Summary of the Options for a Physical Solve and Mesh Solve Coupling

Option 1: $(PM)^n$ - Iteration

- In this option the iteration is between the Physical solve and the Mesh Solve

inputs: starting uniform grid X^0

for $n = 0, 1, \ldots$

① Physical Solve
- Solve the physical problem on mesh X^n to obtain U^n

② mesh solve
- solve the MMPDE to obtain X^{n+1}

The MMPDE: $\nabla \left(M(U^{n-1}) \nabla X^n(\xi, \eta) \right) = (0,0)^T$

Option 1 Results

- This scheme works well and converges (according to Kopteva this makes sense)

Option 2: P + (M iteration) or (PM^k iteration)

- In this option the iteration is between mesh solves

inputs: the starting uniform grid X^0

for $n = 0, 1, \ldots$

① Physical Solve
- Solve the physical problem on X^n to obtain $U^{(n)}$

② for $k = n, n+1, \ldots$

① Compute the mesh density function $M(X^k, U^{(n)}(X^{(k-1)}))$

② Solve the MMPDE for x and y

③ Construct the new mesh $X^{(k+1)}$
Interpolate $U^{(n)}$ onto $X^{(k+1)}$

The MMPDE: $\nabla \left(M(U^{n-1}) \nabla X^n(\xi, \eta) \right) = (0,0)^T$

Option 2 Results

- We chose $M = \sqrt{1 + u_x^2 + u_y^2}$
- The iteration seems to work but diverges

Option 3: A different $(MP)^k$ iteration

- This iteration has the same overall structure as Option 2. However, in this iteration we hope to update U (and consequently M) at every iteration of the solver used in FireDrake
- In the last option, M was behind by one $(M)^k$ iteration. In this option, M will be behind only by one solver iteration in a particular $(M)^k$ iteration

inputs: the starting uniform grid X^0

for $n = 0, 1, \ldots$

① Physical Solve
- Solve the physical problem on X^n to obtain $U^{(n)}$

② for $k = n, n+1, \ldots$

① Compute the mesh density function $M(X^k, U^{(n)}(X^{(k-1)}))$

② Solve the MMPDE for x and y

→ If an iterative method is used to compute $X^{(k+1)}$ we want U to be interpolated at each intermediate $X^{(k+1)}$ approximation

Issues with this option

- U depends on a mesh constructed from x and y solutions
- Consequently, U depends on the background mesh for the particular MMPDE solve
- For this option to work, we would have to be able to update the background mesh for every iteration of the built-in solver → I am not sure how this is done.