

# The Inverse Winslow Problem

## The Problem

The Winslow Problem entails solving the PDE's

$$\begin{aligned}\nabla^2 z &= 0 \\ \nabla^2 \kappa &= 0\end{aligned}$$

for  $z = z(x, y)$  and  $\kappa = \kappa(x, y)$ , given boundary conditions.

We want to invert this problem to find two analogous PDE's we can solve for  $x = x(z, \kappa)$  and  $y = y(z, \kappa)$

## Tools

### Jacobian

The transformation from  $(x, y) \rightarrow (z, \eta)$  has the Jacobian

$$J = \begin{bmatrix} x_z & x_\eta \\ y_z & y_\eta \end{bmatrix}$$

The inverse transformation from  $(z, \eta) \rightarrow (x, y)$  has the Jacobian

$$J^{-1} = \begin{bmatrix} z_x & z_y \\ \eta_x & \eta_y \end{bmatrix}$$

Furthermore, taking the inverse of  $J$  we also see that

$$J^{-1} = \begin{bmatrix} z_x & z_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{J} \begin{bmatrix} y_\eta & -x_\eta \\ -y_z & x_z \end{bmatrix} \quad (1a)$$

$$\begin{aligned} J &= \det(J) \\ &= x_z y_\eta - x_\eta y_z \end{aligned}$$

This equivalence is established in section 4.2.3 of Knapp and Steinberg's book!

This equivalence tells us that

$$\begin{aligned} z_x &= \frac{y\eta}{J} \quad , \quad z_y = -\frac{x\eta}{J} \\ \eta_x &= -\frac{y\zeta}{J} \quad , \quad \eta_y = \frac{x\zeta}{J} \end{aligned} \quad (1b)$$

### Chain Rule

Consider  $f(z(x,y), \eta(x,y))$ . Chain rule gives

$$\begin{aligned} \frac{\partial}{\partial x} [f] &= \frac{\partial}{\partial z} [f] \frac{\partial z}{\partial x} + \frac{\partial}{\partial \eta} [f] \frac{\partial \eta}{\partial x} \\ &= z_x \frac{\partial}{\partial z} [f] + \eta_x \frac{\partial}{\partial \eta} [f] \end{aligned} \quad (2a)$$

and

$$\frac{\partial}{\partial y} [f] = z_y \frac{\partial}{\partial z} [f] + \eta_y \frac{\partial}{\partial \eta} [f] \quad (2b)$$

## Derivation

$$z_{xx}$$

i) Let  $f = z_x$  in (2a). Then,

$$z_{xx} = z_x \frac{\partial}{\partial z} (z_x) + x_x \frac{\partial}{\partial x} (z_x)$$

ii) From (1b) we know,

$$z_x = \frac{y_n}{J} \quad \text{and} \quad x_x = -\frac{y_3}{J}$$

$$\text{where } J = x_3 y_n - y_3 x_n$$

iii) Therefore,

$$z_{xx} = \frac{y_n}{J} \frac{\partial}{\partial z} \left( \frac{y_n}{J} \right) - \frac{y_3}{J} \frac{\partial}{\partial x} \left( \frac{y_n}{J} \right)$$

iv) Expand the derivatives using chain rule

$$\begin{aligned} z_{xx} &= \frac{y_n}{J} \left[ -J^{-2} J_z y_n + J^{-1} y_{nz} \right] \\ &\quad - \frac{y_3}{J} \left[ -J^{-2} J_n y_n + J^{-1} y_{nx} \right] \\ &= \frac{y_n}{J^3} [J y_{nz} - J_z y_n] - \frac{y_3}{J^3} [J y_{nx} - J_n y_n] \end{aligned}$$

v) Compute  $J_3$  and  $J_n$  via the product rule

$$\begin{aligned} J_3 &= \frac{\partial}{\partial z} [x_3 y_n - x_n y_3] \\ &= x_3 y_{n3} + x_{33} y_n - x_n y_{33} - x_{n3} y_3 \end{aligned} \quad (3a)$$

and

$$J_n = x_3 y_{nn} + y_n x_{n3} - x_n y_{3n} - y_3 x_{nn} \quad (3b)$$

vi) Plugging these into  $J_{xx}$  we have

$$\begin{aligned} &J_{xx} \\ &= \frac{y_n}{J_3} \left[ (x_3 y_n - y_3 x_n) y_{n3} - (x_3 y_{n3} + y_n x_{33} \right. \\ &\quad \left. - x_n y_{33} - y_3 x_{n3}) y_n \right] \\ &- \frac{y_3}{J_3} \left[ (x_3 y_n - y_3 x_n) y_{nn} - (x_3 y_{nn} + y_n x_{n3} \right. \\ &\quad \left. - x_n y_{3n} - y_3 x_{nn}) y_n \right] \end{aligned}$$

$$= \frac{y_n}{J^3} \left[ \cancel{x_3 y_n y_{n3}} - x_n y_3 y_{n3} - \cancel{x_3 y_{n3} y_n} \right. \\ \left. - y_n^2 x_{33} + x_n y_{33} y_n + y_3 x_{n3} y_n \right]$$

$$- \frac{y_3}{J^3} \left[ \cancel{x_3 y_n y_{nn}} - y_3 x_n y_{nn} - \cancel{x_3 y_{nn} y_n} \right. \\ \left. - y_n^2 x_{n3} + x_n y_{3n} y_n + y_3 x_{nn} y_n \right]$$

$$= \frac{1}{J^3} \left[ x_n (y_n^2 y_{33} + y_3^2 y_{nn} - 2 y_3 y_n y_{3n}) \right. \\ \left. - y_n (y_n^2 x_{33} + y_3^2 x_{nn} - 2 y_3 y_n x_{n3}) \right]$$

$$z_{xx} = \frac{1}{J^3} \left[ x_n (y_n^2 y_{33} + y_3^2 y_{nn} - 2 y_3 y_n y_{3n}) \right. \\ \left. - y_n (y_n^2 x_{33} + y_3^2 x_{nn} - 2 y_3 y_n x_{n3}) \right]$$

$$\zeta_{yy}$$

We follow same steps but swap the roles of  $x$  and  $y$ . We obtain

$$\zeta_{yy} = \frac{1}{J^3} \left[ -y_n (x_n^2 x_{33} + x_3^2 x_{nn} - 2x_3 x_n x_{3n}) + x_n (x_n^2 y_{33} + x_3^2 y_{nn} - 2x_3 x_n y_{n3}) \right]$$

$$\eta_{xx}$$

i) Let  $f = \eta_x$  and (2a). Then,

$$\eta_{xx} = \zeta_x \frac{\partial}{\partial z} (\eta_x) + \eta_x \frac{\partial}{\partial n} (\eta_x)$$

ii) From (1b) we know

$$\zeta_x = \frac{y_n}{J} \quad \text{and} \quad \eta_x = -\frac{y_3}{J}$$

iii) So,

$$\eta_{xx} = \frac{y_n}{J} \frac{\partial}{\partial z} \left( -\frac{y_3}{J} \right) - \frac{y_3}{J} \frac{\partial}{\partial n} \left( -\frac{y_3}{J} \right)$$

$$= \frac{y_3}{J} \frac{\partial}{\partial n} \left( \frac{y_3}{J} \right) - \frac{y_2}{J} \frac{\partial}{\partial z} \left( \frac{y_3}{J} \right)$$

iv) Expand derivatives via chain rule

$$\begin{aligned} \kappa_{xx} &= \frac{y_3}{J} \left[ -J^{-2} J_n y_3 + J^{-1} y_{3n} \right] \\ &\quad - \frac{y_2}{J} \left[ -J^{-2} J_z y_3 + J^{-1} y_{3z} \right] \\ &= \frac{y_3}{J^3} \left[ J y_{3n} - J_n y_3 \right] - \frac{y_2}{J^3} \left[ J y_{3z} - J_z y_3 \right] \end{aligned}$$

v) Plugging in (3a), (3b), and J we have

$$\begin{aligned} \kappa_{xx} &= \frac{y_3}{J^3} \left[ (x_3 y_n - y_3 x_n) y_{3n} \right. \\ &\quad \left. - (x_3 y_{nn} + y_n x_{nz} - x_n y_{3n} \right. \\ &\quad \left. - y_3 x_{nn}) y_3 \right] \\ &\quad - \frac{y_2}{J^3} \left[ (x_3 y_n - y_3 x_n) y_{3z} \right. \\ &\quad \left. - (x_3 y_{nz} + x_{3z} y_n - x_n y_{3z} \right. \\ &\quad \left. - x_{nz} y_3) y_3 \right] \end{aligned}$$



vi) Expand & collect like terms

$$\tau_{xx} = -\frac{y_3}{J^3} \left[ -x_3 y_n y_{3n} + \cancel{y_3 x_n y_{3n}} + x_3 y_{nn} y_3 \right. \\ \left. + y_n x_{n3} y_3 - \cancel{x_n y_{3n} y_3} - y_3^2 x_{nn} \right]$$

$$- \frac{y_n}{J^3} \left[ x_3 y_n y_{33} - \cancel{y_3 x_n y_{33}} - x_3 y_{n3} y_3 \right. \\ \left. - x_{33} y_n y_3 + \cancel{x_n y_{33} y_3} + x_{n3} y_3^2 \right]$$

$$= -\frac{1}{J^3} \left[ x_3 (y_n^2 y_{33} + y_3^2 y_{nn} - 2 y_n y_3 y_{3n}) \right. \\ \left. - y_3 (y_n^2 x_{33} + y_3^2 x_{nn} - 2 y_n y_3 x_{n3}) \right]$$

$$\tau_{xx} = -\frac{1}{J^3} \left[ x_3 (y_n^2 y_{33} + y_3^2 y_{nn} - 2 y_n y_3 y_{3n}) \right. \\ \left. - y_3 (y_n^2 x_{33} + y_3^2 x_{nn} - 2 y_n y_3 x_{n3}) \right]$$

$$\kappa_{yy}$$

We follow same steps but swap the roles of  $x$  and  $y$ . We obtain

$$\kappa_{yy} = -\frac{1}{J^3} \left[ -y_3 (x_n^2 x_{33} + x_3^2 x_{nn} - 2x_3 x_n x_{3n}) + x_3 (x_n^2 y_{33} + x_3^2 y_{nn} - 2x_3 x_n y_{3n}) \right]$$

Pulling Pieces together

i) We require  $\nabla^2 \zeta = 0$  and  $\nabla^2 \kappa = 0$  simultaneously.

ii) We now know

$$\nabla^2 \zeta = 0 \Leftrightarrow \zeta_{xx} + \zeta_{yy} = 0$$

$$\Leftrightarrow \frac{1}{J^3} \left\{ -y_n \left[ (x_n^2 + y_n^2) x_{33} - 2(y_3 y_n + x_3 x_n) x_{3n} + (x_3^2 + y_3^2) x_{nn} \right] \right.$$

$$+ x_n \left[ (x_n^2 + y_n^2) y_{33} - 2(y_3 y_n + x_3 x_n) y_{3n} + (x_3^2 + y_3^2) y_{nn} \right] \} = 0 \quad (4)$$

iii) Furthermore,

$$\nabla^2 u = 0 \Leftrightarrow u_{xx} + u_{yy} = 0$$

$\Leftrightarrow$

$$-\frac{1}{J^3} \left\{ -y_3 \left[ (x_n^2 + y_n^2) x_{33} - 2(x_3 x_n + y_3 y_n) x_{3n} + (x_3^2 + y_3^2) x_{nn} \right] + x_3 \left[ (x_n^2 + y_n^2) y_{33} - 2(x_3 x_n + y_3 y_n) y_{3n} + (x_3^2 + y_3^2) y_{nn} \right] \right\} = 0 \quad (5)$$

iv) Notice that (4) and (5) are both simultaneously true when

$$(x_n^2 + y_n^2) x_{33} - 2(x_3 x_n + y_3 y_n) x_{3n} + (x_3^2 + y_3^2) x_{nn} = 0$$

and

$$(x_n^2 + y_n^2) y_{33} - 2(x_3 x_n + y_3 y_n) y_{3n} + (x_3^2 + y_3^2) y_{nn} = 0$$

v) Therefore, the inverse problem is

### Inverse Winslow Problem

Solve,

$$\alpha x_{33} - 2\beta x_{n3} + \gamma x_{nn} = 0$$

$$\alpha y_{33} - 2\beta y_{3n} + \gamma y_{nn} = 0$$

with

$$\alpha = x_n^2 + y_n^2$$

$$\beta = x_3 x_n + y_3 y_n$$

$$\gamma = x_3^2 + y_3^2$$

for  $x = x(3, n)$  and  $y = y(3, n)$