IE 525 - Numerical Methods in Finance

Monte Carlo simulation - variance reduction

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Variance reduction

- For i.i.d. $\{X_i, i \geq 1\}$ with $\mu = \mathbb{E}[X_1], \sigma^2 = \mathrm{var}(X_1)$

$$\frac{1}{n}\sum_{i=1}^{n} X_i - \mu \sim \frac{\sigma}{\sqrt{n}} N(0,1) \quad \text{approximately}$$

- Monte Carlo methods converge at rate $1/\sqrt{n}$: need to quadruple sample size to halve standard error
- Might be easier to reduce σ^2 (and hence σ) using **variance reduction** techniques
- In European call example: using **antithetic variates**, standard deviation nearly halves (would need to quadruple sample size to achieve the same standard error in the standard approach)

Control variates

- Interested in $\mathbb{E}[Y]$: generate i.i.d. $\{Y_i \sim Y, i \geq 1\}$. Standard approach: estimate $\mathbb{E}[Y]$ with

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

- Suppose
  - X and Y are correlated
  - $\mathbb{E}[X]$ is known (and hence the error $\mathbb{E}[X] - X_i$ for any $X_i \sim X$)
  - Error of Y_i is *"proportional"* to error of X_i

$$\mathbb{E}[Y] - Y_i \approx b(\mathbb{E}[X] - X_i)$$

- Using X_i's as **control variates**: generate i.i.d. $\{(X_i, Y_i) \sim (X, Y),\ i \geq 1\}$. Let

$$Y_i^b = Y_i + b(\mathbb{E}[X] - X_i)$$

and corresponding estimator

$$\bar{Y}_n^b = \frac{1}{n}\sum_{i=1}^{n}(Y_i + b(\mathbb{E}[X] - X_i))$$

- By law of large numbers,

$$\bar{Y}_n^b \to \mathbb{E}[Y] + b(\mathbb{E}[X] - \mathbb{E}[X]) = \mathbb{E}[Y]$$

- Variance of $\bar{Y}_n^b$ given by

$$\mathrm{var}(\bar{Y}_n^b) = \frac{1}{n}(\sigma_Y^2 + b^2\sigma_X^2 - 2b\sigma_X\sigma_Y\rho_{XY})$$

where $\sigma_Y^2 = \mathrm{var}(Y)$, $\sigma_X^2 = \mathrm{var}(X)$, $\rho_{XY} = \mathrm{corr}(X, Y)$

- In contrast, without control variates ($b = 0$)

$$\mathrm{var}(\bar{Y}_n) = \frac{1}{n}\sigma_Y^2$$

- Better estimation when

$$b^2\sigma_X^2 - 2b\sigma_X\sigma_Y\rho_{XY} < 0$$

- Best performance when (**optimally controlled**)

$$b = b^* = \frac{\mathrm{cov}(X, Y)}{\mathrm{var}(X)}$$

$$\mathrm{var}(\bar{Y}_n^{b^*}) = (1 - \rho_{XY}^2)\mathrm{var}(\bar{Y}_n)$$

- High correlation preferred:

$$\begin{aligned}
\rho_{XY} &= 0.9487, \quad 1 - \rho_{XY}^2 = 10\% \\
\rho_{XY} &= 0.8660, \quad 1 - \rho_{XY}^2 = 25\% \\
\rho_{XY} &= 0.7071, \quad 1 - \rho_{XY}^2 = 50\%
\end{aligned}$$

- 94.87% correlation leads to a standard error that can only be achieved by increasing the sample size by 10 times in the standard approach

- Usually, b^* has to be estimated (e.g., using $J = 1000$ pairs of (X_i, Y_i))

$$\hat{b}_J := \frac{\sum_{i=1}^{J}(X_i - \bar{X}_J)(Y_i - \bar{Y}_J)}{\sum_{i=1}^{J}(X_i - \bar{X}_J)^2}$$

- By law of large numbers, with probability one

$$\hat{b}_J \to b^* = \frac{\mathrm{cov}(X, Y)}{\mathrm{var}(X)}, \quad J \to +\infty$$

Example: pricing European options

- Pricing a European option with payoff $G(S_T)$

$$e^{-rT}\mathbb{E}[G(S_T)]$$

  For a European vanilla call, $G(S_T) = (S_T - K)^+$

- Simple control variates: **asset price itself**

$$\mathbb{E}[S_T] = S_0 e^{(r-q)T}$$

- Computing $\mathbb{E}[Y] := \mathbb{E}[G(S_T)]$ using control variates:

$$Y_i = G(S_T^i), \quad X_i = S_T^i, \quad Y_i^b = Y_i + b(S_0 e^{(r-q)T} - X_i)$$

Example: pricing Asian options

- Pricing a discrete Asian call option with maturity T, strike K, m monitoring times $t_j = j\delta, 1 \le j \le m, \delta = T/m$

$$V = e^{-rT}\mathbb{E}[(\bar{S}_T - K)^+], \quad \bar{S}_T = \frac{1}{m}\sum_{j=1}^{m} S_{t_j}$$

- Simulate n sample paths, for ith sample path, compute

$$Y_i = \left(\frac{1}{m}\sum_{j=1}^{m} S^i_{t_j} - K\right)^+$$

- Correlation between $\left(\bar{S}_T - K\right)^+$ and S_T not strong
- Better control variates: **geometric Asian call**

$$X = \left(\left(\prod_{j=1}^{m} S_{t_j} \right)^{1/m} - K \right)^+$$

- The geometric average is log-normally distributed in BSM

$$\left(\prod_{j=1}^{m} S_{t_j} \right)^{1/m} = S_0 \exp \left((r - q - \frac{1}{2}\sigma^2) \frac{1}{m} \sum_{j=1}^{m} t_j + \frac{\sigma}{m} \sum_{j=1}^{m} B_{t_j} \right)$$

- Since $t_j = j\delta$,

$$\frac{1}{m}\sum_{j=1}^{m} t_j = \frac{1}{2}(m+1)\delta = \frac{1}{2}(T+\delta) := \tilde{T}$$

- Since $\{B_t, t \geq 0\}$ is a standard BM

$$\frac{\sigma}{m}\sum_{j=1}^{m} B_{t_j} = \frac{\sigma}{m}\Big(B_{m\delta} - B_{(m-1)\delta} + 2(B_{(m-1)\delta} - B_{(m-2)\delta}) + \cdots + mB_{\delta}\Big)$$

is normal with mean 0 and variance

$$\frac{\sigma^2}{m^2}(1+4+\cdots m^2)\delta = \frac{\sigma^2}{6m^2}m(m+1)(2m+1)\delta = \frac{2m+1}{3m}\sigma^2\,\tilde{T}$$

- Define $\tilde{\sigma}$ such that

$$\tilde{\sigma}^2 = \frac{2m+1}{3m}\sigma^2$$

and denote

$$\tilde{q} = q + \frac{1}{2}\sigma^2 - \frac{1}{2}\tilde{\sigma}^2$$

then

$$\begin{aligned} \left(\prod_{j=1}^{m} S_{t_j}\right)^{1/m} &\sim S_0 \exp\left((r - q - \frac{1}{2}\sigma^2)\tilde{T} + \tilde{\sigma} B_{\tilde{T}}\right) \\ &= S_0 \exp\left((r - \tilde{q} - \frac{1}{2}\tilde{\sigma}^2)\tilde{T} + \tilde{\sigma} B_{\tilde{T}}\right) := \tilde{S}_{\tilde{T}} \end{aligned}$$

- For the geometric Asian call

$$\mathbb{E}[X] = e^{r\tilde{T}}\mathbb{E}[e^{-r\tilde{T}}(\tilde{S}_{\tilde{T}} - K)^+] = e^{r\tilde{T}} \times \text{BS call price}$$

  where the Black-Scholes call price is obtained with *interest rate r, continuous yield $\tilde{q}$, volatility $\tilde{\sigma}$, strike K, maturity $\tilde{T}$, initial asset price S_0*

- **Control variate method for pricing discrete Asian call options**

$$Y_i^b = Y_i + b(\mathbb{E}[X] - X_i)$$

  where $\mathbb{E}[X]$ is computed as above

Effectiveness

- Effectiveness depends on correlation ρ_{XY}, which may depend on parameters in the problem: e.g., for European call in the BSM ($r = 5\%, \sigma = 0.3, S_0 = 50, T = 0.25$), the lower K, the greater the correlation between $Y = (S_T - K)^+$ and $X = S_T$

K	40	50	60
ρ	99.5%	89.5%	60.4%
$1 - \rho^2$	1%	20%	64%

- For Asian call in the BSM ($r = 5\%$, $\sigma = 0.3$, $S_0 = 50$, $T = 0.25$, $K = 50$, $m = 13$): $Y = \left(\frac{1}{m}\sum_{j=1}^{m} S_{t_j} - K\right)^+$, $X = S_T$ or $(S_T - K)^+$ or $\left(\left(\prod_{j=1}^{m} S_{t_j}\right)^{1/m} - K\right)^+$

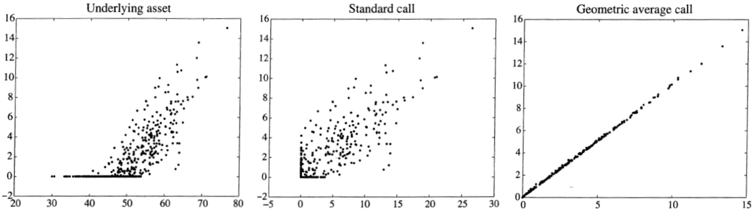

Figure: Correlations are 0.79, 0.85, > 0.99 respectively

- Using geometric Asian call as control is the most efficient

Example: pricing in non-BSM models

- Want to price a contract in the following model

$$dS_t = (r - q)S_t dt + \sigma_t S_t dB_t$$

where σ_t is a function of t and/or S_t as in **local volatility** models or governed by a separate stochastic process as in **stochastic volatility** models

- Using Ito formula, $X_t = \ln(S_t)$ is governed by

$$dX_t = (r - q - \frac{1}{2}\sigma_t^2)dt + \sigma_t dB_t$$

- For $\delta = T/m, t_i = i\delta$, **Euler discretization** leads to

$$X_{t_{i+1}} - X_{t_i} = (r - q - \frac{1}{2}\sigma_{t_i}^2)\delta + \sigma_{t_i}(B_{t_{i+1}} - B_{t_i})$$

or equivalently

$$S_{t_{i+1}} = S_{t_i} \exp\left((r - q - \frac{1}{2}\sigma_{t_i}^2)\delta + \sigma_{t_i}(B_{t_{i+1}} - B_{t_i})\right)$$

- Starting from S_0, $\{S_{t_1}, \cdots, S_{t_m}\}$ can be simulated by replacing $B_{t_{i+1}} - B_{t_i}$ by $\sqrt{\delta} Z_{i+1}$ for $Z_{i+1} \sim N(0, 1)$

- For European call with strike K, one generates replicates for $Y = (S_{t_m} - K)^+$ as above
- For a control, starting from S_0, generate a sequence $\{\hat{S}_{t_1}, \cdots, \hat{S}_{t_m}\}$ as follows:

$$\hat{S}_{t_{i+1}} = \hat{S}_{t_i} \exp\left((r - q - \frac{1}{2}\sigma^2)\delta + \sigma\sqrt{\delta}Z_{i+1}\right)$$

  where σ is a constant and typical value of σ_t.
- Let $X = (\hat{S}_{t_m} - K)^+$. $\mathbb{E}[X]$ is known through Black-Scholes formula. X serves as the control