore

$$C = \left(\frac{1}{\cos(\overline{\alpha}, \overline{\delta})}\right)^2 = 1 + \frac{p+1}{m+1} \qquad \text{and} \qquad R = \frac{||\overline{\alpha}||^2}{||\overline{\delta}||^2} = 1 + \frac{p+1}{m+1}$$

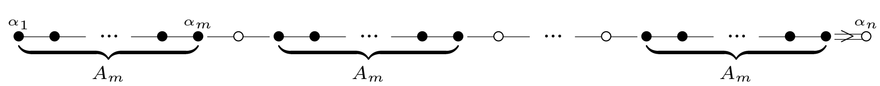
So we have $C = R$ which forces $C = R = 2$ and we recover the condition $m = p$.

Let us also remark that two short roots (that is of type $\pm\overline{e_r}$) or long (that is of type $\pm 2\overline{e_r}$) (the length being relative to the length of roots $\pm(\overline{e_s} \pm \overline{e_t})$) are necessarily proportional or orthogonal. This observation excludes the occurrence of a root system of type F_4. Combining these observations, we see that except if $m = 0$ (trivial case where the projection is the identity), we obtain maximal subsystems of type BC (in particular non reduced).

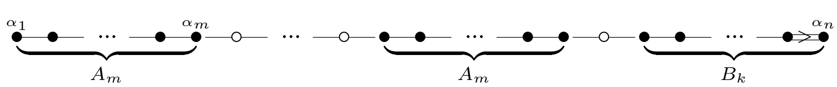
Case 2 ($k \geq 1$) : $e_n \in \Theta$. The projection on the orthogonal complement of e_n gives a system B_{n-1} and reiterating this procedure when Θ contains B_k, we recover the case 1 previously treated for B_{n-k}. In conclusion, we have proven :

Lemma 2.2. *The maximal subsystems are of type B or BC. These contain the subsystems of type B, C or D of the same rank. Let us assume e_n belongs to a connected component of length k (then of type B_k), with $k \geq 0$ (the case $k = 0$ is the case in which e_n does not belong to Θ). Then, the set Σ_Θ contains a system of rank equal to the dimension d of $\mathfrak{a}_\Theta$ if the other components are all of the same length m (and type A_m), the intervals between any of these components being of length one with $n - k = (m+1)d$. The projected system is of type BC_d except if $m = 0$ in which case we obtain B_{n-k}.*

Case 1 : $k = 0$, $n = d(m+1)$: the projected system is of type BC_d if $m \geq 1$.



Case 2 : $k \geq 1$, $n - k = d(m+1)$: the projected system is of type BC_d.



This corresponds to a partition of the ordered basis Ξ of cardinal n in a union of $d+1$ ordered parts

$$\Xi = \Xi_1 \cup \cdots \cup \Xi_{d+1}$$

where

$$\Xi_r = \{e_{(r-1)(m+1)+1} \cdots e_{r(m+1)}\} \qquad \textit{for } 1 \leq r \leq d \textit{ and } \qquad \Xi_{d+1} = \{e_{d(m+1)+1} \cdots e_{d(m+1)+r}\}.$$

2.2.1 The case where Σ is of type C_n

In this case the basis of a is formed with the e_i for $i \in \{1, \cdots, n\}$ and the elements of Σ are the $\pm 2e_i$ and the $\pm e_i \pm e_j$; moreover Δ is constituted of the $\alpha_i = e_i - e_{i+1}$ for $i \leq n-1$ and of $\alpha_n = 2e_n$. The set

Θ is an union of irreducible components all of type A except for at most one of type C_k. We distinguish two cases whether e_n belongs or not to Θ.

Case 1 ($k=0$) : $2e_n \notin \Theta$. In this case Θ is an union of components of type A. As in the case of Σ of type A_n, let us consider three vectors e_r, e_s and e_t whose projections (which are non-zero) $\overline{e_r}$, $\overline{e_s}$ et $\overline{e_t}$ are distinct and associated to three components of Θ of type A_m, A_p and A_q and roots $\alpha = \pm e_i \pm e_j$ and $\beta = \pm e_k \pm e_l$ whose projections are

$$\overline{\alpha} = \pm(\overline{e_r} \pm \overline{e_s}) \qquad \text{and} \qquad \overline{\beta} = \pm(\overline{e_s} \pm \overline{e_t})$$

They will constitute a root system if and only if $m = p = q$. Then we obtain a root system of type C_3 constituted of the $\pm(\overline{e_r} \pm \overline{e_s})$, $\pm(\overline{e_s} \pm \overline{e_t})$, $\pm(\overline{e_r} \pm \overline{e_t})$ and $\pm 2\overline{e_v}$ for $v \in \{r, s, t\}$.

Let us now consider the two roots $\alpha = \pm e_i \pm e_j$ and $\beta = \pm 2e_k$ whose projections write

$$\overline{\alpha} = \pm\overline{e_r} \pm \overline{e_s} \qquad \text{and} \qquad \overline{\beta} = \pm\overline{2e_s}\ .$$

$$||\overline{\alpha}||^2 = \frac{1}{m+1} + \frac{1}{p+1} \qquad \text{and} \qquad ||\overline{\beta}||^2 = \frac{4}{p+1}$$

and therefore

$$\left(<\overline{\alpha}, \overline{\beta}>\right)^2 = \frac{4}{(p+1)^2} \qquad \text{and} \qquad C = \left(\frac{1}{\cos(\overline{\alpha}, \overline{\beta})}\right)^2 = \left(1 + \frac{p+1}{m+1}\right)\ .$$

If we assume $||\overline{\beta}|| \geq ||\overline{\alpha}||$ we have

$$R = \frac{||\overline{\beta}||^2}{||\overline{\alpha}||^2} = \frac{4}{\left(1 + \frac{p+1}{m+1}\right)}$$

and $CR = 4$. All the cases are *a priori* possible.

If $C = 2$ et $R = 2$ then we necessarily have $p = m$. The vectors $\overline{\alpha}$ and $\overline{\beta}$ are the basis of a root system of a type C_2 where $\overline{\beta}$ is the long root. The roots are

$$\pm\overline{\alpha} = \pm(\overline{e_r} - \overline{e_s}) \quad , \quad \pm\overline{\beta} = \pm 2\overline{e_s} \quad , \quad \pm(\overline{\alpha} + \overline{\beta}) = \pm(\overline{e_r} + \overline{e_s}) \quad \text{and} \quad \pm(2\overline{\alpha} + \overline{\beta}) = \pm 2\overline{e_r}\ .$$

The case $C = 4$ and $R = 1$ implies

$$(p+1) = 3(m+1) \qquad \text{and therefore} \qquad p = 3m + 2.$$

Then $||\overline{\alpha}||$ and $||\overline{\beta}||$ constitute the basis of a a root system of type A_2 whose roots are

$$\pm\overline{\alpha} = \pm(\overline{e_r} - \overline{e_s}) \qquad , \qquad \pm\overline{\beta} = \pm 2\overline{e_s} \qquad \text{and} \qquad \pm(\overline{\alpha} + \overline{\beta}) = \pm(\overline{e_r} + \overline{e_s})$$

but the vector $\pm 2\overline{e_r}$ does not contribute to this system.

Finally if $C = 4/3$ we have

$$3(p+1) = (m+1) \qquad \text{and therefore} \qquad m = 3p + 2.$$

This forces $R = 3$ which is a configuration of simple roots for a root system of type G_2 where $\overline{\beta}$ is the long root. However, Σ_Θ does not contain all the necessary roots for such a system ; indeed the root

$$\overline{\beta} + 3\overline{\alpha} = 3\overline{e_r} - \overline{e_s}$$

is not obtained.

Let us assume $||\overline{\alpha}|| \geq ||\overline{\beta}||$ we have $C/R = 4$ and we recover the case $C = 4$, $R = 1$ and therefore $(p+1) = 3(m+1)$.

Case 2 ($k \geq 1$) : $e_n \in \Theta$. The projection on the orthogonal complement of e_n gives a system of type BC_{n-1}. And, reiterating this procedure, we recover the case of BC_{n-k} which can be treated using our previous considerations on B_{n-k} and C_{n-k}.

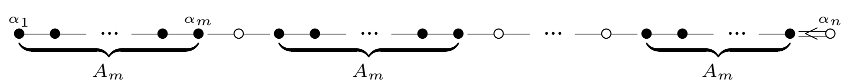
To conclude, we have proved :

Lemma 2.3. *The maximal subsystems are of type A, B, C, D. Let us assume $2e_n$ belongs to a connected component of length k (and type C_k), with $k \geq 1$. The projection on the orthogonal of this component is a root system of type BC_{n-k}. We recover the case where $k = 0$, i.e where e_n does not belong to Θ for a system of type BC.*

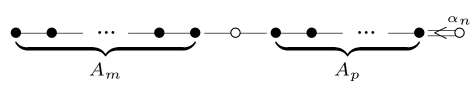
If $d \geq 3$ the set Σ_Θ contains a system of rank equal to the dimension d of $\mathfrak{a}_\Theta$ if the other components are all of the same length $m \geq 0$ (and type A_m), the intervals between any of these components being of length one with $n - k = (m+1)d$, then the projected system is of type BC_d (or C_n if $k = 0$ and $m = 0$, trivial case excluded).

If $d = 2$ we obtain either BC_d when the two components of type A are of length m or A_2 when $(p+1) = 3(m+1)$.

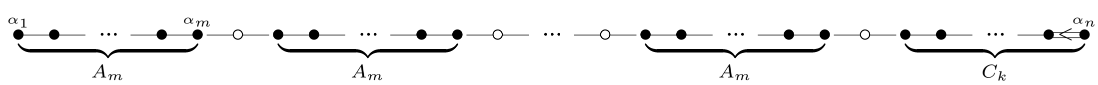
The case $k = 0$, with $n = (m+1)d$ and projected system C_d



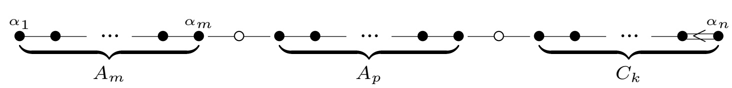
The case $k = 0$, with $p = 3m + 2$ and $n = 4(m+1)$, and projected system containing A_2



The case $k \geq 1$, with $n - k = (m+1)d$ and projected system BC_d



The case $k \geq 1$, with $p = 3m + 2$ and $n - k = 4(m+1)$, the projected system contains A_2



2.2.2 The case where Σ is of type D_n

With the notations analogous to the previous cases the roots are the $\pm e_i \pm e_j$ and Δ is constituted of $\alpha_i = e_i - e_{i+1}$ for $i \leq n-1$ and of $\alpha_n = e_{n-1} + e_n$.

Case 1 : $\alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_{n-1} + e_n$ are in Θ and the orthogonal complement of Θ admits the e_i for $1 \leq i \leq n-2$ as a basis. The projection on the orthogonal of e_n and e_{n-1} contain the $\pm e_i \pm e_j$ along with the roots $\pm e_i$ for i and j between 1 and $n-2$ obtained projecting the $\pm(e_i - e_n)$. We, therefore, obtain the system B_{n-2} already considered above.

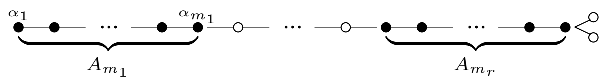
Case 2 : $\alpha_{n-1} = e_{n-1} - e_n$ is in Θ but $e_{n-1} + e_n$ is not. As in the case of root system of type B_n let us consider the three vectors e_r, e_s and e_t whose non-zero projections $\overline{e_r}$, $\overline{e_s}$ et $\overline{e_t}$ are distinct and associated to three components of Θ of type A_m, A_p and A_q. Once projected we find the $\pm\overline{e_r} \pm \overline{e_s}$ and $\pm\overline{e_s} \pm \overline{e_t}$. We also have

$$2\overline{e_r} = \overline{e_r} + \overline{e_{r+1}} = 2\overline{e_{r+1}}$$

if $\alpha_r = e_r - e_{r+1}$ belongs to a connected component of Θ. Therefore Σ_Θ contains a root system of type C_d if all the connected components of Θ are of the same cardinal m with $n = d(m+1)$.

Case 2' : analogous to the case 2 when exchanging e_n with $-e_n$.

Case 3 : Neither $\alpha_{n-1} = e_{n-1} - e_n$ nor $\alpha_n = e_{n-1} + e_n$ are in Θ.



We, therefore, have either an analogous situation to the one treated for A_n, or we consider $\overline{\alpha} = \pm\overline{e_{n-1}}\pm\overline{e_n}$ and $\overline{\beta} = \overline{e_s} \pm \overline{e_{n-1}}$.

In this case we have $e_n = \overline{e_n}$ and therefore with the now familiar notations

$$R = \frac{(1 + (p+1))}{(1 + \frac{p+1}{m+1})} \qquad \text{and} \qquad C = (1 + (p+1))\left(1 + \frac{p+1}{m+1}\right)$$

Therefore

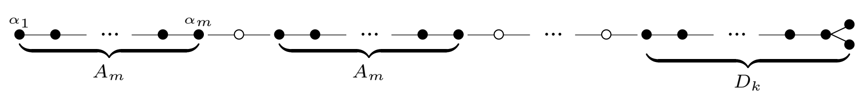
$$\frac{C}{R} = (1 + \frac{p+1}{m+1})^2$$

which forces $R = 1$ and $C = 4$; thus $p = m = 0$. The existence of a system of maximal rank in the projection for a configuration of this sort forces $m_i = 0$ for any i, that is Θ is empty, a case which is possible but trivial hence excluded *a priori*.

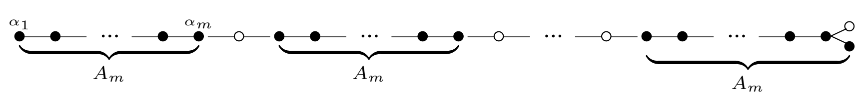
To sum up, we have proven the :

LEMMA 2.4. *For a system of type D the subsystems in the projection are of type A, B, C or D. If $\alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_{n-1} + e_n$ are in Θ and if the others components of Θ are all of type A_m, the interval between two such components are of length one, with $n - k = (m+1)d$, then there exists a system of type BC_d in Σ_Θ. In the case 2 or 2', the projection contains a system of maximal rank of type C_d if all the components are of type A_m and if $n = (m+1)d$.*

The case 1 : $D_k \subset \Theta$ with $k \geq 2$; we recover the case of B_{n-k}.



The case 2 (or 2') : The projection contains a rank maximal system of type C_d if all the components are of type A_m and if $n = (m+1)d$.



3 Exceptional root systems

As opposed to the previous treatment in the context of classical root systems, the case of exceptional groups requires a cumbersome case-by-case analysis, which leads to the following result :

THEOREM (1.2). Let Σ be an irreducible root system of exceptional type (i.e of type E, F_4 or G_2). If Σ_Θ contains an irreducible system of rank d it is necessarily of classical type, except in the case of the orthogonal projection to any single root of the roots of E_8 where E_7 appears in the projection.

As a result of this case-by-case analysis, we also give the most exhaustive description of subsystems of Σ_Θ of rank greater or equal to 2. In most of those cases, we exhibit a basis for the root system of largest rank obtained. We have verified for each case that those subsystems were indeed of largest rank in the projection although we have not written systematically all justifications.

Remark 3.1. Let us explain here two important observations made in the case of exceptional root systems.

1. In almost all cases, in order to obtain a subsystem $\mathcal{S}$ of Σ_Θ of rank d, one has to consider a basis $\Delta_\mathcal{S}$ constituted of *at least some projections of non-simple roots.* This observation contrasts with classical root systems, where as the reader has noticed, $\Delta_\mathcal{S}$ is constituted of projections of simple roots except possibly for the last root of $\Delta_\mathcal{S}$, i.e the one on the extreme right of the Dynkin diagram constituted from those simple roots.

2. In the root systems of type E, the only root systems of rank d appearing in the projection are of type A or D.

The results of Theorem 1.2 rely on the three following Lemmas :

Lemma 3.2. *Let Σ be an irreducible root system. Any two subsystems Θ and Θ' with only one root of Σ of the same length are conjugated under the Weyl group W. If w is a Weyl group element sending Θ on Θ', we have $\Sigma_{\Theta'} = w(\Sigma_\Theta)$.*

Proof. First by an equivalent of the incomplete basis theorem for root systems (see [1], Chap VI, 1, Proposition 24), it is always possible to complete any root to a basis of the root system. Hence it is enough to consider the case of simple roots.
By a classical lemma (see for instance Lemma C in [6], III, 10), all roots of the same length are conjugate under W. Let us consider any two basis' roots of the same length, α, and $\beta = w(\alpha)$. Let us assume we project all vectors in Σ orthogonally to $\Theta = \{\alpha\}$, and $\Theta' = \{\beta\}$. Since the Weyl group is a subgroup of the isometry group of the root system, it preserves lengths and angles. Therefore applying first $w \in W$ to all roots and projecting with respect to β yields the same result as applying $w \in W$ to Σ_Θ. □

Remark 3.3. Under the conditions of Lemma 3.2, if Σ_Θ contains a maximal root system of rank d, then it doesn't depend on the choice of the root generating Θ ; It is enough to determine Σ_Θ for any root in Σ. Indeed the ratios and angles between roots in the maximal root system occurring in Σ_Θ and $w(\Sigma_\Theta)$ are the same.

Lemma 3.4. *Let Σ be the root system of an exceptional group. No root system of type G_2 appears as a subsystem of rank d in Σ_Θ.*

Proof. The conditions to obtain G_2 as a subsystem of rank 2 in Σ_Θ are :

1. The cardinal of $\Delta - \Theta$ to be equal to two.

2. Considering the projections α and β of two roots in Σ, the values of $C = 4/3$ and $R = 3$.

These conditions are verified in the case $E_6, \Theta = \{\alpha_2, \alpha_3\} \cup \{\alpha_5, \alpha_6\}$ studied in Lemma 3.8. The squared norm equals to 6/9 is specific to E_6 and will not appear in the case $E_7, \Theta = \{\alpha_2, \alpha_3\} \cup \{\alpha_5, \alpha_6, \alpha_7\}$ neither in the case $E_8, \Theta = \{\alpha_2, \alpha_3\} \cup \{\alpha_5, \alpha_6, \alpha_7, \alpha_8\}$. It might be possible to obtain one root of squared norm containing a factor 3 (for example, in $E_8, \Theta = \{\alpha_2, \alpha_3\} \cup \{\alpha_1, \alpha_6, \alpha_7, \alpha_8\}$, the root $e_3 + e_4$ has squared norm equal to 4/3). The case to consider are when there is two consecutive roots in the Levi (such as α_2 and α_3) completed by others which are not their immediate neighbours (second branch), but the number of roots in the second branch always lead to inappropriate factors R and C. This observation along with the results in F_4 presented in the Subsection 3.1 yield the result. □

Lemma 3.5. *Let Σ be the root system of an exceptional group. No root system of type F_4 appears as a subsystem of rank d in Σ_Θ.*

Proof. We are looking for ratio $R = 2$ and $C = 2$ and in particular for roots with squared norms equal to 1. We consider here in particular the projections of roots of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ to complete the details given in the case by case analysis below.

In E_6, if $\Theta = \{\alpha_1, \alpha_i\}$, the projections of roots of the form $\frac{1}{2}[e_0 \pm e_1 \pm e_2 \cdots \pm e_6 - e_7]$ give only roots of squared norms 3/2 or 2. In case $\alpha_1 \notin \Theta = \{\alpha_k, \alpha_i\}$, some roots of the form $\frac{1}{2}[e_0 - e_{i,1} + e_{i,2} - e_7]$ or $\frac{1}{2}[e_0 + e_{i,1} - e_{i,2} - e_7]$ (call a root of this sort β) whose squared norms equal one appear in the projection. There are also some roots of the form $\frac{1}{2}[e_0 \pm e_1 \pm e_2 \cdots \pm e_6 - e_7]$ (call a root of this sort α) whose squared norms equal two. Considering the scalar product between α and $\beta :$ they are either orthogonal, or $C = 2$ and $R = 2$. The issue is that in the F_4 basis, one needs two roots of norm 1 whose scalar product is -1/2 ; and their sum needs to appear in the projection. Here making variations on $\pm\frac{1}{2}[e_0 - e_{i,1} + e_{i,2} - e_7]$ is the only option to have roots of norm 1, and they don't satisfy the latter conditions.

In E_7, one considers the roots in Θ to be "two consecutive plus one" (such as $\Theta = \{\alpha_2, \alpha_3, \alpha_7\}$ or $\Theta = \{\alpha_2, \alpha_5, \alpha_6\}$) and then roots of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ have norms 2 or 3/2. Or one considers three consecutive roots (such as $\Theta = \{\alpha_5, \alpha_6, \alpha_7\}$) where roots of norms 1 of the form the

$$\frac{1}{2}[\pm e_i \pm e_j \pm e_k \pm e_l]$$

(for instance $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \pm e_3]$) appear in the projections. Obviously, since they have to be orthogonal to all the roots in Θ and are projections of roots in E_7 (constraints on the number of negative signs), as opposed to the roots of F_4 of this form, not all of the 2^4 roots of this form are obtained !

In E_8, the case of $\Theta = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ (resp. $\Theta = \{\alpha_1, \alpha_3, \alpha_5, \alpha_8\}$) yields roots of norms 1 among the $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \pm e_3]$ (resp. the $\frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_6]$). Again, since they have to be orthogonal to all the roots in Θ and are projections of roots in E_8 (constraints on the number of negative signs), as opposed to the roots of F_4 of this form, not all of the 2^4 roots of this form are obtained ! Furthermore, as there are roots of the form $e_i - e_j$ but no root of the form e_i ; obtaining the 48 roots of an F_4 root system is not possible. □

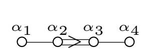
The following lemma will be used to study various cases below.

Lemma 3.6. *If $\overline{\alpha_i} = e_i - \frac{e_{i-1}+e_{i-2}}{2}$ (resp. $\overline{\alpha_i} = \frac{e_i+e_{i+1}}{2} - e_{i-1}$), the roots $\{\overline{\alpha_i}, \overline{\alpha_{i+1}} = \alpha_{i+1}\}$ (resp. $\{\overline{\alpha_i}, \overline{\alpha_{i-1}} = \alpha_{i-1}\}$) cannot be the simple roots of a root system in Σ_Θ.*

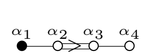
Proof. The squared length of $\overline{\alpha_i}$ is 3/2. The squared length of $\overline{\alpha_{i+1}}$ (resp. $\overline{\alpha_{i-1}}$) is 2. The ratio is 4/3. $\frac{1}{C} = \frac{1}{3/2 \times 2} = 1/3$ therefore $C = 3$. This is not a valid value for C to obtain a root system of rank 2. □

3.1 The case F_4

In this case a has for basis the e_i for $i \in \{1, 2, 3, 4\}$ and the elements of Σ are the $\pm e_i$, the $\pm e_i \pm e_j$ $(i \neq j)$ and the $1/2(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$. Furthermore, Δ is of the form : $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3$ and $\alpha_4 = -1/2(e_1 + e_2 + e_3 + e_4)$. There are ten Θ with 1 or 2 elements, we examine each case separately.



Case 1 : $\Theta = \{\alpha_1\}$



where

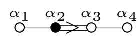
$$\overline{\alpha_4} = \alpha_4$$

$$\overline{\alpha_3} = \alpha_3$$

$$\overline{\alpha_2} = \frac{e_1 + e_2}{2} - e_3$$

The squared norms of $\overline{\alpha_4}$ and $\overline{\alpha_3}$ are 1, whereas the squared norm of $\overline{\alpha_2}$ is $3/2$.
The roots α_3, α_4, and $e_1 + e_2$ form the basis of a root system of type C_3.

Case 2 : $\Theta = \{\alpha_2\}$



where

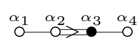
$$\overline{\alpha_1} = e_1 - \frac{e_2 + e_3}{2}$$

$$\overline{\alpha_3} = \frac{e_2 + e_3}{2}$$

$$\overline{\alpha_4} = \alpha_4$$

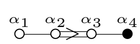
The squared norms of $\overline{\alpha_1}$ and $\overline{\alpha_3}$ are respectively $3/2$ and $1/2$, whereas the squared norm of $\overline{\alpha_4}$ is 1. The roots e_1, α_4, and $e_2 + e_3$ form the basis of a root system of type C_3.

Case 3 : $\Theta = \{\alpha_3\}$



We observe that Σ_Θ contains B_3 with the $\pm e_i$, and the $\pm e_i \pm e_j$ $(i \neq j)$ for i and j in $\{1, 2, 4\}$ as maximal rank subsystem.
The projection of α_4 and the analogous roots $1/2(\pm e_1 \pm e_2 \pm e_4)$ do not contribute to a highest rank subsystem.

Case 4 : $\Theta = \{\alpha_4\}$



We have

$$\overline{e_i} = e_i - \frac{(e_1 + e_2 + e_3 + e_4)}{4}$$

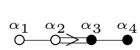
and the projection of $\Delta - \Theta$ is made of

$$\overline{\alpha_1} = e_1 - e_2 \quad , \quad \overline{\alpha_2} = e_2 - e_3 \quad \text{and} \quad \overline{\alpha_3} = e_3 - \frac{(e_1 + e_2 + e_3 + e_4)}{4} = \frac{(3e_3 - e_1 - e_2 - e_4)}{4}$$

whose squared lengths are respectively 2, 2 and 12/16=3/4.

The $\pm(\overline{e_i} - \overline{e_j}) = \pm(e_i - e_j)$ with $i \neq j$ in $\{1, 2, 3\}$ constitute a root system of type A_2.
Since the root $\frac{(e_1 - e_2 + e_3 - e_4)}{2}$ is orthogonal to α_4, it appears in the projection, together with $e_2 - e_3$ they form the basis of a root system B_2 appearing in the projection. Further, $\{e_3 - e_1, e_2 - e_3, \frac{(e_1 - e_2 + e_3 - e_4)}{2}\}$ constitute the basis of a root system of type B_3, and all the sums of this basis' roots occur in the projection. Therefore B_3 is therefore of highest rank in the projection.

Case 5 : $\Theta = \{\alpha_3, \alpha_4\}$

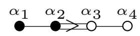


The projection of $\Delta - \Theta$ is :

$$\overline{\alpha_1} = \alpha_1 = e_1 - e_2 \qquad \text{and} \qquad \overline{\alpha_2} = e_2 - \frac{e_1 + e_2 + e_4}{3} = \frac{2e_2 - e_1 - e_4}{3}$$

whose squared lengths are respectively 2 and 6/9=2/3. Therefore $R = 3$ and $C = 4/3$, which is compatible with a root system of type G_2. However, $\overline{\alpha_1} + \overline{\alpha_2} = \frac{2e_1 - e_2 - e_4}{3}$ is not the projection of a root in Σ.

Case 6 : $\Theta = \{\alpha_1, \alpha_2\}$

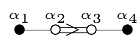


The projection of $\Delta - \Theta$ is :

$$\overline{\alpha_3} = \overline{e_3} = \frac{(e_1 + e_2 + e_3)}{3} \qquad \text{and} \qquad \overline{\alpha_4} = \alpha_4 \;\; \text{whose squared lengths are respectively 1/3 and 1.}$$

C=4/3, R=3, these are the conditions for a configuration of type G_2.
However, we notice that $\overline{\alpha_4} + \overline{\alpha_3}$ and $\overline{\alpha_4} + 2\overline{\alpha_3}$ do not occur in the projection.

Case 7 : $\Theta = \{\alpha_1, \alpha_4\}$

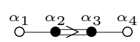


The projection of $\Delta - \Theta$ is :

$$\overline{\alpha_3} = e_3 - \frac{e_1 + e_2 + e_3 + e_4}{4} = \frac{(-e_1 - e_2 - e_4 + 3e_3)}{4} \qquad \text{and} \qquad \overline{\alpha_2} = \frac{(e_1 + e_2)}{2} - e_3$$

whose squared lengths are respectively 3/2 and 3/4. The value of C is 9/8. No root system of rank 2 satisfies this condition. Looking at projections of non-simple roots does not yield any further potential basis of root system in the projection.

Case 8 : $\Theta = \{\alpha_2, \alpha_3\}$

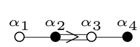


The projection of $\Delta - \Theta$ is :

$$\overline{\alpha_4} = -\frac{e_1 + e_4}{2} \qquad \text{and} \qquad \overline{\alpha_1} = e_1$$

whose squared lengths are respectively 1/2 and 1. The ratio of squared lengths is 2, and the squared scalar product is 1/4, therefore $C = 2$. We observe Σ_Θ contains B_2 with the $\pm e_i$, and the $\pm e_i \pm e_j$ $(i \neq j)$ for i and j in $\{1, 4\}$ as highest rank subsystem.

Case 9 : $\Theta = \{\alpha_2, \alpha_4\}$



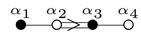
$$\overline{\alpha_1} = e_1 - \frac{e_2 + e_3}{2} \;\; \text{whose squared length is } 3/2,$$

$$\alpha_3 = \frac{e_2 + e_3}{2} - \frac{e_1 + e_2 + e_3 + e_4}{4} \;\; \text{whose squared length is } 1/2 + 1/4 = 3/4,$$

$$1/C = \frac{1/4}{3/2 \times 3/4} = 2/9, \quad \text{hence} \quad C = 9/2$$

Looking at projections of non-simple roots does not yield any further potential basis of root system in the projection. No root system can be obtained in Σ_Θ.

Case 10 : $\Theta = \{\alpha_1, \alpha_3\}$



$$\overline{e_1} = \overline{e_2}$$

$$\overline{\alpha_2} = \overline{e_2} \qquad \text{and} \qquad \overline{\alpha_4} = -\frac{e_1 + e_2 + e_4}{2}$$

The squared length of $\overline{\alpha_4}$ is 3/4.
We have $R = 3/2$ and $1/C = \frac{1/4}{1/2 \times 3/4} = \frac{2}{3}$. Therefore $C = 3/2$ and there is no root system satisfying such condition.
The root $e_1 + e_2$ of squared norm 2 also appears in the projection but the values of C obtained while considering it together with $\overline{\alpha_4}$ or $\overline{\alpha_2}$ is incompatible with any root system. No root system can be obtained in Σ_Θ.

$\Theta = \{..\}$	squared lengths of projected roots	chosen roots to calculate C and R	C and R	root system of highest rank obtained (of rank $\geq$ 2)
α_1	3/2 and 1			C_3
α_2	3/2,1/2,1			C_3
α_3				B_3
α_4	2 and 3/4			B_3
α_3,α_4	2 and 2/3	$\overline{\alpha_1}$,$\overline{\alpha_2}$		None
α_1,α_2	1/3, and 1	$\overline{\alpha_3}$,$\overline{\alpha_4}$	4/3 and 3	None
α_1,α_4	$\overline{\alpha_3}$ has squared length 3/2, $\overline{\alpha_2}$ 3/4	$\overline{\alpha_2}$,$\overline{\alpha_3}$	C=9/8	None
α_2,α_3	1 and 1/2	$\overline{\alpha_1}$,$\overline{\alpha_4}$	C=2	B_2
α_2,α_4	3/2 and 3/4		C=9/2	None
α_1,α_3	1 and 3/4		R=3/2, C=3/2	None

Table 1: Roots system occurring in Σ_Θ for Σ of type F_4

3.2 The case of root systems of type E

We will use the E basis as proposed by Jean-Pierre Labesse (see [7] and an unpublished note) : the details are given below (see also the remark 3.7). We say roots are of type A when they are of the form $\pm(e_i - e_j)$, of type D when they are of the form $\pm(e_i + e_j)$ and of type E when they are of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ with an even number of signs (and some further conditions in E_7 and E_6).

3.2.1 The occurrence of roots of type E in the projection

We need the expression $\alpha_1 = \frac{(e_0 - e_7) + e_1 + e_2 + e_3 - e_4 - e_5 - e_6}{2}$ here to elaborate on the constraints borne by the roots of type E occurring in the projections.
Let us consider a root of type E different from α_1 and call it β, either its scalar product to α_1 is -1, either

it is orthogonal to it.
In E_6, if β is positive since $e_0 - e_7$ is fixed, the product of $e_0 - e_7$ with itself gives 2, and we need the products over all the other indices to sum up to -6. If β is negative, the product of $e_0 - e_7$ with $e_7 - e_0$ gives -2, we need the products over all the other indices to sum up to -2= -4+2, the only option is to have two signs unchanged and four changed within the indices $\{1, \dots, 6\}$.

The second option is to have α_1 orthogonal to β. If β is positive, since the product of $e_0 - e_7$ with itself gives 2, we need the products over all the other indices to sum up to -2, and $-2 = 2-4$. If β is negative, the product of $e_0 - e_7$ with $e_7 - e_0$ gives -2, and then we need the products over all the other indices to sum up to 2, and $2 = 4-2$. Let us consider two examples : the roots $\frac{1}{2}[-e_0 - e_1 + e_2 + e_3 - e_4 - e_5 + e_6 + e_7]$ and $\frac{1}{2}[e_0 + e_1 - e_2 - e_3 + e_4 + e_5 - e_6 - e_7]$ are orthogonal to α_1.
In the case of E_7 and E_8, our sole constraint is that the number of negative signs in the expression of β is 4. To obtain $\langle \alpha_1, \beta \rangle = -1 = \frac{2-6}{4}$, one needs that among the signs in front of the e_i, two signs are the same than in α_1 and six change. It is also possible to have α_1 orthogonal to β when four signs in front of the e_i in the expression of β are different from the signs in the expression of α_1.
One observes, that at most three roots of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ can be orthogonal to each other. Therefore if Θ contains α_1, it is still possible to obtain two roots of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$, orthogonal one to another in the projection.

3.2.2 Occurrence of type D subsystems in the projection

To observe the occurrence of a type D_n root system in the projection, it is easier to work with the conventions of Bourbaki (see the tables at the end of [1]). We recall here the conventions of Bourbaki for E_6 (resp. E_7). We consider the hyperplane $\widetilde{V}$ of $\mathbb{R}^8$ whose points have coordinates ξ_i satisfying $\xi_6 = \xi_7 = -\xi_8$ (resp. orthogonal to $e_7 + e_8$ for E_7).

The positive roots are of the following form :

- $\pm e_i + e_j$ for $1 \leq i < j \leq 5$ (resp. ≤ 6 and along with $(e_8 - e_7)$).
- $\frac{1}{2}[e_8 - e_7 - e_6 \pm e_1 \pm e_2 \cdots \pm e_5]$ with an even number of negative signs. (resp. $\frac{1}{2}[e_8 - e_7 \pm e_1 \pm e_2 \cdots \pm e_5 \pm e_6]$ with an odd number of negative signs).

A system of simple roots is given by :

$$\alpha_1 = \frac{1}{2}[(e_1 + e_8) - (e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7)] \qquad \text{and} \qquad \alpha_i = [e_{i-1} - e_{i-2}] \qquad \text{for } 3 \leq i \leq 6$$

(resp $\alpha_i = [e_{i-1} - e_{i-2}]$ for $3 \leq i \leq 7$)

$$\text{and } \alpha_2 = e_1 + e_2$$

In the case of E_6 and $\Theta = \{\alpha_6\}$ a root system of type D_3 occurs in the projection. This subsystem has a basis made of projections of simple roots in E_6. If $\Theta = \{\alpha_i\}, i \neq \{1\ ;6\}$, we also obtain D_3 subsystem but their basis are not made only of projections of simple roots.

If we consider E_7, the case $\Theta = \{\alpha_7\}$ gives a root system of type D_4 in the projection whose basis is made of projections of simple roots ; the cases $\Theta = \{\alpha_i\}, i \neq \{1\ ;7\}$ let also root systems of type D_4 appear however their basis are not made only of projections of simple roots.

The roots of E_8 are the $\pm e_i \pm e_j$ for $1 \leq i < j \leq 8$ and the $\frac{1}{2}[\pm e_8 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_5 \pm e_6]$ with an even number of negative signs. If Θ contains only one simple root (which is not α_1) we obtain D_5 (when $\Theta = \{\alpha_8\}$, the basis of D_5 is made only of projections of simple roots). The root system D_5 is not the only option ; as one can observe in Section 3.2.5, D_7 occurs when $\Theta = \{\alpha_8\}$ while using another basis for E_8.

3.2.3 The case E_6

We come back to the conventions as established in Jean-Pierre Labesse's unpublished note (see the introduction of this subsection).
We consider the euclidean space $\widetilde{V}$ of dimension 8, equipped with a orthonormal basis indexed by the elements of $\mathbb{Z}/8\mathbb{Z}$

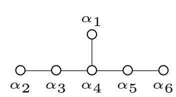
$$\{e_0, e_1, \cdots, e_6, e_7\}$$

such that e_0 will sometimes be denoted e_8. The roots of E_6 are the roots in E_7 orthogonal to $e_7 - e_8 = -(e_7 + e_0)$ (see the definitions of E_8 and E_7 in the next subsections). They are of the following form :

- $\pm(e_i - e_j)$ for $1 \leq i < j \leq 6$ or $i = 0$ and $j = 7$.
- $\pm\frac{1}{2}[(e_0 - e_7) \pm e_1 \pm e_2 \cdots \pm e_6]$

with the same number of $+$ and $-$ sign in the bracket. A system of simple roots is given by

$$\alpha_1 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7] \qquad \text{and} \qquad \alpha_{i+1} = [e_{i+1} - e_i] \qquad \text{for } 1 \leq i \leq 5 \; .$$



Remark 3.7. This depiction is different from the one given in Bourbaki : we have used a subsystem of the system E_8 as defined by Bourbaki, except that ϵ_8 is here e_0 and that we have an order -and therefore simple roots- which is (are) different(s). In particular, in our convention the roles of α_1 and α_2 in the Dynkin diagram are inverted. The correspondence is the following :

$$\text{Our notation} \qquad \longleftrightarrow \quad \text{Bourbaki's notation}$$

$$\alpha_1 = \tfrac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7] \longleftrightarrow \quad \alpha_2 = \epsilon_1 + \epsilon_2$$

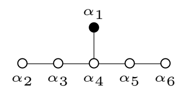
$$\alpha_2 = e_2 - e_1 \longleftrightarrow \quad \alpha_1 = \tfrac{1}{2}[\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8]$$

$$\alpha_{i+1} = e_{i+1} - e_i \longleftrightarrow \quad \alpha_{i+1} = \epsilon_i - \epsilon_{i-1} \qquad \text{for} \qquad 2 \leq i \leq 5$$

With our writing, it is easily seen that there exists an automorphism $\theta(e_i) = -e_{(7-i)}$; sending α_{i+1} on α_{7-i} for $1 \leq i \leq 5$ and it fixes α_1 and α_4.
One notices that under this convention, there are no roots of type D (see the beginning of Subsection 3.2 for this terminology) in the root systems of E_6 and E_7 ; this is why we dealt with the occurrence of type D root systems in the projection earlier on.

Following 3.3, we know that if Θ contains only one root, whatever is this one root, the root system we obtained in the projection should always be the same. Below, we illustrate this result, and exhibit basis for the A_5 root system appearing in the projection whenever Θ contains a simple root of E_6.

Case $\Theta = \{\alpha_1\}$



The projection of $\Delta - \Theta$ is made of the

$$\overline{\alpha_i} = e_i - e_{i-1} \qquad i \text{ in } \{2,3,5,6\} \qquad \text{whose squared lengths are } 2$$

and

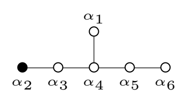
$$\overline{\alpha_4} = e_4 - e_3 + \frac{e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7}{4} \qquad \text{whose squared length is } 3/2.$$

A root system of type A_5 appears in the projection, a basis is given by :

$$\{\alpha_4, \alpha_5, \frac{e_0 - e_7 + e_1 - e_2 - e_3 + e_4 + e_5 - e_6}{2}, \alpha_2, \alpha_3\}$$

.

Case $\Theta = \{\alpha_2\}$



$$\overline{\alpha_3} = e_3 - \frac{[e_1 + e_2]}{2}$$

$$\overline{\alpha_i} = \alpha_i \qquad i \text{ in } \{1,4,5,6\}\ .$$

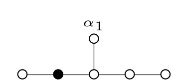
These form a root system of type A_4.

Could we complete this root system to obtain a root system of rank 5?
We are looking for a root β which is orthogonal to any α_i with $i \in \{1, 2, 4, 5\}$, whose scalar product with α_6 is -1 then $\beta = \frac{1}{2}[e_0 - e_1 - e_2 + e_3 + e_4 + e_5 - e_6 - e_7]$ satisfies these conditions ; further the sum (which is the longest root) $\beta + \alpha_1 + \alpha_4 + \alpha_5 + \alpha_6 = e_0 - e_7$ is obtained in the projection.
Therefore a root system of type A_5 is obtained.

Looking at this case in E_7, one also observes that another basis of an A_5 root system in the projection can be obtained from the roots $\{\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_7 = e_7 - e_6\}$.

The symmetrical case with $\Theta = \{\alpha_6\}$ yields a root system of type A_5 whose basis is constituted of $\overline{\alpha_i} = \alpha_i,\ i$ in $\{1,4\}$ and $\beta = \frac{1}{2}[e_0 + e_1 - e_2 - e_3 - e_4 + e_5 + e_6 - e_7]$. The sum of the simple roots yields $e_0 - e_7$, which appears in the projection.

Case $\Theta = \{\alpha_3\}$



$$\overline{\alpha_i} = \alpha_i \qquad i \text{ in } \{1,5,6\} \text{ whose squared length is } 2.$$

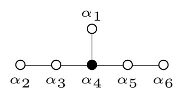
Since $\overline{e_3} = \overline{e_2}$, we have :

$$\overline{\alpha_2} = \frac{e_2 + e_3}{2} - e_1 \text{ and } \overline{\alpha_4} = e_4 - \frac{[e_2 + e_3]}{2} \text{ whose squared lengths are } 3/2.$$

The roots $\overline{\alpha_5}$ and $\overline{\alpha_4}$ cannot form a root system since $C = 3$. The roots $\{\alpha_1 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7]\ ; e_4 - e_1, \alpha_5, \alpha_6, \alpha_7\}$ constitute the basis of an A_5.

The symmetrical case $\Theta = \{\alpha_5\}$ is treated similarly.

Case $\Theta = \{\alpha_4\}$



$$\overline{\alpha_i} = \alpha_i \qquad i \text{ in } \{2,6\} \text{ whose squared length is } 2.$$

Since $\overline{e_4} = \overline{e_3}$, we have :

$$\overline{\alpha_1} = \frac{[e_0 + e_1 + e_2 - e_5 - e_6 - e_7]}{2} \qquad \text{whose squared length is } 3/2,$$

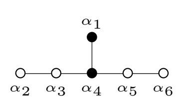
$$\overline{\alpha_3} = \frac{[e_3 + e_4]}{2} - e_2 \qquad \text{and} \qquad \overline{\alpha_5} = e_5 - \frac{1}{2}[e_3 + e_4] \text{ whose squared lengths are } 3/2.$$

Considering the roots $\overline{\alpha_3}$ and $\overline{\alpha_5}$ (or $\overline{\alpha_3}$ and $\overline{\alpha_1}$) , the value of $C = 9$, whereas for $\overline{\alpha_3}$ and $\overline{\alpha_2}$ (resp. $\overline{\alpha_5}$ and $\overline{\alpha_6}$) it is 3.
A root system of type A_5 appears in the projection, its basis is given by

$$\{\alpha_6, e_5 - e_2, e_2 - e_1, \frac{e_0 - e_7 + e_1 - e_2 + e_3 + e_4 - e_5 - e_6}{2}, e_7 - e_0\}$$

.

Case $\Theta = \{\alpha_1, \alpha_4\}$



$$\overline{\alpha_i} = \alpha_i \qquad \text{i in } \{2,6\} \text{ whose squared length is } 2.$$

Since $\overline{e_4} = \overline{e_3}$, we have :

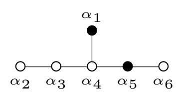
$$\overline{\alpha_3} = \frac{[e_3 + e_4]}{2} - e_2 + \frac{[e_0 + e_1 + e_2 - e_5 - e_6 - e_7]}{6},$$

$$\overline{\alpha_5} = e_5 - \frac{[e_3 + e_4]}{2} + \frac{[e_0 + e_1 + e_2 - e_5 - e_6 - e_7]}{6}.$$

The squared length of $\overline{\alpha_3}$ (resp. $\overline{\alpha_5}$) is $51/32$ and the value of C, considered with respect to $\overline{\alpha_2}$ (resp. $\overline{\alpha_6}$), does not correspond to any root system. Although we could compose a root system from the elements $\{e_0, e_1, e_2, e_5, e_6, e_7\}$, as opposed to the context of E_7, one cannot add $e_7 - e_0$ to α_6 since, expressions of the form $e_0 - e_i, i \neq 7$ (for instance the sum $e_7 - e_0$ and α_6 which equals $e_0 - e_6$) are not roots of E_6. However, we can use root of the form $\frac{1}{2}[e_0 \pm e_1 \pm e_2 \cdots \pm e_6 - e_7]$ orthogonal to α_1. Then, a root system of type A_3 with basis $\{\alpha_6, \frac{1}{2}[e_0 + e_1 - e_2 - e_3 + e_4 + e_5 - e_6 - e_7], e_2 - e_1\}$ appears in the projection.

We now consider the cases $\Theta = \{\alpha_1, \alpha_5\}$ and $\Theta = \{\alpha_1, \alpha_6\}$. The corresponding symmetrical cases, $\Theta = \{\alpha_1, \alpha_3\}$ and $\Theta = \{\alpha_1, \alpha_2\}$, imply clearly the same reasoning and results.

Case $\Theta = \{\alpha_1, \alpha_5\}$ (resp. $\{\alpha_1, \alpha_3\}$)

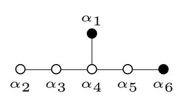


$$\overline{\alpha_4} = \frac{e_4 + e_5}{2} - e_3 + \frac{1}{4}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7] \text{ whose squared length is 2.}$$

The value of C when considering $\overline{\alpha_4}$ and $\overline{\alpha_3}$ is 4.

The roots $\overline{\alpha_2}, \overline{\alpha_3}, \overline{\alpha_4}$ constitute a basis for a root system of type A_3. However, the longest root $\overline{\alpha_4} + \overline{\alpha_3} + \overline{\alpha_2} = \frac{1}{4}[e_0 + e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7]$ is not a projection of a root in Σ. Therefore, a subsystem in Σ_Θ is A_2 with basis $\{\overline{\alpha_2}, \overline{\alpha_3}\}$. It is possible to find a root β (of the form $\pm\frac{1}{2}[(e_0 - e_7) \pm e_1 \pm e_2 \cdots \pm e_6])$ which is orthogonal to α_i for i in $\{1, 3, 5\}$ and such that its scalar product with α_2 is -1 : $\frac{1}{2}[e_0 + e_1 - e_2 - e_3 + e_4 + e_5 - e_6 - e_7]$. The sum of this root with α_2 and α_3 appears in the projection. Therefore the subsystem of highest rank in Σ_Θ is A_3.

Case $\Theta = \{\alpha_1, \alpha_6\}$ (resp. $\{\alpha_1, \alpha_2\}$)



$$\overline{\alpha_i} = \alpha_i \qquad i \text{ in } \{2,3\} \text{ whose squared length is 2.}$$

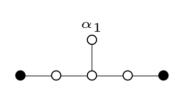
Since $\overline{e_6} = \overline{e_5}$, we have :

$$\overline{\alpha_5} = \frac{1}{2}[e_6 + e_5] - e_4$$

$$\overline{\alpha_4} = e_4 - e_3 + \frac{1}{4}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7].$$

The squared length of $\overline{\alpha_4}$ is 2 + 1/2 - 4.1/4 = 3/2 ; therefore when considering $\overline{\alpha_4}$ and $\overline{\alpha_3}$, $C = 3$. This value of C does not correspond to any root system of rank 2, therefore we need to exclude the possibility of rank 3 system (when completing those two roots with $\overline{\alpha_2}$). A subsystem A_2 has basis given by $\overline{\alpha_3}$ and $\overline{\alpha_2}$.
It is possible to find a root β which is orthogonal to α_i for i in $\{1, 3, 6\}$ and such that its scalar product with α_2 is -1 : $\frac{1}{2}[e_0 + e_1 - e_2 - e_3 - e_4 + e_5 + e_6 - e_7]$; its sum with α_2 and α_3 appears in the projection. Therefore the subsystem of highest rank in Σ_Θ is A_3.

Case $\Theta = \{\alpha_2, \alpha_6\}$



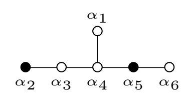
$$\overline{\alpha_i} = \alpha_i \qquad i \text{ in } \{1,4\} \text{ whose squared length is 2,}$$

$$\overline{\alpha_3} = e_3 - \frac{1}{2}[e_1 + e_2],$$

$$\overline{\alpha_5} = \frac{1}{2}[e_6 + e_5] - e_4.$$

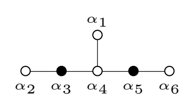
Consider $\overline{\alpha_1}$ and $\overline{\alpha_4}$, their scalar product is -1. $R = 1, C = 4$. This could give us A_2.
The root $\overline{\alpha_1} + \overline{\alpha_4} = \frac{1}{2}[e_0 + e_1 + e_2 - e_3 + e_4 - e_5 - e_6 - e_7]$ appears in the projection. Considering the value of C between (for instance) $\overline{\alpha_5}$ and $\overline{\alpha_4}$ yields 3 which forbids the appearance of a root system of higher rank. However, the root $\frac{1}{2}[e_0 - e_1 - e_2 + e_3 - e_4 + e_5 + e_6 - e_7]$ orthogonal to α_i for $i \in \{1, 2, 6\}$ is the third root which constitute with $\overline{\alpha_1}$ and $\overline{\alpha_4}$ the basis of a root system of type A_3. This subsystem is of highest rank.

Case $\Theta = \{\alpha_2, \alpha_5\}$



Using (in particular) Lemma 3.6, we see that the ratio of lengths of $\overline{\alpha_3}$ and $\overline{\alpha_4}$ is 1 ; whereas their scalar product is -1, and $C = 9/4$. Considering now $\overline{\alpha_1}$ and $\overline{\alpha_4}$, one has a scalar product of -1, a ratio R of 4/3 and $C = 3$. In both cases, the value of C does not correspond to any rank 2 root system. Further, in the projection, we also obtain $\beta = \frac{1}{2}[e_0 - e_3 + e_6 - e_7]$ of squared norm 1 ; and $\beta' = \frac{1}{2}[e_0 - + e_1 + e_2 - e_3 + e_6 - e_7]$ or $\beta'' = \frac{1}{2}[e_0 - e_3 + e_4 + e_5 + e_6 - e_7]$ of squared norm 3/2. When looking at scalar product of, say β' and α_3, we also reach a value of $C = 9/4$. However, it is possible to find a system of type A_3, using $\{e_0, e_7, e_3, e_6\}$ and root of the form $\pm\frac{1}{2}[e_0 - e_7 \pm e_1 \pm e_2 \cdots \pm e_6]$. For instance, a basis is given by : $\{e_7 - e_0, \frac{1}{2}[e_0 - e_7 + e_1 + e_2 - e_5 - e_4 + e_6 - e_3], e_3 - e_6\}$.

Case $\Theta = \{\alpha_3, \alpha_5\}$



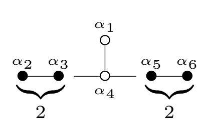
$$\overline{\alpha_4} = \frac{[e_5 + e_4]}{2} - \frac{[e_3 + e_2]}{2},$$

$$\overline{\alpha_2} = \frac{[e_3 + e_2]}{2} - e_1,$$

$$\overline{\alpha_6} = e_6 - \frac{[e_5 + e_4]}{2}.$$

The squared norm of $\overline{\alpha_1}$ is 2, whereas the squared norm of $\overline{\alpha_4}$ is 1. The squared norms of $\overline{\alpha_2}, \overline{\alpha_6}$ is 3/2. The scalar product $\langle \overline{\alpha_1}, \overline{\alpha_4} \rangle = -1$, $R = 2$ and $C = 2$, these roots form the basis of a root system of type B_2.
Notice that $\overline{\alpha_1} + 2\overline{\alpha_4} = \frac{e_0 + e_1 - e_2 - e_3 + e_4 + e_5 - e_6 - e_7}{2}$ (remaining the same when projected since orthogonal to α_3, α_5) is in Σ_Θ and $\overline{\alpha_1} + \overline{\alpha_4}$, projection of $\frac{e_0 + e_1 + e_3 - e_2 - e_4 + e_5 - e_6 - e_7}{2}$ also. Therefore a root system of type B_2 appears in the projection. There also is a A_3 root system whose basis is, for instance, given by : $\{e_7 - e_0, \frac{1}{2}[e_0 - e_7 + e_2 + e_3 - e_5 - e_4 - e_6 + e_1], e_6 - e_1\}$.

Case $\Theta = \{\alpha_2, \alpha_3\} \cup \{\alpha_5, \alpha_6\}$



LEMMA 3.8. *For a system of type E_6, if Θ is the union of two components of type A_2 defined by $\{\alpha_2, \alpha_3\}$ and $\{\alpha_5, \alpha_6\}$, the projection on the orthogonal of Θ contains the basis of a system of G_2 but not the whole system.*

PROOF. The projections are :

$$\overline{\alpha_1} = \alpha_1 \qquad \text{and} \qquad \overline{\alpha_4} = \frac{e_4 + e_5 + e_6}{3} - \frac{e_1 + e_2 + e_3}{3},$$

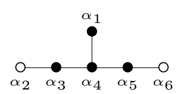
$$\text{where } \ ||\overline{\alpha_1}||^2 = 2 \ \text{ and } \ ||\overline{\alpha_4}||^2 = 6/9 = 2/3$$

and the scalar product is -1. Therefore $R = 3$ and $C = 4/3$ and this is the basis of a root system of type G_2. For the projection to contain a system of type G_2 one would need that we obtain $\overline{\alpha_1} + \overline{\alpha_4}$, $\overline{\alpha_1} + 2\overline{\alpha_4}$, $\overline{\alpha_1} + 3\overline{\alpha_4}$ and $2\overline{\alpha_1} + 3\overline{\alpha_4}$. But

$$\overline{\alpha_1} = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7] = \frac{1}{2}[(e_0 - e_7) - 3\overline{\alpha_4}]$$

and we check easily that $\overline{\alpha_1} + 3\overline{\alpha_4}$ is obtained by varying the signs in the parenthesis. The root $2\overline{\alpha_1} + 3\overline{\alpha_4} = e_0 - e_7$ is also obtained. However, $\overline{\alpha_1} + \overline{\alpha_4}$ (for instance) is not obtained and therefore G_2 is not a subsystem. □

Case $\Theta = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$



$$\overline{e_2 + e_3 + e_4 + e_5} = e_2 + e_3 + e_4 + e_5 \ \text{ since this sum is orthogonal to all } \alpha_i \text{ in } \Theta,$$

$$\overline{\alpha_2} = \overline{e_2} - e_1 = \frac{e_2 + e_3 + e_4 + e_5}{4} - e_1 \text{ whose squared length is } 5/4,$$

$$\overline{\alpha_6} = e_6 - \overline{e_5} \text{ whose squared length is } 5/4.$$

The value of C corresponding to those two roots is 25. Therefore no root system can be obtained in the projection.

Θ= {..}	squared lengths of projected roots	chosen roots to calculate C and R	C and R	root system of highest rank obtained (of rank $\geq$ 2)
α_1	2 and 3/2	$\overline{\alpha_3}$,$\overline{\alpha_4}$	C=3	A_5
α_2 or α_6	2 and 3/2			A_5
α_6				D_3, A_5
α_3 or α_5	2 and 3/2	$\overline{\alpha_5}$,$\overline{\alpha_4}$	C=3/2	A_5
α_4	3/2	($\overline{\alpha_1}$,$\overline{\alpha_3}$; ($\overline{\alpha_2}$,$\overline{\alpha_3}$)	C=9 ; C=3	A_5
α_1,α_4	51/32 and 2			A_3
α_1, α_5 or α_1, α_3	2	$\overline{\alpha_3}$,$\overline{\alpha_4}$	C=4	A_3
α_1,α_6 or α_1, α_2		$\overline{\alpha_3}$,$\overline{\alpha_4}$	C=3	A_3
α_2,α_6	2 and 3/2	$\overline{\alpha_1}$,$\overline{\alpha_4}$;($\overline{\alpha_5}$,$\overline{\alpha_4}$)	C=4, R=1 ; C=3	A_3
α_3,α_5	1, 2, and 3/2	$\overline{\alpha_1}$,$\overline{\alpha_4}$	C=2, R=2	B_2, A_3
$\{\alpha_2, \alpha_3\} \cup \{\alpha_5, \alpha_6\}$				A_2
α_2, α_5 or α_3, α_6		$\{\overline{\alpha_3}, \overline{\alpha_4}\}$ and $\{\overline{\alpha_1}, \overline{\alpha_4}\}$	C=9/4 ; C=3	A_3
$\{\alpha_1, \alpha_2, \alpha_6\}$ or $\{\alpha_1, \alpha_3, \alpha_5\}$				A_2
$\{\alpha_3, \alpha_2, \alpha_5\}$ or $\{\alpha_3, \alpha_5, \alpha_6\}$				A_2
$\{\alpha_2, \alpha_4, \alpha_6\}$				A_2
$\{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$ or $\{\alpha_1, \alpha_3, \alpha_4\}$				None
$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$				None

Table 2: Roots system occurring in Σ_Θ for Σ of type E_6
Root systems of type D occurring, in particular when Θ contains only one root, are not systematically written.

3.2.4 The case E_7

We consider as for E_6, an euclidean space $\widetilde{V}$ of dimension 8, equipped with an orthonormal basis indexed by the elements of $\mathbb{Z}/8\mathbb{Z}$

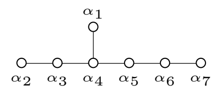
$$\{e_0, e_1, \cdots, e_6, e_7\}$$

The roots of E_7 are the roots of E_8 orthogonal to $\epsilon_7 + \epsilon_8 = (-1/2)(e_0 + e_2 + \cdots + e_7)$ where ϵ_7, ϵ_8 refer to the notations in [1] and the change of basis was given in an unpublished note of Jean-Pierre Labesse. They are of the following form :

- $\pm(e_i - e_j)$ for $0 \leq i < j \leq 7$.
- $\frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6]$

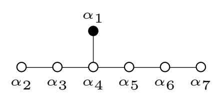
with a number of + signs (and therefore of $-$) in the bracket equal to 4. A root system is given by :

$$\alpha_1 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7] \qquad \text{and} \qquad \alpha_{i+1} = [e_{i+1} - e_i] \qquad \text{pour } 1 \leq i \leq 6 \ .$$



Following 3.3, we know that if Θ contains only one root, whatever is this one root, the root system we obtained in the projection should always be the same. Below, we illustrate this result, and exhibit basis for the A_5 root system appearing in the projection whenever Θ contains a simple root of E_7.

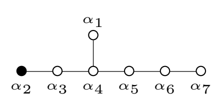
Case $\Theta = \{\alpha_1\}$



$$\overline{\alpha_i} = \alpha_i \qquad \text{for } i \text{ in } \{2,3,5,6,7\},$$
$$\overline{\alpha_4} = e_4 - e_3 + \frac{e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7}{4}.$$

The squared norm of $\overline{\alpha_4}$ is $3/2$, whereas the squared norm of $\overline{\alpha_i}$ for i in $\{2,3,5,6,7\}$ is 2. The system of greatest rank in the projection while restricting only on projection of simple roots is one which is of type A_3 with basis $\overline{\alpha_i}$ for i in $\{5,6,7\}$. The roots $\{\alpha_6, \alpha_7, \frac{e_0+e_1-e_2-e_3-e_4+e_5+e_6-e_7}{2}, e_2 - e_1, e_1 - e_0\}$ constitute the basis of a root system of type A_5, the sums of roots appear in the projection, hence we obtain a root system of type A_5.

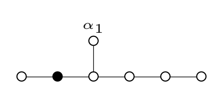
Case $\Theta = \{\alpha_2\}$



The projection of $\Delta - \Theta$ is made of the $\overline{\alpha_i} = e_i - e_{i-1}$, i in $\{1,4,5,6,7\}$ which constitute the basis of a root system of type A_5, and $\overline{\alpha_3} = e_3 - \frac{e_1+e_2}{2}$.

There does not exist any root β, of the form $\frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6]$, which is orthogonal to $\overline{\alpha_i}$ for i in $\{1,2,4,5,6\}$ and whose scalar product with $\overline{\alpha_7}$ is -1, therefore we cannot complete this system to form a system of rank 6 ; hence it is of highest rank.
We could also imagine that a root β (of the form $\pm\frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6]$)-to complete the basis- would be orthogonal to all α_i, i in $\{2,4,5,6,7\}$ and its scalar product with $\overline{\alpha_1} = \alpha_1$ be -1. Such root does not exists. The scalar product is necessarily one.

Case $\Theta = \{\alpha_3\}$

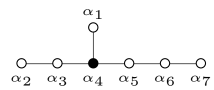


$$\overline{\alpha_i} = \alpha_i \qquad i \text{ in } \{1,5,6,7\} \text{ whose squared length is 2.}$$

Since $\overline{e_3} = \overline{e_2}$, we have : $\overline{\alpha_2} = \frac{e_2+e_3}{2} - e_1$ and $\overline{\alpha_4} = e_4 - \frac{e_2+e_3}{2}$ whose squared length are 3/2.

Roots of the form $\pm(e_i - e_j)$ can be made out of the elements $\{e_0, e_1, e_4, e_5, e_6, e_7\}$; they form a root system of type A_5. It cannot be completed by any root of the form $\frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6]$ without getting a contradiction to the requirement of having 4 positive and negative signs within the bracket.

Case $\Theta = \{\alpha_4\}$



Since $\overline{e_4} = \overline{e_3}$, we have :

$$\overline{\alpha_1} = \frac{[e_0 + e_1 + e_2 - e_5 - e_6 - e_7]}{2} \text{ whose squared length is } 3/2.$$

Moreover,

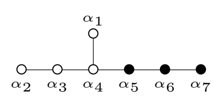
$$\overline{\alpha_3} = \frac{e_3 + e_4}{2} - e_2, \qquad \text{and} \qquad \overline{\alpha_5} = e_5 - \frac{e_3 + e_4}{2} \quad \text{whose squared length are } 3/2.$$

$$\overline{\alpha_i} = \alpha_i \qquad i \text{ in } \{2{,}6{,}7\}.$$

Same reasoning than in the previous case, we obtain an A_5 root system in the projection.

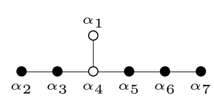
The case $\Theta = \{\alpha_5\}$ and $\Theta = \{\alpha_7\}$ are treated similarly and yield the same result. The ratios of lengths do not allow the occurrence of root systems of type G_2, F_4 and E_6. The roots $\{e_7 - e_6, e_0 - e_7, e_1 - e_0, e_2 - e_1, e_3 - e_2\}$ (resp. $\{e_1 - e_0, e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4\}$) constitute the basis of a root system of type A_5. Since all the sum of any consecutive roots in this basis appear in the projection, we obtain a root system of type A_5 in the projection.

Case $\Theta = \{\alpha_5, \alpha_6, \alpha_7\}$



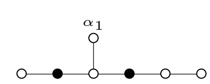
$\overline{\alpha_i} = \alpha_i$ for i in $\{1, 2, 3\}$, whose squared length is 2 ; $\overline{\alpha_4} = \frac{e_4+e_5+e_6+e_7}{4} - e_3$ whose squared length is 5/4.
The ratios of lengths are incompatible with F_4. The roots $\{e_1 - e_0, e_2 - e_1, e_3 - e_2\}$ constitute the basis of the A_3 root system occurring in the projection.

Case $\Theta = \{\alpha_2, \alpha_3\} \cup \{\alpha_5, \alpha_6, \alpha_7\}$



$\overline{\alpha_1} = \alpha_1$ with squared length 2 ; $\overline{\alpha_4} = \frac{e_4+e_5+e_6+e_7}{4} - \frac{e_1+e_2+e_3}{3}$ with squared length 7/12. The ratios of lengths are incompatible with a root system of type G_2, and even any classical root system of rank 2.

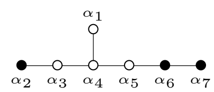
Case $\Theta = \{\alpha_3, \alpha_5\}$



$$\text{Then, } \overline{\alpha_4} = \frac{e_4 + e_5}{2} - \frac{e_2 + e_3}{2} \text{ whose squared length is } 1.$$

The value of C when considering the projected roots $\overline{\alpha_4}$ and $\overline{\alpha_2}$ is 8. The roots $\overline{\alpha_4}$ and $\overline{\alpha_1}$ constitute the basis of a root system of type B_2. Since $\overline{\alpha_4}+\overline{\alpha_1}$ and $2\overline{\alpha_4}+\overline{\alpha_1}$ are obtained in the projection, a system of type B_2 appear. We can complete the basis with e_7-e_0 and e_0-e_1 to obtain a root system of type B_4. Notice also that the roots $\{e_6-e_7, e_7-e_1, e_1-e_0\}$ constitute the basis of the A_3 root system occurring in the projection. To add a root of the form $\frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6]$ to this basis, it would need to have the same sign for all $e_i, i \in \{2,3,4,5\}$ and one other $e_i, i \in \{0,7\}$ or $\{7,1\}$, this contradicts the requirement of having 4 positive and negative signs in the bracket.

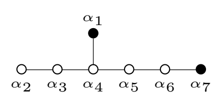
Case $\Theta = \{\alpha_2\} \cup \{\alpha_6, \alpha_7\}$



$$\overline{\alpha_5} = \frac{e_5+e_6+e_7}{3} - e_4 \qquad \text{whose squared length is } 4/3.$$
$$\overline{\alpha_3} = e_3 - \frac{e_2+e_1}{2} \qquad \text{whose squared length is } 3/2.$$
$$\overline{\alpha_i} = \alpha_i, \text{ for } i\text{=1 or 4}, \text{ whose squared length is 2.}$$

The ratios of lengths are incompatible with F_4 or even any classical system of rank 4. Using Lemma 3.6, it is clear that no root system can be obtained from $\overline{\alpha_3}, \overline{\alpha_4}, \overline{\alpha_5}$. Therefore the only root system one can obtain is A_2 with basis $\overline{\alpha_i} = \alpha_i$ for $i = 1$ and 4.

Case $\Theta = \{\alpha_1, \alpha_7\}$.

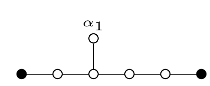


The roots $\{e_1-e_0, e_2-e_1, \frac{1}{2}[e_0+e_1-e_2-e_3+e_4+e_5-e_6-e_7], e_3-e_2\}$ constitute the basis of the D_4 root system occurring in the projection.

For the cases $\Theta = \{\alpha_1, \alpha_i\}$, with i in $\{4,5\}$, we obtain A_3 root systems. A basis is given by $\{\alpha_2, \alpha_3, \frac{1}{2}[-e_0+e_1+e_2-e_3-e_4-e_5+e_6+e_7]\}$.

The case $\Theta = \{\alpha_1, \alpha_2\}$ (resp. $\Theta = \{\alpha_1, \alpha_3\}$) gives rise to a root systems of type A_4 (resp. A_3) in the projection (the argumentation is similar to the one for E_6). It is not possible to find a root whose scalar product with α_7 is -1 and which is orthogonal to $\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6$ (resp to $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$) to complete this A_4 (resp. A_3) to an A_5 (resp. A_4). Therefore, the root system A_4 (resp. A_3) is of highest rank in the projection.

Case $\Theta = \{\alpha_2, \alpha_7\}$

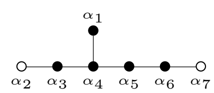


Treated similarly than the case $E_6, \Theta = \{\alpha_2, \alpha_6\}$. We use Lemma 3.6 with $\{\overline{\alpha_3}, \overline{\alpha_4}\}$ and $\{\overline{\alpha_6}, \overline{\alpha_5}\}$. The roots $\{\overline{\alpha_5}, \overline{\alpha_4}, \overline{\alpha_1}\}$ constitute the basis of a root system of type A_3. It is not possible to find a root of the form $\frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6]$ which is orthogonal to any α_i for $i \in \{1,2,4,7\}$ and whose scalar product with $\overline{\alpha_5} = \alpha_5$ is -1 ; The root system of type A_3 is of highest rank in the projection.

Case $\Theta = \{\alpha_2, \alpha_4, \alpha_6\}$
We have $\overline{\alpha_3} = \frac{e_3+e_4}{2} - \frac{e_2+e_1}{2}$ and $\overline{\alpha_5} = \frac{e_5+e_6}{2} - \frac{e_3+e_4}{2}$ of squared norms equal to one. Further $C = 4$, and $R = 1$ hence this give us the basis of an A_2. We can complete this with $\frac{1}{2}[-e_0-e_7+e_1+e_2+e_3+e_4-e_5-e_6]$, which has scalar product -1 with $\overline{\alpha_5}$ and $\frac{1}{2}[+e_0+e_7+e_1+e_2-e_3-e_4-e_5-e_6]$, which has scalar product -1 with $\overline{\alpha_3}$. The sum of these basis roots is $\frac{e_2+e_1}{2} - \frac{e_5+e_6}{2}$ and therefore appears in the projection. The case $\Theta = \{\alpha_2, \alpha_4, \alpha_6\}$ only gives a A_2.

Case $\Theta = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$ is treated similarly than in E_6. The squared norm of $\overline{\alpha_2}$ and $\overline{\alpha_6}$ is $5/4$; while $C = 25$; hence roots $\overline{\alpha_2}$ and $\overline{\alpha_6}$ do not form a root system. Also, notice that among the roots formed from the vectors $\{e_1, e_0, e_6, e_7\}$ only $e_6 - e_7$ is orthogonal to α_1.

Case $\Theta = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$



Although the vectors $\{e_1, e_0, e_7\}$ could constitute some roots of type A, only one of them, $e_1 - e_0$, is orthogonal to α_1.

The cases of $\Theta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\Theta = \{\alpha_2, \alpha_3, \alpha_4\}$ are treated simultaneously. We have a root system of type A_2 and basis α_6, α_7. We cannot add a root of the form $\frac{1}{2}[\pm e_0 \pm e_7 \pm e_1 \pm e_2 \cdots \pm e_6]$ since it would have to get the same sign for all $e_i, i \in \{1, 2, 3, 4\}$ and one $e_i, i \in \{6, 5\}$ or $\{6, 7\}$, this contradicts the requirement of having 4 positive and negative signs in the bracket.

$\Theta = \{..\}$	squared lengths of projected roots	chosen roots to calculate C and R	C and R	root system of highest rank obtained (of rank $\geq$ 2)
α_1	2 and 5/2			A_5
α_2	2 and 3/2			A_5
α_3	2 and 3/2	$\overline{\alpha_5}$,$\overline{\alpha_4}$	C=3/2	A_5
α_4	2 and 3/2	($\overline{\alpha_1}$,$\overline{\alpha_3}$; ($\overline{\alpha_2}$,$\overline{\alpha_3}$)	C=9 ; C=3	A_5
α_5	2 and 3/2	$\overline{\alpha_3}, \overline{\alpha_4}$	$C = 3$	A_5
α_6				A_5
α_7				A_5 or D_4
$\{\alpha_2\} \cup \{\alpha_6,\alpha_7\}$	4/3,3/2 and 2			A_2
$\alpha_5, \alpha_6, \alpha_7$	2, 5/4			A_2
$\{\alpha_2, \alpha_3\} \cup \{\alpha_5, \alpha_6, \alpha_7\}$	2 and 7/2			None
α_3,α_5	1, 2, and 3/2	$\overline{\alpha_1}$,$\overline{\alpha_4}$	C=2, R=2	B_4
α_1, α_4				A_3
α_1,α_5	2 and 3/2	$\overline{\alpha_3}$,$\overline{\alpha_4}$	C=4	A_3
α_1,α_2				A_4
α_1,α_3				A_3
α_1, α_7				D_4
α_2, α_7	2 and 3/2			A_3
$\alpha_1, \alpha_3, \alpha_4, \alpha_5$, or $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$				None
$\alpha_2, \alpha_3, \alpha_4$ or $\alpha_1, \alpha_3, \alpha_2, \alpha_4$				A_2
$\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ or $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$				None
$\alpha_1, \alpha_2, \alpha_6$ or $\alpha_1, \alpha_3, \alpha_5$				A_3
α_2, α_5 or α_2, α_6, or α_3, α_6				A_3
$\alpha_1, \alpha_2, \alpha_3, \alpha_6, \alpha_7$				None
$\alpha_1, \alpha_2, \alpha_4, \alpha_6$				A_2
$\alpha_2, \alpha_4, \alpha_6$				A_4

Table 3: Roots system occurring in Σ_Θ for Σ of type E_7
Root systems of type D occurring, in particular when Θ contains only one root, are not systematically written.

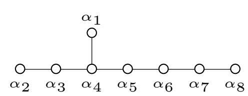
3.2.5 The case E_8

The positive roots are of the following form :

- $\pm e_i \pm e_j$ for $0 \leq i < j \leq 7$.
- $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ with an even number of negative signs.

A system of simple roots is given by :

$$\alpha_1 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7] \qquad \text{and} \qquad \alpha_i = e_i - e_{i-1} \qquad \text{for } 2 \leq i \leq 8$$

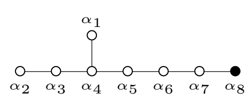


3.2.6 Cases with Θ containing only one element

Let us assume Θ contains $\alpha_k = e_k - e_{k-1}$ (resp. α_1 or α_8). Let $i, j \neq k, k-1$ (resp. $i, j \neq 3, 4$ or $i, j \neq 7, 6$). In the projection, the roots of the form $\pm e_i \pm e_j$ have squared norms equal to 2. The projections of roots of the form $\pm e_i \pm e_k$, $\pm e_i \pm e_{k-1}$ and $\frac{1}{2}[\pm e_0 \ldots \pm e_{k-2} \ldots \pm e_{k+1} \ldots \pm e_7]$ have squared norms equal to $3/2$. The ratio of lengths do not allow F_4 and G_2 since the squared norms of projected roots are 2 or $3/2$. Since there are no roots of norms 1, or 4, by the remark in Subsection 1.2, the ratio of lengths allow only the occurrence of root systems of type A and D.

Following 3.3, we know the nature of the root system of maximal rank occurring in the projection does not depend on the choice of root in Θ. This explains the statement of Theorem 1.2. Indeed,

Case $\Theta = \{\alpha_8\}$



We obtain $\alpha_1 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7]$ and the α_i with $2 \leq i \leq 6$ which generates the E_6, but also $\beta = -\frac{1}{2}[e_0 - e_1 - e_2 - e_3 - e_4 - e_5 + e_6 - e_7]$. The Dynkin diagram associated to $(\alpha_1, \ldots, \alpha_6, \beta)$ is the one of E_7.

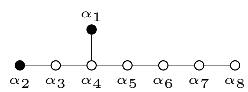
Remark 3.9. This phenomenon is specific to E_8 ; Recall that with the conventions of [1] the roots of E_7 are the roots of E_8 orthogonal to the root $\epsilon_7 + \epsilon_8$ of E_8. The roots of E_6 are the orthogonal in E_7 to $\pi = \epsilon_6 + \epsilon_7 + 2\epsilon_8$ which is *not* a root in E_7. Hence the phenomenon observed in the previous point does not occur : we cannot obtain E_6 when projecting orthogonally to a unique (simple) root in E_7.

We also give the basis of an A_7, and D_6 in the case $\Theta = \{\alpha_2\}$.
The roots $\overline{\alpha_i} = \alpha_i$ for $i \in \{1, 4, 5, 6, 7, 8\}$ together with $\frac{1}{2}[e_0 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7]$ form the basis of a root system of type A_7. One can check that the sum of these roots is $e_1 + e_2$ which appears in

the projection.
Let us consider the occurrence of type D root system in this case. With this choice of basis, we only get a D_6 in the projection. The roots $\pm e_i \pm e_j$ for $i < j$ in $\{0,3,4,5,6,7\}$ constitute the roots of D_6. The root $\frac{1}{2}[e_0+e_1+e_2+e_3-e_4-e_5-e_6-e_7]$ is orthogonal to all $\overline{\alpha_i} = \alpha_i$ for $i \in \{\alpha_4,\ldots,\alpha_7\}$ but it cannot be orthogonal to e_0+e_7 and e_7-e_0 which could constitute the two other extremal roots of the Dynkin diagram. Adding roots of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$, orthogonal between themselves and with all others but one in the basis, as extremal roots of the Dynkin diagram also leads to contradiction : The sum of all the basis roots does not appear in the projection.

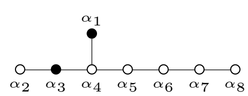
3.2.7

Case $\Theta = \{\alpha_1, \alpha_2\}$



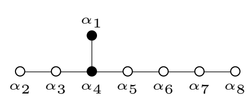
The inverse of the projections $\overline{\alpha_i} = \alpha_i,\ i \in \{5,6,7,8\}$ form the basis of a root system of type A_4. We are looking for a root β of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ (with an even number of negative signs) which is orthogonal to all $\alpha_i,\ i \in \{1,2,5,6,7\}$ and whose scalar product with $\overline{\alpha_8} = \alpha_8$ is -1. The root $\beta = \frac{1}{2}[e_0+e_7+e_1+e_2+e_3+e_4+e_5+e_6]$ satisfies this condition. This root completes the basis for a root system of type A_5. The sum of the simple roots is $\frac{1}{2}[e_0+e_7-e_1-e_2+e_3+e_4+e_5+e_6]$ which appears in the projection. One could also complete the above basis for A_4 with the root e_0-e_3 to obtain a root system of type A_5.

Case $\Theta = \{\alpha_1, \alpha_3\}$



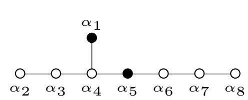
The roots $\overline{\alpha_i} = \alpha_i$ for $i \in \{5,\cdots,8\}$ along with e_0-e_1 and e_1-e_2 form the basis of a root system of type A_6.

Case $\Theta = \{\alpha_1, \alpha_4\}$



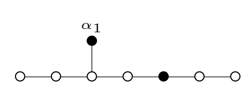
The roots $\overline{\alpha_i} = \alpha_i$ for $i \in \{6,7,8\}$ along with e_0-e_1, e_1-e_2, and e_2-e_3 form the basis of a root system of type A_6.

Case $\Theta = \{\alpha_1, \alpha_5\}$



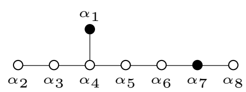
The roots $\overline{\alpha_i} = \alpha_i$ for $i \in \{7,8\}$ along with e_0-e_1, e_1-e_2 and e_2-e_3 form the basis of a root system of type A_5.

Case $\Theta = \{\alpha_1, \alpha_6\}$



The roots $\overline{\alpha_i} = \alpha_i$ for $i \in \{2,3\}$ along with $e_1 - e_0$, $e_0 + e_7$ form the basis of a root system of type A_4. It is not possible to find a root of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ (with an even number of negative signs) which is orthogonal to all $\alpha_i,\ i \in \{1,2,3,6\}$, and $e_1 - e_0$ (resp. $\alpha_i,\ i \in \{1,2,6\}$, $e_1 - e_0$, and $e_0 + e_7$) and whose scalar product with $e_7 + e_0$ (resp $e_3 - e_2$) is -1.

Case $\Theta = \{\alpha_1, \alpha_7\}$



The roots $e_i - e_{i+1}$ for $i \in \{0,1,2\}$, $e_3 + e_4$ and $e_5 - e_4$ form the basis of a A_5 root system.

For $\Theta = \{\alpha_1, \alpha_8\}$
The roots $-e_5 - e_6, \frac{1}{2}[e_0 + e_1 - e_2 - e_3 - e_4 + e_5 + e_6 - e_7], e_2 - e_1, e_3 - e_2$ form the basis of a A_4 root system.

Remark 3.10. It is clear that in the context where Θ contains α_1, adding a root of the form $e_{i+1} + e_i$ in the basis obtained in the projection -so that it is attached to $e_{i+2} - e_{i+1}$ next to $e_{i+1} - e_i$ in the Dynkin diagram corresponding to this basis- is not possible since both $e_{i+1} + e_i$ and $e_{i+1} - e_i$ cannot be orthogonal to α_1. Therefore, it is not possible to obtain a root system of type D_n in the projection.

$\Theta = \{\alpha_1, \alpha_2, \alpha_3\}$
The roots $\overline{\alpha_i} = \alpha_i$ for $i \in \{5, \ldots, 8\}$ form a basis of type A_4. Let us add a root of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ to this basis. Let us assume this root has scalar product with α_8 equals to -1, therefore e_0 and e_7 get a + sign. Since this root is orthogonal to α_i for $i \in \{5,6,7\}$, it forces a + sign on $\{e_4, e_5, e_6\}$ too. Further this root has to be orthogonal to α_1 and therefore it is : $\frac{1}{2}[e_0 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7]$.

$\Theta = \{\alpha_1, \alpha_3, \alpha_4\}$
As in the previous point, the roots $\overline{\alpha_i} = \alpha_i$ for $i \in \{6,7,8\}$ and $\frac{1}{2}[e_0 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7]$ form the basis of a root system of type A_4.

$\Theta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$
As in the previous point, the roots $\overline{\alpha_i} = \alpha_i$ for $i \in \{6,7,8\}$ and $\frac{1}{2}[e_0 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7]$ form the basis of a root system of type A_4. It is immediate to notice the impossibility to complete this root system to a D_4.

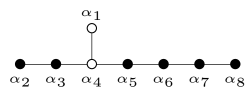
$\Theta = \{\alpha_6, \alpha_7, \alpha_8\}$
A basis of an A_4 root system is given by $\overline{\alpha_i} = \alpha_i$ for $i \in \{2,3,4\}$ and α_1 ; to add a root of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$, the latter has to be orthogonal to α_i for $i \in \{2,3,4,6,7,8\}$ and has scalar product with α_2 equal -1. Such root does not exists in E_8. Let us see if we can complete this basis to obtain a root system of type D_5. By the Remark 3.10, it is not possible to add $e_2 + e_1$ since it is not orthogonal to α_1, a root in the basis of D_5. However, the root $\frac{1}{2}[-e_0 + e_1 + e_2 + e_3 - e_4 + e_5 + e_6 + e_7]$, whose scalar product with α_4 is -1 appears as the fifth basis root for D_5. The sum of the basis' roots is $e_3 + e_2$ which appears in the projection. Therefore a root system of type D_5 is obtained.

$\Theta = \{\alpha_1, \alpha_6, \alpha_7, \alpha_8\}$
A basis of an A_3 root system is given by $e_4 + e_1, e_2 - e_1$ and $e_3 - e_2$.

$\Theta = \{\alpha_2, \alpha_3\}$

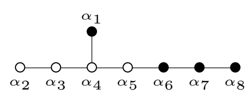
The roots $\overline{\alpha_i} = \alpha_i$ for $i \in \{5, \ldots, 8\}$ form a basis of type A_4. We can add the root of the form $\beta = \frac{1}{2}[e_0 - e_1 - e_2 - e_3 + e_4 - e_5 - e_6 - e_7]$ to complete this basis and obtain a root system of type A_5. However, adding $e_0 - e_7$ or some root of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ to one extreme of the Dynkin diagram to obtain a D_6 is not possible. In the first case, $e_0 - e_7$ is not orthogonal to β ; in the second the desired root should be orthogonal to all α_i in $\{2, 3, 6, 7, 8\}$ and β and has scalar product with α_5 equals to 1. Such root does not exist in E_8.

Case $\Theta = \{\alpha_2, \alpha_3\} \cup \{\alpha_5, \alpha_6, \alpha_7, \alpha_8\}$



$\overline{\alpha_1} = \alpha_1$ and $\overline{\alpha_4} = \frac{e_2+e_3+e_4+e_5+e_6}{5} - \frac{e_1+e_2+e_3}{3}$ and the squared length of $\overline{\alpha_4}$ is $1/3 + 1/5 = 8/15$.
The ratio of lengths of projected roots is not compatible with G_2 and neither with any root system of classical type and rank 2.

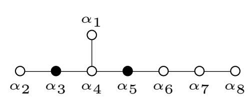
Case $\Theta = \{\alpha_1, \alpha_6, \alpha_7, \alpha_8\}$



$$\overline{\alpha_i} = \alpha_i,\ i \text{ in } \{2,3\} \text{ whose squared length are 2.}$$
$$\overline{\alpha_3} = e_3 - \frac{1}{2}[e_2 + e_1] \text{ whose squared length is } 3/2.$$
$$\overline{\alpha_5} = \frac{1}{4}[e_5 + e_6 + e_7 - e_0] - e_4 \text{ whose squared length is } 5/4.$$

The ratios of lengths do not allow the occurrence of F_4. The root $e_4 + e_1$ constitutes the third basis root of a root system of type A_3 together with α_2 and α_3.

Case $\Theta = \{\alpha_3, \alpha_5\}$



$$\overline{\alpha_4} = \frac{1}{2}[e_4 + e_5] - \frac{1}{2}[e_2 + e_3] \text{ whose squared length is 1.}$$
$$\overline{\alpha_1} = \alpha_1$$

These form the basis of a root system of type B_2, further $\overline{\alpha_4} + \overline{\alpha_1}$ and $2\overline{\alpha_4} + \overline{\alpha_1}$ appear in the projection. We can complete this basis with $e_6 - e_7, e_0 + e_7, -e_1 - e_0$, for instance, to form the basis of a root system B_5 ; since the sums of basis' roots appear in the projection, we obtain B_5 in the projection. If one adds the root α_1 to Θ, although one still obtain a projection of root (α_4) of squared norm equal to 1, one cannot complete it to form a root system of type B.

A root system of type A_3 is also obtained from the basis $\{\overline{\alpha_7}, \overline{\alpha_8}, e_1 - e_0\}$.

A similar result can be obtained if one considers $\Theta = \{\alpha_4, \alpha_6\}$. Then

$$\overline{\alpha_5} = \frac{1}{2}[e_6 + e_5] - \frac{1}{2}[e_3 + e_4] \text{ whose squared length is 1.}$$

An appropriate root $\beta = \frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 + e_3 + e_4 - e_6 - e_5 \pm e_7]$ is playing the role of α_1. This basis of type B_2 can be completed to obtain a B_5 in the projection.

From the case $\Theta = \{\alpha_1, \alpha_4\}$, one easily deduces the cases where Θ equals $\{\alpha_1, \alpha_2, \alpha_4\}$ or where we obtain a root system of type A_3 from $\overline{\alpha_i} = \alpha_i$, $i \in \{6, 7, 8\}$.

$\Theta = \{\alpha_3, \alpha_4\}$. The projected roots $\overline{\alpha_i} = \alpha_i$ $i \in \{6, 7, 8\}$ and the root $e_0 - e_1$ along with the root $\frac{1}{2}[-e_0 + e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7]$ form the basis of a root system of type A_5.

Similarly for the case where Θ is $\{\alpha_2, \alpha_3, \alpha_4\}$. The projected roots $\overline{\alpha_i} = \alpha_i$ $i \in \{6, 7, 8\}$ along with the root $\frac{1}{2}[e_0 - e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7]$ form the basis of a root system of type A_4.

In the case of $\Theta = \{\alpha_2, \alpha_5\}$ (resp. $\Theta = \{\alpha_3, \alpha_4, \alpha_5\}$) the $\pm e_i \pm e_j$ with $i, j \in \{0, 3, 6, 7\}$ (resp. with $i, j \in \{0, 1, 6, 7\}$) form a D_4. If $\Theta = \{\alpha_1, \alpha_2, \alpha_5\}$, this observation is no longer valid. We obtain a A_3 in the projection with, for instance, the basis $\alpha_7, \alpha_8, e_0 - e_3$. Idem for $\Theta = \{\alpha_1, \alpha_3, \alpha_5\}$, with the basis $\alpha_7, \alpha_8, e_0 - e_1$.

Let us consider the case of $\Theta = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$. The roots $\overline{\alpha_2}$ and $\overline{\alpha_6}$ have squared norm equal to 5/4. A root system of type A_3 occurs with basis α_7, α_8 and $e_0 - e_1$. Adding a root of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ to this basis is not possible since its scalar product with $e_0 - e_1$ or α_7 shall be -1, while it is orthogonal to all α_i in Θ ; such root with an even number of negative signs does not exist.

$$\Theta = \{\alpha_1, \alpha_3, \alpha_5, \alpha_8\}$$

$$e_2 = e_3 \; ; e_5 = e_4, e_0 = -e_7$$

The roots $\pm\frac{1}{2}[e_0 - e_7 + e_6 - e_1]$ have squared norms equal to 1. It is possible to add $\pm(e_6 - e_1)$ or $\pm(e_7 - e_0)$ to get a basis of B_2. However, it is not possible to add a root of the form $\frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \cdots \pm e_6 \pm e_7]$ of squared norm 2 since the couples e_2, e_3 and e_4, e_5 need to have the same sign, and it has to be orthogonal to α_1 ; and this is incompatible with the requirement of having scalar product equal -1 with (resp. being orthogonal to) $\pm\frac{1}{2}[e_0 - e_7 + e_6 - e_1]$ or $\pm(e_6 - e_1)$ (resp. $\pm(e_7 - e_0)$).

$\Theta = \{..\}$	squared lengths of projected roots	chosen roots to calculate C and R	C and R	root system of highest rank obtained (of rank ≥ 2)
α_8 or any (simple) root of E_8	2 and 3/2			E_7 and therefore D_7, A_7
$\{\alpha_1,\alpha_4\}$ or $\{\alpha_1,\alpha_2,\alpha_4\}$ or $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$				A_4
α_1,α_5				A_5
$\alpha_1,\alpha_5,\ \alpha_2$				A_4
$\{\alpha_1,\alpha_5,\alpha_3\}$ or $\{\alpha_1,\alpha_3,\alpha_4,\alpha_5\}$				A_3
α_2,α_5 and α_6,α_7 or α_8 or any combination of those three				A_3
$\{\alpha_1,\alpha_2\}$ or $\{\alpha_1,\alpha_2,\alpha_3\}$				A_5
α_1,α_3				A_5
$\alpha_5,\alpha_6,\alpha_7,\alpha_8$				A_3
$\{\alpha_1,\alpha_3\} \cup \{\alpha_5,\alpha_6,\alpha_7,\alpha_8\}$	2 and 7/10			None
$\alpha_2,\alpha_3,\alpha_5$	1, 2, and 3/2	$\overline{\alpha_1},\overline{\alpha_4}$	C=2, R=2	A_3
α_3,α_5	1, 2, and 3/2		C=2, R=2	B_5
$\{\alpha_3,\alpha_4,\alpha_5\}$ or $\{\alpha_2,\alpha_5\}$				D_4
$\alpha_1,\alpha_6,\alpha_7,\alpha_8$	3/2 and 2			A_3
α_3,α_4				A_5
$\{\alpha_3,\alpha_4,\alpha_2\}$ or $\{\alpha_3,\alpha_4,\alpha_1\}$				A_4
$\alpha_3,\alpha_4,\alpha_5$				A_4
$\alpha_2,\alpha_5,\alpha_6$				A_4
$\{\alpha_2,\alpha_4\}$ or $\{\alpha_2,\alpha_4,\alpha_1\}$ or $\{\alpha_2,\alpha_4,\alpha_3,\alpha_1\}$				A_4
$\{\alpha_1,\alpha_8\}$ or $\{\alpha_1,\alpha_6\}$				A_4
α_1,α_7				A_5
$\alpha_2,\alpha_3,\alpha_4,\alpha_5$				A_3
$\alpha_6,\alpha_7,\alpha_8$				D_5
$\alpha_1,\alpha_3,\alpha_5,\alpha_8$				B_2

Table 4: Roots system occurring in Σ_Θ for Σ of type E_8
Root systems of type D occurring, in particular when Θ contains only one root, are not written systematically.

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