IE 525 - Numerical Methods in Finance

Monte Carlo simulation - Introduction

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Type: Average Price Options

Contract Unit	On expiration of a call option, the value will be the difference between the average daily settlement price during the calendar month of the first nearby underlying Light Sweet Crude Oil Futures and the strike price multiplied by 1,000 barrels, or zero, whichever is greater. On expiration of a put option, the value will be the difference between the strike price and the average daily settlement price during the calendar month of the first nearby underlying Light Sweet Crude Oil Futures multiplied by 1,000 barrels, or zero, whichever is greater.

Black-Scholes-Merton model

- In BSM, asset price is modeled by **stochastic differential equation**

$$dS_t = (\mu - q)S_t dt + \sigma S_t dB_t$$

or more intuitively

$$\frac{dS_t}{S_t} = (\mu - q)dt + \sigma dB_t$$

- μ: mean rate of return of the asset; q: dividend yield of the asset; σ: **volatility** of the asset
- $\{B_t, t \geq 0\}$: a standard **Brownian motion**

Deterministic case

- **When there is no stochastic term ($\sigma = 0$),** solve the ODE

$$dS_t = (\mu - q)S_t dt \quad \Leftrightarrow \quad S_t = S_0 e^{(\mu - q)t}$$

- Consider a foreign currency. S_t: exchange rate (USD price of the currency), q: the interest rate of the currency.
- Invest S_0 USDs, buy 1 unit of the currency at time 0, deposit at rate q (continuously compounded). At time t, you get e^{qt} units of the currency. Its USD value is

$$S_t e^{qt} = S_0 e^{\mu t}$$

  So μ is the rate of return of the currency.
- Define the log return process $X_t = \ln(S_t / S_0)$, $dX_t = (\mu - q)dt$

Case with a BM term

- **When $\sigma > 0$,** Ito's formula from stochastic calculus leads to

$$dX_t = (\mu - q - \frac{1}{2}\sigma^2)dt + \sigma dB_t$$

Or equivalently,

$$X_t = \ln(S_t/S_0) = (\mu - q - \frac{1}{2}\sigma^2)t + \sigma B_t,$$

$$S_t = S_0 \exp\left((\mu - q - \frac{1}{2}\sigma^2)t + \sigma B_t\right)$$

- In BSM, S_t is a **geometric Brownian motion**

Brownian motion

- $\{B_t, t \geq 0\}$ is a standard Brownian motion if $B_0 = 0$, and for any $0 \leq s < t \leq u < v$,
  - (**independent increments**) $B_v - B_u$ and $B_t - B_s$ are independent
  - (**stationary increments**) the distribution of $B_t - B_s$ only depends on $t - s$
  - (**normality**) $B_t \sim N(0, t)$
- Increments of a BM are normal: $B_t - B_s \sim N(0, t - s)$

Volatility

- Volatility σ = standard deviation of log return per unit time

$$\sigma^2 = \mathrm{var}(\ln(S_1/S_0))$$

- Note that

$$(\mu - q - \frac{1}{2}\sigma^2)t + \sigma B_t \sim N((\mu - q - \frac{1}{2}\sigma^2)t, \sigma^2 t)$$

Therefore, S_t has a **lognormal distribution**:

$$\mathbb{E}[S_t] = S_0 e^{(\mu - q)t}$$

μ is the mean rate of return

Risk neutral valuation

- To price derivatives, you compute discounted expected payoff
- Derivatives are risky assets. What discount rate to use?
- **Risk neutral valuation**: for pricing derivatives, you can switch to the **risk neutral** world, where all assets earn the risk free interest rate

$$S_t = S_0 \exp\left((r - q - \frac{1}{2}\sigma^2)t + \sigma B_t\right)$$

  Derivative price is the expected payoff (using the above S) discounted at the **risk free interest rate** r
- Refer to the binomial model for intuitions

European vanilla options

- For a **European call** option with maturity T and strike price K, the payoff at maturity: $(S_T - K)^+$
- Call price at time zero is given by

$$c = \mathbb{E}[e^{-rT}(S_T - K)^+]$$

- In BSM, S_T is lognormal. c admits closed-form solution - the **Black-Scholes formula**

$$c = S_0 e^{-qT} N(d_+) - K e^{-rT} N(d_-)$$

$$d_\pm = \frac{\ln(S_0/K) + (r - q \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$N()$ is the standard normal cdf

Asian options

- What's the price of a **discrete Asian call** option with maturity T, strike price K, and number of monitoring intervals m?
- Option payoff at maturity: $(\bar{S}_T - K)^+$

$$\bar{S}_T = \frac{1}{m}\sum_{i=1}^{m} S_{i\Delta}, \quad \Delta = T/m$$

- Risk neutral valuation:

$$V = \mathbb{E}[e^{-rT}(\bar{S}_T - K)^+]$$

- The average of lognormal r.v.'s is not lognormal. V doesn't admit a closed-form solution

- Consider the case with $m = 3$. Denote $Y_t = (r - q - \frac{1}{2}\sigma^2)t$ $+\sigma B_t$, $S_t = S_0 e^{Y_t}$

$$\frac{1}{3}(S_\Delta + S_{2\Delta} + S_{3\Delta}) = \frac{1}{3} S_0 (e^{Y_\Delta} + e^{Y_{2\Delta}} + e^{Y_{3\Delta}})$$

- $(Y_\Delta, Y_{2\Delta}, Y_{3\Delta})$ is multivariate normal. Denote its joint density by $f(y_1, y_2, y_3)$

$$V = e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{3} S_0 (e^{y_1} + e^{y_2} + e^{y_3}) - K \right)^+$$

$$\times f(y_1, y_2, y_3) dy_1 dy_2 dy_3$$

- As m increases, the above becomes more difficult to compute

Monte Carlo methods

- In financial applications, often need to compute

$$\mu = \mathbb{E}[X]$$

  where $\mathbb{E}[X]$ has no analytical expression. For the Asian call option, $X = e^{-rT}(\bar{S}_T - K)^+$
- Simulate i.i.d. copies of X: $\{X_i, i \geq 1\}$. Approximate μ by the sample mean

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- $\bar{X}_n$ is an **unbiased estimator** of μ: $\mathbb{E}[\bar{X}_n] = \mu$

Law of large numbers

- (**Strong law of large numbers**) Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed random variables, with finite mean μ. Then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to \mu, a.s.$$

- LLN guarantees the convergence of the Monte Carlo method
- $\bar{X}_n$ is a **strongly consistent** estimator of μ

Central limit theorem

- (**Lindeberg-Lévy central limit theorem**) Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed random variables, with mean μ and finite variance σ^2. Then

$$\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sigma/\sqrt{n}} \Rightarrow N(0,1)$$

- By CLT, the Monte Carlo method converges at rate $1/\sqrt{n}$
- Attractive for **high dimensional problems** (where other numerical methods often fail)

Confidence interval

- Let z_α be the $1-\alpha$ quantile of $N(0,1)$. By CLT,

$$\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \;\; \text{approximately}$$

$$\mathbb{P}(|\bar{X}_n-\mu| \le z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) \approx 1-\alpha$$

$$\mathbb{P}\left(\mu \in [\bar{X}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}]\right) \approx 1-\alpha$$

- $\alpha = 0.05, z_{\alpha/2} = 1.96$: with probability approximately 95%, the true mean μ is in the interval $\bar{X}_n \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

Standard error

- $\sigma/\sqrt{n}$ is the **standard error of the mean**, i.e., the standard deviation of $\bar{X}_n$
- Standard error measures the **precision** of the Monte Carlo estimation
- Estimate the unknown σ by the sample standard deviation of $\{X_1, \cdots, X_n\}$

$$s_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}$$

- Report both $\bar{X}_n$ and the estimated standard error $s_n/\sqrt{n}$

Computing s_n

- Avoid storing $\{X_1, \cdots, X_n\}$ when computing s_n

$$
\begin{aligned}
s_n^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \\
&= \frac{1}{n-1}\sum_{i=1}^{n}(X_i^2 - 2\bar{X}_n X_i + \bar{X}_n^2) \\
&= \frac{1}{n-1}\Big(\sum_{i=1}^{n} X_i^2 - 2\bar{X}_n \sum_{i=1}^{n} X_i + \sum_{i=1}^{n}\bar{X}_n^2\Big) \\
&= \frac{1}{n-1}\Big(\sum_{i=1}^{n} X_i^2 - 2n\bar{X}_n^2 + n\bar{X}_n^2\Big) \\
&= \frac{n}{n-1}\Big(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}_n^2\Big)
\end{aligned}
$$

Algorithm

- Algorithm (estimate $\mu = \mathbb{E}[X]$ using Monte Carlo simulation)

  Generate x_1*, let* $\bar{x} = x_1, \bar{y} = x_1^2$
  For $k = 2 : n$
      Generate x_k
      Update sample mean: $\bar{x} = (1 - \frac{1}{k})\bar{x} + \frac{1}{k}x_k$
      Update $\bar{y}$*:* $\bar{y} = (1 - \frac{1}{k})\bar{y} + \frac{1}{k}x_k^2$
  End
  Compute $se = \sqrt{\frac{1}{n-1}(\bar{y} - \bar{x}^2)}$*. Report* $\bar{x}$ *and se*

- Here $\bar{x}$ denotes the average of $\{x_1, x_2, \cdots\}$, $\bar{y}$ denotes the average of $\{x_1^2, x_2^2, \cdots\}$

- For the discrete Asian call, need to simulate $S_{\Delta}, \cdots, S_{m\Delta}$ to obtain one $\bar{S}_T$

$$S_{i\Delta} = S_0 \exp\left((r - q - \frac{1}{2}\sigma^2)i\Delta + \sigma B_{i\Delta}\right)$$

- **What's wrong with the following**: since $B_{i\Delta} \sim \sqrt{i\Delta}N(0,1)$. Simulate standard normal random variables $Z_1, \cdots, Z_m$. Let

$$S_{i\Delta} = S_0 \exp\left((r - q - \frac{1}{2}\sigma^2)i\Delta + \sigma\sqrt{i\Delta}Z_i\right),$$

and then compute $\bar{S}_T = \frac{1}{m}\sum_{i=1}^{m} S_{i\Delta}$

Use independent increment property

- Note that

$$S_{i\Delta} = S_{(i-1)\Delta} \exp\left((r - q - \frac{1}{2}\sigma^2)\Delta + \sigma(B_{i\Delta} - B_{(i-1)\Delta}) \right)$$

- $B_{i\Delta} - B_{(i-1)\Delta} \sim \sqrt{\Delta} N(0,1)$ and are independent
- Algorithm (simulate discounted payoff for discrete Asian call)

Start from the initial asset price S_0
For $i = 1 : m$
  Generate z_i from $N(0,1)$
  Compute $S_{i\Delta} = S_{(i-1)\Delta} \exp\left((r - q - \frac{1}{2}\sigma^2)\Delta + \sigma\sqrt{\Delta} z_i \right)$
End
Compute $\bar{S}_T = \frac{1}{m}\sum_{i=1}^{m} S_{i\Delta}$ and $e^{-rT}(\bar{S}_T - K)^+$