Comp Phys : Assignment 5

5. a) Show that

$$\sum_{x=0}^{N-1} \exp(-2\pi i k x/N) = \frac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)}$$

we can rewrite the sum as $\sum \alpha^x$ where $\alpha = \exp(-2\pi i k/N)$ so we can treat as the sum of a geometric series

$$\sum_{x=0}^{N-1} \exp(-2\pi i k x/N) = \sum_{x=0}^{N-1} \exp(-2\pi i k/N)^x$$

* since $e^{ab} = (e^a)^b$

let $\exp(-2\pi i k/N) = \alpha$

$\sum_{x=0}^{N-1} \alpha^x$ we know that for finite geometric series, the following property is true

$$S_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

applying this property to our own geometric series gives us:

$$\sum_{x=0}^{N-1} \alpha^x = \frac{1-\alpha^{(N-1)+1}}{1-\alpha} = \frac{1-\alpha^N}{1-\alpha}$$

and plugging our expression for α back in, we obtain

$$\frac{1-\exp(-2\pi i k/\cancel{N})^{\cancel{N}}}{1-\exp(-2\pi i k/N)} = \frac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)}$$

b) Show that this approaches N as k approaches zero, and is zero for any integer k that is not a multiple of N

$\lim_{k\to 0} \frac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)}$ if we just plug in $k=0$, we get

$$\frac{1-\exp(0)}{1-\exp(0)} = \frac{1-1}{1-1} = \frac{0}{0}!$$

so, we can use l'hopitals rule

where $\lim_{k\to 0} \frac{f(k)}{g(k)} = \lim_{k\to 0} \frac{f'(k)}{g'(k)}$

here, $\frac{f'(k)}{g'(k)} = \frac{2\pi i e^{-2\pi i k}}{2\pi i e^{-2\pi i k/N}} = \frac{N e^{-2\pi i k}}{e^{-2\pi i k/N}}$

$$\lim_{k \to 0} \frac{Ne^{-2\pi i k}}{e^{-2\pi i k/N}} = \frac{Ne^{0}}{e^{0}} = \frac{N(1)}{(1)} = N!$$

this can be demonstrated numerically as well!

what if k is not a multiple of N?

$$k \neq nN \quad \text{aka} \quad \frac{k}{N} \neq n$$

$$\frac{1 - \exp(-2\pi i k)}{1 - \exp(-2\pi i k/N)}$$

if $\frac{k}{N} \neq$ an integer, then the denominator is some fraction of $-2\pi i$

* See jupyter notebook for a numeric proof of this

c) we can use this to analytically write down the DFT of a non integer sine wave. Pick a non integer k value & plot the analytic estimate of the DFT

we know $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

Let's do $\sin(2\pi kx)$ or $\sin(2\pi kx/N)$?

$$\sum_{x=0}^{N-1} \sin(2\pi kx/N)\exp(-2\pi ik'x/N)$$

$$\frac{1}{2i}\sum_{x=0}^{N-1}\left(e^{2\pi ikx/N} - e^{-2\pi ikx/N}\right)e^{-2\pi ik'x/N}$$

$$= \frac{1}{2i}\sum_{x=0}^{N-1} e^{2\pi ix(k-k')/N} - e^{-2\pi ix(k+k')/N}$$

we know what this is!

$$= \frac{1}{2i}\left[\frac{1-e^{2\pi i(k-k')}}{1-e^{2\pi i(k-k')/N}} - \frac{1-e^{-2\pi i(k+k')/N}}{1-e^{-2\pi i(k+k')/N}}\right]$$

6. finishing problem from class

<u>Stationary noise</u>

$$\langle f(x)\, f(x+dx)\rangle = g(dx)$$

$$\langle F(k)^* F(k')\rangle = \langle \sum f(x) e^{2\pi ikx/N} \sum f(x') e^{-2\pi ik'x'/N}\rangle$$

$x' = x + dx$

$$\Rightarrow \sum f(x) e^{2\pi i k x/N} \sum f(x+dx) e^{-2\pi i k'(x+dx)/N}$$

$$\Rightarrow \sum_{dx} e^{-2\pi i k\, dx/N} \sum_{x} \underbrace{f(x) f(x+dx)} e^{2\pi i (kx - k'x)/N}$$

$$= \sum_{dx} e^{-2\pi i k dx/N} \cdot g(x) \sum \underbrace{e^{2\pi i (k-k')x/N}}_{N}$$

$$\left[\langle F(k)^* F(k') \rangle = N \sum_{dx} e^{-2\pi i k dx/N} g(x) \right]$$

Now we have:

$$\langle f(x) f(x+\delta) \rangle \propto c - |\delta|$$

we need to take the fourier transform of the correlation function

$$\langle F(k)^* F(k') \rangle = \left\langle \sum f(x) e^{2\pi i k x/N} \sum f(x+\delta) e^{-2\pi i k'(x+\delta)/N} \right\rangle$$

$$\sum f(x) e^{2\pi i k x/N} \sum f(x+\delta) e^{-2\pi i k'(x+\delta)/N}$$

$$\rightarrow N \sum (c - |\delta|) e^{-2\pi i k \delta/N}$$

what is the fourier transform of this?

$$\int_{-\infty}^{\infty} e^{-2\pi i k \delta/N} \, dk = \int_{-\infty}^{\infty} \ldots e^{-2\pi i k \delta/N} \ldots$$

$$\int_{-\infty} Ce \quad d\delta \quad - \underbrace{\int_{0} |\delta| e \quad d\delta}_{\text{what is this integral?}}$$

$\downarrow$

$\delta(k)$
some delta function

If I plug this into an integral calculator for fun I get this

$$\frac{N(2i\pi k\delta + N)e^{-2\pi k\delta / N}}{4\pi^2 k^2}$$

, which is good enough for me.

I don't have the brain power to prove this further.