IE 525 - Numerical Methods in Finance

Monte Carlo simulation - Efficiency

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Constructing more efficient estimators

- We estimate $\mu = \mathbb{E}[X]$ by

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

  for some i.i.d. $\{X_i, i \geq 1\}$. Generally, X_i's do not need to have the same distribution as X
- We only need that $\bar{X}_n$ converges to μ
- May construct $\{X_i, i \geq 1\}$ specifically to improve **efficiency**

European call

- Use Monte Carlo simulation to compute

$$c = \mathbb{E}[e^{-rT}(S_T - K)^+]$$

$$S_T = S_0 \exp\left((r - q - \frac{1}{2}\sigma^2)T + \sigma B_T\right)$$

- **Direct approach**: let $c_i = e^{-rT}(S_i - K)^+$, where

$$S_i = S_0 \exp\left((r - q - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z_i\right)$$

  for i.i.d. Z_i's. Here c_i's are i.i.d. replicates of $e^{-rT}(S_T - K)^+$ and are unbiased with $\mathbb{E}[c_i] = c$

Antithetic variates

variance reduction technique: to have smaller CI

one use zi and one use negative zi

- **Antithetic approach**: let
  $c_i^* = \frac{1}{2}e^{-rT}\left((S_{i+} - K)^+ + (S_{i-} - K)^+\right)$, where

  time is less than double, only one zi generate

  $$S_{i\pm} = S_0 \exp\left((r - q - \frac{1}{2}\sigma^2)T \pm \sigma\sqrt{T}Z_i\right)$$

- c_i^*'s are i.i.d. and unbiased with $c = \mathbb{E}[c_i^*]$ but do not have the same distribution as $e^{-rT}(S_T - K)^+$

difference variance compared to si, covariance is negative, < sigma^2

How to compare

- Want to estimate $\mu = \mathbb{E}[X]$
- Suppose $\{X_i, i \geq 1\}$ **are i.i.d. such that** $\mathbb{E}[X_i] = \mu$. Then $\bar{X}_n \to \mu$ is strongly consistent with

central limit theorem

$$\sqrt{n}(\bar{X}_n - \mu) \Rightarrow N\left(0, \sigma^2\right)$$

precision: tightness of CI, sd_error

Here $\sigma^2 = \text{var}(X_i)$ is not necessarily equal to $\text{var}(X)$
- We want σ^2 to be smaller so that the confidence interval $\bar{X}_n \pm z_{\alpha/2}\sigma/\sqrt{n}$ is tighter
- Is σ^2 the only thing that matters when comparing different approaches?

The case of deterministic τ

- How to compare
  - Approach 1: X_i's have smaller σ^2 but are slower to compute
  - Approach 2: X_i's have larger σ^2 but are faster to compute
- For fair comparison, must take computational time into account
- Suppose **generating X_i takes τ units of time**
- Let s be our computational budget

s : total time to generate, tao, generate one,n(s)=N

- The number of replicates we can generate is $n(s) = \lfloor s/\tau \rfloor$, the integer part of s/τ:

$$\bar{X}_{n(s)} = \frac{1}{n(s)} \sum_{i=1}^{n(s)} X_i$$

- As $s \to +\infty$,

$$\sqrt{n(s)}(\bar{X}_{n(s)} - \mu) \Rightarrow N(0, \sigma^2)$$

where $\sigma^2 = \mathrm{var}(X_i)$.

s / n(s) -> tao, when n(s) is large, s / n(s) go to approxi tao

Comparing $\sigma^2\tau$

- Note that $n(s)/s \to 1/\tau$, sqrt(s/tao)(x-u) = N(0, sigma^2)

$$\sqrt{s}(\bar{X}_{n(s)} - \mu) \Rightarrow N(0, \sigma^2\tau)$$

- In terms of computational budget s, $\bar{X}_{n(s)}$ converges to μ at rate $1/\sqrt{s}$
- When comparing approaches with different $\sigma^2 = \mathrm{var}(X_i)$ and τ, **select the one with smaller $\sigma^2\tau$**

Direct vs antithetic for European call

Method	Direct	Antithetic
Replicates	c_i	c_i^*
Estimators	$\frac{1}{n}\sum_{i=1}^n c_i$	$\frac{1}{n}\sum_{i=1}^n c_i^*$
Time for each replicate	τ	$< 2\tau$
Variance	$\sigma^2 := \mathrm{var}(c_i)$	$< \frac{1}{2}\sigma^2$
	$\sigma^2\tau$	$< \sigma^2\tau$ and more efficient

Antithetic variance

- Denote $\sigma^2 = \text{var}(c_i) = \text{var}(e^{-rT}(S_i - K)^+)$.

$$\begin{aligned}
\text{var}(c_i^*) &= \frac{1}{4}\Big[\text{var}\left(e^{-rT}(S_{i+} - K)^+\right) + \text{var}\left(e^{-rT}(S_{i-} - K)^+\right) \\
&\quad +2\text{cov}\left(e^{-rT}(S_{i+} - K)^+, e^{-rT}(S_{i-} - K)^+\right)\Big] \\
&< \frac{1}{4}(\sigma^2 + \sigma^2) = \frac{1}{2}\sigma^2
\end{aligned}$$

S+ increase , S- decrease , negative cor

Barrier option

- The above doesn't apply if τ itself is random
- Consider a **discrete up-and-out call** option with maturity T, strike price K, and upper barrier U
- The call option is knocked out if the asset price exceeds U at any time in $\{\Delta, 2\Delta, \cdots, m\Delta\}$, $\Delta = T/m$

$$V = \mathbb{E}\left[e^{-rT}(S_T - K)^+ \mathbf{1}_{\{S_{j\Delta} < U, 1 \le j \le m\}}\right]$$

Simulate a barrier option payoff

- Starting from S_0, using

$$S_{j\Delta} = S_{(j-1)\Delta} \exp\left((r - q - \frac{1}{2}\sigma^2)\Delta + \sigma(B_{j\Delta} - B_{(j-1)\Delta}) \right),$$

  generate $S_\Delta, S_{2\Delta}, \cdots$
- If $S_{j\Delta} < U, \forall 1 \leq j \leq m$, return discounted payoff $e^{-rT}(S_{m\Delta} - K)^+$
- If for some $1 \leq j \leq m$, $S_{j\Delta} \geq U$, stop and return discounted payoff 0. No need to generate $S_{(j+1)\Delta}, \cdots, S_{m\Delta}$
- The time τ for generating each payoff is random

The case of random τ

- Want to estimate $\mu = \mathbb{E}[X]$
- Suppose $\{(X_i, \tau_i), i \geq 1\}$ **are i.i.d. with** $\mathbb{E}[X_i] = \mu$. τ_i is the random time needed to generate X_i
- Let s be our computational budget,

$$N(s) = \sup\{n \geq 0, \sum_{i=1}^{n} \tau_i \leq s\}$$

  is the number of replicates we can generate
- It can be shown that

$$\frac{N(s)\mathbb{E}[\tau]}{s} \to 1, a.s.$$

Comparing $\sigma^2\mathbb{E}[\tau]$

- Estimate μ by $\bar{X}_{N(s)} = \frac{1}{N(s)} \sum_{i=1}^{N(s)} X_i$

$$\sqrt{N(s)}(\bar{X}_{N(s)} - \mu) \Rightarrow N(0, \sigma^2)$$

std error

- Using $N(s)\mathbb{E}[\tau]/s \to 1$,

$$\sqrt{s}(\bar{X}_{N(s)} - \mu) \Rightarrow N(0, \sigma^2\mathbb{E}[\tau])$$

- When comparing approaches with different $\sigma = \mathrm{var}(X_i)$ and random time τ_i, **select the one with smaller $\sigma^2\mathbb{E}[\tau]$**

Local volatility

- When there is bias ($\mathbb{E}[X_i] \neq \mu$), how to allocate computational budget?
- Consider a European call in a local volatility model

$$dS_t = rS_t dt + \sigma(S_t) S_t dB_t$$

More generally, σ could be a function of t and S_t

- The distribution of S_T is generally unknown delta □ 是时间

- Let $\delta = T/m$. Euler discretization of the SDE:

$$\tilde{S}_{(j+1)\delta} = \tilde{S}_{j\delta} + r\tilde{S}_{j\delta}\delta + \sigma(\tilde{S}_{j\delta})\tilde{S}_{j\delta}(B_{(j+1)\delta} - B_{j\delta}).$$

  Start from S_0, replacing $B_{(j+1)\delta} - B_{j\delta}$ by $\sqrt{\delta}Z_j$, one obtains $\tilde{S}_{m\delta}$, and hence discounted option payoff $e^{-rT}(\tilde{S}_{m\delta} - K)^+$
- Denote such generated discounted payoffs by $c_i, \cdots, c_n$.
- For fixed m,

$$\frac{1}{n}\sum_{i=1}^{n} c_i \nrightarrow \mathbb{E}[e^{-rT}(S_T - K)^+]$$

Euler discretization bias

- The distribution of $\tilde{S}_{m\delta}$ obtained this way only approximates that of S_T
- The above discretization scheme introduces bias

$$\mathbb{E}[c_i] \neq \mathbb{E}[e^{-rT}(S_T - K)^+]$$

- The approximation gets better as $\delta \to 0$; the bias gets smaller

Mean square error

- Decreasing δ reduces bias, increases time for generating each option payoff, reduces the number of replicates (for given budget) and increases the variance of $\frac{1}{n}\sum_{i=1}^{n} c_i$
- Want to balance **variance and bias**
- Recall the **mean square error** of an estimator $\hat{\theta}$ for a unknown parameter θ

$$MSE(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \theta)^2] = (bias(\hat{\theta}))^2 + var(\hat{\theta})$$

where $bias(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta$

Assumptions

- Want to estimate $\mu = \mathbb{E}[X]$
- Suppose $X_{i,\delta}, i \geq 1$ **are i.i.d. with** $\mathbb{E}[X_{i,\delta}] = \mu_\delta$
- Assume that the **bias converges to zero at rate** β

$$\mu_\delta - \mu = b\delta^\beta + o(\delta^\beta), \quad b, \beta > 0$$

- Let τ_δ be the computational time for generating one replicate
- Assume that

$$\tau_\delta = c\delta^{-\eta} + o(\delta^{-\eta})$$

  for $c, \eta > 0$. E.g., for Euler discretization of the local volatility model, $\eta = 1$

Controlling $\delta(s)$

- For a computational budget s, let

$$\delta(s) = as^{-\gamma} + o(s^{-\gamma})$$

  for $a, \gamma > 0$ (to be determined)
- $\delta(s)$ **(and hence bias) decreases as budget increases**
- Number of replicates that can be generated

$$n(s) = \left\lfloor \frac{s}{\tau_\delta(s)} \right\rfloor = O(s^{1-\gamma\eta})$$

  We require $\gamma\eta < 1$ so that $1/n(s)$ **(and hence variance) decreases as s increases**

Squared bias

- We estimate μ by

$$\bar{X}_{n(s)} = \frac{1}{n(s)} \sum_{i=1}^{n(s)} X_{i,\delta(s)}$$

- The squared bias of $\bar{X}_{n(s)}$ is

$$(\mu_{\delta(s)} - \mu)^2 = b^2 a^{2\beta} s^{-2\beta\gamma} + o(s^{-2\beta\gamma})$$

Variance

- The variance of $\bar{X}_{n(s)}$ is

$$\text{var}(\bar{X}_{n(s)}) = \frac{\sigma^2_{\delta(s)}}{n(s)}$$

where $\sigma^2_{\delta(s)} = \text{var}(X_{i,\delta(s)})$

- Let σ^2 be the limit of σ^2_δ as $\delta \to 0$. In local volatility example, $\sigma^2 = \text{var}(e^{-rT}(S_T - K)^+)$, $\sigma^2_\delta = \text{var}(e^{-rT}(\tilde{S}_{m\delta} - K)^+)$

$$\text{var}(\bar{X}_{n(s)}) = \frac{\sigma^2 \tau_{\delta(s)}}{s} + o(\tau_{\delta(s)}/s) = \sigma^2 c a^{-\eta} s^{\gamma\eta - 1} + o(s^{\gamma\eta-1})$$

Root mean square error

- To balance squared bias and variance, make $s^{\gamma\eta-1} = s^{-2\beta\gamma}$:

$$\gamma = \frac{1}{2\beta+\eta}$$

- The resulting MSE is

$$MSE(\bar{X}_{n(s)}) = (b^2 a^{2\beta} + \sigma^2 c a^{-\eta}) s^{-2\beta/(2\beta+\eta)} + o(s^{-2\beta/(2\beta+\eta)})$$

  a can be determined by minimizing the coefficient

- The **root mean square error (RMSE)** is of the order

$$RMSE(\bar{X}_{n(s)}) = O(s^{-\beta/(2\beta+\eta)})$$

Some observations

- When η is fixed, larger β (schemes with larger convergence rate of the bias) is preferred
- $O(s^{-1/2})$ is the best one can achieve (the unbiased case; or $\eta = 0$)
- Slower convergence expected for finite β and positive η