Homotopy Charge, Anomalies and non-Fermi Liquids

G. La Nave

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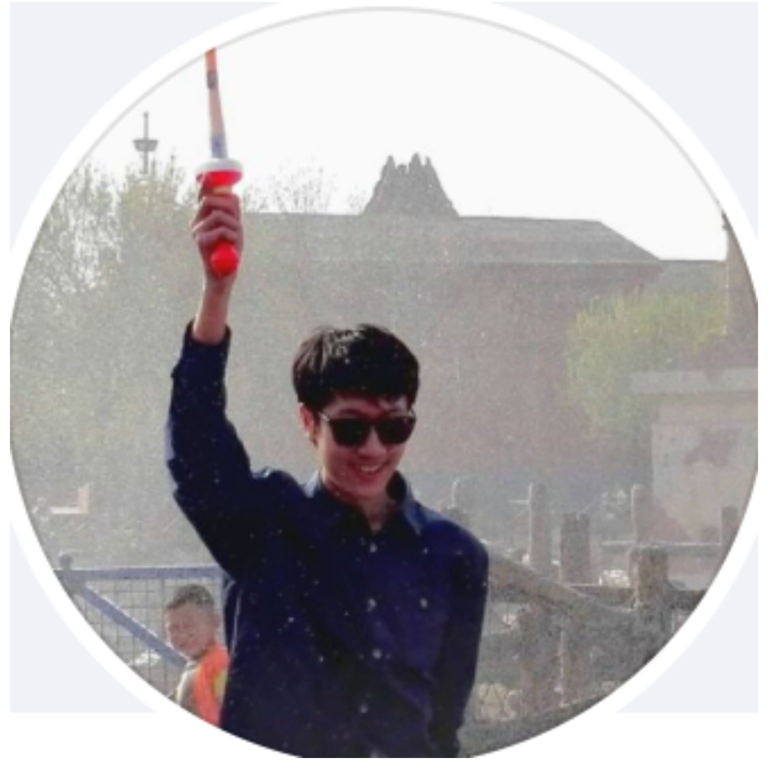
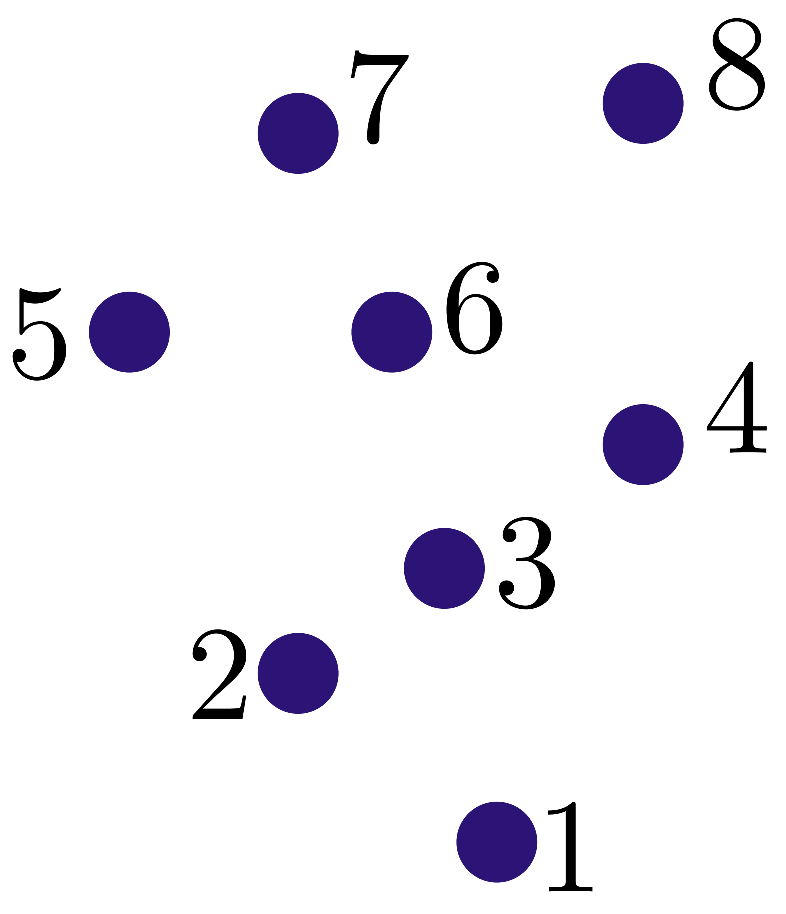


3 ways of
counting
charges

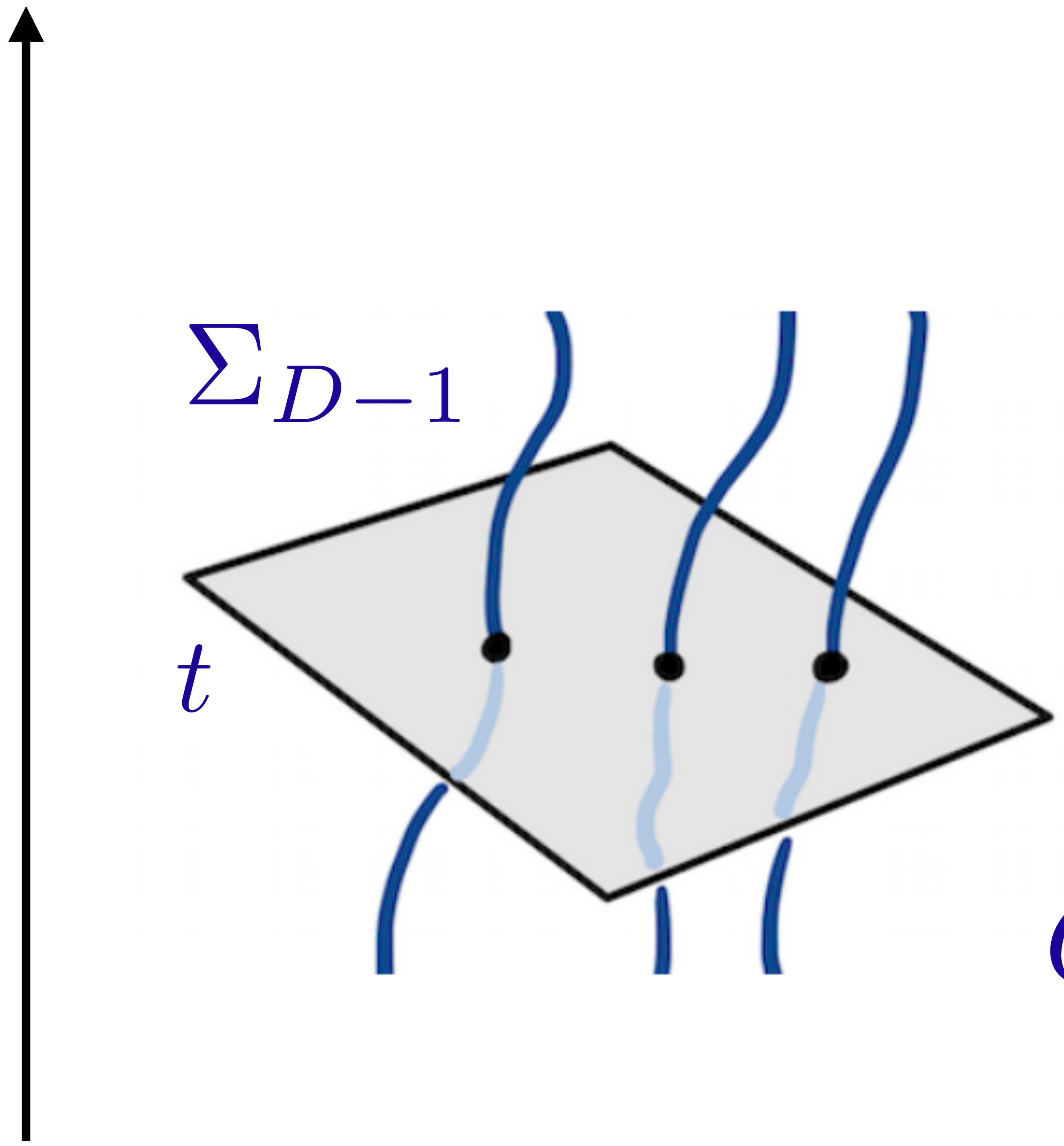
1.)



is there a more efficient way?

2.)

Maxwell/Nöther $U(1)$



$\partial_\mu j^\mu = 0$ $\quad$ $d * j = 0$

$$Q = \int_{\Sigma_{D-1}} *j$$

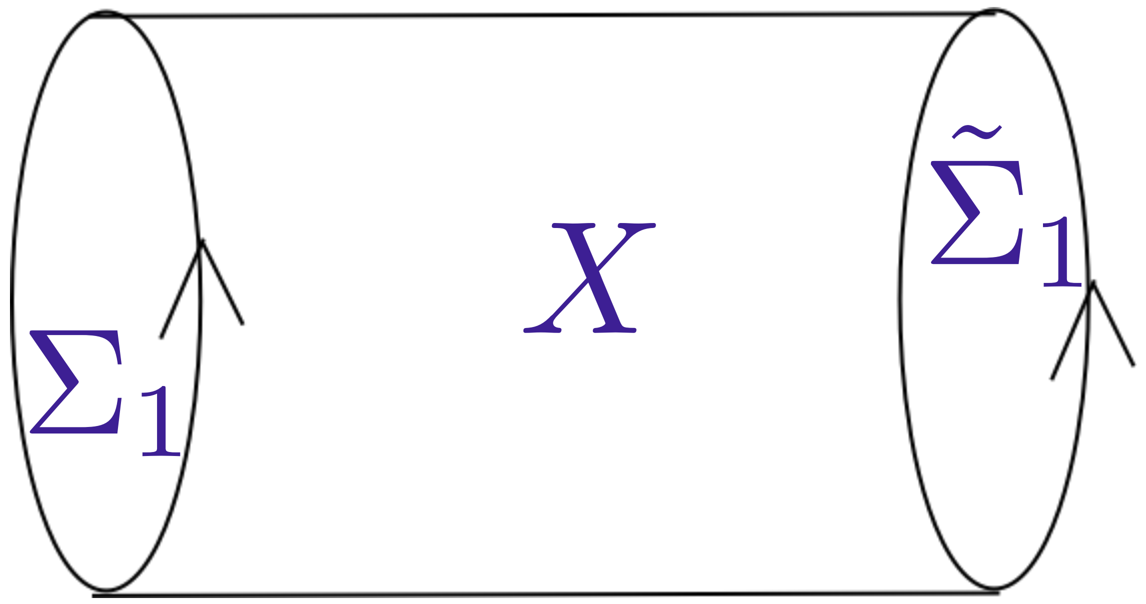
charge

independent of Σ_{D-1}

topological

$$\int_{\Sigma_1} *j + \int_{\tilde{\Sigma}_1} *j = \int_X d * j = 0$$

Stokes′ theorem



$$\int_{\Sigma_1} *j = \int_{\tilde{\Sigma}_1} *j$$

current determined by homotopy class

$$[Q_\Sigma, H] = 0$$

$$\dot{Q} = 0$$

world lines terminate

in a charge operator

$$U_g = e^{igQ}$$

symmetry operator

$$\mathcal{O}(x) \to U_g \mathcal{O}(x) U_g^\dagger$$

gauge conserved current

$$F = dA$$

$$S = \frac{1}{4e^2}\int d^4x F_{\mu\nu}F^{\mu\nu} + j^\mu A_\mu$$

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

$$\Downarrow$$

$$\partial_\mu j^\mu = 0$$

$$Q = \oint A \cdot dl$$

$$\int_S E \cdot dS \qquad \int_S B \cdot dS$$

$$U(1)_E \times U(1)_M$$

generalization: higher-form symmetries

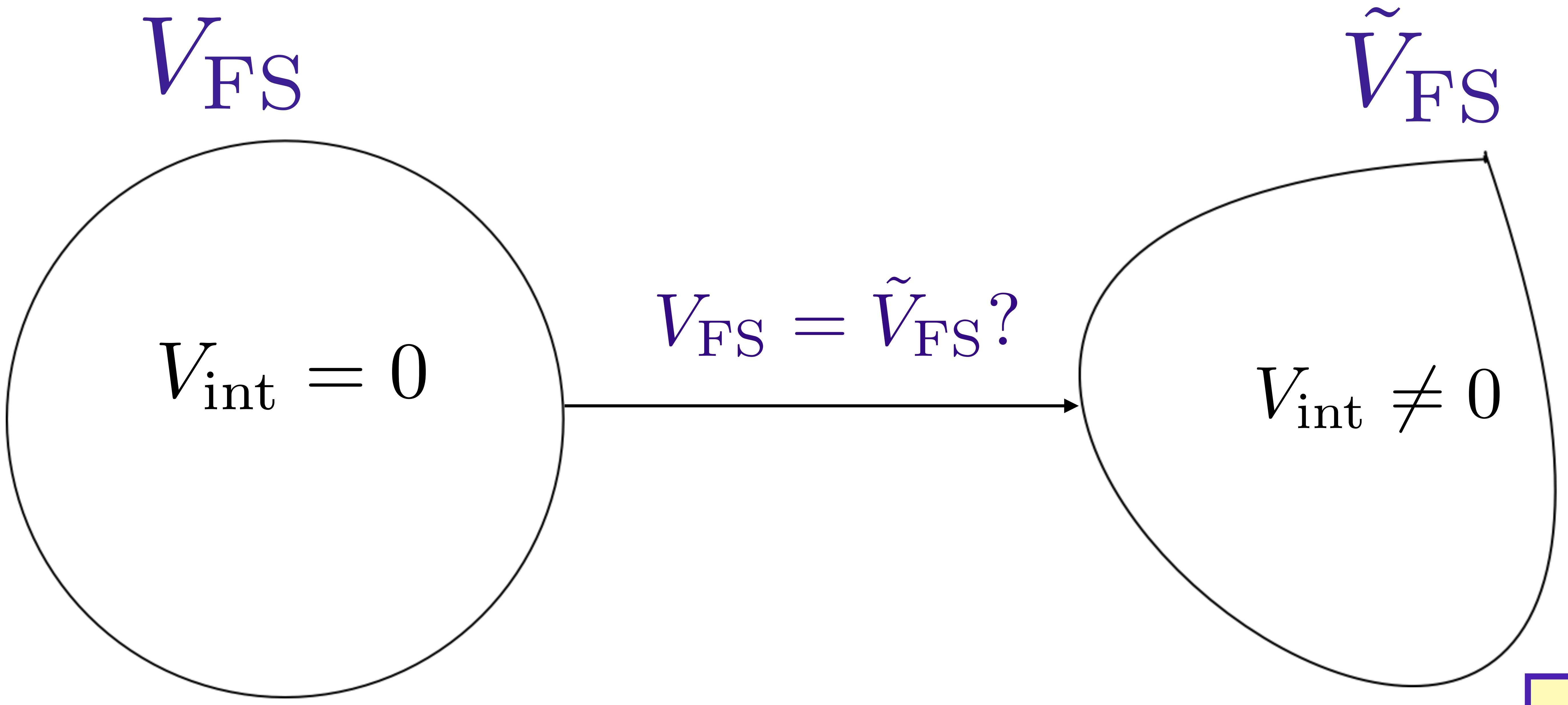
$$\partial^\mu j_{(\mu\nu\kappa)} = 0$$

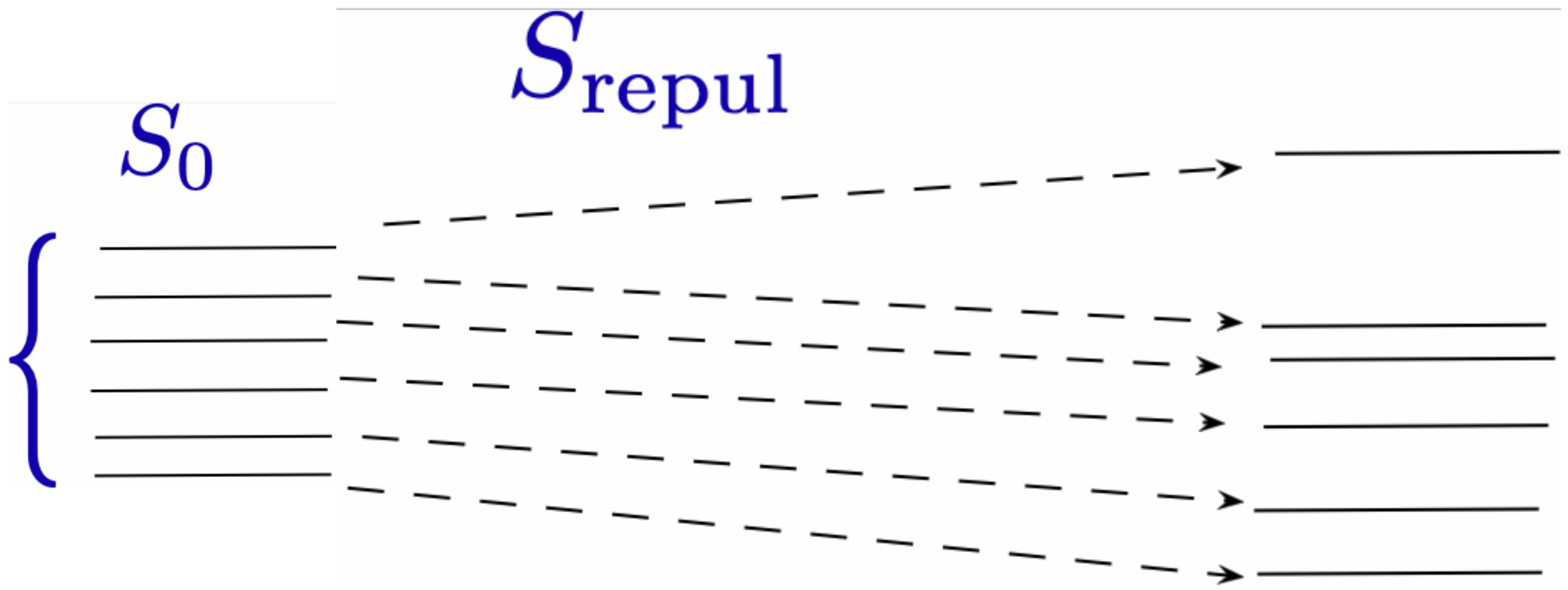
spin

$$\partial^\mu J_{[\mu\nu\kappa\rho\cdots]} = 0$$

fermionic current

interlude: Luttinger's problem



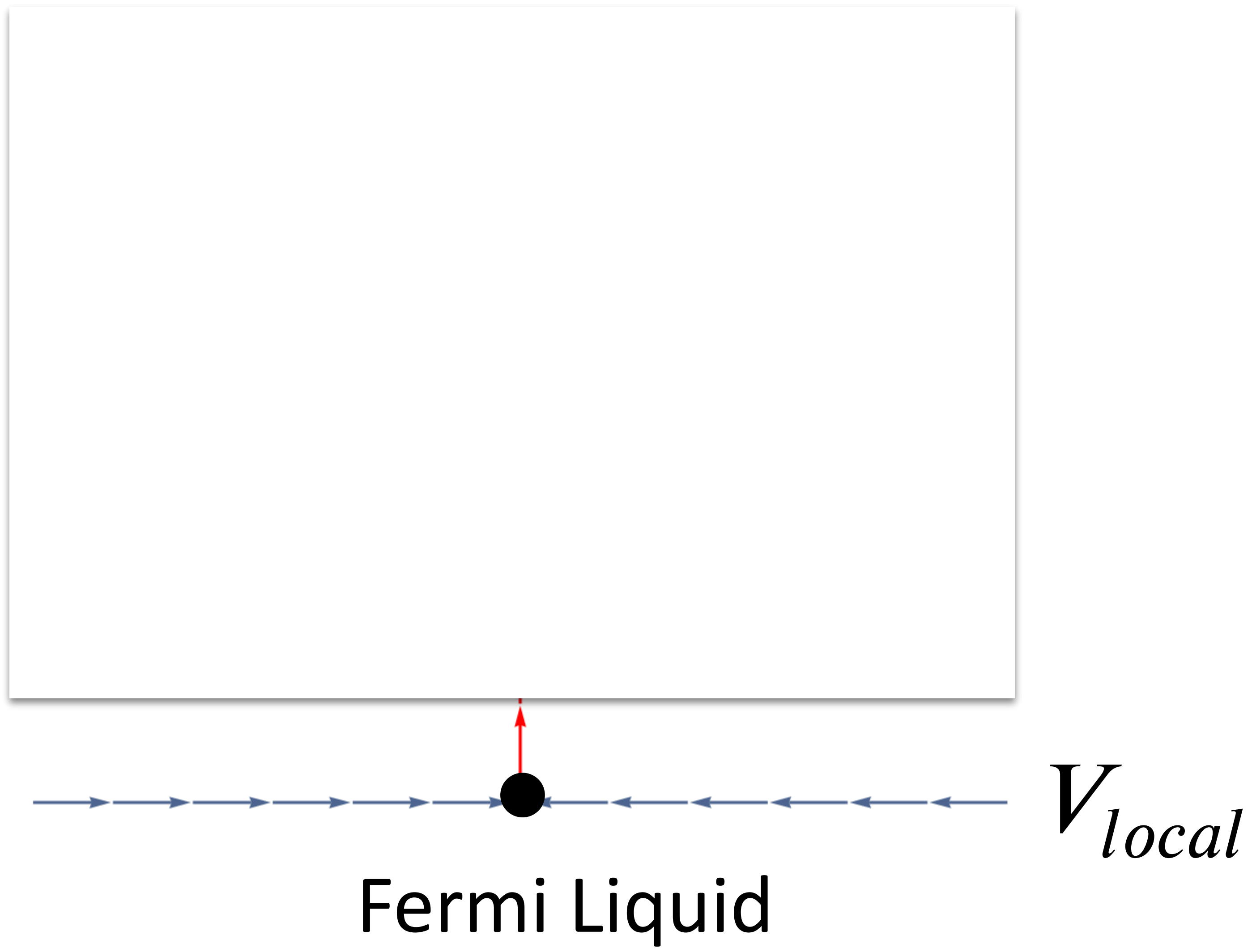


Landau claim

1-1 correspondence

$$gV_{\text{int}}$$

$$\frac{dg}{d\ln E} = 0$$



3.)

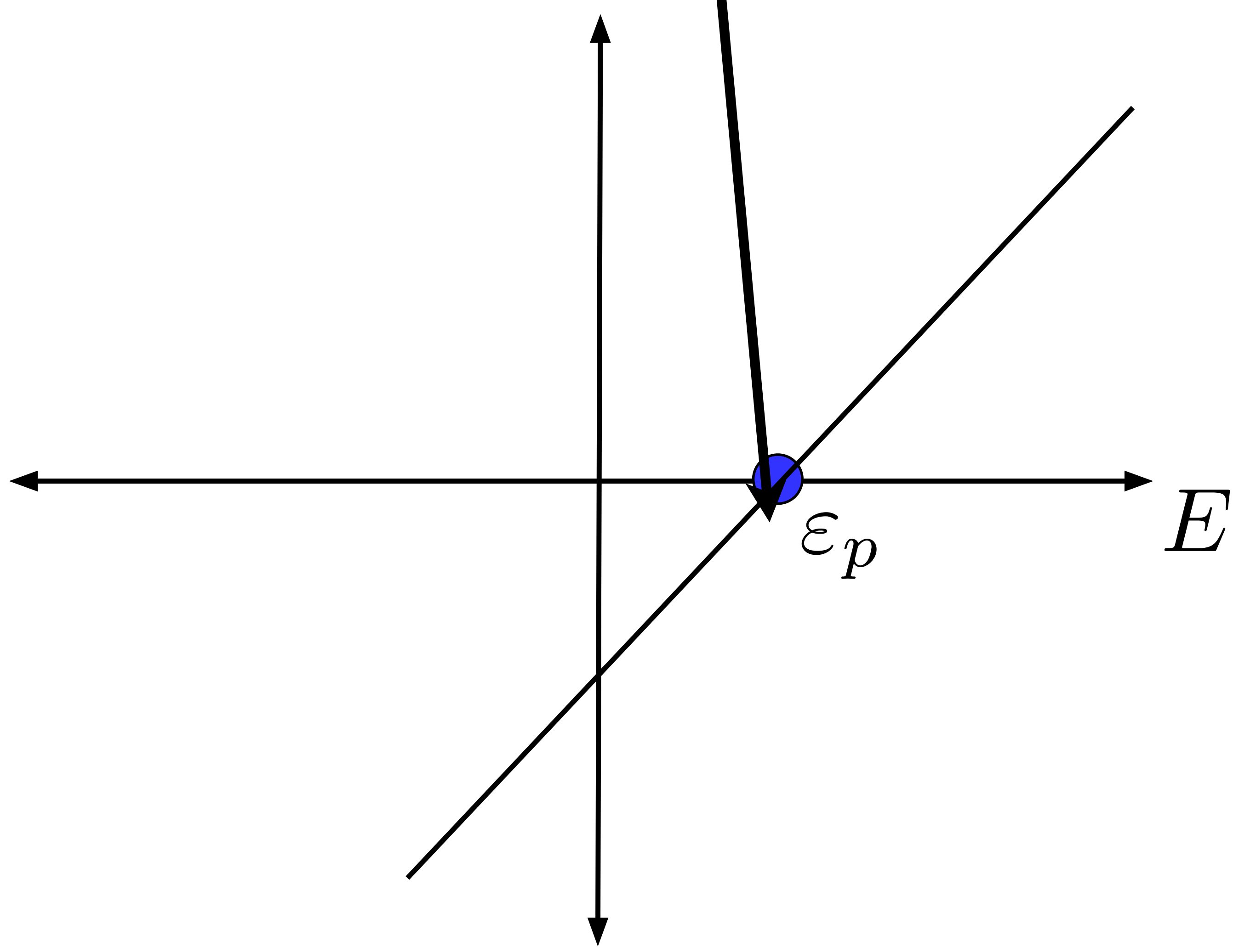
Luttinger counting `theorem'

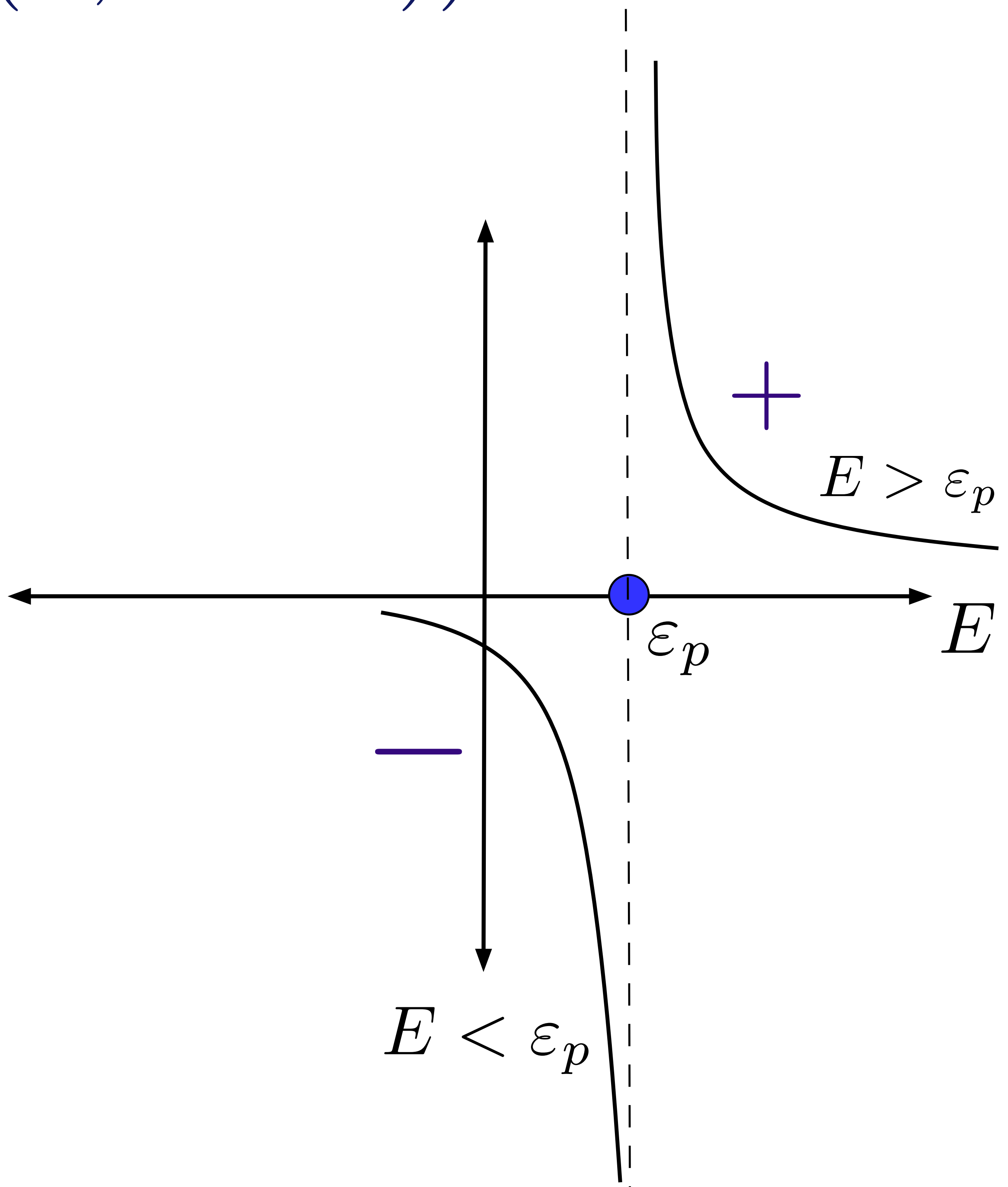
$$G(E) = \frac{1}{E - \varepsilon_p}$$

$$n = 2\sum_{\mathbf{k}} \Theta(\Re G(\mathbf{k}, \omega = \mathbf{0}))$$

zero-crossing

$\mathrm{Det}G(\omega = 0, \vec{p}) = 0$





counting poles (qp)

$$Q_{\mathrm{Lutt}} = \int_{ReG(\omega,0,p)>0} \frac{d^d p}{(2\pi)^d}$$

can Q_{Lutt} be gauged?

equivalence to

Maxwell/Nöther

??

$$n = -2i \int \frac{d^d\mathbf{p}}{(2\pi)^d} \lim_{t\to 0^+} \int \frac{d\omega}{2\pi} G(\omega, \mathbf{p}) e^{i\omega t}$$

$\boxed{\text{insert unity}}$

$$1 = \frac{\partial}{\partial\omega} G^{-1} + \frac{\partial}{\partial\omega} \Sigma(\omega, \mathbf{p})$$

$$n = -2i \lim_{t\to 0^+} \int \frac{d^d\mathbf{p}}{(2\pi)^d} \int \frac{d\omega}{2\pi} \left(\frac{\partial}{\partial\omega} \ln G^{-1}(\omega, \mathbf{p}) + G(\omega, \mathbf{p}) \frac{\partial}{\partial\omega} \Sigma(\omega, \mathbf{p}) \right) e^{i\omega t}$$

$$= I_1 - I_2$$

$$I_2 = 0$$

$$n = I_1$$

$I_2 = 0?$

$$I_2 \equiv -i \int \frac{d^n k}{(2\pi)^n} \int_\gamma \frac{d\omega}{2\pi} \left(G(\omega, k) \frac{\partial \Sigma(\omega, k)}{\partial \omega} \right)$$

$$\Sigma \delta G - \Sigma G$$

integrate by parts

Luttinger-Ward functional

$$\frac{\partial}{\partial G_{\alpha\beta}(\vec{k}, i\omega_\nu)} \Phi[G] = \frac{1}{\beta} \Sigma_{\alpha\beta}(\vec{k}, i\omega_\nu)$$

$$I_2 = i \oint_\gamma \delta(\Phi - Tr[\Sigma G]) = 0$$

if LWF is analytic

$$\delta\Phi[G] = \frac{1}{V}\sum_{k,\sigma}\int \frac{d\omega}{2\pi}\Sigma_\sigma(\omega,k)\delta G_\sigma$$

symmetry

$$\Big\downarrow \; G_0^{-1} - G^{-1}$$

$$\delta\Phi[G] = \frac{1}{V}\sum_{k,\sigma}\int_\gamma \frac{d\omega}{2\pi}(G_0^{-1}\delta G - \delta \ln G^{-1})$$

singular

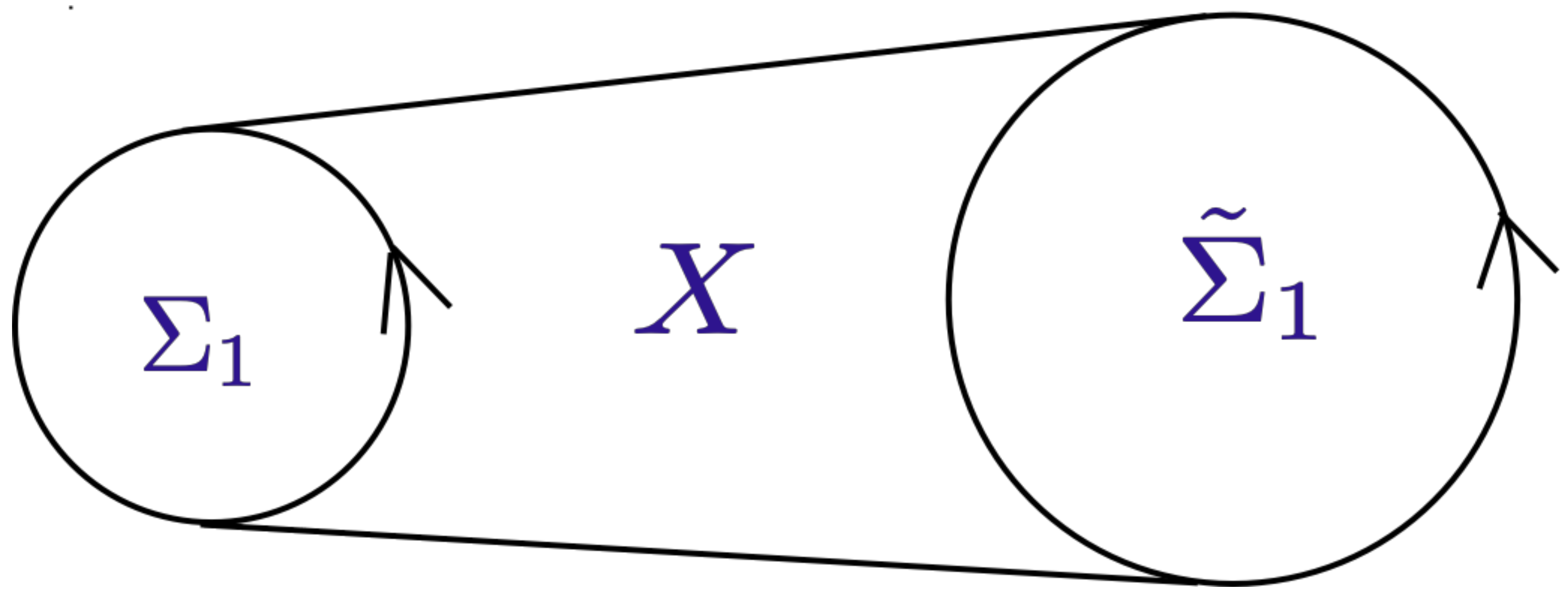
Luttinger theorem fails

gauging: does an anomaly lurk?

$$\oint \delta\Phi(G) = 0$$

$\boxed{\text{analogous to}}$ $d * j$

$$\int_{\Sigma_1} *j + \int_{\tilde{\Sigma}_1} *j = \int_X d * j = 0$$



$$j = \delta\Phi$$

current

conserved

if

Φ is analytic

LW functional as a generalized symmetry

how does it enter the path integral?

$$\Gamma[G] = \Phi[G] - Tr[\Sigma\, G] + Tr\log(-G)$$

Kadanoff-Baym

$$Z[G] = C\,\exp\{\Phi[G] - \mathrm{Tr}[\Sigma\,\mathrm{G}] + \mathrm{Tr}\log(-\mathrm{G})\}$$

operator insertion = generalized symmetry

$$Q_\gamma = e^{\int_\gamma \delta\Phi[G]}$$

`Wilson loop'

$$Q_\gamma = $$

homotopy charge

analyticity

no analyticity

$$n = \frac{N}{V}$$

ABJ anomaly

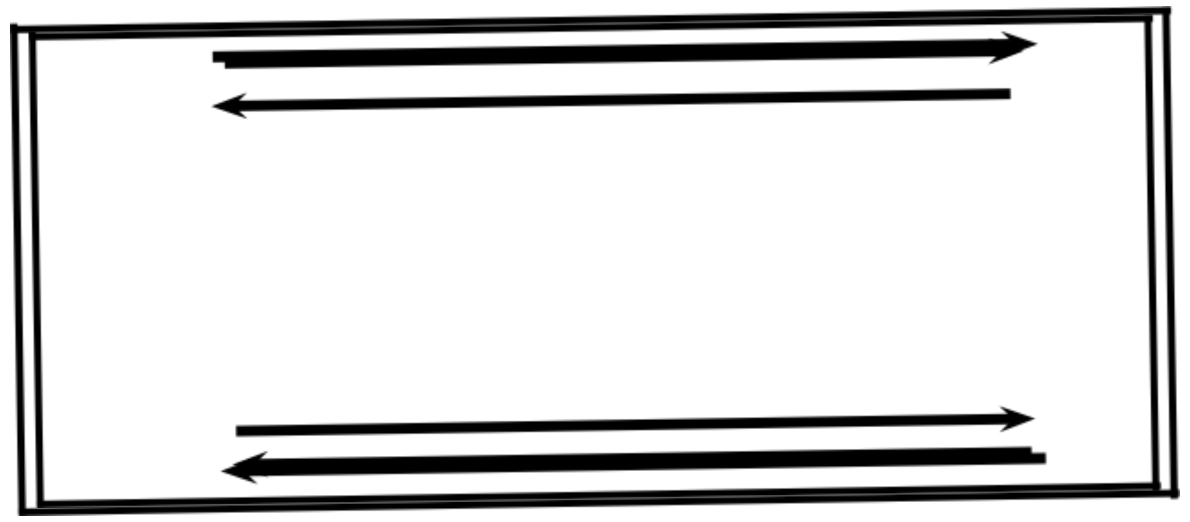
$$Q_\gamma \neq \frac{N}{V}$$

LW functional cannot be gauged

ABJ anomaly
(chiral anomaly)

$$\not{D} = \gamma^\mu \nabla_{A,\mu}$$

$$\nabla_A, \mu = \partial_\mu + ieA_\mu$$



quantum theory cannot be gauged

$$S = i \int d^4x \bar{\psi} \not{D} \psi$$

global symmetries
$$\begin{cases} \psi \to e^{i\alpha}\psi & \qquad j^\mu = \bar{\psi}\gamma^\mu\psi \\ \psi \to e^{i\alpha\gamma^5}\psi & \qquad j_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \end{cases}$$

classically

$$\partial_\mu j^\mu = \partial_\mu j_A^\mu = 0$$

NFT

measure

$$\mathcal{D}\psi\mathcal{D}\bar{\psi}$$

$$\psi \to \psi + \underbrace{i\epsilon\gamma^5\psi}_{\delta\psi}$$

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \to \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{iW_{\rm sing}} e^{iW_{\rm smooth}}$$

$$\partial_\mu j^\mu_A = \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \neq 0$$

axial current not conserved

$$Q_\gamma = e^{\int_\gamma \delta\Phi[G]}$$

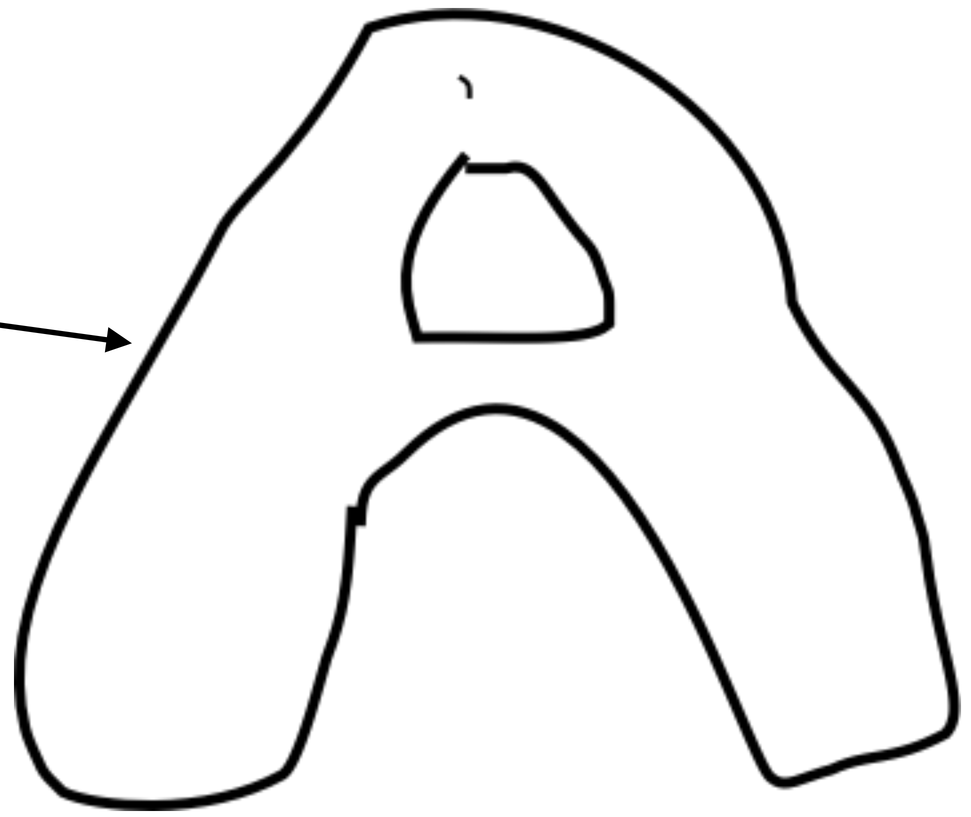
$$Q_\gamma = \boxed{\text{homotopy charge}}$$

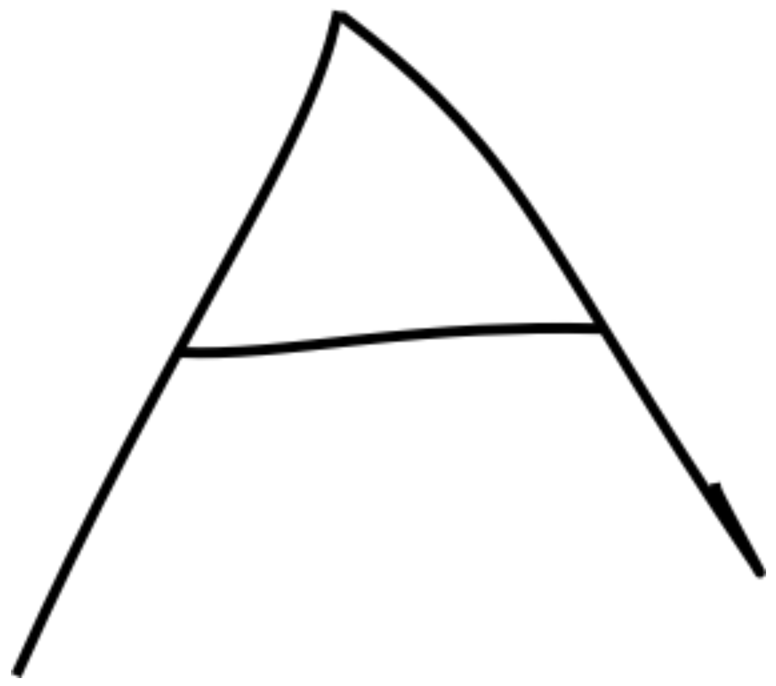
homotopy equivalence=symmetry

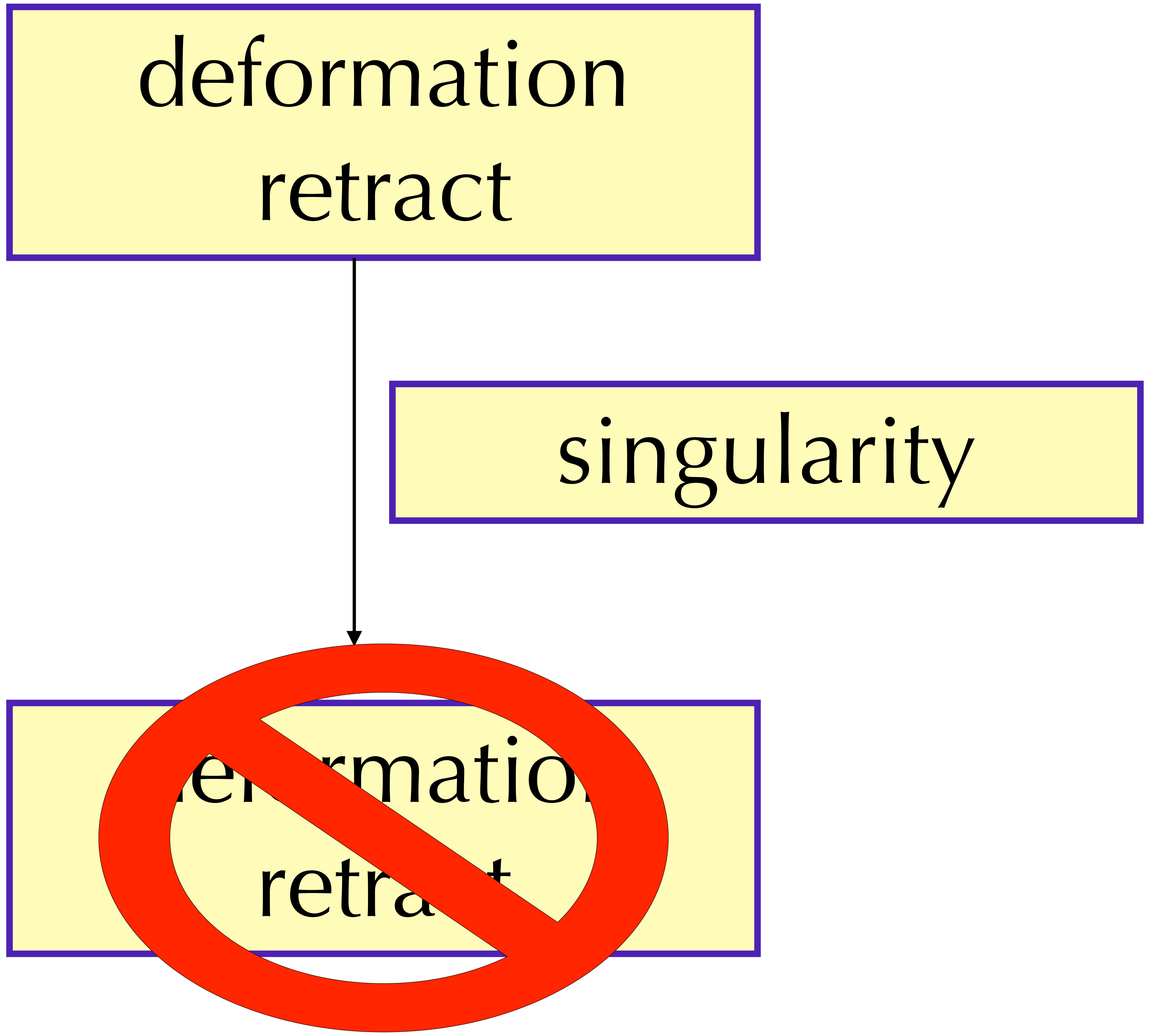
$$\gamma_1(x) \xrightarrow[I \times I]{\gamma} \gamma_2(x)$$

$$\begin{tikzcd} & {[0,1]} \arrow[dl] \arrow[dr] & \\ \gamma(0,x) = \gamma_1(x) & & \gamma(1,x) = \gamma_2(x) \end{tikzcd}$$

lose weight=
deformation retract






deformation retract
singularity

$$M \times \Omega_1(M)$$

$$\Omega_1(M) = \{\gamma : [0,1] \to M, \;\; \gamma(0) = \gamma(1)\}$$

$$Z = \int_F \mathcal{D}\phi \mathcal{D}\gamma e^{S(\phi)}$$

$$\partial_\mu j^\mu \neq 0$$

$$\partial_\mu j^\mu = 0$$

$$\mathcal{Z} = \int_{F \times \Omega^1(M)} \mathcal{D}\phi \mathcal{D}\gamma e^{iS(\phi) + i\int_\gamma j}$$

$$Z_{\text{ext}} = \int_{F \times \pi_1(M)} \mathcal{D}\phi \mathcal{D}\gamma e^{iS(\phi) + i\int_\gamma j}$$

$$Q_\gamma \neq n$$

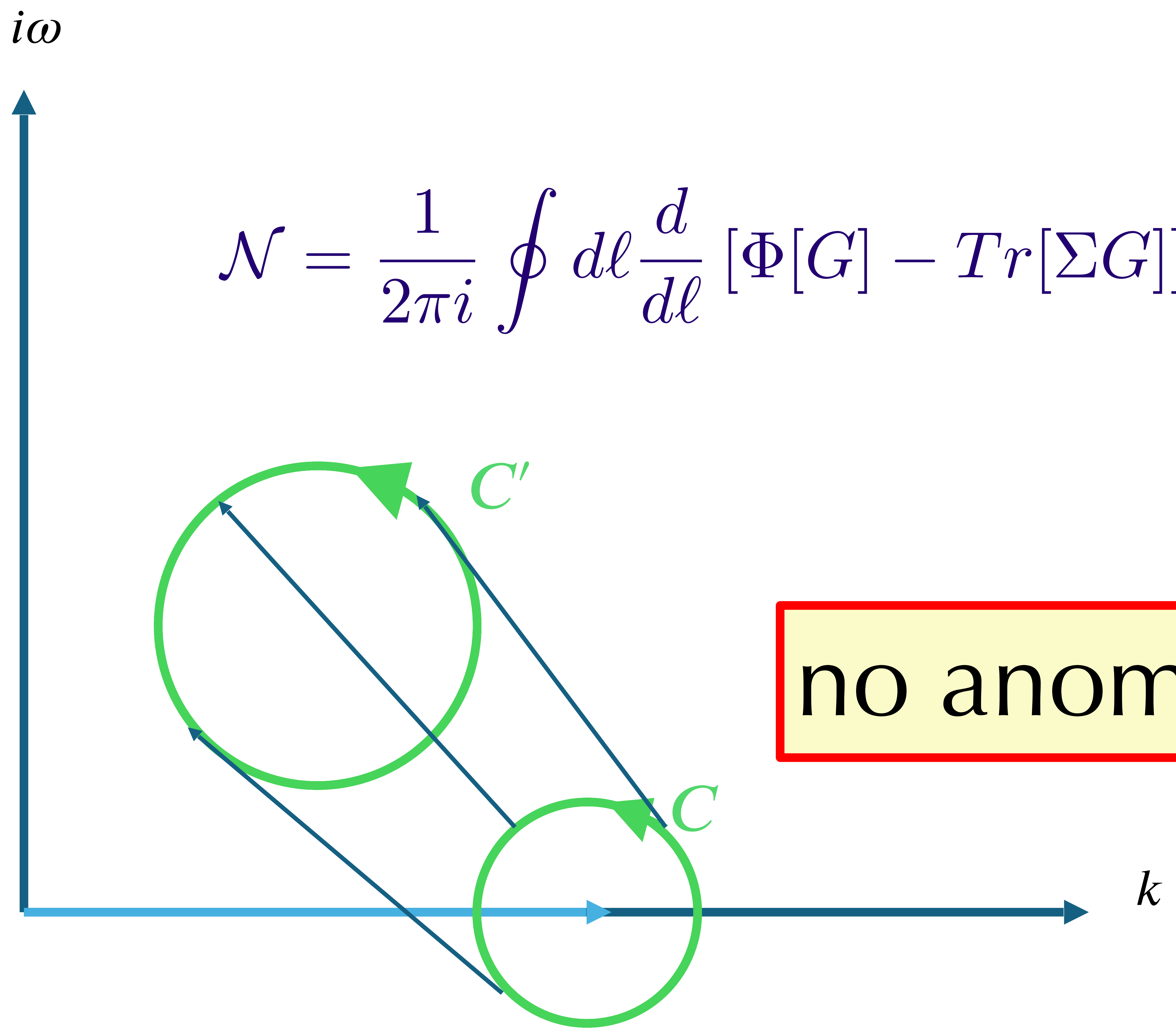
$$\pi_1(M) = \Omega^1(M)/\equiv$$

homotopy equiv.

$$Q_\gamma = n$$

Generalized Symmetry

$$\mathcal{N} = \frac{1}{2\pi i}\oint d\ell \frac{d}{d\ell}\left[\Phi[G] - Tr[\Sigma G]\right] = \mathbb{Z}$$

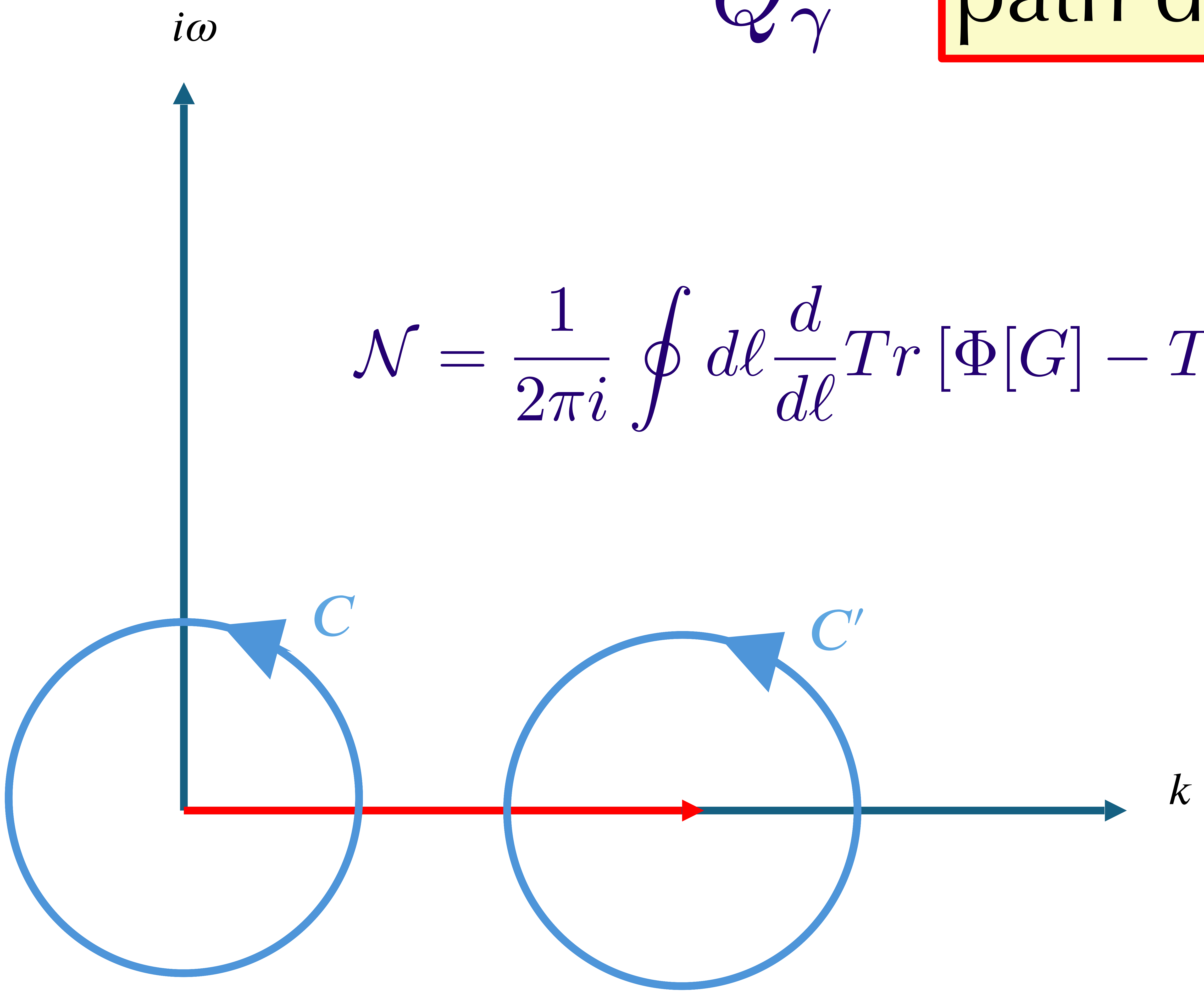


no anomaly

Anomaly

Q_γ **path dependent**

$$\mathcal{N} = \frac{1}{2\pi i} \oint d\ell \frac{d}{d\ell} Tr \left[\Phi[G] - Tr[\Sigma G] \right] \neq n_e$$



$$\Sigma \to \infty$$

$$\delta\Phi[G] = \frac{1}{V} \sum_{k,\sigma} \int \frac{d\omega}{2\pi} \Sigma_\sigma(\omega, k) \delta G_\sigma$$

undefined: singularity

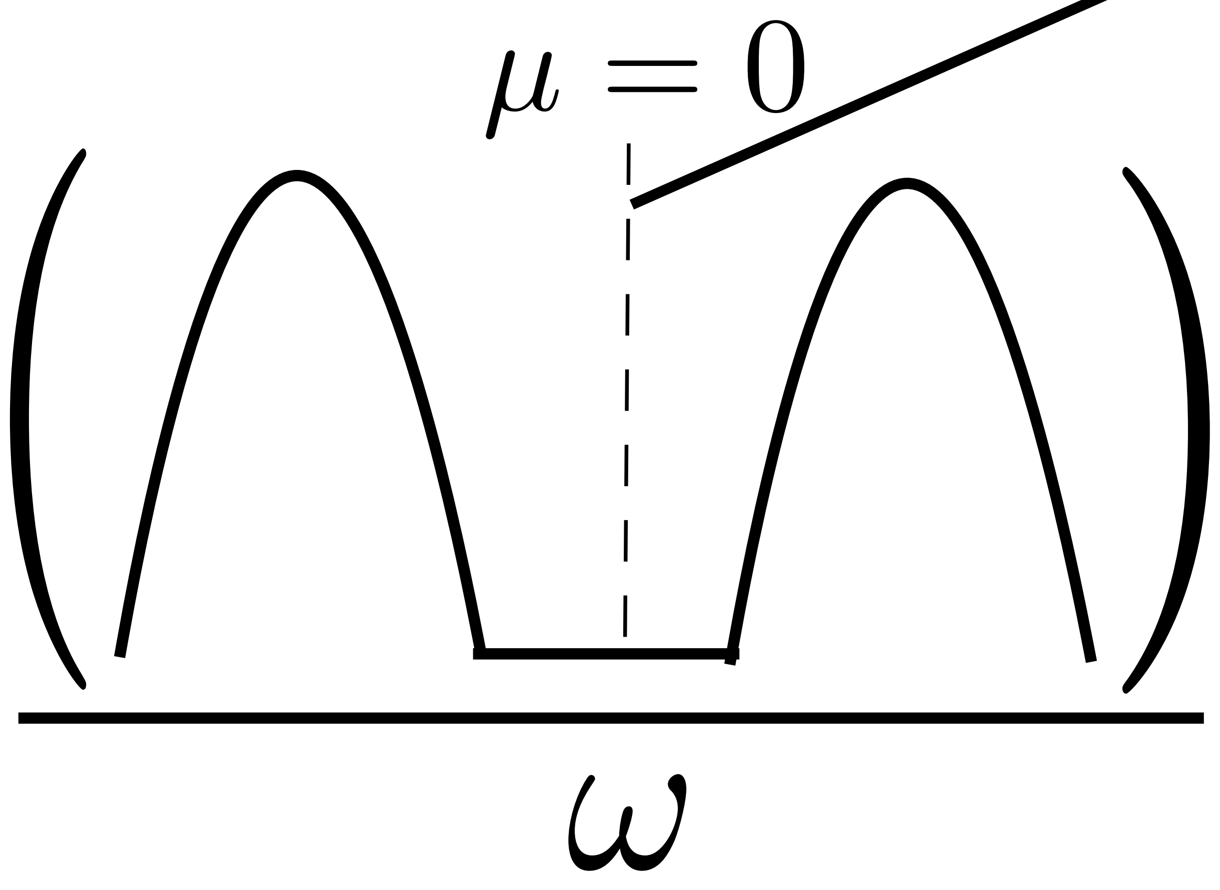
$$G(E) = \frac{1}{E - \varepsilon_p - \Sigma(E, p)}$$

Green function zeros

How do zeros obtain?

Im G=0

$$\mathrm{Re}G(0,p) = \int_{-\infty}^{\infty}$$



$$d\omega$$

Kramers -Kronig

= below gap+above gap

= 0

$$\mathrm{DetReG}(k,\omega=0)=0$$ (single band)

single pole

two poles

$$\frac{1}{E-\alpha}.$$

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Luttinger theorem for a spin-density-wave state

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(received 23 September 1997; accepted in final form 18 December 1997)

PACS. 71.10Pm – Fermions in reduced dimensions (anyons, composite fermions, Luttinger liquid, etc.).

Abstract. – We obtained the analog for the Luttinger relation for a commensurate spin-density-wave state. We show that while the relation between the area of the occupied states and the density of particles gets modified in a simple and predictable way when the system becomes ordered, a perturbative consideration of the Luttinger theorem does not work due to the presence of an anomaly similar to chiral anomaly in quantum electrodynamics.

could not identify which symmetry is broken?

Breaking Z_2: Charge Gap?

$$c_{p\uparrow} = \underbrace{c_{p\uparrow}(1 - n_{p\downarrow})}_{\xi_{p\uparrow}} + \underbrace{c_{p\uparrow} n_{p\downarrow}}_{\eta_{p\uparrow}}$$

$$\begin{pmatrix} \xi_{p\uparrow} \\ \eta_{p\uparrow} \end{pmatrix} \xrightarrow{Z_2} \begin{pmatrix} \eta_{p\uparrow} \\ \xi_{p\uparrow} \end{pmatrix}$$

interchange: degeneracy

$\cancel{Z_2}$

degeneracy broken $\rightarrow$ energy gap

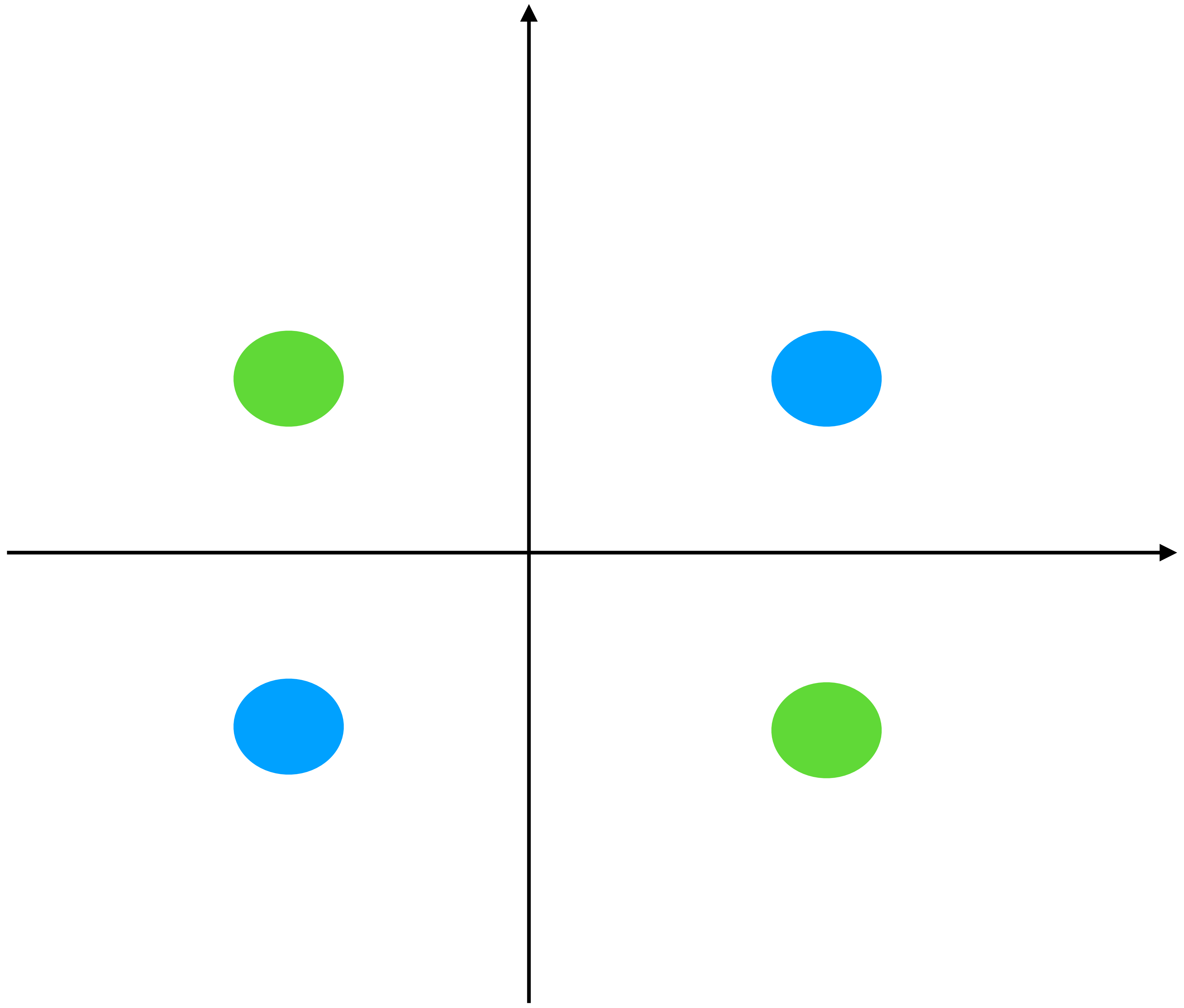
electron splits into two excitations
(`fractionalization')

$$G(z,k) = \frac{Z_1}{z - \xi(k) - U_1(k)} + \frac{Z_2}{z - \xi(k) - U_2(k)}$$

$$\Big\downarrow \; z = \xi + Z_1 U_2 + Z_2 U_1$$

$$G = 0(\Sigma = \infty)$$

location of poles

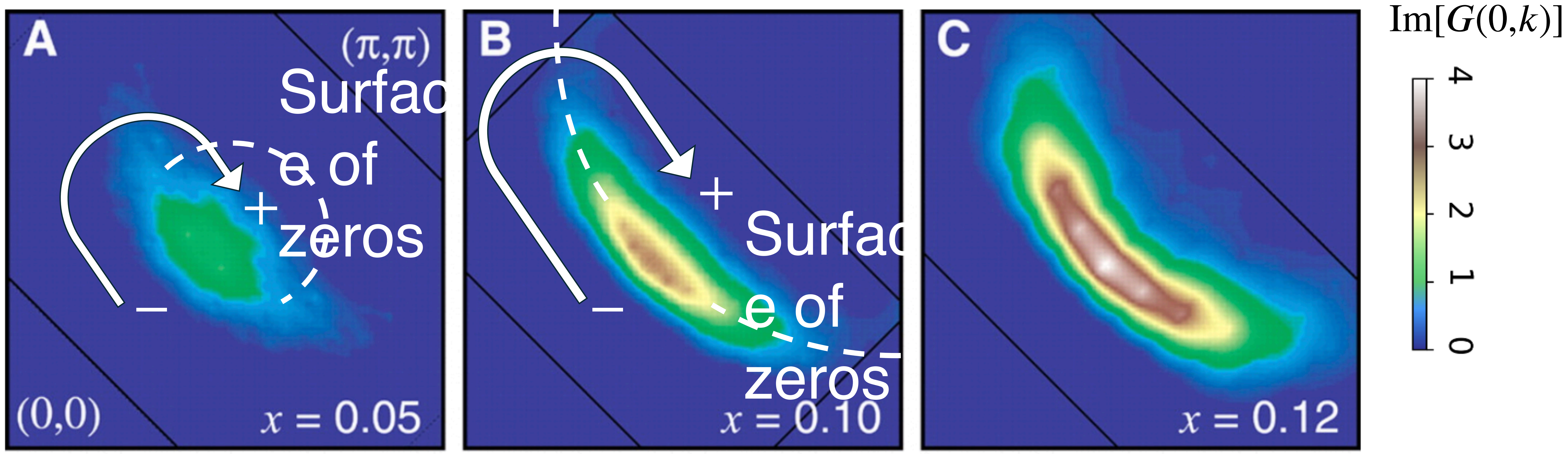


$$I_2 \neq 0$$

$$Q_\gamma \neq n$$

pseudogap

Fermi Arcs from ARPES

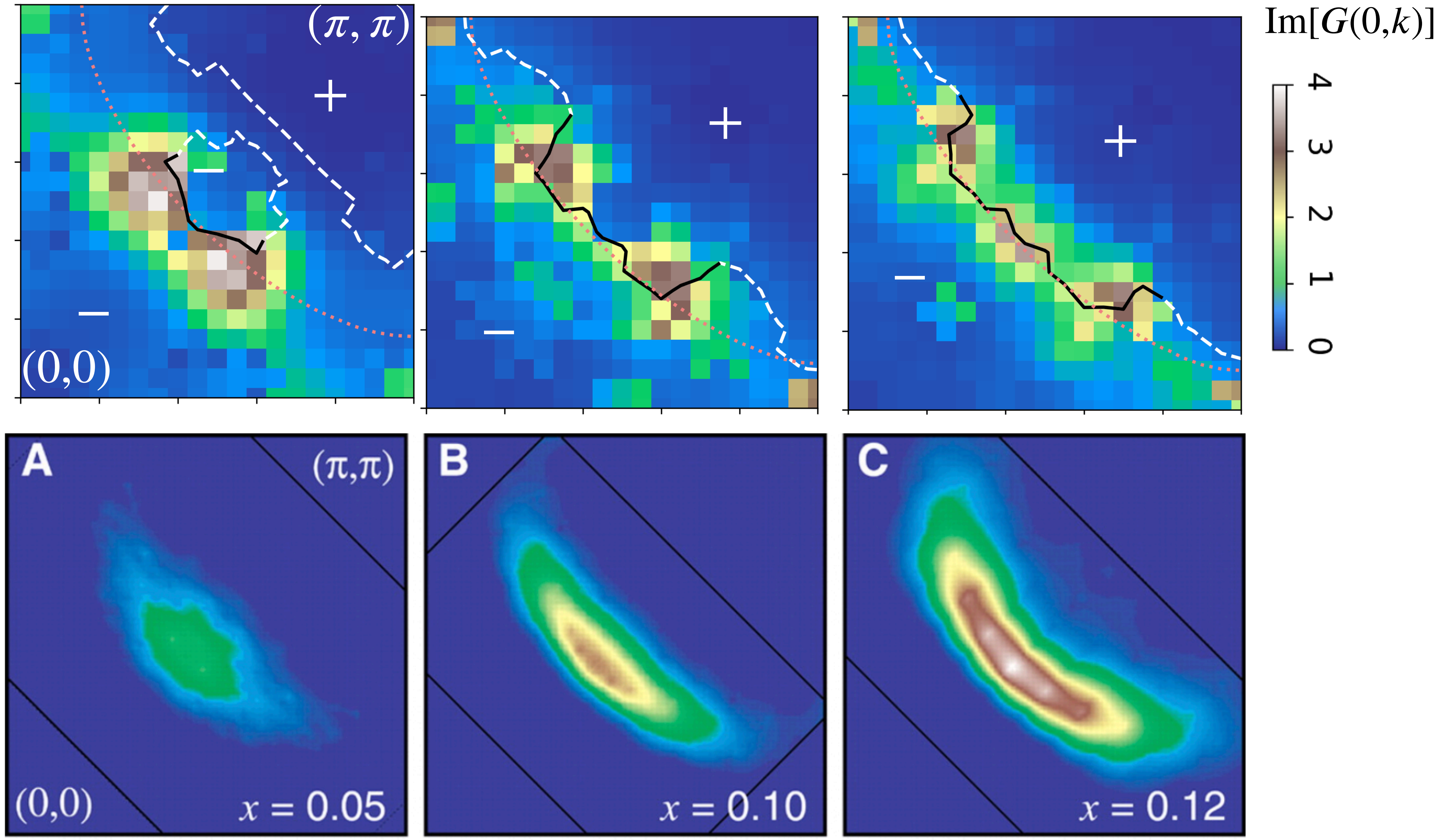


Shen, Kyle M., et al. *Science* 307.5711 (2005).

16 momentum mixing HK (MMHK) vs ARPES

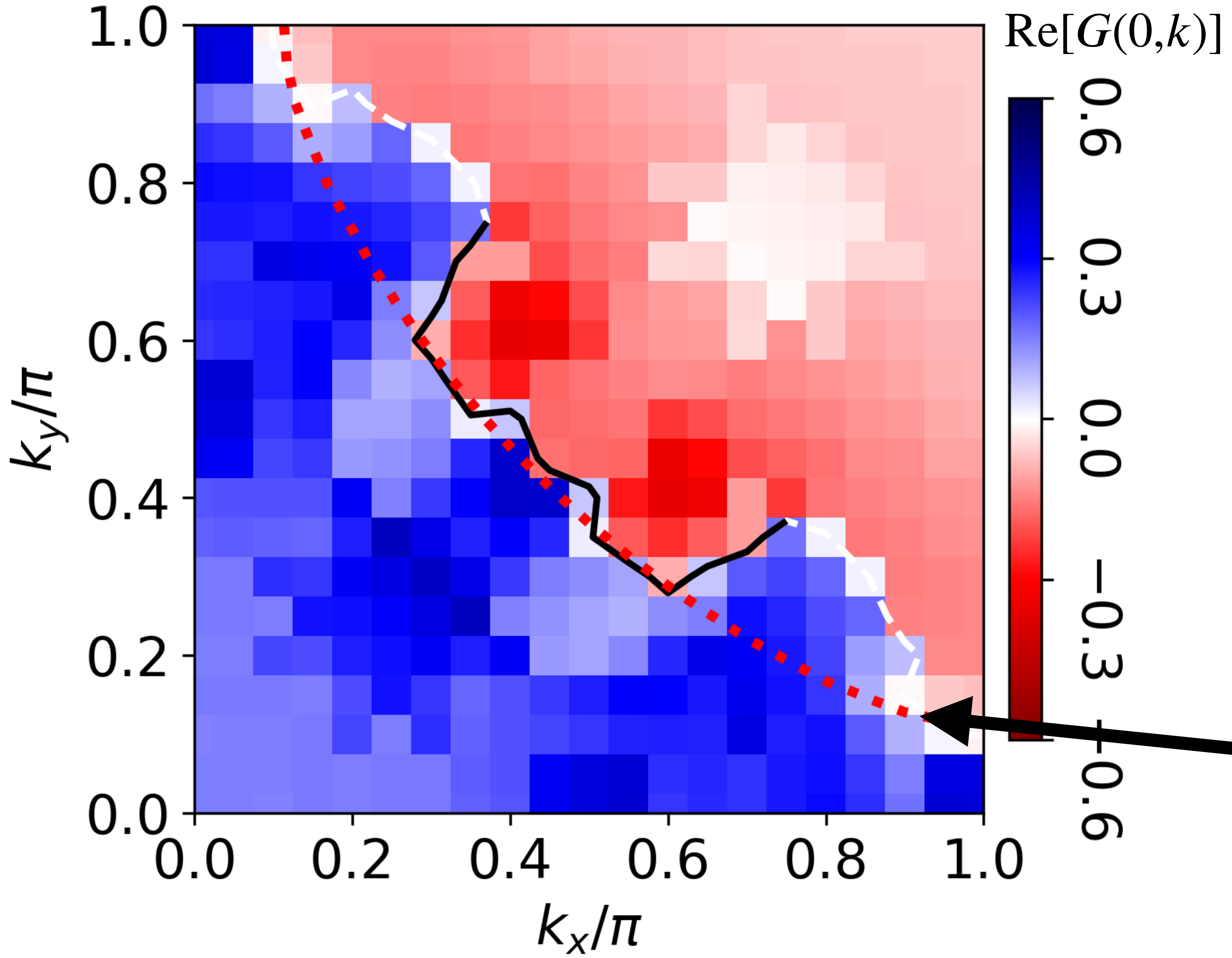
Theory
Hole-doped
t′=-0.25
U = 8

ARPES



Shen, Kyle M., et al. *Science* 307.5711 (2005).

Luttinger count $\neq$ particle density



$$\int_{Re[G(0,k)]>0} d^d k$$

$$\neq$$

Non-interacting volume (number of particles)

net of local algebras
(Casini)

$$A(R_1 \cup R_2) = \mathcal{A}(R_1) \bigvee \mathcal{A}(R_2)$$

additivity

$$\mathcal{A}(R_1) \subseteq \mathcal{A}(R_2) \; R_1 \subseteq R_2$$

isotonia

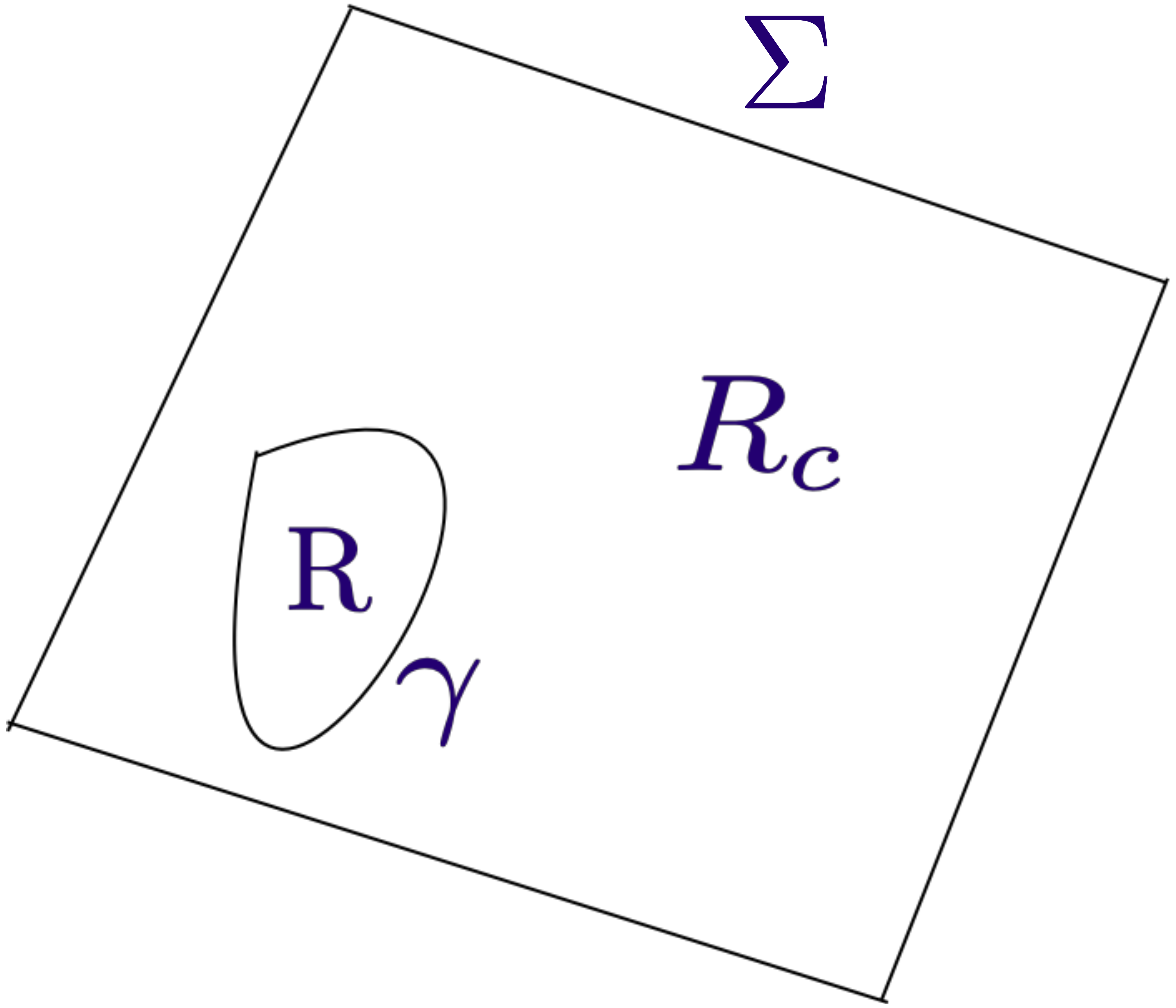
$$\mathcal{A}(R) \subseteq (\mathcal{A}(R'))'$$

causality

$$\mathcal{A}(R')' = \mathcal{A}(R) \bigvee \{a\}$$

non-locally generated operators in R

$$\mathrm{R_c} = \Sigma - \mathrm{R}$$



$$Q_\gamma = e^{i\int_\gamma \omega^p} = Q_{\gamma \in R_c}$$

$$\big\downarrow \; [Q_\gamma, H] \neq 0$$

$$Q_\gamma \neq Q_{\gamma \in R_c}$$

anomaly of
generalized symmetry

the complement space
contains non-locally
generated operators

non-Fermi liquids

1D: LL

all circles are equiv.

$$Q_\gamma = n$$

D>1

$$Q_\gamma \neq n$$

charges are non-local