The Inverse Winslow Problem

The Problem

The Winslow Problem entails solving the PDE's

$$\nabla^2 \xi = 0$$
$$\nabla^2 \eta = 0$$

for $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$, given boundary conditions.

We want to invert this problem to find two analagous PDE's we can solve for $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$

Tools

Jacobian

The transformation from $(x,y) \to (\xi,\eta)$ has the Jacobian

$$J = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}$$

The inverse transformation from $(\xi,\eta) \to (x,y)$ has the Jacobian

$$J^{-1} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix}$$

Furthermore, taking the inverse of J we also see that

$$J^{-1} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{J} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix} \quad (1a)$$

$$J = \det(J) \\ = x_\xi y_\eta - x_\eta y_\xi$$

This equivalence is established in section 4.2.3 of Knupp and Steinbergs book!

This equivalence tells us that

$$\xi_x = \frac{y_\eta}{J}, \quad \xi_y = -\frac{x_\eta}{J} \tag{1b}$$

$$\eta_x = -\frac{y_\xi}{J}, \quad \eta_y = \frac{x_\xi}{J}$$

Chain Rule

Consider $f(\xi(x,y), \eta(x,y))$. Chain rule gives

$$\frac{\partial}{\partial x}[f] = \frac{\partial}{\partial \xi}[f]\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}[f]\frac{\partial \eta}{\partial x} \tag{2a}$$

$$= \xi_x \frac{\partial}{\partial \xi}[f] + \eta_x \frac{\partial}{\partial \eta}[f]$$

and

$$\frac{\partial}{\partial y}[f] = \xi_y \frac{\partial}{\partial \xi}[f] + \eta_y \frac{\partial}{\partial \eta}[f] \tag{2b}$$

Derivation

ξ_{xx}

i) Let $f = \xi_x$ in (2a). Then,

$$\xi_{xx} = \xi_x \frac{\partial}{\partial \xi}(\xi_x) + \eta_x \frac{\partial}{\partial \eta}(\xi_x)$$

ii) From (1b) we know,

$$\xi_x = \frac{y_\eta}{J} \quad \text{and} \quad \eta_x = -\frac{y_\xi}{J}$$

where $J = x_\xi y_\eta - y_\xi x_\eta$

iii) Therefore,

$$\xi_{xx} = \frac{y_\eta}{J} \frac{\partial}{\partial \xi}\left(\frac{y_\eta}{J}\right) - \frac{y_\xi}{J} \frac{\partial}{\partial \eta}\left(\frac{y_\eta}{J}\right)$$

iv) Expand the derivatives using chain rule

$$\xi_{xx} = \frac{y_\eta}{J}\left[-J^{-2} J_\xi y_\eta + J^{-1} y_{\eta\xi}\right] - \frac{y_\xi}{J}\left[-J^{-2} J_\eta y_\eta + J^{-1} y_{\eta\xi}\right]$$

$$= \frac{y_\eta}{J^3}\left[J y_{\eta\xi} - J_\xi y_\eta\right] - \frac{y_\xi}{J^3}\left[J y_{\eta\eta} - J_\eta y_\eta\right]$$

v) Compute J_ξ and J_η via the product rule

$$J_\xi = \frac{\partial}{\partial \xi}\left[x_\xi y_\eta - x_\eta y_\xi \right] \qquad (3a)$$

$$= x_\xi y_{\eta\xi} + x_{\xi\xi} y_\eta - x_\eta y_{\xi\xi} - x_{\eta\xi} y_\xi$$

and

$$J_\eta = x_\xi y_{\eta\eta} + y_\eta x_{\eta\xi} - x_\eta y_{\xi\eta} - y_\xi x_{\eta\eta} \qquad (3b)$$

vi) Plugging these into ξ_{xx} we have

$$\xi_{xx}$$

$$= \frac{y_\eta}{J^3}\Big[(x_\xi y_\eta - y_\xi x_\eta) y_{\eta\xi} - (x_\xi y_{\eta\xi} + y_\eta x_{\xi\xi} - x_\eta y_{\xi\xi} - y_\xi x_{\eta\xi}) y_\eta \Big]$$

$$- \frac{y_\xi}{J^3}\Big[(x_\xi y_\eta - y_\xi x_\eta) y_{\eta\eta} - (x_\xi y_{\eta\eta} + y_\eta x_{\eta\xi} - x_\eta y_{\xi\eta} - y_\xi x_{\eta\eta}) y_\eta \Big]$$

$$= \frac{y_\eta}{J^3}\Big[\cancel{x_\zeta y_\eta y_{\eta\zeta}} - x_\eta y_\zeta y_{\eta\zeta} - \cancel{x_\zeta y_{\eta\zeta} y_\eta}$$

$$- y_\eta^2 x_{\zeta\zeta} + x_\eta y_{\zeta\zeta} y_\eta + y_\zeta x_{\eta\zeta} y_\eta\Big]$$

$$- \frac{y_\zeta}{J^3}\Big[\cancel{x_\zeta y_\eta y_{\eta\eta}} - y_\zeta x_\eta y_{\eta\eta} - \cancel{x_\zeta y_{\eta\eta} y_\eta}$$

$$- y_\eta^2 x_{\eta\zeta} + x_\eta y_{\zeta\eta} y_\eta + y_\zeta x_{\eta\eta} y_\eta\Big]$$

$$= \frac{1}{J^3}\Big[x_\eta\left(y_\eta^2 y_{\zeta\zeta} + y_\zeta^2 y_{\eta\eta} - 2\, y_\zeta y_\eta y_{\zeta\eta}\right)$$

$$- y_\eta\left(y_\eta^2 x_{\zeta\zeta} + y_\zeta^2 x_{\eta\eta} - 2 y_\zeta y_\eta x_{\eta\zeta}\right)\Big]$$

$$\boxed{\zeta_{xx} = \frac{1}{J^3}\Big[x_\eta\left(y_\eta^2 y_{\zeta\zeta} + y_\zeta^2 y_{\eta\eta} - 2 y_\zeta y_\eta y_{\zeta\eta}\right) - y_\eta\left(y_\eta^2 x_{\zeta\zeta} + y_\zeta^2 x_{\eta\eta} - 2 y_\zeta y_\eta x_{\eta\zeta}\right)\Big]}$$

ζ_{yy}

We follow same steps but swap the roles of x and y. We obtain

$$\boxed{\zeta_{yy} = \frac{1}{J^3}\Big[-y_\eta\left(x_\eta^2 x_{\zeta\zeta} + x_\zeta^2 x_{\eta\eta} - 2x_\zeta x_\eta x_{\zeta\eta}\right) + x_\eta\left(x_\eta^2 y_{\zeta\zeta} + x_\zeta^2 y_{\eta\eta} - 2x_\zeta x_\eta y_{\eta\zeta}\right)\Big]}$$

η_{xx}

i) Let $f = \eta_x$ and (2a). Then,

$$\eta_{xx} = \zeta_x \frac{\partial}{\partial\zeta}(\eta_x) + \eta_x \frac{\partial}{\partial\eta}(\eta_x)$$

ii) From (1b) we know

$$\zeta_x = \frac{y_\eta}{J} \quad \text{and} \quad \eta_x = \frac{-y_\zeta}{J}$$

iii) So,

$$\eta_{xx} = \frac{y_\eta}{J}\frac{\partial}{\partial\zeta}\left(-\frac{y_\zeta}{J}\right) - \frac{y_\zeta}{J}\frac{\partial}{\partial\eta}\left(-\frac{y_\zeta}{J}\right)$$

$$= \frac{y_\xi}{J}\frac{\partial}{\partial \eta}\left(\frac{y_\xi}{J}\right) - \frac{y_\eta}{J}\frac{\partial}{\partial \xi}\left(\frac{y_\xi}{J}\right)$$

iv) Expand derivatives via chain rule

$$\eta_{xx} = \frac{y_\xi}{J}\left[-J^{-2} J_\eta y_\xi + J^{-1} y_{\xi\eta}\right] - \frac{y_\eta}{J}\left[-J^{-2} J_\xi y_\xi + J^{-1} y_{\xi\xi}\right]$$

$$= \frac{y_\xi}{J^3}\left[J y_{\xi\eta} - J_\eta y_\xi\right] - \frac{y_\eta}{J^3}\left[J y_{\xi\xi} - J_\xi y_\xi\right]$$

v) Plugging in (3a), (3b), and J we have

$$\eta_{xx} = \frac{y_\xi}{J^3}\Big[(x_\xi y_\eta - y_\xi x_\eta) y_{\xi\eta} - (x_\xi y_{\eta\eta} + y_\eta x_{\eta\xi} - x_\eta y_{\xi\eta} - y_\xi x_{\eta\eta}) y_\xi\Big]$$
$$- \frac{y_\eta}{J^3}\Big[(x_\xi y_\eta - y_\xi x_\eta) y_{\xi\xi} - (x_\xi y_{\eta\xi} + x_{\xi\xi} y_\eta - x_\eta y_{\xi\xi} - x_{\eta\xi} y_\xi) y_\xi\Big]$$

vi) Expand & collect like terms

$$\eta_{xx} = -\frac{y_\xi}{J^3}\Big[-x_\xi y_\eta y_{\xi\eta} + y_\eta \cancel{x_\eta y_{\xi\eta}} + x_\xi y_{\eta\eta} y_\xi + y_\eta x_{\eta\xi} y_\xi - \cancel{x_\eta y_{\xi\eta}}\, y_\xi - y_\xi^2 x_{\eta\eta}\Big]$$

$$- \frac{y_\eta}{J^3}\Big[x_\xi y_\eta y_{\xi\xi} - y_\xi \cancel{x_\eta y_{\xi\xi}} - x_\xi y_{\eta\xi} y_\xi - x_{\xi\xi} y_\eta y_\xi + \cancel{x_\eta y_{\xi\xi}}\, y_\xi + x_{\eta\xi} y_\xi^2\Big]$$

$$= \frac{-1}{J^3}\Big[x_\xi\left(y_\eta^2 y_{\xi\xi} + y_\xi^2 y_{\eta\eta} - 2 y_\eta y_\xi y_{\xi\eta}\right) - y_\xi\left(y_\eta^2 x_{\xi\xi} + y_\xi^2 x_{\eta\eta} - 2 y_\eta y_\xi x_{\xi\eta}\right)\Big]$$

$$\boxed{\eta_{xx} = -\frac{1}{J^3}\Big[x_\xi\left(y_\eta^2 y_{\xi\xi} + y_\xi^2 y_{\eta\eta} - 2 y_\xi y_\eta y_{\xi\eta}\right) - y_\xi\left(y_\eta^2 x_{\xi\xi} + y_\xi^2 x_{\eta\eta} - 2 y_\xi y_\eta x_{\eta\xi}\right)\Big]}$$

η_{yy}

We follow same steps but swap the roles of x and y. We obtain

$$\eta_{yy} = -\frac{1}{J^3}\Big[-y_\xi\left(x_\eta^2 x_{\xi\xi} + x_\xi^2 x_{\eta\eta} - 2x_\xi x_\eta x_{\xi\eta}\right) + x_\xi\left(x_\eta^2 y_{\xi\xi} + x_\xi^2 y_{\eta\eta} - 2x_\xi x_\eta y_{\eta\xi}\right)\Big]$$

Pulling Pieces together

i) We requre $\nabla^2 \xi = 0$ and $\nabla^2 \eta = 0$ simultaneously.

ii) We now know

$$\nabla^2 \xi = 0 \iff \xi_{xx} + \xi_{yy} = 0$$

$$\iff \frac{1}{J^3}\Big\{-y_\eta\Big[(x_\eta^2 + y_\eta^2)\,x_{\xi\xi} - 2\,(y_\xi y_\eta + x_\xi x_\eta)x_{\xi\eta} + (x_\xi^2 + y_\xi^2)\,x_{\eta\eta}\Big]$$

$$+ x_\eta \Big[(x_\eta^2 + y_\eta^2) y_{\xi\xi} - 2(y_\xi y_\eta + x_\xi x_\eta) y_{\xi\eta} + (x_\xi^2 + y_\xi^2) y_{\eta\eta} \Big] \Big\} = 0 \tag{4}$$

iii) Furthermore,

$$\nabla^2 \eta = 0 \iff \eta_{xx} + \eta_{yy} = 0$$

$\iff$

$$-\frac{1}{J^3} \Big\{ -y_\xi \Big[(x_\eta^2 + y_\eta^2) x_{\xi\xi} - 2(x_\xi x_\eta + y_\xi y_\eta) x_{\xi\eta} + (x_\xi^2 + y_\xi^2) x_{\eta\eta} \Big]$$

$$+ x_\xi \Big[(x_\eta^2 + y_\eta^2) y_{\xi\xi} - 2(x_\xi x_\eta + y_\xi y_\eta) y_{\xi\eta} + (x_\xi^2 + y_\xi^2) y_{\eta\eta} \Big] \Big\} = 0 \tag{5}$$

iv) Notice that (4) and (5) are both simultaneously true when

$$(x_\eta^2 + y_\eta^2)\, x_{\xi\xi} - 2\,(x_\xi x_\eta + y_\xi y_\eta)\, x_{\xi\eta} + (x_\xi^2 + y_\xi^2)\, x_{\eta\eta} = 0$$

and

$$(x_\eta^2 + y_\eta^2)\, y_{\xi\xi} - 2\,(x_\xi x_\eta + y_\xi y_\eta)\, y_{\xi\eta} + (x_\xi^2 + y_\xi^2)\, y_{\eta\eta} = 0$$

v) Therefore, the inverse problem is

Inverse Winslow Problem

Solve,

$$\alpha\, x_{\xi\xi} - 2\beta\, x_{\eta\xi} + \gamma\, x_{\eta\eta} = 0$$

$$\alpha\, y_{\xi\xi} - 2\beta\, y_{\xi\eta} + \gamma\, y_{\eta\eta} = 0$$

with

$$\alpha = x_\eta^2 + y_\eta^2$$

$$\beta = x_\xi x_\eta + y_\xi y_\eta$$

$$\gamma = x^2 + y_\xi^2$$

for $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$