IE 525 - Numerical Methods in Finance

Monte Carlo simulation - American Options

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- CBOE most active options for Monday, June 22, 2015

Market Data Get Quote Enter Symbol

Overview | Most Active Options | Futures List | Portfolio

Most Active Options

Contract Name	Type	Ticker	Last	Chg	Chg %	Volume
SPY\15H21\225.0	Call	SPY	0.13	-0.01	-6.67%	52,840
F\15G17\6.0	Call	F	8.90	0.00	0.00%	50,000
CHK\15G17\13.0	Call	CHK	0.10	-0.04	-26.67%	47,315
SPY\15F26\212.0	Call	SPY	1.00	0.21	31.82%	45,401
$\boxed{\text{AAPL\15F26\129.0}}$	Call	AAPL	0.27	0.04	15.38%	43,221
SPY\15F26\214.5	Call	SPY	0.10	-0.02	-20.00%	43,087

Contract Name	Type	Ticker	Last	Chg	Chg %	Volume
SPY\15R30\205.0	Put	SPY	0.18	-0.34	-62.96%	61,867
SPY\15R26\212.0	Put	SPY	1.09	-1.14	-47.90%	61,260
CHK\15S17\10.0	Put	CHK	0.04	-0.03	-27.27%	55,772
F\15S17\6.0	Put	F	0.00	0.00	0.00%	50,000

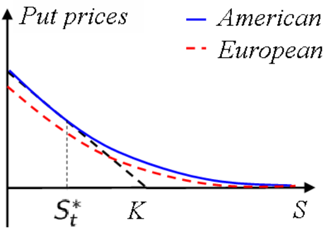
American vs European

- All the above options are American style
- **American style** options can be exercised at any time at or before maturity
  - Stock options: Apple, Ford, Chesapeake
  - ETF options: SPY
  - Some index options: OEX (S&P 100)
  - Need to determine optimal exercise policy & value corresponding to the optimal policy
- **European style** options can be exercised at maturity only
  - Important index options: S&P 500 index options
  - XEO: S&P 100
- Path dependence vs exercise style: European/American style vanilla/barrier/Asian options

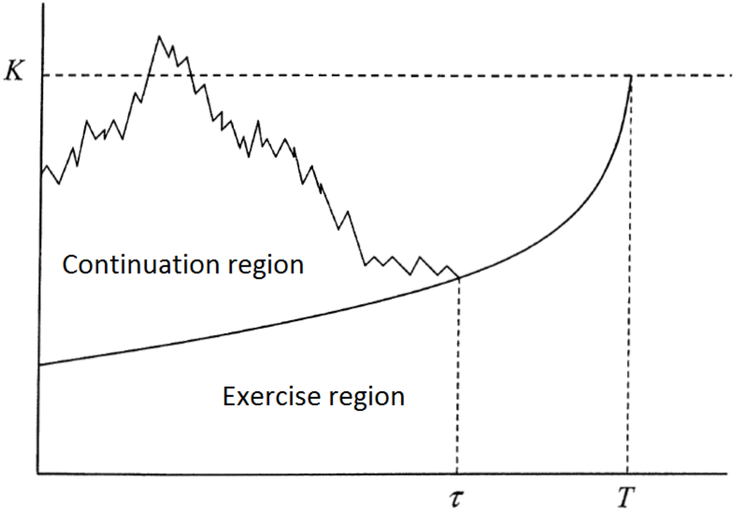
- Consider American vanilla put with strike K and maturity T
- **Early exercise may be optimal for American puts**
- Suppose at time $0 \leq t \leq T$, the underlying asset price S is close to zero. Early exercise could be optimal
    - Exercise the put and receive almost K immediately and start earning interest
    - Exercise later to receive at most K

Early exercise boundary

- Find S_t^*: put should be exercised if time t asset price $S < S_t^*$; hold otherwise; indifferent if $S = S_t^*$



- The **early exercise boundary**: $\{S_t^* : 0 \leq t \leq T\}$ (what's S_T^*? for $t_1 < t_2$, which of $S_{t_1}^*$ and $S_{t_2}^*$ is smaller?)


K
Continuation region
Exercise region
τ
T

American put valuation

- Value of American put with strike K and maturity T:

$$V_0(S) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau}(K - S_\tau)^+ | S_0 = S]$$

where $\mathcal{T}$ is the set of stoping times taking values in $[0, T]$

- **Optimal stopping** time is of the following form

$$\tau^* = \inf\{t \geq 0 : S_t \leq S_t^*\}$$

where $\{S_t^*, 0 \leq t \leq T\}$ is the early exercise boundary

- When stock price becomes sufficiently low, exercise is optimal

Numerical valuation

- In the BSM model, American option price solves a system of partial differential inequalities; discretized using finite difference/finite element
- In models with jumps, solve **partial integro-differential inequalities**
- American options that involve multi assets are hard to evaluate
- Monte Carlo simulation attractive for multi-dimensional American option valuation problems

- The option is of **Bermudan style** if permissible exercise times are in $\{t_1, \cdots, t_m\}$
- Bermudan put price

$$V_0(S) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau}(K - S_\tau)^+ | S_0 = S],$$

where $\mathcal{T}$ is the set of stopping times taking values in $\{t_1, \cdots, t_m\}$. Optimal stopping time:

$$\tau^* = \inf\{t_i : S_i \leq S_i^*\}$$

where S_i is the asset price at time t_i

- Converges to American put price as $m \to +\infty$

Dynamic programming

- Let $\delta = T/m, t_i = i\delta, i = 1, \cdots, m$
- Bermudan put price V_0 can be computed recursively

$$V_m(S) = (K - S)^+,$$

$$V_i(S) = \max\{(K-S)^+, \mathbb{E}[e^{-r\delta} V_{i+1}(S_{i+1}) | S_i = S]\}, 0 \leq i \leq m-1$$

- At any time $i\delta$, when the underlying asset price is S, one exercises if the put payoff is greater than the **continuation value** $\mathbb{E}[e^{-r\delta} V_{i+1}(S_{i+1}) | S_i = S]$
- The main task is computing the continuation value

Regression based approximation

- Recursion:

$$V_i(S) = \max\{(K - S)^+, e^{-r\delta}\mathbb{E}[V_{i+1}(S_{i+1})|S_i = S]\}$$

- The **conditional expectation** $\mathbb{E}[V_{i+1}(S_{i+1})|S_i = S]$ is a function of S. Approximate it by

$$\mathbb{E}[V_{i+1}(S_{i+1})|S_i = S] = \beta_{i,1}\psi_1(S) + \cdots + \beta_{i,K}\psi_K(S)$$

where $\psi_1(S), \cdots, \psi_K(S)$ are **basis functions**, $\beta_{i,1}, \cdots, \beta_{i,K}$ are constants to be estimated

Regression

- Multiple linear regression model at time $i\delta$:

$$V_{i+1}(S_{i+1}) = \beta_{i,1}\psi_1(S_i) + \cdots + \beta_{i,K}\psi_K(S_i) + \epsilon,$$

  where ϵ is a zero mean error term. Given n pairs of (S_i, S_{i+1}), estimate the coefficients
- Denote

$$\beta_i = \begin{pmatrix} \beta_{i,1} \\ \vdots \\ \beta_{i,K} \end{pmatrix}, Y_i = \begin{pmatrix} V_{i+1}(S_{i+1,1}) \\ \vdots \\ V_{i+1}(S_{i+1,n}) \end{pmatrix}, X_i = \begin{pmatrix} \psi_1(S_{i,1}) & \cdots & \psi_K(S_{i,1}) \\ \vdots & & \vdots \\ \psi_1(S_{i,n}) & \cdots & \psi_K(S_{i,n}) \end{pmatrix}$$

  where $\{(S_{i,j}, S_{i+1,j}), j = 1, \cdots, n\}$ are the n stock price pairs

- OLS estimate
$$\beta_i = (X_i^\top X_i)^{-1} X_i^\top Y_i$$
- n stock price paths are simulated. This provides the needed stock price pairs for estimating each β_i
- V_m is known - the payoff function. With the recursion, when approximating

$$\mathbb{E}[V_{i+1}(S_{i+1})|S_i = S] = \beta_{i,1}\psi_1(S) + \cdots + \beta_{i,K}\psi_K(S),$$

  V_{i+1} is known already from a previous step
- The basis functions could be polynomials: $1, S, S^2, \cdots, S^{K-1}$

Implementation

- Given a stock price model (BSM, jump diffusion, Lévy, etc.)
- Given initial asset price S, put strike price K, put maturity T, risk free interest rate r
- Divide $[0, T]$ into m equal subintervals, each with length $\delta = T/m$
- Simulate n stock price paths:

$$\text{1st path: } (S, S_{1,1}, S_{2,1}, \cdots, S_{m,1})$$

$$\text{2nd path: } (S, S_{1,2}, S_{2,2}, \cdots, S_{m,2})$$

$$\vdots$$

$$\text{nth path: } (S, S_{1,n}, S_{2,n}, \cdots, S_{m,n})$$

- Compute time 0 option price $V_0(S)$ recursively:

$$V_m(S) = (K - S)^+,$$

$$V_i(S) = \max\left((K - S)^+, e^{-r\delta} C_i(S)\right),\ 0 \le i \le m-1$$

$$C_i(S) = \mathbb{E}[V_{i+1}(S_{i+1}) | S_i = S]$$

- $V_{m-1}(S) = \max\left((K - S)^+, e^{-r\delta} C_{m-1}(S)\right)$ where

bls

$$e^{-r\delta} C_{m-1}(S) = e^{-r\delta}\mathbb{E}[(K - S_m)^+ | S_{m-1} = S]$$

is computed using European put price formula (Black-Scholes formula in BSM, or by Fourier transform methods in jump diffusion or Lévy models) is not BLS

- $V_{m-2}(S) = \max((K-S)^+, e^{-r\delta} C_{m-2}(S))$ where

$$\begin{aligned} C_{m-2}(S) &= \mathbb{E}[V_{m-1}(S_{m-1}) | S_{m-2} = S] \\ &\approx \beta_{m-2,1}\psi_1(S) + \cdots + \beta_{m-2,K}\psi_K(S) \end{aligned}$$

where $\beta_{m-2} = (\beta_{m-2,1}, \cdots, \beta_{m-2,K})^\top$ is obtained via linear regression: $\beta_{m-2} = (X_{m-2}^\top X_{m-2})^{-1} X_{m-2}^\top Y_{m-2}$

$$Y_{m-2} = \begin{pmatrix} V_{m-1}(S_{m-1,1}) \\ \vdots \\ V_{m-1}(S_{m-1,n}) \end{pmatrix}, X_{m-2} = \begin{pmatrix} \psi_1(S_{m-2,1}) & \cdots & \psi_K(S_{m-2,1}) \\ \vdots & & \vdots \\ \psi_1(S_{m-2,n}) & \cdots & \psi_K(S_{m-2,n}) \end{pmatrix}$$

where Y_{m-2} is computed using Black-Scholes formula or Fourier transform

- $V_{m-3}(S) = \max((K-S)^+, e^{-r\delta} C_{m-3}(S))$ where

$$\begin{aligned} C_{m-3}(S) &= \mathbb{E}[V_{m-2}(S_{m-2}) | S_{m-3} = S] \\ &\approx \beta_{m-3,1}\psi_1(S) + \cdots + \beta_{m-3,K}\psi_K(S) \end{aligned}$$

where $\beta_{m-3} = (\beta_{m-3,1}, \cdots, \beta_{m-3,K})^\top$ is obtained via linear regression: $\beta_{m-3} = (X_{m-3}^\top X_{m-3})^{-1} X_{m-3}^\top Y_{m-3}$

$$Y_{m-3} = \begin{pmatrix} V_{m-2}(S_{m-2,1}) \\ \vdots \\ V_{m-2}(S_{m-2,n}) \end{pmatrix}, X_{m-3} = \begin{pmatrix} \psi_1(S_{m-3,1}) & \cdots & \psi_K(S_{m-3,1}) \\ \vdots & & \vdots \\ \psi_1(S_{m-3,n}) & \cdots & \psi_K(S_{m-3,n}) \end{pmatrix}$$

where

$$\begin{aligned} V_{m-2}(S_{m-2,j}) &= \max\Big((K - S_{m-2,j})^+, \\ &\quad e^{-r\delta}(\beta_{m-2,1}\psi_1(S_{m-2,j}) + \cdots + \beta_{m-2,K}\psi_K(S_{m-2,j}))\Big) \end{aligned}$$

- Option price at time 0 when asset price is S:

$$\begin{aligned} V_0(S) &= \max\left((K-S)^+, e^{-r\delta}\mathbb{E}[V_1(S_1)|S_0=S]\right) \\ &\approx \max\left((K-S)^+, e^{-r\delta}\frac{1}{n}\sum_{j=1}^{n} V_1(S_{1,j})\right) \end{aligned}$$

where

$$\begin{aligned} V_1(S_{1,j}) &= \max\Big((K-S_{1,j})^+, \\ &\qquad e^{-r\delta}(\beta_{1,1}\psi_1(S_{1,j}) + \cdots + \beta_{1,K}\psi_K(S_{1,j}))\Big) \end{aligned}$$