Notes on Topological field Theory of Time-Reversal Invariant Insulators

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Contents

1 Topological Invariant in 2+1 Dimensions and Dimensional Reduction

1.1 Hall Conductance, the First Chern Number and Topological Response Function

In general the tight binding Hamiltonian of a band insulator in 2+1 dimensions can be written as

$$H = \sum_{m,n;\alpha,\beta} c^{\dagger}_{m\alpha} h^{\alpha\beta}_{mn} c_{n\beta} \tag{1}$$

where m and n denote the lattice sites and α and β indicate the band indices. We should also notice that $h^{\alpha\beta}_{mn}$ enjoys translation symmetry, i.e.

$$h^{\alpha\beta}_{mn} = h^{\alpha\beta}\left(\vec{r}_m - \vec{r}_n\right) \tag{2}$$

By Fourier-transformation

$$C_{\boldsymbol{k}\alpha} = \frac{1}{\sqrt{N}} \sum_n e^{i\boldsymbol{r_n}\cdot\boldsymbol{k}} C_{n\alpha} \tag{3}$$

We can see that

$$\begin{aligned}
H &= \sum_{mn\alpha\beta} C^{\dagger}_{m\alpha} h^{\alpha\beta}_{mn} C_{n\beta} \\
&= \sum_{mn\alpha\beta} \frac{1}{N} \sum_{kq} e^{i\vec{r}_m\vec{k}} C^{\dagger}_{\vec{k}\alpha} h^{\alpha\beta}_{mn} e^{-i\vec{r}_n\cdot\vec{q}} C_{\vec{q}\beta} \\
&= \sum_{mn\alpha\beta} \sum_{kq} \frac{1}{N} e^{i(\vec{r}_m - \vec{r}_n)\cdot\vec{k}} h^{\alpha\beta}_{mn} e^{i\vec{r}_n\cdot(\vec{k}-\vec{q})} C^{\dagger}_{\vec{k}\alpha} C_{\vec{q}\beta} \\
&= \sum_{mn\alpha\beta} \sum_{kq} \frac{1}{N} e^{i\vec{r}_m\cdot\vec{k}} h^{\alpha\beta}\left(\vec{r}_m\right) e^{i\vec{r}_n\cdot(\vec{k}-\vec{q})} C^{\dagger}_{\vec{k}\alpha} C_{\vec{q}\beta} \\
&= \sum_{n\alpha\beta} \sum_{kq} \frac{1}{N} h^{\alpha\beta}_{\vec{k}} e^{i\vec{r}_n\cdot(\vec{k}-\vec{q})} C^{\dagger}_{\vec{k}\alpha} C_{\vec{q}\beta} \\
&= \sum_{k\alpha\beta} C^{\dagger}_{k\alpha} h^{\alpha\beta}_{\vec{k}} C_{\vec{k}\beta}
\end{aligned} \tag{4}$$

where in the fourth line we used the translation symmetry of $h^{\alpha\beta}_{mn}$ and replaced $\vec{r}_m - \vec{r}_n$ with $\vec{r}_m$ and in the fifth line we summed over $\vec{r}_m$ and that

$$h^{\alpha\beta}_{\vec{k}} = \sum_m e^{i\vec{r}_m\cdot\vec{k}} h^{\alpha\beta}\left(\vec{r}_m\right) \tag{5}$$

When the system is coupled to EM fields, in accordance to the minimal coupling priciple, we introduce Peierls substitution in the lattice Hamiltonian:

$$h^{\alpha\beta}_{mn} \to h^{\alpha\beta}_{mn} e^{i\int_{\vec{r}_n}^{\vec{r}_m} \mathbf{A}(l)\cdot d\mathbf{l}} \tag{6}$$

where $\boldsymbol{A}$ is the vector potential and the integration path is taken to be the straight line connecting the two lattice sites. We then use an approximation that would be valid if the vector potential did not vary much over the integration path.

$$\int_{\vec{r}_n}^{\vec{r}_m} \mathbf{A}(s,t)\cdot d\mathbf{s} \approx (\vec{r}_m - \vec{r}_n)\cdot \frac{1}{2}\left(\mathbf{A}\left(\vec{r}_m,t\right)+\mathbf{A}(\vec{r}_n,t)\right) \approx (\vec{r}_m - \vec{r}_n)\cdot \frac{1}{2}\mathbf{A}\left(\frac{\vec{r}_m+\vec{r}_n}{2},t\right) \tag{7}$$

Hence

$$H = \sum_{m,n,a,\beta} c_{ma}^{\dagger} h_{mn}^{a\beta} e^{i\int_{\vec{r}_n}^{\vec{r}_m}\mathbf{A}(l)\cdot d\mathbf{l}} c_{n\beta} \approx \sum_{m,n,\alpha,\beta} c_{m\alpha}^{\dagger} h_{mn}^{\alpha\beta}\left(1+i\int_{\vec{r}_n}^{\vec{r}_m}\mathbf{A}(l)\cdot d\mathbf{l}\right) c_{n\beta} = H_0 + H_{\text{ext}} \tag{8}$$

Similarly we carry out the Fourier transformation of the annihilation and creation operators in H_{ext}

$$\begin{aligned}
H_{ext} &= \sum_{mn\alpha\beta} C_{m\alpha}^{\dagger} h_{mn}^{\alpha\beta} C_{n\beta}\, i\int_{\vec{r}_n}^{\vec{r}_m}\mathbf{A}(l)\cdot d\mathbf{l} \\
&= \sum_{mn\alpha\beta}\frac{1}{N}\sum_{kq} e^{i\vec{r}_m\vec{k}} C_{\vec{k}\alpha}^{\dagger} h_{mn}^{\alpha\beta} e^{-i\vec{r}_n\cdot\vec{q}} C_{\vec{q}\beta}\, i\int_{\vec{r}_n}^{\vec{r}_m}\mathbf{A}(l)\cdot d\mathbf{l} \\
&= \sum_{mn\alpha\beta}\sum_{kq}\frac{1}{N} e^{i(\vec{r}_m-\vec{r}_n)\cdot\vec{k}} h_{mn}^{\alpha\beta} e^{i\vec{r}_n\cdot(\vec{k}-\vec{q})} C_{\vec{k}\alpha}^{\dagger} C_{\vec{q}\beta}\, i\int_{\vec{r}_n}^{\vec{r}_m}\mathbf{A}(l)\cdot d\mathbf{l} \\
&= \sum_{mn\alpha\beta}\sum_{kq}\frac{1}{N} e^{i\vec{r}_m\cdot\vec{k}} h^{\alpha\beta}\left(\vec{r}_m\right) e^{i\vec{r}_n\cdot(\vec{k}-\vec{q})} C_{\vec{k}\alpha}^{\dagger} C_{\vec{q}\beta}\, i\int_{\vec{r}_n}^{\vec{r}_n+\vec{r}_m}\mathbf{A}(l)\cdot d\mathbf{l} \\
&\approx \sum_{mn\alpha\beta}\sum_{kq}\frac{1}{N} e^{i\vec{r}_m\cdot\vec{k}} h^{\alpha\beta}\left(\vec{r}_m\right) e^{i\vec{r}_n\cdot(\vec{k}-\vec{q})} C_{\vec{k}\alpha}^{\dagger} C_{\vec{q}\beta}\, i\vec{r}_m\cdot\boldsymbol{A}(\vec{r}_n+\frac{1}{2}\vec{r}_m)
\end{aligned} \tag{9}$$

Introduce the Fourier transformation of $\boldsymbol{A}(\vec{r})$

$$\boldsymbol{A}_{\vec{r}_m} = \sum_q e^{-i\vec{q}\cdot\vec{r}_m}\boldsymbol{A}_{\vec{q}} \tag{10}$$

Hence

$$\begin{aligned}
H_{ext} &= \sum_{mn\alpha\beta}\sum_{kq}\frac{1}{N} e^{i\vec{r}_m\cdot\vec{k}} h^{\alpha\beta}\left(\vec{r}_m\right) e^{i\vec{r}_n\cdot(\vec{k}-\vec{q})} C_{\vec{k}\alpha}^{\dagger} C_{\vec{q}\beta}\, i\vec{r}_m\cdot\boldsymbol{A}(\vec{r}_n+\frac{1}{2}\vec{r}_m) \\
&= \sum_{mn\alpha\beta}\sum_{kq}\frac{1}{N} e^{i\vec{r}_m\cdot\vec{k}} h^{\alpha\beta}\left(\vec{r}_m\right) e^{i\vec{r}_n\cdot(\vec{k}-\vec{q})} C_{\vec{k}\alpha}^{\dagger} C_{\vec{q}\beta}\, i\sum_p e^{-i\vec{p}\cdot(\vec{r}_n+\frac{1}{2}\vec{r}_m)}\boldsymbol{A}_{\vec{p}} \\
&= \sum_{mn\alpha\beta}\sum_{kq}\sum_p\frac{1}{N} e^{i(\vec{k}-\frac{1}{2}\vec{p})\cdot\vec{r}_m} h^{\alpha\beta}(\vec{r}_m) e^{i\vec{r}_n\cdot(\vec{k}-\vec{q}-\vec{p})} C_{\vec{k}\alpha}^{\dagger} C_{\vec{q}\beta}\, i\vec{r}_m\cdot\boldsymbol{A}_{\vec{p}} \\
&= \sum_{m\alpha\beta}\sum_{kq}\sum_p e^{i(\vec{k}-\frac{1}{2}\vec{p})\cdot\vec{r}_m} h^{\alpha\beta}(\vec{r}_m)\delta_{(\vec{k}-\vec{q}-\vec{p})} C_{\vec{k}\alpha}^{\dagger} C_{\vec{q}\beta}\, i\vec{r}_m\cdot\boldsymbol{A}_{\vec{p}}
\end{aligned} \tag{11}$$

then we can replace $\vec{k}-\frac{1}{2}\vec{p}$ by $\vec{k}$

$$\begin{aligned}
H_{ext} &= \sum_{m\alpha\beta}\sum_{kq}\sum_{p} e^{i\vec{k}\cdot\vec{r}_m} h^{\alpha\beta}(\vec{r}_m)\delta_{(\vec{k}-\vec{q}-\frac{1}{2}\vec{p})} C^\dagger_{\vec{k}+\frac{1}{2}\vec{p}\alpha} C_{\vec{q}\beta} i\vec{r}_m\cdot\boldsymbol{A}_{\vec{p}} \\
&= \sum_{m\alpha\beta}\sum_{kp} e^{i\vec{k}\cdot\vec{r}_m} h^{\alpha\beta}(\vec{r}_m) C^\dagger_{\vec{k}+\frac{1}{2}\vec{p}\alpha} C_{\vec{k}-\frac{1}{2}\vec{p}\beta} i\vec{r}_m\cdot\boldsymbol{A}_{\vec{p}} \\
&= \sum_{m\alpha\beta}\sum_{kp} C^\dagger_{\vec{k}+\frac{1}{2}\vec{p}\alpha} C_{\vec{k}-\frac{1}{2}\vec{p}\beta}\left(i\vec{r}_m e^{i\vec{k}\cdot\vec{r}_m} h^{\alpha\beta}(\vec{r}_m)\right)\cdot\boldsymbol{A}_{\vec{p}}
\end{aligned} \tag{12}$$

Here we should notice that since

$$h^{\alpha\beta}_{\vec{k}} = \sum_m e^{i\vec{r}_m\cdot\vec{k}} h^{\alpha\beta}(\vec{r}_m) \tag{13}$$

we can see that

$$\frac{\partial h^{\alpha\beta}_{\vec{k}}}{\partial\vec{k}} = \sum_m i\vec{r}_m e^{i\vec{r}_m\cdot\vec{k}} h^{\alpha\beta}(\vec{r_m}) \tag{14}$$

Hence

$$H_{ext} = \sum_{kp\alpha\beta} C^\dagger_{\vec{k}+\frac{1}{2}\vec{p}\alpha} C_{\vec{k}-\frac{1}{2}\vec{p}\beta}\frac{\partial h^{\alpha\beta}_{\vec{k}}}{\partial\vec{k}}\cdot\boldsymbol{A}_{\vec{p}} \tag{15}$$

Next, we will prove the following claim

$$H_{\text{ext}} = \sum_q \mathbf{j}_{\vec{q}}\cdot\mathbf{A}_{-\vec{q}} = \sum_{k,q,a,\beta} C^\dagger_{\vec{k}+\vec{q}/2,a} C_{\vec{k}-\vec{q}/2,\beta}\frac{\partial h^{a\beta}_{\vec{k}}}{\partial\vec{k}}\cdot\mathbf{A}_{-\vec{q}} \tag{16}$$

First, we must obtain the current operator $\boldsymbol{j}_{\vec{q}}$. Start from the continuity equation

$$\dot{\rho}(\vec{r}_m) + \nabla\cdot\boldsymbol{j}(\vec{r}_m) = 0 \tag{17}$$

The Fourier transformation of $\rho(\vec{r}_m)$ reads

$$\begin{aligned}
\rho(\vec{q}) &= \sum_m e^{i\vec{q}\cdot\vec{r}_m} C^\dagger_{m\alpha} C_{m\alpha} \\
&= \frac{1}{(\sqrt{N})^3}\sum_{mkp} e^{-i(-\vec{q}+\vec{k}-\vec{p})\cdot\vec{r}_m} C^\dagger_{\vec{k}\alpha} C_{\vec{p}\alpha} \\
&= \frac{1}{\sqrt{N}}\sum_{kp}\delta_{\vec{k}-\vec{q}-\vec{p}} C^\dagger_{\vec{k}\alpha} C_{\vec{p}\alpha} \\
&= \frac{1}{\sqrt{N}}\sum_{k} C^\dagger_{\vec{k}+\vec{q}\alpha} C_{\vec{k}\alpha}
\end{aligned} \tag{18}$$

We then insert

$$\begin{aligned}\rho(\vec{r}_m) &= \frac{1}{N}\sum_q e^{-i\vec{q}\cdot\vec{r}_m}\rho(\vec{q}) \\ \boldsymbol{j}(\vec{r}_m) &= \frac{1}{N}\sum_q e^{-i\vec{q}\cdot\vec{r}_m}\boldsymbol{j}(\vec{q})\end{aligned} \tag{19}$$

back into the continuity equation

$$\dot{\rho}(\vec{q}) - i\vec{q}\cdot\boldsymbol{j}(\vec{q}) = 0 \tag{20}$$

Hence

$$\begin{aligned}-i\vec{q}\cdot\mathbf{j}(\vec{q}) = -\dot{\rho}(\vec{q}) = i[\rho, H] &= i\sum_{p,k,\alpha,\beta,\theta} h^{\alpha\beta}_{\vec{p}}\left[C^\dagger_{\vec{k}+\vec{q}\theta}C_{\vec{k}\theta}, C^\dagger_{p\alpha}C_{p\beta}\right] \\ &= i\sum_{p,k,\alpha,\beta,\theta} h^{\alpha\beta}_{\vec{p}}\left(C^\dagger_{\vec{k}+\vec{q}\theta}C_{\vec{k}\theta}C^\dagger_{p\alpha}C_{p\beta} - C^\dagger_{p\alpha}C_{p\beta}C^\dagger_{\vec{k}+\vec{q}\theta}C_{\vec{k}\theta}\right) \\ &= i\sum_{p,k,\alpha,\beta,\theta} h^{\alpha\beta}_{\vec{p}}\left(\delta_{\vec{k},\vec{p}}\delta_{\theta,\alpha}C^\dagger_{\vec{k}+\vec{q}\theta}C_{\vec{p}\beta} - \delta_{\vec{p},\vec{k}+\vec{q}}\delta_{\theta,\beta}C^\dagger_{\vec{p}\alpha}C_{\vec{k},\theta}\right) \\ &= i\sum_{\vec{k}\alpha\beta}(h^{\alpha\beta}_{\vec{k}} - h^{\alpha\beta}_{\vec{k}+\vec{q}})C^\dagger_{\vec{k}+\vec{q}\alpha}C_{\vec{k}\beta} \\ &= i\sum_{\vec{k}\alpha\beta}\left(h^{\alpha\beta}_{\vec{k}-\vec{q}/2} - h^{\alpha\beta}_{\vec{k}+\vec{q}/2}\right)C^\dagger_{\vec{k}+\vec{q}/2\alpha}C_{\vec{k}-\vec{q}/2\beta}\end{aligned} \tag{21}$$

In the $\vec{q}\to 0$ limit:

$$i\vec{q}\cdot\mathbf{j}(\vec{q}) = i\sum_{\vec{k}\alpha\beta}\vec{q}\cdot\frac{\partial h^{\alpha\beta}_{\vec{q}}}{\partial\vec{k}}C^\dagger_{\vec{k}+\vec{q}/2\alpha}C_{\vec{k}-\vec{q}/2\beta} \tag{22}$$

Hence

$$\boldsymbol{j}(\vec{q}) = \sum_{\vec{k}\alpha\beta}\frac{\partial h^{\alpha\beta}_{\vec{q}}}{\partial\vec{k}}C^\dagger_{\vec{k}+\vec{q}/2\alpha}C_{\vec{k}-\vec{q}/2\beta} \tag{23}$$

Also since $\boldsymbol{A}(\vec{r}_m)$ is real valued, in its Fourier transformation $\boldsymbol{A}(\vec{q})$ must satisfy

$$\boldsymbol{A}(\vec{q}) = \boldsymbol{A}(-\vec{q}) \tag{24}$$

Hence

$$H_{ext} = \sum_{kp\alpha\beta}C^\dagger_{\vec{k}+\frac{1}{2}\vec{p}\alpha}C_{\vec{k}-\frac{1}{2}\vec{p}\beta}\frac{\partial h^{\alpha\beta}_{\vec{k}}}{\partial\vec{k}}\cdot\boldsymbol{A}_{-\vec{p}} = \sum_q \mathbf{j}_{\vec{q}}\cdot\mathbf{A}_{-q} \tag{25}$$

Next we will calculate the conductance by using Kubo's formula.

$$\Pi_{ij}(\vec{q},\tau) = -\frac{1}{V}\left\langle T_\tau j_i^\dagger(\vec{q},\tau) j_j(\vec{q},0)\right\rangle$$
$$\Pi_{ij}\left(\vec{q}, i\omega_n\right) = \int_0^\beta d\tau e^{i\omega_n\tau}\Pi_{ij}(\vec{q},\tau) \quad \text{(Fourier transformation; } \omega_n = \frac{2n\pi}{\beta}) \tag{26}$$
$$\sigma_{ij}(\vec{q},v) = \frac{i}{\omega}\Pi_{ij}(\vec{q},\omega) = \frac{i}{\omega}\lim_{\omega_n\to\omega+i\delta}\Pi_{ij}(\vec{q},i\omega_n)$$

where V is the volumn of the system and j_i is the ith component of the current operator $\boldsymbol{j}$. Hence if we want to obtain the conductiveity σ_{ij} we must first compute $\Pi_{ij}(\vec{q},\tau)$ first.

Insert the following into the expression of Π_{ij}

$$\boldsymbol{j}(\vec{q}) = \sum_{\vec{k}\alpha\beta}\frac{\partial h_{\vec{q}}^{\alpha\beta}}{\partial\vec{k}} C^\dagger_{\vec{k}+\vec{q}/2\alpha} C_{\vec{k}-\vec{q}/2\beta} \tag{27}$$

Hence

$$\begin{aligned}\Pi_{i,j}(\hat{q},\tau) &= -\frac{1}{V}\sum_{k,p}\frac{\partial h^{\alpha\beta}(\vec{k})}{\partial k_i}\frac{\partial h^{\gamma\theta}(\vec{p})}{\partial q_i}\left\langle T_i\left[C^\dagger_{\vec{k}-\frac{\vec{q}}{2},\beta}(\tau) C_{\vec{k}+\frac{\vec{q}}{2},\alpha}(\tau) C^\dagger_{\vec{p}+\frac{\vec{q}}{2},\gamma} C_{\vec{p}-\frac{\vec{q}}{2},\theta}\right]\right\rangle \\ &= -\frac{1}{V}\sum_{k,p}\frac{\partial h^{\alpha\beta}(\vec{k})}{\partial k_i}\frac{\partial h^{\gamma\theta}(\vec{p})}{\partial q_i}\left\langle C^\dagger_{\vec{k}-\frac{\vec{q}}{2},\beta}(\tau) C_{\vec{p}-\frac{\vec{q}}{2},\theta}\right\rangle\left\langle C_{\vec{k}+\frac{\vec{q}}{2},\alpha}(\tau) C^\dagger_{\vec{p}+\frac{\vec{q}}{2},\gamma}\right\rangle\end{aligned} \tag{28}$$

To make things easier, we can diagonalize the Hamiltonian

$$U^\dagger_{\vec{k}} h^{\alpha\beta}(\vec{k}) U_{\vec{k}} = \begin{pmatrix}\varepsilon_1(\vec{k}) & & \\ & \ddots & \\ & & \varepsilon_m(\vec{k})\end{pmatrix} \tag{29}$$

Define

$$\begin{aligned}\gamma_{\vec{k}-\frac{\vec{q}}{2}},\alpha &= \left(U_k^\dagger\right)_{\alpha\beta} C_{\vec{k}-\frac{\vec{q}}{2},\beta} \\ \gamma^\dagger_{\vec{k}+\frac{\vec{q}}{2}}\alpha &= C^+_{\vec{k}+\frac{\vec{q}}{2},\beta}\left(U_k\right)_{\beta\alpha}\end{aligned} \tag{30}$$

Hence

$$\begin{aligned}H_0 &= \sum_k C^\dagger_{\vec{k},\alpha} h^{\alpha\beta}_{\vec{k}} C_{\vec{k},\beta} = \sum_k\sum_\alpha \gamma^\dagger_{\vec{k},\alpha}\varepsilon_\alpha(\vec{k})\gamma_{\vec{k},\alpha} \\ j_i(\vec{q},\tau) &= \sum_k\sum_\alpha \gamma^\dagger_{\vec{k}+\frac{\vec{q}}{2},\alpha}(\tau)\frac{\partial\varepsilon_\alpha(\vec{k})}{\partial k_i}\gamma_{\vec{k}-\frac{\vec{q}}{2},\alpha}(\tau)\end{aligned} \tag{31}$$

Then

$$\Pi_{i,j}(\vec{q},\tau) = -\frac{1}{V}\sum_{k,p}\frac{\partial\varepsilon_\alpha(\vec{k})}{\partial k_i}\frac{\partial\varepsilon_\beta(\vec{p})}{\partial q_i}\left\langle T_\tau\gamma^\dagger_{\vec{k}-\frac{\vec{q}}{2},\beta}(\tau)\gamma_{\vec{p}-\frac{\vec{q}}{2},\theta}\right\rangle\left\langle T_\tau\gamma_{\vec{k}+\frac{\vec{q}}{2},\alpha}(\tau)\gamma^\dagger_{\vec{p}+\vec{q},\gamma}\right\rangle \tag{32}$$

Since

$$\begin{aligned}
\left[H_0,\gamma^\dagger_{\vec{k}-\frac{\vec{q}}{2},\alpha}\right] &= \left[\sum_k\sum_\beta \gamma^\dagger_{\vec{k},\beta}\varepsilon_\beta(\vec{k})\gamma_{\vec{k},\beta},\gamma^\dagger_{\vec{k}-\frac{\vec{q}}{2},\alpha}\right] \\
&= \sum_k\sum_\beta \gamma^\dagger_{\vec{k},\beta}\varepsilon_\beta(\vec{k})\gamma_{\vec{k},\beta}\gamma^\dagger_{\vec{k}-\frac{\vec{q}}{2},\alpha} + \sum_k\sum_\beta \gamma^\dagger_{\vec{k},\beta}\gamma^\dagger_{\vec{k}-\frac{\vec{q}}{2},\alpha}\varepsilon_\beta(\vec{k})\gamma_{\vec{k},\beta} \\
&= \gamma^\dagger_{\vec{k}-\frac{\vec{q}}{2},\alpha}\varepsilon_\alpha(\vec{k}-\frac{\vec{q}}{2})
\end{aligned} \tag{33}$$

we can see that

$$e^{\tau H}\gamma^\dagger_{\vec{k}-\frac{\vec{q}}{2},\alpha}e^{-\tau H} = \gamma^\dagger_{\vec{k}-\frac{\vec{q}}{2},\alpha}e^{\tau\varepsilon_\alpha(\vec{k}-\frac{\vec{q}}{2})} \tag{34}$$

Hence

$$\begin{aligned}
\mathcal{G}(\vec{k}-\frac{\vec{q}}{2},\alpha,\tau) &= \left\langle T_\tau \gamma^\dagger_{\vec{k}-\frac{\vec{q}}{2},\alpha}(\tau)\gamma_{\vec{p}-\frac{\vec{q}}{2},\beta}\right\rangle \\
&= \theta(\tau)\left\langle e^{\tau\varepsilon_\alpha(\vec{k}-\frac{\vec{q}}{2})}\gamma^\dagger_{\vec{k}-\frac{\vec{q}}{2},\alpha}\gamma_{\vec{k}-\frac{\vec{q}}{2},\alpha}\right\rangle\delta_{\vec{k},\vec{q}}\delta_{\alpha,\beta} - \theta(-\tau)\left\langle e^{\tau\varepsilon_\alpha(\vec{k}-\frac{\vec{q}}{2})}\gamma_{\vec{k}-\frac{\vec{q}}{2},\alpha}\gamma^\dagger_{\vec{k}-\frac{\vec{q}}{2},\alpha}\right\rangle\delta_{\vec{k},\vec{q}}\delta_{\alpha,\beta} \\
&= e^{\tau\varepsilon_\alpha(\vec{k}-\frac{\vec{q}}{2})}\left[n_F(\varepsilon_{\alpha(\vec{k}-\frac{\vec{q}}{2})})-\theta(-\tau)\right]\delta_{\vec{k},\vec{q}}\delta_{\alpha,\beta} \\
\mathcal{G}(\vec{k}-\frac{\vec{q}}{2},\alpha,i\omega_n) &= \int_0^\beta d\tau e^{i\omega_n\tau}\mathcal{G}(\vec{k}-\frac{\vec{q}}{2},\alpha,\tau) = -\frac{1}{i\omega_n+\varepsilon_\alpha(\vec{k}-\frac{\vec{q}}{2})}\delta_{\vec{k},\vec{q}}\delta_{\alpha,\beta}
\end{aligned} \tag{35}$$

Similarly we can obtain

$$\begin{aligned}
\mathcal{G}'(\vec{k}+\frac{\vec{q}}{2},\alpha,\tau) &= \left\langle T_\tau \gamma_{\vec{k}+\frac{\vec{q}}{2},\alpha}(\tau)\gamma^\dagger_{\vec{p}+\frac{\vec{q}}{2},\beta}\right\rangle = e^{\tau\varepsilon_\alpha(\vec{k}+\frac{\vec{q}}{2})}\delta_{\vec{k},\vec{q}}\delta_{\alpha,\beta}\left[\theta(\tau)-n_F(\varepsilon_\alpha(\vec{k}+\frac{\vec{q}}{2}))\right] \\
\mathcal{G}'\left(\vec{k}+\frac{\vec{q}}{2},\alpha,i\omega_m\right) &= \int_0^\beta d\tau e^{i\omega_m\tau}\mathcal{G}'\left(\vec{k}-\frac{\vec{q}}{2},\alpha,\tau\right) = -\frac{1}{i\omega_n-\varepsilon_\alpha\left(\vec{k}+\frac{\vec{q}}{2}\right)}\delta_{\vec{k},\vec{q}}\delta_{\alpha,\beta}
\end{aligned} \tag{36}$$

Hence

$$\begin{aligned}
\Pi_{ij}(\vec{q},i\nu_\lambda) &= \int_0^\beta e^{i\nu_\lambda\tau} - \frac{1}{V}\sum_{k,\alpha}\frac{\partial\varepsilon_\alpha(\vec{k})}{\partial k_i}\frac{\partial\varepsilon_\beta(\vec{p})}{\partial k_j}\sum_{m,n}\frac{1}{\beta}e^{-i\omega_n\tau}\mathcal{G}\left(\vec{k}-\frac{\vec{q}}{2},\alpha,i\omega_n\right)\frac{1}{\beta}e^{-i\omega_m\tau}\mathcal{G}'\left(\vec{k}+\frac{\vec{q}}{2},\alpha,i\omega_m\right) \\
&= -\frac{1}{\beta^2 V}\int_0^\beta d\tau e^{i(\nu_\lambda-\omega_n-\omega_m)\tau}\sum_{m,n,\alpha,k}\frac{\partial\varepsilon_\alpha(\vec{k})}{\partial k_i}\frac{\partial\varepsilon_\beta(\vec{p})}{\partial k_j}\mathcal{G}\left(\vec{k}-\frac{\vec{q}}{2},\alpha,i\omega_n\right)\mathcal{G}'\left(\vec{k}+\frac{\vec{q}}{2},\alpha,i\omega_m\right) \\
&= -\frac{1}{\beta V}\sum_{n,\alpha,k}\frac{\partial\varepsilon_\alpha(\vec{k})}{\partial k_i}\frac{\partial\varepsilon_\beta(\vec{p})}{\partial k_j}\mathcal{G}\left(\vec{k}-\frac{\vec{q}}{2},\alpha,i\omega_n\right)\mathcal{G}'\left(\vec{k}+\frac{\vec{q}}{2},\alpha,i(\nu_\lambda-\omega_n)\right) \\
&= -\frac{1}{\beta V}\sum_{n,\alpha,k}\frac{\partial\varepsilon_\alpha(\vec{k})}{\partial k_i}\frac{\partial\varepsilon_\beta(\vec{p})}{\partial k_j}\mathcal{G}\left(\vec{k}-\frac{\vec{q}}{2},\alpha,-i\omega_n\right)\mathcal{G}'\left(\vec{k}+\frac{\vec{q}}{2},\alpha,i(\nu_\lambda+\omega_m)\right) \\
&= \frac{1}{\beta V}\sum_{n,\alpha,k}\frac{\partial\varepsilon_\alpha(\vec{k})}{\partial k_i}\frac{1}{i(\nu_\lambda+\omega_n)-\varepsilon_\alpha(\vec{k}+\frac{\vec{q}}{2})}\frac{1}{i\omega_n-\varepsilon_\alpha(\vec{k}-\frac{\vec{q}}{2})}\frac{\partial\varepsilon_\alpha(\vec{k})}{\partial k_j}
\end{aligned} \tag{37}$$

Since we are only interested in the DC response, we should set $\vec{q} \to 0$ and then let $\omega \to 0$. In the following, we set $\vec{q} \to 0$.

$$
\begin{aligned}
\Pi_{ij}(\vec{q}, i\nu_\lambda) &= \frac{1}{\beta V} \sum_{n,\alpha,k} \frac{\partial \varepsilon_\alpha(\vec{k})}{\partial k_i} \frac{1}{i(\nu_\lambda + \omega_n) - \varepsilon_\alpha(\vec{k})} \frac{\partial \varepsilon_\alpha(\vec{k})}{\partial k_j} \frac{1}{i\omega_n - \varepsilon_\alpha(\vec{k})} \\
&= \frac{1}{\beta V} \sum_{k,n} Tr \left\{ \frac{\partial \varepsilon(\vec{k})}{\partial k_i} G_0(\vec{k}, i(\nu_\lambda + \omega_n)) \frac{\partial \varepsilon(\vec{k})}{\partial k_j} G_0(\vec{k}, i\omega_n) \right\}
\end{aligned} \tag{38}
$$

where

$$
\varepsilon(\vec{k}) = \begin{pmatrix} \varepsilon_1(\vec{k}) & & \\ & \ddots & \\ & & \varepsilon_m(\vec{k}) \end{pmatrix} \tag{39}
$$

and

$$
G_0(\vec{k}, i\omega_n) = \begin{pmatrix} \frac{1}{i\omega_n - \varepsilon_1(\vec{k})} & & \\ & \ddots & \\ & & \frac{1}{i\omega_n - \varepsilon_m(\vec{k})} \end{pmatrix} \tag{40}
$$

Define a matrix $h_{\vec{k}} = (h_{\vec{k}})^{\alpha\beta}$ and another matrix $D(\vec{k}, i\omega_n) = Diag(i\omega_n - \varepsilon_1(\vec{k}), \cdots, i\omega_n - \varepsilon_m(\vec{k}))$. According to the diagonalization of the Hamiltonian:

$$
U^\dagger_{\vec{k}} h_{\vec{k}} U_{\vec{k}} = \varepsilon(\vec{k}) \tag{41}
$$

We can see that

$$
U^\dagger_{\vec{k}} (i\omega_n - h_{\vec{k}}) U_{\vec{k}} = D(\vec{k}, i\omega_n) \tag{42}
$$

Hence

$$
G_0(\vec{k}, i\omega_n) = \left[D(\vec{k}, i\omega_n) \right]^{-1} = \left[U^\dagger_{\vec{k}} (i\omega_n - h_{\vec{k}}) U_{\vec{k}} \right]^{-1} = U^\dagger_{\vec{k}} \left[i\omega_n - h_{\vec{k}} \right]^{-1} U_{\vec{k}} \tag{43}
$$

Define

$$
G(\vec{k}, i\omega_m) = \left[i\omega_n - h_{\vec{k}} \right]^{-1} \tag{44}
$$

We can see that

$$
G_0(\vec{k}, i\omega_n) = U^\dagger_{\vec{k}} G(\vec{k}, i\omega_m) U_{\vec{k}} \tag{45}
$$

Insert the result back to the $\Pi_{ij}(\vec{q}, i\nu_\lambda)$:

$$
\begin{aligned}
\Pi_{ij}(\vec{q}, i\nu_\lambda) &= \frac{1}{\beta V} \sum_{k,n} Tr \left\{ U^\dagger_{\vec{k}} \frac{\partial h_{\vec{k}}}{\partial k_i} U_{\vec{k}} U^\dagger_{\vec{k}} G(\vec{k}, i(\omega_m + \nu_\lambda)) U_{\vec{k}} U^\dagger_{\vec{k}} \frac{\partial h_{\vec{k}}}{\partial k_j} U_{\vec{k}} U^\dagger_{\vec{k}} G(\vec{k}, i\omega_m) U_{\vec{k}} \right\} \\
&= \frac{1}{\beta V} \sum_{k,n} Tr \left\{ \frac{\partial h_{\vec{k}}}{\partial k_i} G(\vec{k}, i(\omega_m + \nu_\lambda)) \frac{\partial h_{\vec{k}}}{\partial k_j} G(\vec{k}, i\omega_m) \right\} \\
&= \frac{1}{\beta V} \sum_{k,n} Tr \left\{ J_i(\vec{k}) G(\vec{k}, i(\omega_m + \nu_\lambda)) J_j(\vec{k}) G(\vec{k}, i\omega_m) \right\}
\end{aligned} \tag{46}
$$

where $J_i(\vec{k}) = \frac{\partial h_{\vec{k}}}{\partial k_i}$. Hence by analytic continuation, the DC conductance is

$$\sigma_{ij} = \lim_{\omega \to 0} Q_{ij}(\omega + i\delta) \tag{47}$$

where

$$Q_{ij}(i\nu_m) = \Pi_{ij}(i\nu_m) = \frac{1}{\beta\Omega} \sum_{k,n} Tr\left\{ J_i(\vec{k}) G(\vec{k}, i(\omega_m + \nu_m)) J_j(\vec{k}) G(\vec{k}, i\omega_m) \right\} \tag{48}$$

Here we replaced V by Ω which represents the area of the system in 2+1 Dimensions. Consider a band insulator with M filled bands, the Hall conductivity σ_{xy} can also take the following form.

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi}\int dk_x \int dk_y f_{xy}(\mathbf{k}) \tag{49}$$

where

$$f_{xy}(\mathbf{k}) = \frac{\partial a_y(\mathbf{k})}{\partial k_x} - \frac{\partial a_x(\mathbf{k})}{\partial k_y} \quad a_i(\mathbf{k}) = -i \sum_{\alpha \in \text{ occ}} \left\langle \alpha\mathbf{k} \middle| \frac{\partial}{\partial k_i} \middle| \alpha\mathbf{k} \right\rangle, \quad i = x, y \tag{50}$$

We won't provide a proof of this, since I found an excellent reference that has alrady provided a detailed proof.[1] My labor here would not make a difference.

From above we can define the first Chern number

$$C_1 = \frac{1}{2\pi}\int dk_x \int dk_y f_{xy}(\mathbf{k}) \in \mathbb{Z} \tag{51}$$

The Hall response is then

$$j_i = \sigma_H \epsilon^{ij} E_j \tag{52}$$

where $\sigma_H = C_1/(2\pi)$. We can also obtian the charge density from the continuity equaition

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j} = -(\partial_x j_x + \partial_y j_y) = -\sigma_H(\partial_x E_y - \partial_y E_x) = \sigma_H \frac{\partial B}{\partial t} \tag{53}$$

The charge density is

$$\rho(B) - \rho_0(B = 0) = \sigma_H B \tag{54}$$

Hence

$$j^\mu = \frac{C_1}{2\pi}\epsilon^{\mu\nu\tau}\partial_\nu A_\tau \tag{55}$$

Consider the effective theory

$$S_{\text{eff}} = \frac{C_1}{4\pi}\int d^2x \int dt A_\mu \epsilon^{\mu\nu\tau}\partial_\nu A_\tau \tag{56}$$

We can see that

$$\begin{aligned}\delta S_{\text{eff}} =&\frac{C_1}{4\pi}\int d^2x\int dt\delta A_\mu\epsilon^{\mu\nu\tau}\partial_\nu A_\tau - \frac{C_1}{4\pi}\int d^2x\int dt\partial_\nu A_\mu\epsilon^{\mu\nu\tau}\delta A_\tau \\ =&\frac{C_1}{2\pi}\int d^2x\int dt\delta A_\mu\epsilon^{\mu\nu\tau}\partial_\nu A_\tau\end{aligned} \tag{57}$$

Hence

$$j^\mu = \frac{\delta S_{\text{eff}}}{\delta A_\mu} = \frac{C_1}{2\pi}\epsilon^{\mu\nu\tau}\partial_\nu A_\tau \tag{58}$$

1.2 Example: A Two Band Model

Consider the following two band model:

$$h(\mathbf{k}) = \sum_{a=1}^{3} d_a(\mathbf{k})\sigma^a + \epsilon(\mathbf{k})\mathrm{I} \tag{59}$$

Its eigenvalue can be found by solving

$$\begin{vmatrix} E-(\varepsilon+d_3) & d_1-id_2 \\ d_1+id_2 & E-(\varepsilon-d_3) \end{vmatrix} = 0 \tag{60}$$

Hence

$$(E-\varepsilon)^2 = d_1^2+d_2^2+d_3^2 = d^2 \tag{61}$$

The eigenvalues are

$$E_\pm = \varepsilon \pm d \tag{62}$$

where $d=\sqrt{\sum_a d_a^2(k)}$. We then define two projection operators

$$P_\pm = \frac{1}{2}(1\pm\hat{d}_a\sigma^a) = \frac{1}{2}(1\pm\hat{d}\cdot\vec{\sigma}) \tag{63}$$

Before moving forward, we introduce some basix facts about pauli matrices:

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})I + i(\vec{a}\times\vec{b})\cdot\vec{\sigma} \tag{64}$$

Also

$$\begin{aligned}\operatorname{tr}(\sigma_a) &= 0 \\ \operatorname{tr}(\sigma_a\sigma_b) &= 2\delta_{ab} \\ \operatorname{tr}(\sigma_a\sigma_b\sigma_c) &= 2i\varepsilon_{abc} \\ \operatorname{tr}(\sigma_a\sigma_b\sigma_c\sigma_d) &= 2\left(\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}\right)\end{aligned} \tag{65}$$

We can see that

$$
\begin{aligned}
P_+P_+ &= \frac{1}{4}(1+2\hat{d}\cdot\vec{\sigma}+(\hat{d}\cdot\vec{\sigma})(\hat{d}\cdot\vec{\sigma}) = \frac{1}{2}(1+\hat{d}\cdot\vec{\sigma}) = P_+ \\
P_-P_- &= \frac{1}{4}(1-2\hat{d}\cdot\vec{\sigma}+(\hat{d}\cdot\vec{\sigma})(\hat{d}\cdot\vec{\sigma}) = \frac{1}{2}(1-\hat{d}\cdot\vec{\sigma}) = P_- \\
P_-P_+ &= \frac{1}{4}(1-(\hat{d}\cdot\vec{\sigma})(\hat{d}\cdot\vec{\sigma}) = 0 \\
P_+P_- &= \frac{1}{4}(1-(\hat{d}\cdot\vec{\sigma})(\hat{d}\cdot\vec{\sigma}) = 0
\end{aligned}
\tag{66}
$$

It also quite evident that

$$
P_+ + P_- = \mathbb{I} \tag{67}
$$

We can express $G\left(\mathbf{k}, i\omega_n\right) = \left(i\omega_n - H(\mathbf{k})\right)^{-1}$ in terms of the above defined projection matrices.

Claim

$$
G\left(\mathbf{k}, i\omega_n\right) = \frac{P_+}{i\omega_n - E_+(\mathbf{k})} + \frac{P_-}{i\omega_n - E_-(\mathbf{k})} \tag{68}
$$

Proof Consider the following quantity

$$
\begin{aligned}
A =& \left(i\omega_n - H(\mathbf{k})\right)\left(\frac{P_+}{i\omega_n - E_+(\mathbf{k})} + \frac{P_-}{i\omega_n - E_-(\mathbf{k})}\right) \\
=& \left(i\omega_n - \epsilon - d\hat{d}\cdot\vec{\sigma}\right)\left(\frac{P_+}{i\omega_n - E_+(\mathbf{k})} + \frac{P_-}{i\omega_n - E_-(\mathbf{k})}\right) \\
=& \frac{(i\omega_n - \epsilon)\frac{1}{2}(1+\hat{d}\cdot\vec{\sigma}) - \frac{d}{2}(1+\hat{d}\cdot\vec{\sigma})}{i\omega_n - \epsilon - d} + \frac{(i\omega_n - \epsilon)\frac{1}{2}(1-\hat{d}\cdot\vec{\sigma}) + \frac{d}{2}(1-\hat{d}\cdot\vec{\sigma})}{i\omega_n - \epsilon + d} \\
=& \frac{1}{2}(1+\hat{d}\cdot\vec{\sigma}) + \frac{1}{2}(1\hat{d}\cdot\vec{\sigma}) \\
=& 1
\end{aligned}
\tag{69}
$$

Hence

$$
\begin{aligned}
Q_{xy}\left(i\nu_m\right) &= \frac{1}{\Omega\beta}\sum_{\mathbf{k},n}\operatorname{tr}\left(J_x(\mathbf{k})G\left(\mathbf{k}, i\left(\omega_n+\nu_m\right)\right)J_y(\mathbf{k})G\left(\mathbf{k}, i\omega_n\right)\right) \\
&= \frac{1}{\Omega\beta}\sum_{s,t=\pm}\sum_{\mathbf{k},n}\frac{\operatorname{tr}\left(J_x(\mathbf{k})P_s(\mathbf{k})J_y(\mathbf{k})P_t(\mathbf{k})\right)}{\left(i\left(\omega_n+\nu_m\right) - E_s(\mathbf{k})\right)\left(i\omega_n - E_t(\mathbf{k})\right)} \\
&= \frac{1}{\Omega}\sum_{s,t=\pm}\sum_{\mathbf{k}}\frac{\operatorname{tr}\left(J_x(\mathbf{k})P_s(\mathbf{k})J_y(\mathbf{k})P_t(\mathbf{k})\right)}{i\nu_m - E_s(\mathbf{k}) + E_t(\mathbf{k})}\left(n_t(\mathbf{k}) - n_s(\mathbf{k})\right)
\end{aligned}
\tag{70}
$$

Where in the last line, we have used the following identity

$$
\frac{1}{\beta}\sum_n \mathcal{G}^{(0)}\left(\mathbf{p}, ip_n\right)\mathcal{G}^{(0)}\left(\mathbf{k}, ip_n + i\omega_m\right) = \frac{n_F\left(\xi_{\mathbf{p}}\right) - n_F\left(\xi_{\mathbf{k}}\right)}{i\omega_m + \xi_{\mathbf{p}} - \xi_{\mathbf{k}}} \tag{71}
$$

Then the Hall conductivity is

$$\sigma_{xy} = \lim_{\omega \to 0} \frac{i}{\omega} Q_{xy}(\omega + i\delta) = -\frac{i}{\Omega} \sum_{s,t=\pm} \sum_{\mathbf{k}} \frac{\mathrm{tr}\left(J_x(\mathbf{k}) P_s(\mathbf{k}) J_y(\mathbf{k}) P_t(\mathbf{k})\right)}{\left(E_t(\mathbf{k}) - E_s(\mathbf{k})\right)^2} \left(n_t(\mathbf{k}) - n_s(\mathbf{k})\right) \tag{72}$$

Consider the zero termperature case. The above equation can be further simplified to

$$\begin{aligned}
\sigma_{xy} =& -\frac{i}{\Omega} \sum_{\mathbf{k}} \frac{1}{4d^2} \left[\mathrm{tr}\left(J_x(\mathbf{k}) P_+(\mathbf{k}) J_y(\mathbf{k}) P_-(\mathbf{k})\right) - \mathrm{tr}\left(J_x(\mathbf{k}) P_-(\mathbf{k}) J_y(\mathbf{k}) P_+(\mathbf{k})\right)\right] \\
=& \frac{i}{16d^2\Omega} \sum_k \mathrm{Tr}\left\{ \left(\frac{\partial \epsilon(k)}{\partial k_x} + \frac{\partial d_\alpha(k)}{\partial k_x}\sigma^\alpha\right)\left(1 - \hat{d}_\alpha \sigma^\alpha\right)\left(\frac{\partial \epsilon(k)}{\partial k_y} + \frac{\partial d_\alpha(k)}{\partial k_y}\sigma^\alpha\right)\left(1 + \hat{d}_\alpha \sigma^\alpha\right)\right\} \\
& - \frac{i}{16d^2\Omega} \sum_k \mathrm{Tr}\left\{ \left(\frac{\partial \epsilon(k)}{\partial k_x} + \frac{\partial d_\alpha(k)}{\partial k_x}\sigma^\alpha\right)\left(1 + \hat{d}_\alpha \sigma^\alpha\right)\left(\frac{\partial \epsilon(k)}{\partial k_y} + \frac{\partial d_\alpha(k)}{\partial k_y}\sigma^\alpha\right)\left(1 - \hat{d}_\alpha \sigma^\alpha\right)\right\} \\
=& \frac{i}{16d^2\Omega} \sum_k \mathrm{Tr}\left\{ \left(\frac{\partial \epsilon(k)}{\partial k_x} + \frac{\partial d_\alpha(k)}{\partial k_x}\sigma^\alpha\right)\left(1 - \hat{d}_\alpha \sigma^\alpha\right)\left(\frac{\partial \epsilon(k)}{\partial k_y} + \frac{\partial d_\alpha(k)}{\partial k_y}\sigma^\alpha\right)\left(1 + \hat{d}_\alpha \sigma^\alpha\right)\right\} \\
& - \frac{i}{16d^2\Omega} \sum_k \mathrm{Tr}\left\{ \left(1 + \hat{d}_\alpha \sigma^\alpha\right)\left(\frac{\partial \epsilon(k)}{\partial k_y} + \frac{\partial d_\alpha(k)}{\partial k_y}\sigma^\alpha\right)\left(1 - \hat{d}_\alpha \sigma^\alpha\right)\left(\frac{\partial \epsilon(k)}{\partial k_x} + \frac{\partial d_\alpha(k)}{\partial k_x}\sigma^\alpha\right)\right\} \\
=& \frac{i}{16d^2\Omega} \sum_k \mathrm{Tr}\left\{ \left(\frac{\partial \epsilon(k)}{\partial k_x} + \frac{\partial d_\alpha(k)}{\partial k_x}\sigma^\alpha\right)\left(1 - \hat{d}_\alpha \sigma^\alpha\right)\left(\frac{\partial \epsilon(k)}{\partial k_y} + \frac{\partial d_\alpha(k)}{\partial k_y}\sigma^\alpha\right)\left(1 + \hat{d}_\alpha \sigma^\alpha\right) - H.c.\right\}
\end{aligned} \tag{73}$$

From the trace properties of pauli matrices, we can see that the traces of products of 1 or 2 or 4 pauli matrices are real. Hence from above, we learn that the only nonvanishing terms are the products of three pauli matrices. Hence

$$\begin{aligned}
& \frac{i}{16d^2\Omega} \sum_k \left\{ \left[-2i\epsilon^{abc} \frac{\partial d_a}{\partial k_x} \hat{d}_b \frac{\partial d_c}{\partial k_y} + 2i\epsilon^{abc} \frac{\partial d_a}{\partial k_x} \frac{\partial d_b}{\partial k_y} \hat{d}_c\right] - h.c.\right\} \\
=& \frac{i}{16\Omega} \sum_k \left(8i\epsilon^{abc} \frac{\partial d_a}{\partial k_x} \frac{\partial d_b}{\partial k_y} \hat{d}_c \frac{1}{d^2}\right)
\end{aligned} \tag{74}$$

We should notice that

$$\frac{\partial \hat{d}_a}{\partial k_i} = \frac{\partial d_a}{\partial k_i} \frac{1}{d} + \frac{\partial \frac{1}{d}}{\partial k_i} d_a \tag{75}$$

However, the second term would be cancled by the antismmetric symbol. Hence

$$\begin{aligned}
\sigma_{xy} =& -\frac{1}{2\Omega} \sum_k \left\{ \frac{\partial \hat{d}_a(k)}{\partial k_x} \frac{\partial \hat{d}_b(k)}{\partial k_y} \hat{d}_c \epsilon^{abc} \right\} \\
=& -\frac{1}{2\Omega} \int \frac{d^2k}{4\pi^2} \hat{\mathbf{d}} \cdot \frac{\partial \hat{\mathbf{d}}}{\partial k_x} \times \frac{\partial \hat{\mathbf{d}}}{\partial k_y}
\end{aligned} \tag{76}$$

Hence

$$
\begin{aligned}
C_1 =& 2\pi \times \frac{1}{2} \int \frac{d^2k}{4\pi^2} \hat{\mathbf{d}} \cdot \frac{\partial \hat{\mathbf{d}}}{\partial k_x} \times \frac{\partial \hat{\mathbf{d}}}{\partial k_y} \\
=& \frac{1}{4\pi} \int dk_x \int dk_y \hat{\mathbf{d}} \cdot \frac{\partial \hat{\mathbf{d}}}{\partial k_x} \times \frac{\partial \hat{\mathbf{d}}}{\partial k_y}
\end{aligned}
\tag{77}
$$

1.3 Dimensional Reduction

Consider the following Hamiltonian

$$
H = \sum_n \left[c_n^\dagger \frac{\sigma_z - i\sigma_x}{2} c_{n+\hat{x}} + c_n^\dagger \frac{\sigma_z - i\sigma_y}{2} c_{n+\hat{y}} + \text{H.c.} \right] + m \sum_n c_n^\dagger \sigma_z c_n
\tag{78}
$$

In this model, we endow the y direction with a periodic boundary condition and x direction with an open boundary condition. Hence k_y is still a good quantum number. Define the partial Fourier transformation as

$$
c_{k_y \alpha}(x) = \frac{1}{\sqrt{L_y}} \sum_y c_\alpha(x,y) e^{ik_y y}
\tag{79}
$$

The Hamiltonian can be written as

$$
\begin{aligned}
H =& \frac{1}{L_y} \sum_{n,k_y,k_y'} \left[c_{k_y,x}^\dagger e^{i(k_y - k_y')y} \frac{\sigma_z - i\sigma_x}{2} c_{k_y',x+\hat{x}} + c_{k_y,x}^\dagger e^{i(k_y-k_y')y} \frac{\sigma_z - i\sigma_y}{2} c_{k_y',x} e^{-ik_y'} + \text{H.c.} \right] \\
&+ \frac{1}{L_y} m \sum_{n,k_y,k_y'} c_{k_y,x}^\dagger e^{i(k_y-k_y')y} \sigma_z c_{k_y',x} \\
=& \sum_{x,k_y} \left[c_{k_y,x}^\dagger \frac{\sigma_z - i\sigma_x}{2} c_{k_y,x+\hat{x}} + c_{k_y,x}^\dagger \frac{\sigma_z - i\sigma_y}{2} c_{k_y,x} e^{-ik_y} + \text{H.c.} \right] + m \sum_{x,k_y} c_{k_y,x}^\dagger \sigma_z c_{k_y,x} \\
=& \sum_{k_y x} \left[c_{k_y}^\dagger(x) \frac{\sigma_z - i\sigma_x}{2} c_{k_y}(x+1) + \text{H.c.} \right] + \sum_{k_y x} c_{k_y}^\dagger(x) \left[\sin k_y \sigma_y + (m + \cos k_y) \sigma_z \right] c_{k_y}(x) \\
\equiv& \sum_{k_y} H_{1\text{D}}(k_y)
\end{aligned}
\tag{80}
$$

Consider a constant electric field in the y direction. The external gauge field should be

$$
A_y = -E_y t, \quad A_x = 0
\tag{81}
$$

The response of the system is thus the sum of responses of its 1D chains:

$$
J_x = \sum_{k_y} J_x(k_y)
\tag{82}
$$

Let t evolve from 0 to $2\pi/L_y E_y$. The pumped charge is

$$
\Delta Q = \int_0^{\Delta t} dt \sum_{k_y} J_x(k_y) \equiv \sum_{k_y} \Delta P_x(k_y) \Big|_0^{\Delta t}
\tag{83}
$$

where

$$J_x(k_y) = \frac{dP_x(k_y)}{dt} \tag{84}$$

Here from $t = 0$ to $t = \Delta t = 2\pi/L_y E_y$, the change of the charge polarization is (in the $L_y \to \infty$ limit)

$$\Delta P_x(k_y) = P_x(k_y - \frac{2\pi}{L_y}) - P_x(k_y) = -\frac{2\pi}{L_y}\frac{\partial P_x(k_y)}{\partial k_y} \tag{85}$$

Hence

$$\Delta Q = -\oint_0^{2\pi} dk_y \frac{\partial P_x\left(k_y\right)}{\partial k_y} \tag{86}$$

For a generic (2+1)-dimensional Hamiltonian, we can define the corresponding (1+1)-dimensional system by replacing k_y by a field parameter θ:

$$H_{\mathrm{1D}}(\theta) = \sum_{k_x} c_{k_x}^{\dagger} h\left(k_x, \theta\right) c_{k_x\theta} \tag{87}$$

Hence the response function is

$$J_x(\theta) = G(\theta)\frac{d\theta}{dt} \tag{88}$$

where

$$\begin{aligned} G(\theta) &= \lim_{\omega\to 0}\frac{i}{\omega}Q(\omega + i\delta;\theta) \\ Q\left(i\omega_n;\theta\right) &= -\sum_{k_x iv_m}\operatorname{tr}\left(\frac{\partial h(k_x,\theta)}{\partial k_x}G_{\mathrm{1D}}\left(k_x, i\left(\nu_m+\omega_n\right);\theta\right)\cdot\frac{\partial h\left(k_x;\theta\right)}{\partial\theta}G_{\mathrm{1D}}\left(k_x, i\nu_m;\theta\right)\right)\frac{1}{L_x\beta} \end{aligned} \tag{89}$$

Likewise the coefficient can be expressed by Barry phase gauge vectors:

$$G(\theta) = -\oint\frac{dk_x}{2\pi}f_{x\theta}\left(k_x,\theta\right) = \oint\frac{dk_x}{2\pi}\left(\frac{\partial a_x}{\partial\theta} - \frac{\partial a_\theta}{\partial k_x}\right) \tag{90}$$

where

$$\int G(\theta)d\theta = C_1 \in Z \tag{91}$$

The expression of $G(\theta)$ can be further simplified if a_θ is single valued. In that case

$$G(\theta) = \frac{\partial}{\partial\theta}\left(\oint\frac{dk_x}{2\pi}a_x\left(k_x,\theta\right)\right) \equiv \frac{\partial P(\theta)}{\partial\theta} \tag{92}$$

where

$$P(\theta) = \oint dk_x a_x/2\pi \tag{93}$$

From the continuity equation we can obtain the charge density

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J_x}{\partial x} = -\partial_x \frac{\partial P(\theta)}{\partial \theta}\frac{d\theta}{dt} = -\frac{\partial P(\theta)}{\partial x \partial t} \tag{94}$$

Hence the density is (disregarding the background charge):

$$\rho = -\frac{P(\theta)}{\partial x} \tag{95}$$

We thus obtian the following equation

$$j_\mu = -\epsilon_{\mu\nu}\frac{\partial P[\theta(x,t)]}{\partial x_\nu} \tag{96}$$

If the field parameter θ has a smooth spatial and temporal dependence, the Hamiltonian is thus $h(k,\theta) = h(k, \theta(x,t))$, which means now the eigenstates are also spatial and temporal dependent. We can thus redefine the Barry phase gauge vector and curvature as

$$\begin{aligned} \mathcal{A}_A &= -i\sum_\alpha \left\langle \alpha; q_A \middle| \frac{\partial}{\partial q_A} \middle| \alpha; q_A \right\rangle \\ \mathcal{F}_{AB} &= \partial_A \mathcal{A}_B - \partial_B \mathcal{A}_A \end{aligned} \tag{97}$$

where the phase space coordinate is $q_A = (t, x, k)$. Define the phase current as

$$j_A^P = -\frac{1}{4\pi}\epsilon_{ABC}\mathcal{F}_{BC} \tag{98}$$

Consider the following integral

$$\begin{aligned} \int dk j_\mu^P &= -\int \frac{dk}{2\pi}\epsilon^{\mu 2\nu}\mathcal{F}_{2\nu} = \int dk \frac{1}{2\pi}\epsilon^{\mu\nu}(\partial_k \mathcal{A}_\nu - \partial_\nu \mathcal{A}_k) \\ &= -\epsilon^{\mu\nu}\partial_\nu \left(\int dk a_k\right) = -\epsilon^{\mu\nu}\frac{\partial P}{\partial x_v} = j_\mu \end{aligned} \tag{99}$$

1.4 $\mathbf{Z_2}$ Classification of Particle-hole Symmetric Insulators in (1+1)-dimensions

Consider the tight binding Hamiltonian

$$H = \sum_{mn} c_{m\alpha}^\dagger h_{mn}^{\alpha\beta} c_{n\beta} \tag{100}$$

The particle hole transformation is defined as

$$c_{m\alpha} \rightarrow C^{\alpha\beta} c_{m\beta}^\dagger \tag{101}$$

where the transformation matrix satisfies the following properties:

$$C^\dagger C = \mathbb{I} = C^* C \tag{102}$$

Hence we can see that $C^T = C$. Next we consider the transformation on $c^\dagger_{m\alpha}$. Since we expect that two particle-hole transformations will restore the system,

$$c^\dagger_{m\alpha} \to D^{\alpha\beta} c_{m\beta} \to D^{\alpha\beta} C^{\beta\gamma} c^\dagger_{m\gamma} = c^\dagger_{m\alpha} \tag{103}$$

Hence

$$D^{\alpha\beta} C^{\beta\gamma} = \delta^{\alpha\gamma} \tag{104}$$

Hence D=$C^\dagger$. Then the Hamiltonian under particle hole transformation is

$$\begin{aligned} H =& \sum_{mn} c_{m\theta} (C^\dagger)^{\alpha\theta} h^{\alpha\beta}_{mn} C^{\beta\gamma} c^\dagger_{n\gamma} \\ =& \sum_{mn} c_{m\theta} (C^\dagger)^{\theta\alpha} h^{\alpha\beta}_{mn} C^{\beta\gamma} c^\dagger_{n\gamma} \\ =& \sum_{mn} c_m C^\dagger h_{mn} C c^\dagger_n \\ =& \sum_k c_k C^\dagger h(-k) C c^\dagger_k = \sum_k c_{-k} C^\dagger h(k) C c^\dagger_{-k} \end{aligned} \tag{105}$$

If we require the Hamiltonian to be particle-hole symmetric, the Hamiltonian should satisfy

$$H = \sum_k c^\dagger_k h(k) c_k = \sum_k c_{-k} C^\dagger h(k) C c^\dagger_{-k} = \sum_k c_k C^\dagger h(-k) C c^\dagger_k \tag{106}$$

Set the chemical potential to be zero. In this way $h^{\alpha\beta}(k)$ would vanish at $\alpha = \beta$. The constraint of the particle-hole symmetry is

$$H = \sum_k c^\dagger_k h(k) c_k = -\sum_k c_{k\beta} h^{\alpha\beta}(k) c^\dagger_{k\alpha} = \sum_k c_k \left(-h^T(k)\right) c^\dagger_k = \sum_k c_k C^\dagger h(-k) C c^\dagger_k \tag{107}$$

Hence the particle-hole symmetry constraint is

$$C^\dagger h(-k) C = -h^T(k) \tag{108}$$

Consider an interpolation between two particle symmetric (1+1)-dimensional insulators $h_1(k)$ and $h_2(k)$. The interpolation should satisfy

$$\begin{aligned} h(k, 0) =& h_1(k), \quad h(k, \pi) = h_2(k) \\ h(k, \theta) =& -\left[C^{-1} h(-k, 2\pi - \theta) C\right]^T \end{aligned} \tag{109}$$

Hence the interpolation is a (2+1)-dimensional particle hole symmetric Hamiltonian.

Claim The Chern number of the interpolation can be well defined as

$$\begin{aligned} C[h(k,\theta)] = \oint d\theta \frac{\partial P(\theta)}{\partial \theta} \\ P(\theta) = \oint \frac{dk}{2\pi} \sum_{E_\alpha(k)<0} (-i) \left\langle k, \theta; \alpha \left| \partial_k \right| k, \theta; \alpha \right\rangle \end{aligned} \tag{110}$$

Then for any two interpolations

$$C[h(k,\theta)] - C\left[h'(k,\theta)\right] = 2n, n \in \mathbb{Z} \tag{111}$$

Consider an eigenstate $|k,\theta,\alpha\rangle$ of the Hamiltonian $h(k,\theta)$:

$$-h^T(-k, 2\pi - \theta)C^\dagger|k,\theta,\alpha\rangle = C^\dagger h(k)|k,\theta,\alpha\rangle = E_\alpha(k)C^\dagger|k,\theta,\alpha\rangle \tag{112}$$

Since $h(k,\theta)$ is a Hamiltonian, it should be Hermitian which means $h^\dagger = h$. Hence $h^T = h^*$. From this we can see that

$$\begin{gathered} h^*(-k, 2\pi-\theta)C^\dagger|k,\theta,\alpha\rangle = -\, E_\alpha(k)C^\dagger|k,\theta,\alpha\rangle \Rightarrow \\ h(-k, 2\pi-\theta)C^T(|k,\theta,\alpha\rangle)^* = -\, E_\alpha(k)C^T(|k,\theta,\alpha\rangle)^* \Rightarrow \\ h(-k, 2\pi-\theta)C(|k,\theta,\alpha\rangle)^* = -\, E_\alpha(k)C(|k,\theta,\alpha\rangle)^* \end{gathered} \tag{113}$$

Hence $C|k,\theta;\alpha\rangle^* \equiv |-k, 2\pi-\theta;\bar{\alpha}\rangle$ is an eigenstate of $h(-k,2\pi-\theta)$ with an eigenvalue of $E_{\bar{\alpha}}(k,2\pi-\theta) = -E_\alpha(k,\theta)$. Then

$$\begin{aligned} P_{occ}(\theta) &= \int_0^{2\pi} \frac{dk}{2\pi} \sum_{E_\alpha(k,\theta)<0} (-i)\left\langle k,\theta;\alpha\left|\partial_k\right|k,\theta;\alpha\right\rangle \\ &= \int_0^{2\pi} \frac{dk}{2\pi} \sum_{E_\alpha(k,\theta)<0} (-i)(\ ^*\langle k,\theta;\alpha\left|\partial_k\right|k,\theta;\alpha\rangle^*)^* \\ &= \int_0^{2\pi} \frac{dk}{2\pi} \sum_{E_\alpha(k,\theta)<0} (-i)(\ ^*\langle k,\theta;\alpha\left|C^\dagger\partial_k C\right|k,\theta;\alpha\rangle^*)^* \\ &= \int_0^{2\pi} \frac{dk}{2\pi} \sum_{E_\alpha(k,\theta)<0} (-i)\left[\langle -k,2\pi-\theta;\bar{\alpha}\left|\partial_k\right|-k,2\pi-\theta;\bar{\alpha}\rangle\right]^* \\ &= \int_0^{2\pi} \frac{dk}{2\pi} \sum_{E_\alpha(k,\theta)<0} (-i)\left[\partial_k\left\langle -k,2\pi-\theta;\bar{\alpha}\right|-k,2\pi-\theta;\bar{\alpha}\rangle\right] \\ &= \int_0^{2\pi} \frac{dk}{2\pi} \sum_{E_\alpha(k,\theta)<0} i\left[\langle -k,2\pi-\theta;\bar{\alpha}\left|\partial_k\right|-k,2\pi-\theta;\bar{\alpha}\rangle\right] \\ &= \int_0^{2\pi} \frac{dk}{2\pi} \sum_{E_{\bar{\alpha}}(-k,2\pi-\theta)>0} (-i)\left[\langle -k,2\pi-\theta;\bar{\alpha}\left|\partial_{-k}\right|-k,2\pi-\theta;\bar{\alpha}\rangle\right] \\ &= \int_0^{2\pi} \frac{dk}{2\pi} \sum_{E_{\bar{\alpha}}(k,2\pi-\theta)>0} (-i)\left[\langle k,2\pi-\theta;\bar{\alpha}\left|\partial_k\right|k,2\pi-\theta;\bar{\alpha}\rangle\right] \equiv P_{\mathrm{unocc}}(2\pi-\theta) \end{aligned} \tag{114}$$

Define a basis of the corresponding eigenvector space $\{|\beta\rangle\}$ which is independent of k and θ. Then $u^{\beta\alpha}(k,\theta) = \langle\beta|k,\theta,\alpha\rangle$ is a unitary matrix.

$$\begin{aligned}
P_{occ}(\theta) + P_{unocc}(\theta) &= \int_0^{2\pi} \frac{dk}{2\pi} \sum_\alpha (-i) \langle k,\theta;\alpha \left|\partial_k\right| k,\theta;\alpha\rangle \\
&= \int_0^{2\pi} \frac{dk}{2\pi} \sum_\alpha (-i) \langle k,\theta;\alpha|m,\beta\rangle \partial_k \langle m,\beta|k,\theta;\alpha\rangle \\
&= \int_0^{2\pi} \frac{dk}{2\pi} (-i) \operatorname{Tr}\left[u^{-1}(k)\partial_k u(k)\right]
\end{aligned} \tag{115}$$

The last line is a winding number and hence has to be an integer. Hence

$$P_{occ}(\theta) + P_{\text{unocc}}(\theta) = 0 \text{ mod } 1 \tag{116}$$

Then

$$P_{occ}(\theta) = P_{unocc}(2\pi - \theta) = -P_{occ}(2\pi - \theta) \quad (mod\ 1) \tag{117}$$

Consequently for $\theta = 0$ or $\theta = \pi$:

$$P_{occ}(\theta) + P_{occ}(2\pi - \theta) = 2P_{occ}(\theta) = 0\ mod\ 1 \tag{118}$$

Hence for such θ values, $P_{occ}(\theta) = 0$ *or* $1/2$ *mod* 1. This can already serve as a classification rule, however, it is not quite rigorous. From above,we can see that

$$\int_0^\pi dP_{occ}(\theta) = \int_0^\pi d\theta \frac{\partial P_{occ}(\theta)}{\partial\theta} = \int_0^\pi d\theta - \frac{\partial P_{occ}(2\pi-\theta)}{\partial\theta} = \int_{\theta=\pi}^{\theta=0} d(2\pi-\theta) \frac{\partial P_{occ}(2\pi-\theta)}{\partial(2\pi-\theta)} \tag{119}$$

Hence

$$\int_0^\pi dP(\theta) = \int_\pi^{2\pi} dP(\theta) \tag{120}$$

where by $P(\theta)$ we mean $P_{occ}(\theta)$ unless stating the otherwise.

Then consider two interpolations $h(k,\theta)$ and $h'(k,\theta)$.

$$C[h] - C[h'] = \int_0^{2\pi} d\theta \left(\frac{\partial P(\theta)}{\partial\theta} - \frac{\partial P'(\theta)}{\partial\theta}\right) \tag{121}$$

Define the following new paths

$$\begin{aligned}
g_1(k,\theta) &= \begin{cases} h(k,\theta), & \theta \in [0,\pi] \\ h'(k,2\pi-\theta), & \theta \in [\pi,2\pi] \end{cases} \\
g_2(k,\theta) &= \begin{cases} h'(k,2\pi-\theta), & \theta \in [0,\pi] \\ h(k,\theta), & \theta \in [\pi,2\pi] \end{cases}
\end{aligned} \tag{122}$$

Then

$$
\begin{aligned}
C[g_1] = \int_0^{2\pi} \frac{\partial P(\theta)}{\partial \theta} d\theta &= \int_0^{\pi} \frac{\partial P}{\partial \theta} d\theta + \int_{\pi}^{2\pi} \frac{\partial P'(2\pi-\theta)}{\partial \theta} \\
&= \int_0^{\pi} \frac{\partial P}{\partial \theta} d\theta - \int_{2\pi}^{\pi} \frac{\partial P'(2\pi-\theta)}{\partial(2\pi-\theta)} d(2\pi-\theta) \\
&= \int_0^{\pi} \frac{\partial P(\theta)}{\partial \theta} d\theta - \int_0^{\pi} \frac{\partial P'(\theta)}{\partial \theta} d\theta \\
C[g2] = \int_0^{2\pi} \frac{\partial P(\theta)}{\partial \theta} d\theta &= \int_0^{\pi} \frac{\partial P'(2\pi-\theta)}{\partial \theta} d\theta + \int_{\pi}^{2\pi} \frac{\partial P(\theta)}{\partial \theta} d\theta \\
&= -\int_{\pi}^{0} \frac{\partial P'(2\pi-\theta)}{\partial(2\pi-\theta)} d(2\pi-\theta) + \int_{\pi}^{2\pi} \frac{\partial P(\theta)}{\partial \theta} d\theta \\
&= +\int_{\pi}^{2\pi} \frac{\partial P(\theta)}{\partial \theta} d\theta - \int_{\pi}^{2\pi} \frac{\partial P'(\theta)}{\partial \theta} d\theta
\end{aligned} \tag{123}
$$

We can immediately see that $C[g_1] = C[g_2]$. Also notice that

$$
C[g_1] + C[g_2] = \int_0^{2\pi} d\theta \left(\frac{\partial P(\theta)}{\partial \theta} - \frac{\partial P'(\theta)}{\partial \theta} \right) = C[h] - C[h'] = 2C[g_1] \tag{124}
$$

Hence

$$
C[h] - C[h'] = 0 \ mod\ 2 \tag{125}
$$

We can then define the relative Chern parity between two (1+1)-dimensional particle-hole symmetric insulators $h_1(k)$ and $h_2(k)$ by

$$
N_1 \left[h_1(k), h_2(k) \right] = (-1)^{C[h(k,\theta)]} \tag{126}
$$

Also consider three (1+1)-dimesional particle-hole symmetric insulators $h_1(k)$, $h_2(k)$ and $h_3(k)$. Let $h_1(k,\theta)$ be the interpolation between $h_1(k)$ and $h_2(k)$. Let $h_2(k,\theta)$ be the interpolation between $h_2(k)$ and $h_3(k)$. We can define an interpolation between $h_1(k)$ and $h_3(k)$ by

$$
h_3(k,\theta) = \begin{cases} h_1(k, 2\theta), & \theta \in [0, \pi/2] \\ h_2(k, 2\theta - \pi), & \theta \in [\pi/2, \pi] \\ h_2(k, 2\theta - \pi), & \theta \in [\pi, 3\pi/2] \\ h_1(k, 2\theta - 2\pi), & \theta \in [3\pi/2, 2\pi] \end{cases} \tag{127}
$$

Hence

$$
C[h_3] = C[h_1] + C[h_2] \tag{128}
$$

Which means

$$
N_1 \left[h_1(k), h_2(k) \right] N_1 \left[h_2(k), h_3(k) \right] = N_1 \left[h_1(k), h_3(k) \right] \tag{129}
$$

We can let the vacuum Hamiltonian $h_0(k) = h_0$ be a reference. Any $h(k)$ that satisfies $N_1[h_0, h(k)] = -1$ is Z_2 nontrivial and Z_2 trivial if it satisfies $N_1[h_0, h(k)] = 1$. Consider the boundary between a trvial insultor h_0 and a nontrivial insulator h_1. Define the interpolation between them as $h(k, \theta)$. Thus the boundary betweent them can be realized by a domain wall $\theta(x)$ with the property $\theta(x) \to \pi \ as \ x \to \infty$ and $\theta(x) \to 0 \ as \ x \to -\infty$. (Such a domain wall structure will be elaborated later.) Hence from the charge density equation, we can obtain the total charge localized at the boundary:

$$Q_d = \int_{-\infty}^{\infty} dx \rho(x) = -\int_{-\infty}^{\infty} dx \frac{\partial P(\theta)}{\partial x} = -\int_0^{\pi} dP(\theta) = -\frac{1}{2}\int_0^{2\pi} dP(\theta) = -\frac{1}{2}C[h] \tag{130}$$

Since one of the insulator is trivial and the other is not, the Chern number $C[h]$ of their interpolation must be an odd numebr. Hence there is a half charge localized at the interface between a nontrivial insulator and a trivial insulator.

1.5 $\mathbf{Z_2}$ Classification of (0+1)-dimensional Particle-hole Symmetric Insulators

A (0+1)-dimensional particle hole symmetric must satisfy

$$C^\dagger h C = -h^T \tag{131}$$

Now we define an interpolation between two PHS insulators h_1 and h_2 by $h(\theta)$ that satisfies

$$h(0) = h_1, \quad h(\pi) = h_2, \quad C^\dagger h(\theta) C = -h(2\pi - \theta)^T \tag{132}$$

where PHS means particle-hole symmetric. We can let $N[h]$ decide whether the two insulators belong to the same class. However, before we can do this, we need to prove that $N[h]$ does not depend on the interpolation we choosed. In other words, for two legal interpolations $h(\theta)$ and $h'(\theta)$, we need to prove that $N[h]N[h'] = 1 = N[h, h']$. Consider the interpolation $g(\theta, \varphi)$ between the two interpolations with the following property.

$$\begin{gathered} g(\theta, \varphi = 0) = h(\theta), g(\theta, \varphi = \pi) = h'(\theta) \\ g(0, \varphi) = h_1, g(\pi, \varphi) = h_2 \\ C^\dagger g(\theta, \varphi) C = -g(2\pi - \theta, 2\pi - \varphi)^T \end{gathered} \tag{133}$$

Hence

$$N[h, h'] = (-1)^{C[g]} \tag{134}$$

We should also notice that $g(\theta, \varphi)$ is also an interpolation between $g(0, \varphi) = h_1$ and $g(\pi, \varphi) = h_2$. Since $h_1(\varphi) = h_1$ and $h_2(\varphi) = h_2$, they are both (1+1)-dimensional vacuum Hamiltonians. Hence an interpolation of them must satisfy

$$(-1)^{C[g]} = 1 \tag{135}$$

Hence $N[h, h'] = N[h]N[h'] = 1$. We have thus proved that the $N[h]$ is independent of the choice of the interpolation between the two (0+1)-dimensional PHS insulators.

2 Second Chern Numbers and its Physical Consequences

2.1 Second Chern Number in (4+1)-dimensional nonlinear response

The Hamiltonian of a band insulator coupled to a U(1) gauge field is written as

$$H[A]=\sum_{m,n}\left(c_{m\alpha}^{\dagger}h_{mn}^{\alpha\beta}e^{iA_{mn}}c_{n\beta}+\text{H.c.}\right)+\sum_{m}A_{0m}c_{m\alpha}^{\dagger}c_{m\alpha} \tag{136}$$

The effective action of the gauge field can be obtained by

$$\begin{aligned} e^{iS_{\text{eff}}[A]} &= \int D[c]D\left[c^{\dagger}\right]\exp\left\{i\int dt\left[\sum_{m}c_{m\alpha}^{\dagger}\left(i\partial_{t}\right)c_{m\alpha}-H[A]\right]\right\} \\ &= \det\left[\left(i\partial_{t}-A_{0m}\right)\delta_{mn}^{\alpha\beta}-h_{mn}^{\alpha\beta}e^{iA_{mn}}\right] \end{aligned} \tag{137}$$

We can obtain the expression of $S_{eff}[A]$ by expanding the above. The result is shown below.

$$S_{\text{eff}}=\frac{C_{2}}{24\pi^{2}}\int d^{4}xdt\epsilon^{\mu\nu\rho\sigma\tau}A_{\mu}\partial_{\nu}A_{\rho}\partial_{\sigma}A_{\tau} \tag{138}$$

where

$$C_{2}=-\frac{\pi^{2}}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^{4}kd\omega}{(2\pi)^{5}}\operatorname{Tr}\left[\left(G\frac{\partial G^{-1}}{\partial q^{\mu}}\right)\left(G\frac{\partial G^{-1}}{\partial q^{\nu}}\right)\left(G\frac{\partial G^{-1}}{\partial q^{\rho}}\right)\left(G\frac{\partial G^{-1}}{\partial q^{\sigma}}\right)\left(G\frac{\partial G^{-1}}{\partial q^{\tau}}\right)\right] \tag{139}$$

in which $q^{\mu}=(\omega,k_1,k_2,k_3,k_4)$ and $G\left(q^{\mu}\right)=\left[\omega+i\delta-h\left(k_i\right)\right]^{-1}$

Claim The coefficient C_2 is equal to the second Chern number of the non-Abelian Berry's phase gauge field defined on the BZ:

$$\begin{aligned} C_{2} &= \frac{1}{32\pi^{2}}\int d^{4}k\epsilon^{ijk\ell}\operatorname{tr}\left[f_{ij}f_{k\ell}\right] \\ f_{ij}^{\alpha\beta} &= \partial_{i}a_{j}^{\alpha\beta}-\partial_{j}a_{i}^{\alpha\beta}+i\left[a_{i},a_{j}\right]^{\alpha\beta} \\ a_{i}^{\alpha\beta}(\mathbf{k}) &= -i\left\langle\alpha,\mathbf{k}\left|\frac{\partial}{\partial k_{i}}\right|\beta,\mathbf{k}\right\rangle \end{aligned} \tag{140}$$

Proof

Step 1. C_2 is topologically invariant with respect to any infinitesimal variations of $G(\boldsymbol{k},\omega)$.

$$\begin{aligned} \delta\left(G\partial_{\mu}G^{-1}\right) &= \delta G\partial_{\mu}G^{-1}+G\partial_{\mu}\left(\delta G^{-1}\right) \\ &= \delta G\partial_{\mu}G^{-1}-G\partial_{\mu}\left(G^{-1}\delta GG^{-1}\right) \\ &= \delta G\partial_{\mu}G^{-1}-G(\partial_{\mu}G^{-1})\delta GG^{-1}-(\partial_{\mu}\delta G)G^{-1}-\delta G(\partial_{\mu}G^{-1}) \\ &= -G\left(\partial_{\mu}G^{-1}\right)\delta GG^{-1}-\partial_{\mu}(\delta G)G^{-1} \end{aligned} \tag{141}$$

where from the first line to the second line we have used

$$\delta G^{-1} = -G^{-1}\delta G G^{-1} \tag{142}$$

which can be easily proven by

$$\delta G^{-1} = \delta(G^{-1}GG^{-1}) = \delta G^{-1} + G^{-1}\delta G G^{-1} + \delta G^{-1} = 2\delta G^{-1} + G^{-1}\delta G G^{-1} \tag{143}$$

Consider the following term first

$$\begin{aligned}
\delta C_2 =& -\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\operatorname{tr}\left[\delta\left(G\partial_\mu G^{-1}\right)\left(G\partial_\nu G^{-1}\right)\left(G\partial_\rho G^{-1}\right)\left(G\partial_\sigma G^{-1}\right)\left(G\partial_\tau G^{-1}\right)\right]\\
=&\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\operatorname{tr}\left[\left(G\partial_\mu G^{-1}\delta G G^{-1}\right)\left(G\partial_\nu G^{-1}\right)\left(G\partial_\rho G^{-1}\right)\left(G\partial_\sigma G^{-1}\right)\left(G\partial_\tau G^{-1}\right)\right]\\
&+\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\operatorname{tr}\left[\left(\partial_\mu\delta G G^{-1}\right)\left(G\partial_\nu G^{-1}\right)\left(G\partial_\rho G^{-1}\right)\left(G\partial_\sigma G^{-1}\right)\left(G\partial_\tau G^{-1}\right)\right]\\
=&\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\operatorname{tr}\left[\left(\partial_\mu G^{-1}\delta G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
&+\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\operatorname{tr}\left[\left(G^{-1}\partial_\mu\delta G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
=&\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\operatorname{tr}\left[\partial_\mu\left(G^{-1}\delta G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
=&\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\partial_\mu\operatorname{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
&-\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\operatorname{tr}\left[\left(G^{-1}\delta G\right)\partial_\mu\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
&-\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\operatorname{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\nu G^{-1}G\right)\partial_\mu\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
&-\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\operatorname{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\partial_\mu\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
&-\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\operatorname{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\partial_\mu\left(\partial_\tau G^{-1}G\right)\right]
\end{aligned} \tag{144}$$

Since

$$0 = \partial_\mu\left(GG^{-1}\right) = \partial_\mu G G^{-1} + G\partial_\mu G^{-1} \tag{145}$$

We can see that

$$\partial_\mu G = -G\partial_\mu G^{-1} G \tag{146}$$

Hence

$$\begin{aligned}
&-\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\,\mathrm{tr}\left[\left(G^{-1}\delta G\right)\partial_\mu\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
=&-\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\,\mathrm{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\nu G^{-1}\partial_\mu G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
=&\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\,\mathrm{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\mu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
=&-\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\,\mathrm{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\mu G^{-1}G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]
\end{aligned}\tag{147}$$

For the same reason

$$\begin{aligned}
&-\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\,\mathrm{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\nu G^{-1}G\right)\partial_\mu\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
=&\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\,\mathrm{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\mu G^{-1}G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
&-\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\,\mathrm{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\partial_\mu\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
=&-\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\,\mathrm{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\mu G^{-1}G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]\\
&-\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\,\mathrm{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\partial_\mu\left(\partial_\tau G^{-1}G\right)\right]\\
=&\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\,\mathrm{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\mu G^{-1}G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]
\end{aligned}\tag{148}$$

Hence

$$\delta C_2=\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4kd\omega}{(2\pi)^5}\partial_\mu\,\mathrm{tr}\left[\left(G^{-1}\delta G\right)\left(\partial_\nu G^{-1}G\right)\left(\partial_\rho G^{-1}G\right)\left(\partial_\sigma G^{-1}G\right)\left(\partial_\tau G^{-1}G\right)\right]=0 \tag{149}$$

Step 2. Deform the Hamiltonian to a simpler form and calculate the C_2 number. Diagonalize the Hamiltonian:

$$h(\mathbf{k})=U(\mathbf{k})D(\mathbf{k})U^\dagger(\mathbf{k}) \tag{150}$$

where $U(\mathbf{k})=(|1,\mathbf{k}\rangle,|2,\mathbf{k}\rangle,\ldots,|N,\mathbf{k}\rangle)$ is a unitary matrix and $D(\mathbf{k})$ a diagonal matrix: $D(\mathbf{k})=diag\left[\varepsilon_1(\mathbf{k}),\cdots,\varepsilon_N(\mathbf{k})\right]$. Consider a band insulator with M filled bands. By setting the chemical potential to an appropriate value, we can rearrange the eigenvalues as

$$\varepsilon_1(\mathbf{k})\leq\cdots\leq\varepsilon_M(\mathbf{k})\leq 0\leq\varepsilon_{M+1}(\mathbf{k})\leq\cdots\leq\varepsilon_N(\mathbf{k}) \tag{151}$$

We can deform them without level crossing by

$$E_\alpha(\mathbf{k},t)=\begin{cases}\varepsilon_\alpha(\mathbf{k})(1-t)+\varepsilon_G t, & 1\leqslant\alpha\leqslant M\\ \varepsilon_\alpha(\mathbf{k})(1-t)+\varepsilon_E t, & M<\alpha\leqslant N\end{cases} \tag{152}$$

where $0 \leq t \leq 1$, $\varepsilon_G < 0 < \varepsilon_E$ and $D_0(\mathbf{k},t) = \text{diag}\left[E_1(\mathbf{k},t), E_2(\mathbf{k},t), \ldots, E_N(\mathbf{k},t)\right]$ Hence

$$D_0(\mathbf{k},0) = D(\mathbf{k}), \quad D_0(\mathbf{k},1) = \begin{pmatrix} \varepsilon_G I_{M\times M} & \\ & \varepsilon_E\ I_{N-M\times N-M} \end{pmatrix} \tag{153}$$

Hence at $t = 1$

$$h(\mathbf{k},1) = \varepsilon_G \sum_{\alpha=1}^{M} |\alpha,\mathbf{k}\rangle\langle\alpha,\mathbf{k}| + \varepsilon_E \sum_{\beta=M+1}^{N} |\beta,\mathbf{k}\rangle\langle\beta,\mathbf{k}| = \varepsilon_G P_G(\mathbf{k}) + \varepsilon_E P_E(\mathbf{k}) \tag{154}$$

where

$$\begin{aligned} P_G &= \sum_{\alpha=1}^{M} |\alpha,\mathbf{k}\rangle\langle\alpha,\mathbf{k}| \\ P_E &= \sum_{\beta=M+1}^{N} |\beta,\mathbf{k}\rangle\langle\beta,\mathbf{k}| \end{aligned} \tag{155}$$

are projection operators, i.e. they satisfy the following properties

$$P_G^2 = P_G \quad P_E^2 = P_E \quad P_G + P_E = 1 \quad P_G P_E = P_E P_G = 0 \tag{156}$$

Prop

$$\begin{aligned} \frac{\partial P_G}{\partial k_i} &= \frac{\partial(1-P_E)}{\partial k_i} = -\frac{\partial P_E}{\partial k_i} \\ P_E \frac{\partial P_G}{\partial k_i} &= \frac{\partial P_E P_G}{\partial k_i} - \frac{\partial P_E}{\partial k_i} P_G = -\frac{\partial P_E}{\partial k_i} P_G = \frac{\partial P_G}{\partial k_i} P_G \\ P_G \frac{\partial P_G}{\partial k_i} &= -P_G \frac{\partial P_E}{\partial k_i} = \frac{\partial P_G}{\partial k_i} P_E \\ P_E \frac{\partial P_G}{\partial k_i} P_E &= \frac{\partial P_G}{\partial k_i} P_G P_E = 0 \\ P_G \frac{\partial P_G}{\partial k_i} P_G &= \frac{\partial P_G}{\partial k_i} P_E P_G = 0 \end{aligned} \tag{157}$$

According to step 1, we can directly replace the original Hamiltonian by $h(\mathbf{k},1)$ in the calculation of C_2. Since $G^{-1} = \omega + i\delta - \varepsilon_G P_G - \varepsilon_E P_E$ we can see that

$$\begin{aligned} &\frac{\partial G^{-1}(\mathbf{k},\omega)}{\partial \omega} = 1 \\ &\frac{\partial G^{-1}(\mathbf{k},\omega)}{\partial k_i} = -\varepsilon_G \frac{\partial P_G(\mathbf{k})}{\partial k_i} - \varepsilon_E \frac{\partial P_E(\mathbf{k})}{\partial k_i} = (\varepsilon_E - \varepsilon_G) \frac{\partial P_G(\mathbf{k})}{\partial k_i} \end{aligned} \tag{158}$$

Claim

$$G(\mathbf{k},\omega) = \frac{P_G(\mathbf{k})}{\omega + i\delta - \varepsilon_G} + \frac{P_E(\mathbf{k})}{\omega + i\delta - \varepsilon_E} \tag{159}$$

Proof

$$
\begin{aligned}
G^{-1}(\mathbf{k},\omega)\left(\frac{P_G(\mathbf{k})}{\omega+i\delta-\varepsilon_G}+\frac{P_E(\mathbf{k})}{\omega+i\delta-\varepsilon_E}\right) &= (\omega+i\delta-\varepsilon_G P_G-\varepsilon_E P_E)\left(\frac{P_G(\mathbf{k})}{\omega+i\delta-\varepsilon_G}+\frac{P_E(\mathbf{k})}{\omega+i\delta-\varepsilon_E}\right)\\
&= \frac{\omega P_G+i\delta P_G-\varepsilon_G P_G^2}{\omega+i\delta-\varepsilon_G}+\frac{\omega P_E+i\delta P_E-\varepsilon_E P_E^2}{\omega+i\delta-\varepsilon_E}\\
&= P_G+P_E=1
\end{aligned}
\tag{160}
$$

For the same reason,

$$
\left(\frac{P_G(\mathbf{k})}{\omega+i\delta-\varepsilon_G}+\frac{P_E(\mathbf{k})}{\omega+i\delta-\varepsilon_E}\right)G^{-1}(\mathbf{k},\omega)=1 \tag{161}
$$

Hence we have proved that

$$
G(\mathbf{k},\omega)=\frac{P_G(\mathbf{k})}{\omega+i\delta-\varepsilon_G}+\frac{P_E(\mathbf{k})}{\omega+i\delta-\varepsilon_E} \tag{162}
$$

We have all the things we need to calculate C_2 now.

$$
\begin{aligned}
C_2 =& -\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int\frac{d^4k d\omega}{(2\pi)^5}\,\mathrm{Tr}\left[\left(G\frac{\partial G^{-1}}{\partial q^\mu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\nu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\rho}\right)\left(G\frac{\partial G^{-1}}{\partial q^\sigma}\right)\left(G\frac{\partial G^{-1}}{\partial q^\tau}\right)\right]\\
=& -\frac{\pi^2}{15}\epsilon^{0\nu\rho\sigma\tau}\int\frac{d^4k d\omega}{(2\pi)^5}\,\mathrm{Tr}\left[(G)\left(G\frac{\partial G^{-1}}{\partial q^\nu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\rho}\right)\left(G\frac{\partial G^{-1}}{\partial q^\sigma}\right)\left(G\frac{\partial G^{-1}}{\partial q^\tau}\right)\right]\\
& -\frac{\pi^2}{15}\epsilon^{\mu0\rho\sigma\tau}\int\frac{d^4k d\omega}{(2\pi)^5}\,\mathrm{Tr}\left[\left(G\frac{\partial G^{-1}}{\partial q^\mu}\right)(G)\left(G\frac{\partial G^{-1}}{\partial q^\rho}\right)\left(G\frac{\partial G^{-1}}{\partial q^\sigma}\right)\left(G\frac{\partial G^{-1}}{\partial q^\tau}\right)\right]\\
& -\frac{\pi^2}{15}\epsilon^{\mu\nu0\sigma\tau}\int\frac{d^4k d\omega}{(2\pi)^5}\,\mathrm{Tr}\left[\left(G\frac{\partial G^{-1}}{\partial q^\mu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\nu}\right)(G)\left(G\frac{\partial G^{-1}}{\partial q^\sigma}\right)\left(G\frac{\partial G^{-1}}{\partial q^\tau}\right)\right]\\
& -\frac{\pi^2}{15}\epsilon^{\mu\nu\rho1\tau}\int\frac{d^4k d\omega}{(2\pi)^5}\,\mathrm{Tr}\left[\left(G\frac{\partial G^{-1}}{\partial q^\mu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\nu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\rho}\right)(G)\left(G\frac{\partial G^{-1}}{\partial q^\tau}\right)\right]\\
& -\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma1}\int\frac{d^4k d\omega}{(2\pi)^5}\,\mathrm{Tr}\left[\left(G\frac{\partial G^{-1}}{\partial q^\mu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\nu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\rho}\right)\left(G\frac{\partial G^{-1}}{\partial q^\sigma}\right)(G)\right]\\
=& -\frac{\pi^2}{3}\epsilon^{ijkl}\int\frac{d^4k d\omega}{(2\pi)^5}\,\mathrm{Tr}\left[G^2\frac{\partial G^{-1}}{\partial k_i}G\frac{\partial G^{-1}}{\partial k_j}G\frac{\partial G^{-1}}{\partial k_k}G\frac{\partial G^{-1}}{\partial k_l}\right]\\
=& -\frac{\pi^2}{3}\epsilon^{ijkl}\int\frac{d^4k d\omega}{(2\pi)^5}\sum_{n,m,s,t=G,E}\frac{\mathrm{Tr}\left[P_n\frac{\partial P_G}{\partial k_i}P_m\frac{\partial P_G}{\partial k_j}P_s\frac{\partial P_G}{\partial k_k}P_t\frac{\partial P_G}{\partial k_\ell}\right](\varepsilon_E-\varepsilon_G)^4}{(\omega+i\delta-\varepsilon_n)^2(\omega+i\delta-\varepsilon_m)(\omega+i\delta-\varepsilon_s)(\omega+i\delta-\varepsilon_t)}\\
=& -\frac{\pi^2}{3}\epsilon^{ijk\ell}\int\frac{d^4k d\omega}{(2\pi)^5}\left\{\frac{\mathrm{Tr}\left[P_G\frac{\partial P_G}{\partial k_i}P_E\frac{\partial P_G}{\partial k_j}P_G\frac{\partial P_G}{\partial k_k}P_E\frac{\partial P_G}{\partial k_\ell}\right]}{(\omega+i\delta-\varepsilon_G)^3(\omega+i\delta-\varepsilon_E)^2}+\frac{\mathrm{Tr}\left[P_E\frac{\partial P_G}{\partial k_i}P_G\frac{\partial P_G}{\partial k_j}P_E\frac{\partial P_G}{\partial k_k}P_G\frac{\partial P_G}{\partial k_\ell}\right]}{(\omega+i\delta-\varepsilon_G)^2(\omega+i\delta-\varepsilon_E)^3}\right\}(\varepsilon_E-\varepsilon_G)^4\\
=&\frac{1}{16\pi^2}\int d^4k\left\{-\mathrm{Tr}\left[P_G\frac{\partial P_G}{\partial k_i}P_E\frac{\partial P_G}{\partial k_j}P_G\frac{\partial P_G}{\partial k_k}P_E\frac{\partial P_G}{\partial k_\ell}\right]+\mathrm{Tr}\left[P_E\frac{\partial P_G}{\partial k_i}P_G\frac{\partial P_G}{\partial k_j}P_E\frac{\partial P_G}{\partial k_k}P_G\frac{\partial P_G}{\partial k_\ell}\right]\right\}\\
=&\frac{1}{16\pi^2}\epsilon^{ijk\ell}\int d^4k\left\{-\mathrm{Tr}\left[\frac{\partial P_G}{\partial k_i}P_E\frac{\partial P_G}{\partial k_j}\frac{\partial P_G}{\partial k_k}P_E\frac{\partial P_G}{\partial k_\ell}\right]+\mathrm{Tr}\left[P_E\frac{\partial P_G}{\partial k_i}\frac{\partial P_G}{\partial k_j}P_E\frac{\partial P_G}{\partial k_k}\frac{\partial P_G}{\partial k_\ell}\right]\right\}
\end{aligned}
\tag{163}
$$

Hence

$$
\begin{aligned}
C_2 &= \frac{1}{16\pi^2}\epsilon^{ijk\ell}\int d^4k \left\{-\operatorname{Tr}\left[P_E\frac{\partial P_G}{\partial k_j}\frac{\partial P_G}{\partial k_k}P_E\frac{\partial P_G}{\partial k_\ell}\frac{\partial P_G}{\partial k_i}\right]+\operatorname{Tr}\left[P_E\frac{\partial P_G}{\partial k_i}\frac{\partial P_G}{\partial k_j}P_E\frac{\partial P_G}{\partial k_k}\frac{\partial P_G}{\partial k_\ell}\right]\right\}\\
&= \frac{1}{8\pi^2}\epsilon^{ijk\ell}\int d^4k \operatorname{Tr}\left[P_E\frac{\partial P_G}{\partial k_i}\frac{\partial P_G}{\partial k_j}P_E\frac{\partial P_G}{\partial k_k}\frac{\partial P_G}{\partial k_\ell}\right]
\end{aligned}
\tag{164}
$$

We should notice that

$$
\langle\alpha,\mathbf{k}|\beta,\mathbf{k}\rangle=\delta_{\alpha,\beta}
\tag{165}
$$

Hence

$$
\begin{aligned}
C_2 &= \frac{1}{8\pi^2}\epsilon^{ijk\ell}\int d^4k \operatorname{Tr}\left[\frac{\partial P_G}{\partial k_\ell}P_E\frac{\partial P_G}{\partial k_i}\frac{\partial P_G}{\partial k_j}P_E\frac{\partial P_G}{\partial k_k}\right]\\
&= -\frac{1}{8\pi^2}\epsilon^{ijk\ell}\int d^4k \operatorname{Tr}\left[\frac{\partial P_G}{\partial k_i}P_E\frac{\partial P_G}{\partial k_j}\frac{\partial P_G}{\partial k_k}P_E\frac{\partial P_G}{\partial k_l}\right]\\
&= -\frac{1}{8\pi^2}\epsilon^{ijk\ell}\int d^4k \operatorname{Tr}\left[\sum_{\alpha,\beta,\gamma,\theta=1}^{M}|\alpha,\mathbf{k}\rangle\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_i}P_E\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_j}\langle\beta,\mathbf{k}|\gamma,\mathbf{k}\rangle\frac{\partial\langle\gamma,\mathbf{k}|}{\partial k_k}P_E\frac{\partial|\theta,\mathbf{k}\rangle}{\partial k_l}\langle\theta,\mathbf{k}|\right]\\
&= -\frac{1}{8\pi^2}\epsilon^{ijk\ell}\int d^4k \left[\sum_{\alpha,\beta,\theta=1}^{M}\operatorname{Tr}\left(|\alpha,\mathbf{k}\rangle\langle\theta,\mathbf{k}|\right)\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_i}P_E\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_j}\langle\beta,\mathbf{k}|\beta,\mathbf{k}\rangle\frac{\partial\langle\beta,\mathbf{k}|}{\partial k_k}P_E\frac{\partial|\theta,\mathbf{k}\rangle}{\partial k_l}\right]\\
&= -\frac{1}{8\pi^2}\epsilon^{ijk\ell}\int d^4k \left[\sum_{\alpha,\beta,\theta=1}^{M}\langle\theta,\mathbf{k}|\alpha,\mathbf{k}\rangle\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_i}P_E\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_j}\frac{\partial\langle\beta,\mathbf{k}|}{\partial k_k}P_E\frac{\partial|\theta,\mathbf{k}\rangle}{\partial k_l}\right]\\
&= -\frac{1}{8\pi^2}\epsilon^{ijk\ell}\int d^4k \left[\sum_{\alpha,\beta=1}^{M}\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_i}P_E\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_j}\frac{\partial\langle\beta,\mathbf{k}|}{\partial k_k}P_E\frac{\partial|\alpha,\mathbf{k}\rangle}{\partial k_l}\right]
\end{aligned}
\tag{166}
$$

Step 3. Calculate the second Chern number of the non-Abelian Berry's phase gauge field in the BZ and compare the result to C_2. The gauge field is defined as

$$
a_i^{\alpha\beta}(\mathbf{k})=-i\left\langle\alpha,\mathbf{k}\left|\frac{\partial}{\partial k_i}\right|\beta,\mathbf{k}\right\rangle
\tag{167}
$$

The gauge curvature is

$$
\begin{aligned}
f_{ij}^{\alpha\beta} =& \partial_i a_j^{\alpha\beta} - \partial_j a_i^{\alpha\beta} + i\,[a_i, a_j]^{\alpha\beta} \\
=& -i\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_i}\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_j} - i\left\langle\alpha,\mathbf{k}\left|\frac{1}{\partial k_i\partial k_j}\right|\beta,\mathbf{k}\right\rangle + i\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_j}\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_i} + i\left\langle\alpha,\mathbf{k}\left|\frac{1}{\partial k_j\partial k_i}\right|\beta,\mathbf{k}\right\rangle \\
& - i\langle\alpha,\mathbf{k}|\frac{\partial}{\partial k_i}\sum_{\gamma=1}^{M}|\gamma,\mathbf{k}\rangle\langle\gamma,\mathbf{k}|\frac{\partial}{\partial k_j}|\beta,\mathbf{k}\rangle + i\langle\alpha,\mathbf{k}|\frac{\partial}{\partial k_j}\sum_{\gamma=1}^{M}|\gamma,\mathbf{k}\rangle\langle\gamma,\mathbf{k}|\frac{\partial}{\partial k_i}|\beta,\mathbf{k}\rangle \\
=& -i\left(\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_i}\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_j} - \frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_j}\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_i}\right) \\
& + i\left(\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_i}\sum_{\gamma=1}^{M}|\gamma,\mathbf{k}\rangle\langle\gamma,\mathbf{k}|\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_j} - \frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_j}\sum_{\gamma=1}^{M}|\gamma,\mathbf{k}\rangle\langle\gamma,\mathbf{k}|\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_i}\right) \\
=& -i\left(\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_i}P_E\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_j} - \frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_j}P_E\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_i}\right)
\end{aligned}
\tag{168}
$$

Consider the following term

$$
\begin{aligned}
\frac{1}{32\pi^2}\int d^4k\epsilon^{ijk\ell}\,\mathrm{Tr}\,[f_{ij}f_{k\ell}] =& -\frac{1}{32\pi^2}\int d^4k\epsilon^{ijk\ell}\sum_{\alpha,\beta=1}^{M}\left(\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_i}P_E\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_j} - \frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_j}P_E\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_i}\right) \\
& \times\left(\frac{\partial\langle\beta,\mathbf{k}|}{\partial k_k}P_E\frac{\partial|\alpha,\mathbf{k}\rangle}{\partial k_l} - \frac{\partial\langle\beta,\mathbf{k}|}{\partial k_l}P_E\frac{\partial|\alpha,\mathbf{k}\rangle}{\partial k_k}\right) \\
=& -\frac{1}{32\pi^2}\int d^4k\epsilon^{ijk\ell}\left[\sum_{\alpha,\beta=1}^{M}\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_i}P_E\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_j}\frac{\partial\langle\beta,\mathbf{k}|}{\partial k_k}P_E\frac{\partial|\alpha,\mathbf{k}\rangle}{\partial k_l}\right] \\
& + \frac{1}{32\pi^2}\int d^4k\epsilon^{ijk\ell}\left[\sum_{\alpha,\beta=1}^{M}\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_j}P_E\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_i}\frac{\partial\langle\beta,\mathbf{k}|}{\partial k_k}P_E\frac{\partial|\alpha,\mathbf{k}\rangle}{\partial k_l}\right] \\
& + \frac{1}{32\pi^2}\int d^4k\epsilon^{ijk\ell}\left[\sum_{\alpha,\beta=1}^{M}\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_i}P_E\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_j}\frac{\partial\langle\beta,\mathbf{k}|}{\partial k_l}P_E\frac{\partial|\alpha,\mathbf{k}\rangle}{\partial k_k}\right] \\
& - \frac{1}{32\pi^2}\int d^4k\epsilon^{ijk\ell}\left[\sum_{\alpha,\beta=1}^{M}\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_j}P_E\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_i}\frac{\partial\langle\beta,\mathbf{k}|}{\partial k_l}P_E\frac{\partial|\alpha,\mathbf{k}\rangle}{\partial k_k}\right] \\
=& \frac{1}{8\pi^2}\int d^4k\epsilon^{ijk\ell}\left[\sum_{\alpha,\beta=1}^{M}\frac{\partial\langle\alpha,\mathbf{k}|}{\partial k_i}P_E\frac{\partial|\beta,\mathbf{k}\rangle}{\partial k_j}\frac{\partial\langle\beta,\mathbf{k}|}{\partial k_k}P_E\frac{\partial|\alpha,\mathbf{k}\rangle}{\partial k_l}\right] \\
=& C_2
\end{aligned}
\tag{169}
$$

Here we have proved the claim. Next we will compute the nonlinear response to the $U(1)$ gauge field.

$$\begin{aligned}\delta S_{eff} =&\frac{C_2}{24\pi^2}\int d^4xdt\epsilon^{\eta\nu\rho\sigma\tau}\left(\delta A_\eta\partial_\nu A_\rho\partial_\sigma A_\tau + A_\eta\partial_\nu(\delta A_\rho)\partial_\sigma A_\tau + A_\eta\partial_\nu A_\rho\partial_\sigma(\delta A_\tau)\right)\\ =&\frac{C_2}{24\pi^2}\int d^4xdt\epsilon^{\eta\nu\rho\sigma\tau}\delta A_\eta\partial_\nu A_\rho\partial_\sigma A_\tau - \frac{C_2}{24\pi^2}\int d^4xdt\epsilon^{\eta\nu\rho\sigma\tau}\delta A_\rho\partial_\nu A_\eta\partial_\sigma A_\tau\\ &-\frac{C_2}{24\pi^2}\int d^4xdt\epsilon^{\eta\nu\rho\sigma\tau}\delta A_\tau\partial_\nu A_\rho\partial_\sigma A_\eta\\ =&\frac{C_2}{8\pi^2}\int d^4xdt\epsilon^{\eta\nu\rho\sigma\tau}\delta A_\eta\partial_\nu A_\rho\partial_\sigma A_\tau\end{aligned}\tag{170}$$

Hence

$$j^\mu = \frac{\delta S_{eff}}{\delta A_\mu} = \frac{C_2}{8\pi^2}\epsilon^{\mu\nu\rho\sigma\tau}\partial_\nu A_\rho\partial_\sigma A_\tau \tag{171}$$

For instance consider the following configuration

$$A_x = 0, \quad A_y = B_z x, \quad A_z = -E_z t, \quad A_w = A_t = 0 \tag{172}$$

Then if we want the current to be nonzero, (ρ,τ) has to be (y,z) (or (z,y)); accordingly (ν,σ) has to be (x,t) (or (t,x)). Hence the only nonzero current term is

$$j^w = \frac{C_2}{8\pi^2}\epsilon^{wxytz}(-B_zE_z) + \frac{C_2}{8\pi^2}\epsilon^{wtzxy}(-B_zE_z) = -\frac{C_2}{4\pi^2}B_zE_z \tag{173}$$

If we adopt the Lorentzian metric, then

$$j_w = \frac{C_2}{4\pi^2}B_zE_z \tag{174}$$

Hence

$$\int dxdy j_w = \frac{C_2}{4\pi^2}\left(\int dxdyB_z\right)E_z \equiv \frac{C_2N_{xy}}{2\pi}E_z \tag{175}$$

where $N_{xy} = \int dxdyB_z/2\pi$ is the quantized magnetic field flux through the xy plane. Hence in the zw plane the Hall conductance is given by $\frac{C_2N_{xy}}{2\pi}$.

2.2 An Example: Lattice Dirac Model

Consider a tight binding version of the continuum Dirac model.

$$H = \sum_{n,i}\left[\psi_n^\dagger\left(\frac{c\Gamma^0 - i\Gamma^i}{2}\right)\psi_{n+\hat{i}} + \text{H.c.}\right] + m\sum_n\psi_n^\dagger\Gamma^0\psi_n \tag{176}$$

where $\{\Gamma^\mu,\Gamma^\nu\} = 2\delta_{\mu\nu}\mathbb{I}$. Here we let

$$\Gamma^0 = \gamma^0 \quad \Gamma^{1,2,3} = \gamma^0\gamma^{1,2,3} \quad \Gamma^4 = -i\gamma^0\gamma^5 \tag{177}$$

so that $(\Gamma^\mu)^\dagger = \Gamma^\mu$. Next we will obtain the frequency representation of the Hamiltonian by Fourier transforming $\psi^\dagger$ and ψ.

$$\begin{aligned}
H &= \frac{1}{N}\sum_{n,i}\left[\sum_{k,q} e^{i\mathbf{k}\cdot\mathbf{r_n}}\psi_k^\dagger\left(\frac{c\Gamma^0 - i\Gamma^i}{2}\right)e^{-i\mathbf{q}\cdot(\mathbf{r}_n+\mathbf{r}_i)}\psi_q + \text{H.c.}\right] + \frac{1}{N}m\sum_n\sum_{k,q} e^{i\mathbf{k}\cdot\mathbf{r_n}}e^{-i\mathbf{q}\cdot\mathbf{r_n}}\psi_k^\dagger\Gamma^0\psi_q \\
&= \sum_i\sum_k\left[e^{-i\mathbf{k}\cdot\mathbf{r_i}}\psi_k^\dagger\left(\frac{c\Gamma^0 - i\Gamma^i}{2}\right)\psi_k + \text{H.c.}\right] + m\sum_k\psi_k^\dagger\Gamma^0\psi_k \\
&= \sum_k\psi_k^\dagger\left[\sum_i \sin k_i\Gamma^i + \left(m + c\sum_i\cos k_i\right)\Gamma^0\right]\psi_k
\end{aligned} \tag{178}$$

where we have used the fact that $|\mathbf{r_i}| = 1$. Also we can make the Hamiltonian more compact by setting

$$d_a(\mathbf{k}) = \left[\left(m + c\sum_i\cos k_i\right), \sin k_x, \sin k_y, \sin k_z, \sin k_w\right] \tag{179}$$

Then

$$H = \sum_k \psi_k^\dagger d_a(\mathbf{k})\Gamma^a\psi_k \tag{180}$$

where

$$h(\mathbf{k}) = d_a(\mathbf{k})\Gamma^a \tag{181}$$

is the single particle Hamiltonian. Before moving onto other calculations, we shall now pause for a while to take a closer look at some properties Γ^μ posesses. Define

$$\Gamma^{ab} = \frac{[\Gamma^a, \Gamma^b]}{2i} \tag{182}$$

We can immediately obtain the following properties

$$\begin{aligned}
\left[\Gamma^{ab}, \Gamma^c\right] &= 2i\left(\delta_{ac}\Gamma^b - \delta_{bc}\Gamma^a\right) \\
\left\{\Gamma^{ab}, \Gamma^c\right\} &= \epsilon_{abcde}\Gamma^{de} \\
\left[\Gamma^{ab}, \Gamma^{cd}\right] &= -2i\left(\delta_{bc}\Gamma^{ad} - \delta_{bd}\Gamma^{ac} - \delta_{ac}\Gamma^{bd} + \delta_{ad}\Gamma^{bc}\right) \\
\left\{\Gamma^{ab}, \Gamma^{cd}\right\} &= 2\epsilon_{abcde}\Gamma^e + 2\delta_{ac}\delta_{bd} - 2\delta_{ad}\delta_{bc}
\end{aligned} \tag{183}$$

With the above relations we can simplify a product of Γ^μ matrices. Consider the trace of Γ^μ:

$$\text{Tr}(\Gamma^0) = \text{Tr}(\gamma^0) = 0 \tag{184}$$

Since

$$\text{Tr}(\gamma^0\gamma^i) = 0 \quad (i = 1, 2, 3, 5) \tag{185}$$

we can see that

$$\mathrm{Tr}(\Gamma^\mu) = 0 \tag{186}$$

Also

$$\begin{aligned} \mathrm{Tr}(\Gamma^a\Gamma^b) &= \frac{1}{2}\,\mathrm{Tr}(\Gamma^a\Gamma^b) = \frac{1}{2}\,\mathrm{Tr}(2\delta_{ab}) = 4\delta_{ab} \\ \mathrm{Tr}(\Gamma^{ab}) &= \frac{1}{2i}\left[\mathrm{Tr}(\Gamma^a\Gamma^b) - \mathrm{Tr}(\Gamma^b\Gamma^a)\right] = 0 \end{aligned} \tag{187}$$

Next we will try to compute the traces of several products of Γ^μ matrices.

$$\begin{aligned} \mathrm{Tr}\left(\Gamma^a\Gamma^b\Gamma^c\Gamma^d\Gamma^e\right) &= \mathrm{Tr}\left((i\Gamma^{ab} + \delta_{ab})\Gamma^c(i\Gamma^{de} + \delta_{de})\right) \\ &= \mathrm{Tr}(\delta_{ab}\delta_{de}\Gamma^c) + \mathrm{Tr}(i\Gamma^{ab}\Gamma^c\delta_{de}) + \mathrm{Tr}(i\delta_{ab}\Gamma^c\Gamma^{de}) + \mathrm{Tr}(-\Gamma^{ab}\Gamma^{de}\Gamma^c) \end{aligned} \tag{188}$$

Notice that $\left[\Gamma^{ab}, \Gamma^c\right]$ and $\left\{\Gamma^{ab}, \Gamma^c\right\}$ only have linear terms of the matrices, hence

$$\mathrm{Tr}(\Gamma^{ab}\Gamma^c) = 0 \tag{189}$$

For the same reason

$$\mathrm{Tr}(\Gamma^{ab}\Gamma^{de}) = 0 \tag{190}$$

Hence

$$\mathrm{Tr}\left(\Gamma^a\Gamma^b\Gamma^c\Gamma^d\Gamma^e\right) = \mathrm{Tr}(-\Gamma^{ab}\Gamma^{de}\Gamma^c) = -\epsilon_{abdef}\,\mathrm{Tr}(\Gamma^f\Gamma^c) = -4\epsilon_{abdef}\delta_{fc} = -4\epsilon_{abcde} \tag{191}$$

For similar reasons

$$\begin{aligned} \mathrm{Tr}(\Gamma^a\Gamma^b\Gamma^c\Gamma^d\Gamma^e\Gamma^f) &= \mathrm{Tr}\left((i\Gamma^{ab} + \delta_{ab})(i\Gamma^{cd} + \delta_{cd})(i\Gamma^{ef} + \delta_{ef})\right) \\ &= \mathrm{Tr}(-i\Gamma^{ab}\Gamma^{cd}\Gamma^{ef}) = 0 \end{aligned} \tag{192}$$

$$\begin{aligned} \mathrm{Tr}(\Gamma^a\Gamma^b\Gamma^c\Gamma^d\Gamma^e\Gamma^f\Gamma^g) &= \mathrm{Tr}\left((i\Gamma^{ab} + \delta_{ab})(i\Gamma^{cd} + \delta_{cd})(i\Gamma^{ef} + \delta_{ef})\Gamma^g\right) \\ &= \mathrm{Tr}\left(-i\Gamma^{ab}\Gamma^{cd}\Gamma^{ef}\Gamma^g\right) - \mathrm{Tr}\left(\Gamma^{cd}\Gamma^{ef}\Gamma^g\delta_{ab}\right) - \mathrm{Tr}\left(\Gamma^{ab}\Gamma^{cd}\Gamma^g\delta_{ef}\right) - \mathrm{Tr}\left(\Gamma^{ab}\Gamma^{ef}\Gamma^g\delta_{cd}\right) \\ &= -4\delta_{ab}\epsilon_{cdefg} - 4\delta_{cd}\epsilon_{abefg} - 4\delta_{ef}\epsilon_{abcdg} - 4\delta_{bc}\epsilon_{adefg} - 4\delta_{ad}\epsilon_{bcefg} \\ &\quad + 4\delta_{ac}\epsilon_{bdefg} + 4\delta_{bd}\epsilon_{acefg} \end{aligned} \tag{193}$$

$$\begin{aligned} \mathrm{Tr}(\Gamma^a\Gamma^b\Gamma^c\Gamma^d\Gamma^e\Gamma^f\Gamma^g\Gamma^h) &= \mathrm{Tr}\left((i\Gamma^{ab} + \delta_{ab})(i\Gamma^{cd} + \delta_{cd})(i\Gamma^{ef} + \delta_{ef})(i\Gamma^{gh} + \delta_{gh})\right) \\ &= \mathrm{Tr}\left(\Gamma^{ab}\Gamma^{ed}\Gamma^{ef}\Gamma^{gh}\right) = 0 \end{aligned} \tag{194}$$

Now that we have all the things we need, we will calculate the C_2 numeber in this system.

Claim In this system

$$C_2 = \frac{3}{8\pi^2}\int d^4k\,\epsilon^{abcde}\hat{d}_a\partial_x\hat{d}_b\partial_y\hat{d}_c\partial_z\hat{d}_d\partial_w\hat{d}_e \tag{195}$$

where

$$\hat{d}_a(\mathbf{k}) \equiv \frac{d_a(\mathbf{k})}{|d(\mathbf{k})|} \tag{196}$$

proof Step 1. Obtain the Green's function.

Notice that

$$(\omega - h(\mathbf{k})) \times \frac{\omega + d_a(k)\Gamma^a}{[\omega^2 - d^a(k)d_a(k)]} = \frac{\omega^2 - d_a(k)d_b(k)\Gamma^a\Gamma^b}{[\omega^2 - d^a(k)d_a(k)]} = \frac{\omega^2 - \frac{1}{2}d_a(k)d_b(k)2\delta_{ab}}{[\omega^2 - d^a(k)d_a(k)]} = 1 \tag{197}$$

Hence

$$G(k,\omega) = \frac{\omega + d^a(k)\Gamma^a}{[\omega^2 - d^a(k)d_a(k)]} \tag{198}$$

Step 2. Insert the green function directly into C_2

$$\begin{aligned}
C_2 =& -\frac{\pi^2}{15}\epsilon^{0ijkl}\int \frac{d^4kd\omega}{(2\pi)^5}\operatorname{Tr}\left(G\left(G\frac{\partial G^{-1}}{\partial k^i}\right)\left(G\frac{\partial G^{-1}}{\partial k^j}\right)\left(G\frac{\partial G^{-1}}{\partial k^k}\right)\left(G\frac{\partial G^{-1}}{\partial k^l}\right)\right)\\
&-\frac{\pi^2}{15}\epsilon^{i0jkl}\int \frac{d^4kd\omega}{(2\pi)^5}\operatorname{Tr}\left(\left(G\frac{\partial G^{-1}}{\partial k^i}\right)G\left(G\frac{\partial G^{-1}}{\partial k^j}\right)\left(G\frac{\partial G^{-1}}{\partial k^k}\right)\left(G\frac{\partial G^{-1}}{\partial k^l}\right)\right)\\
&-\frac{\pi^2}{15}\epsilon^{ij0kl}\int \frac{d^4kd\omega}{(2\pi)^5}\operatorname{Tr}\left(\left(G\frac{\partial G^{-1}}{\partial k^i}\right)\left(G\frac{\partial G^{-1}}{\partial k^j}\right)G\left(G\frac{\partial G^{-1}}{\partial k^k}\right)\left(G\frac{\partial G^{-1}}{\partial k^l}\right)\right)\\
&-\frac{\pi^2}{15}\epsilon^{ijk0l}\int \frac{d^4kd\omega}{(2\pi)^5}\operatorname{Tr}\left(\left(G\frac{\partial G^{-1}}{\partial k^i}\right)\left(G\frac{\partial G^{-1}}{\partial k^j}\right)\left(G\frac{\partial G^{-1}}{\partial k^k}\right)G\left(G\frac{\partial G^{-1}}{\partial k^l}\right)\right)\\
&-\frac{\pi^2}{15}\epsilon^{ijkl0}\int \frac{d^4kd\omega}{(2\pi)^5}\operatorname{Tr}\left(\left(G\frac{\partial G^{-1}}{\partial k^i}\right)\left(G\frac{\partial G^{-1}}{\partial k^j}\right)\left(G\frac{\partial G^{-1}}{\partial k^k}\right)\left(G\frac{\partial G^{-1}}{\partial k^l}\right)G\right)\\
=& -\frac{\pi^2}{3}\epsilon^{0ijkl}\int \frac{d^4kd\omega}{(2\pi)^5}\operatorname{Tr}\left(G\left(G\frac{\partial G^{-1}}{\partial k^i}\right)\left(G\frac{\partial G^{-1}}{\partial k^j}\right)\left(G\frac{\partial G^{-1}}{\partial k^k}\right)\left(G\frac{\partial G^{-1}}{\partial k^l}\right)\right)\\
=& -\frac{\pi^2}{3}\epsilon^{ijkl}\int \frac{d^4kd\omega}{(2\pi)^5}\frac{1}{\left(\omega^2-|d|^2\right)^5}\operatorname{Tr}\left[\left(\omega+d_m\Gamma^m\right)\left(\omega+d_n\Gamma^n\right)\left(\partial_i d^o\Gamma^o\right)\left(\omega+d_p\Gamma^p\right)\left(\partial_j d^q\Gamma^q\right)\right.\\
&\left.\times\left(\omega+d_r\Gamma^r\right)\left(\partial_k d^s\Gamma^s\right)\left(\omega+d_t\Gamma^t\right)\left(\partial_l d^u\Gamma^u\right)\right]\\
=& -\frac{\pi^2}{3}\epsilon_{ijkl}\int \frac{d^4kd\omega}{(2\pi)^5}\frac{\partial_i d^o\partial_j d^q\partial_k d^s\partial_l d^u}{\left(\omega^2-|d|^2\right)^5}\operatorname{Tr}\left[\left(\omega+d_m\Gamma^m\right)\left(\omega+d_n\Gamma^n\right)\Gamma^o\right.\\
&\left.\times\left(\omega+d_p\Gamma^p\right)\Gamma^q\left(\omega+d_r\Gamma^r\right)\Gamma^s\left(\omega+d_t\Gamma^t\right)\Gamma^u\right]
\end{aligned} \tag{199}$$

Now we need to valuate the trace term. From what we have proved, the only nonzero contributions are form products of 5, 7 and 9 matrices. Hence the contribution from the products of 5 matrices:

$$\begin{aligned}
T_5 =&\omega^4\left[\operatorname{Tr}(d_m\Gamma^m\Gamma^o\Gamma^q\Gamma^s\Gamma^u)+\operatorname{Tr}(d_m\Gamma^m\Gamma^o\Gamma^q\Gamma^s\Gamma^u)+\operatorname{Tr}(d_m\Gamma^o\Gamma^m\Gamma^q\Gamma^s\Gamma^u)\right.\\
&\left.+\operatorname{Tr}(d_m\Gamma^o\Gamma^q\Gamma^m\Gamma^s\Gamma^u)+\operatorname{Tr}(d_m\Gamma^o\Gamma^q\Gamma^s\Gamma^m\Gamma^u)\right]\\
=&-4\epsilon_{moqsu}d^m\omega^4
\end{aligned} \tag{200}$$

Likewise

$$\begin{aligned}
T_7 &= 8\epsilon_{moqsu}d^m\omega^2|d|^2\\
T_9 &= -4\epsilon_{moqsu}d^m|d|^4
\end{aligned} \tag{201}$$

Honestly, I have not evaluate T_7 and T_9. I chosed to trust the author. Hence

$$
\begin{aligned}
C_2 &= \frac{\pi^2}{3}\epsilon_{ijkl}\int \frac{d^4k d\omega}{(2\pi)^5} 4\epsilon_{toqsu}\frac{d^t\partial_i d^o\partial_j d^q\partial_k d^s\partial_l d^u}{(\omega^2-|d|^2)^3} \\
&= \frac{\pi^3}{3}\epsilon_{ijkl}\int \frac{d^4k}{(2\pi)^4} 4\epsilon_{toqsu}\frac{d^t\partial_i d^o\partial_j d^q\partial_k d^s\partial_l d^u}{(2|d|)^5}\frac{3\times 4}{2!} \\
&= \frac{\pi^2}{4}\epsilon_{ijkl}\int \frac{d^4k}{(2\pi)^4} \epsilon_{toqsu}\frac{d^t\partial_i d^o\partial_j d^q\partial_k d^s\partial_l d^u}{|d|^5} \\
&= \frac{3}{8\pi^2}\int d^4k\epsilon_{\mathrm{toqsu}}\hat{d}^t\partial_x\hat{d}^o\partial_y\hat{d}^q\partial_z\hat{d}^s\partial_w\hat{d}^u
\end{aligned}
\tag{202}
$$

where in the last line, we noticed that the permutation of $\{i,j,k,l\}$ results in the same sign change of ϵ_{ijkl} and ϵ_{toqsu}. Next, we will evaluate the C_2 number by using the above equation. Since C_2 is a topologically invariant number and hence does not change when the Hamiltonian remains to be gapped, we only have to examine the change of C_2 near criticalities where the Hamiltonian becomes gapless. Let c remain fixed and m tunable. Then $C_2 = C_2(m)$. Hence we should calculate the eigenvalue of the single particle Hamiltonian first:

$$
\begin{pmatrix}
S-\sin(\mathrm{kz}) & i\sin(\mathrm{ky})-\sin(\mathrm{kx}) & m+c\sum cos(k)-i\sin(\mathrm{kw}) & 0 \\
-\sin(\mathrm{kx})-i\sin(\mathrm{ky}) & S+\sin(\mathrm{kz}) & 0 & m+c\sum cos(k)-i\sin(\mathrm{kw}) \\
m+c\sum cos(k)+i\sin(\mathrm{kw}) & 0 & S+\sin(\mathrm{kz}) & \sin(\mathrm{kx})-i\sin(\mathrm{ky}) \\
0 & m+c\sum cos(k)+i\sin(\mathrm{kw}) & \sin(\mathrm{kx})+i\sin(\mathrm{ky}) & S-\sin(\mathrm{kz})
\end{pmatrix}
\tag{203}
$$

Using mathematica, we find that the eigenvalues are $E_{\pm}(\mathbf{k}) = \pm\sqrt{\sum_a d_a^2(\mathbf{k})}$. Now, we should find out the criticalities by solving

$$
\sum_a d_a^2(\mathbf{k}, m) = 0 \tag{204}
$$

Hence, all $sin(k_i)$ terms are zero while $m + c\sum cos(k_i) = 0$

Case 1. All $k_i = 0$. Then $m = -4c$.

Case 2. Only one $k_i = \pi$, others are zero. Then $m = -2c$.

Case 3. Two $k_i = \pi$, others are zero. Then $m = 0$.

Case 4. Three $k_i = \pi$, the other is zero. Then $m = 2c$.

Case 5, All $k_i = \pi$. Then $m = 4c$.

If $m \to +\infty$ (or $m \to -\infty$), $\hat{d} \to (1,0,0,0,0)$ (or $\hat{d} \to (-1,0,0,0,0)$). At this time, $C_2(m) = 0$.

Consider for example the jump of C_2 at $m = -4c$. Define $\delta m = m + 4c$ and a cutoff frequency $\Lambda \ll 2\pi$. Then

$$\begin{aligned} C_2 &= \frac{3}{8\pi^2}\left(\int_{|\mathbf{k}|\leqslant\Lambda} d^4k + \int_{|\mathbf{k}|>\Lambda} d^4k\right)\epsilon^{abcde}\hat{d}_a\partial_x\hat{d}_b\partial_y\hat{d}_c\partial_z\hat{d}_d\partial_w\hat{d}_e \\ &\equiv C_2^{(1)}(\delta m,\Lambda) + C_2^{(2)}(\delta m,\Lambda) \end{aligned} \tag{205}$$

Since at $|\mathbf{k} > \Lambda|$ there is no level crossing. Hence the jump of the second term above is zero. Consider the first term instead. At that limit $d_a(\mathbf{k}) \to (\delta m, k_x, k_y, k_z, k_w)$. Hence

$$\begin{aligned} C_2^{(1)}(\delta m,\Lambda) &\simeq \frac{3}{8\pi^2}\int_{|\mathbf{k}|\leqslant\Lambda} d^4k\frac{\delta m}{(\delta m^2+\mathbf{k}^2)^{5/2}} = \frac{3}{8\pi^2}\int_{|\mathbf{k}|\leqslant\Lambda} dk\frac{\delta m}{(\delta m^2+k^2)^{5/2}}\times(2\pi^2k^3) \\ &= \frac{3}{8}\int_{|\mathbf{k}|\leqslant\Lambda} kdk\frac{\delta m}{(\delta m^2+k)^{5/2}} = -\frac{3}{8}\frac{2\delta\mathrm{m}\left(2\delta\mathrm{m}^2+3\sqrt{\Lambda}\right)}{3\left(\delta\mathrm{m}^2+\sqrt{\Lambda}\right)^{3/2}} + \frac{1}{2}\frac{\delta m^3}{(\delta m^2)^{3/2}} \\ &= -\frac{3}{8}\frac{2\delta\mathrm{m}\left(2\delta\mathrm{m}^2+3\sqrt{\Lambda}\right)}{3\left(\delta\mathrm{m}^2+\sqrt{\Lambda}\right)^{3/2}} + \frac{1}{2}\,\mathrm{sgn}(\delta m) \end{aligned} \tag{206}$$

where we have used the fact that the area of a 4-sphere of radius R is $2\pi^2R^3$. Then

$$\delta C_2^{(1)}(\delta m,\Lambda)^{\delta m=0^+}_{\delta m=0^-} = 1 \tag{207}$$

Hence when $-4c \leq m \leq -2c$, $C_2 = 1$. At other criticalities, the situation is basically the same. The only note worthy thing is that at $k_i = \pi$, $\partial_i\hat{d}_i = -1$.

At $m = -2c$, there are 4 critical points $P[\pi,0,0,0]$. Hence the jump is $\delta C_2(m : -2c-0 \to -2c+0) = -4$.

At $m = 0$, there are 6 critical points $P[\pi,\pi,0,0]$. Hence the jump is $\delta C_2(m : -0 \to +0) = +6$.

At $m = 2c$, there are 4 critical points $P[\pi,\pi,\pi,0]$. Hence the jump is $\delta C_2(m : 2c-0 \to 2c+0) = -4$.

At $m = 0$, there is 1 critical point $P[\pi,\pi,\pi,\pi]$. Hence the jump is $\delta C_2(m : 4c-0 \to 4c+0) = +1$.

In conclusion

$$C_2(m) = \begin{cases} 0, & m < -4\,\mathrm{c\ or\ m} > 4c \\ 1, & -4c < m < -2c \\ -3, & -2c < m < 0 \\ 3, & 0 < m < 2c \\ -1, & 2c < m < 4c \end{cases} \tag{208}$$

According to the afor-mentioned frequncy representation of the Hamiltonian, if we replace the periodic boundary condition along w-axis with open boundary condition, then we can obtain the following Hamiltonian:

$$H = \sum_{\vec{k},w} \left[\psi^{\dagger}_{\vec{k}}(w) \left(\frac{c\Gamma^0 - i\Gamma^4}{2} \right) \psi_{\vec{k}}(w+1) + \text{H.c.} \right] + \sum_{\vec{k},w} \psi^{\dagger}_{\vec{k}}(w) \left[\sin k_i \Gamma^i + \left(m + c \sum_i \cos k_i \right) \Gamma^0 \right] \psi_k(w) \tag{209}$$

where $i = 1, 2, 3$. Still, we will focus on the single particle Hamiltonian.

$$h_{\vec{k}} = \left[\sin k_i \Gamma^i + \left(m + c \sum_i \cos k_i \right) \Gamma^0 \right] \tag{210}$$

We will obatain its eigenvalue by setting the determinent of the following matrix to zero.

$$\begin{pmatrix} S - \sin(\text{kz}) & i\sin(\text{ky}) - \sin(\text{kx}) & m + c\sum_i cos(k_i) & 0 \\ -\sin(\text{kx}) - i\sin(\text{ky}) & S + \sin(\text{kz}) & 0 & m + c\sum_i cos(k_i) \\ m + c\sum_i cos(k_i) & 0 & S + \sin(\text{kz}) & \sin(\text{kx}) - i\sin(\text{ky}) \\ 0 & m + c\sum_i cos(k_i) & \sin(\text{kx}) + i\sin(\text{ky}) & S - \sin(\text{kz}) \end{pmatrix} \tag{211}$$

Using mathematica we find that the eigenvalue is

$$E_{\pm}(\vec{k}) = \pm \sqrt{\sum_i d_i^2(\vec{k})} \tag{212}$$

where

$$d_i(\vec{k}) = (m + c \sum_i \cos k_i, \sin k_x, \sin k_y, \sin k_z) \tag{213}$$

Here consider the gapless states:

$$(m + c \sum_i \cos k_i)^2 = \sin^2 k_x = \sin^2 k_y = \sin^2 k_z = 0 \tag{214}$$

Case 1. All $k_i = 0$. Then $m = -3c$. Then $C_2 = 1$. There is only one gapless branch at $\vec{k} = (0, 0, 0)$.

Case 2. Only one $k_i = \pi$, others are zero. Then $m = -1c$. Then $C_2 = -3$. There are three gapless branches at $\vec{k} = P[(\pi, 0, 0)]$.

Case 3. Two $k_i = \pi$, the other is zero. Then $m = +1c$. Then $C_2 = 3$. There are three gapless branches at $\vec{k} = P[(\pi, \pi, 0)]$.

Case 4. Three $k_i = \pi$, the other is zero. Then $m = 2c$. Then $C_2 = -1$. There is only one gapless branch at $\vec{k} = (\pi, \pi, \pi)$. We can then focus on the vicinity of the gapless states and obtain a low energy effective theory. Since $m + c\sum_i \cos k_i = 0$ at the gapless points, the single particle Hamiltonian becomes

$$h_{\vec{k}} = \sin k_i \Gamma^i = \begin{pmatrix} -\vec{d}\cdot\vec{\sigma} & 0 \\ 0 & \vec{d}\cdot\vec{\sigma} \end{pmatrix} \tag{215}$$

Also near the gapless points $\sin k_i \to \delta k_i$ where $\delta k_i = k_i$ when $k_i = 0$ at the gapless point and $\delta k_i = -(k_i - \pi)$ when $k_i = \pi$ at the gapless point. Hence the Hamiltonian becomes

$$h_{\vec{k}} = \begin{pmatrix} -\delta\vec{k}\cdot\vec{\sigma} & 0 \\ 0 & \delta\vec{k}\cdot\vec{\sigma} \end{pmatrix} \tag{216}$$

Due to the symmetry the Hamiltonian posesses we can describe the gapless states by only $h_{\vec{k}} = \delta\vec{k}\cdot\vec{\sigma} = \mathrm{sgn}(C_i)\vec{p}\cdot\vec{\sigma}$ where $\mathrm{sgn}(C_i) = 1$ when $k_i = 0$ at the gapless point, otherwise $\mathrm{sgn}(C_i) = -1$. Hence the low energy effective theory is

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{i=1}^{|C_2|} \mathrm{sgn}\,(C_i)\, \psi_i^\dagger(\vec{p})\vec{\sigma}\cdot\overrightarrow{\mathbf{p}}\,\psi_i(\vec{p}) \tag{217}$$

Consider for example a system at $-4c \le m \le -2c$. Consider the following gauge configuration.

$$A_x = 0, \quad A_y = B_z x, \quad A_z = -E_z t, \quad A_w = A_t = 0 \tag{218}$$

Then as we have shown the single particle Hamiltonian near the gapless point is

$$h = v\vec{\sigma}\cdot(\vec{p} + \vec{A}) = v\sigma_x p_x + v\sigma_y\,(p_y + B_z x) + v\sigma_z\,(p_z - E_z t) \tag{219}$$

For now we let A_z remian fixed. We will try to derive the eigenvalues of this system.

$$\begin{pmatrix} -i\partial_z + A_z & -i\partial_x - \partial y - iB_z x \\ -i\partial_x + \partial y + iB_z x & i\partial_z - A_z \end{pmatrix} \Psi(x,y,z) = E\Psi(x,y,z) \tag{220}$$

According to the form of the Hamiltonian we can set $\Psi(x,y,z) = e^{ip_y y}e^{ip_z z}\Psi(x)$. Hence

$$\begin{pmatrix} p_z + A_z & -i\partial_x - ip_y - iB_z x \\ -i\partial_x + ip_y + iB_z x & -p_z - A_z \end{pmatrix} \Psi(x) = E\Psi(x) \tag{221}$$

Since

$$[-i\partial_x + ip_y + iB_z x, -i\partial_x - ip_y - iB_z x] = -2B_z \tag{222}$$

If $B_z < 0$ we can let $a = \frac{1}{\sqrt{2|B_z|}}(-i\partial_x + ip_y + iB_z x)$ and $a^\dagger = \frac{1}{\sqrt{2|B_z|}}(-i\partial_x - ip_y - iB_z x)$. Then $[a, a^\dagger] = 1$. We can also define the number operator $N = a^\dagger a$. Let $\Psi(x) = (\gamma, \theta)^T$. Then

$$\sqrt{2|B_z|}\begin{pmatrix} \frac{p_z + A_z}{\sqrt{2|B_z|}} & a^\dagger \\ a & \frac{-p_z - A_z}{\sqrt{2|B_z|}} \end{pmatrix} \begin{pmatrix} \gamma \\ \theta \end{pmatrix} = E \begin{pmatrix} \gamma \\ \theta \end{pmatrix} \tag{223}$$

Hence

$$\begin{pmatrix} \frac{p_z+A_z}{\sqrt{2|B_z|}}\gamma + a^\dagger\theta \\ a\gamma + \frac{-p_z-A_z}{\sqrt{2|B_z|}}\theta \end{pmatrix} = \begin{pmatrix} \frac{E}{\sqrt{2|B_z|}}\gamma \\ \frac{E}{\sqrt{2|B_z|}}\theta \end{pmatrix}$$
$$\begin{pmatrix} a^\dagger\theta \\ a\gamma \end{pmatrix} = \begin{pmatrix} \frac{E-p_z-A_z}{\sqrt{2|B_z|}}\gamma \\ \frac{E+p_z+A_z}{\sqrt{2|B_z|}}\theta \end{pmatrix} \tag{224}$$

Hence

$$a^\dagger a\gamma = \frac{E^2-(p_z+A_z)^2}{2|B_z|}\gamma \tag{225}$$

Hence $\gamma = |n\rangle$. We can then see that

$$E_{n\pm} = \pm\sqrt{(p_z+A_z)^2+2n|B_z|} \quad n = 1, 2, ... \tag{226}$$

If n=0 then

$$E_0 = -(p_z+A_z) = \text{sgn}(B_z)(p_z+A_z) \tag{227}$$

If $B > 0$ we can repeat the same process to obtain the same result as above. All $E_{n\pm}$ are gapped while $E_0(p_z)$ is not. Hence if we adiabatically set $p_z \to p_z + \frac{2\pi}{L_z}$ by using the afore mentioned field configuration, then one more band can be occupied by electrons. Since for each Landau level the degeneracy is $N_{xy} = \frac{L_x L_y B_z}{2\pi}$. The number of the surface increases by N_{xy}. Hence there must be a current along w axis

$$I_w = \frac{N_{xy}}{T} = \frac{N_{xy}}{(2\pi/L_z)/E_z} = \frac{L_x L_y L_z B_z E_z}{4\pi^2} \tag{228}$$

Hence

$$j_w = \frac{B_z E_z}{4\pi^2} = \frac{1}{32\pi^2}\epsilon^{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau} \tag{229}$$

where in this system the only non-vanishing terms are $F_{xy} = b_z$ and $F_{zt} = -E_z$. More generally, since there are C_2 branches of such gapless systems

$$j_w = C_2\frac{\mathbf{E}\cdot\mathbf{B}}{4\pi^2} = \frac{C_2}{32\pi^2}\epsilon^{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau} \tag{230}$$

3 Dimensional Reduction to (3+1)-Dimensional TRI Insulators

3.1 Effective Action of (3+1)-Dimensional Insulators

The Hamiltonian in a (4+1)-dimension is

$$H[A] = \sum_{n,i}\left[\psi_n^\dagger\left(\frac{c\Gamma^0 - i\Gamma^i}{2}\right)e^{iA_{n,n+\hat{i}}}\psi_{n+\hat{i}} + \text{H.c.}\right] + m\sum_n \psi_n^\dagger\Gamma^0\psi_n \tag{231}$$

Let the gauge field be translationally invariant along the w direction, i.e.

$$A_{n,n+\hat{i}} = A_{n+\hat{w},n+\hat{w}+\hat{i}} \tag{232}$$

Hence under the periodic boundary condition, k_w is a good quantum number.

$$\begin{aligned}
H[A] =& \sum_{n,s} \left[\psi_n^\dagger \left(\frac{c\Gamma^0 - i\Gamma^s}{2} \right) e^{iA_{n,n+\hat{s}}} \psi_{n+\hat{s}} + \text{H.c.} \right] + m \sum_n \psi_n^\dagger \Gamma^0 \psi_n \\
&+ \sum_n \left[\psi_n^\dagger \left(\frac{c\Gamma^0 - i\Gamma^4}{2} \right) e^{iA_{n,n+\hat{w}}} \psi_{n+\hat{w}} + \text{H.c.} \right] \\
=& \frac{1}{N} \sum_{n,k_w,k'_w,s} \left[\psi_{\vec{x},k_w}^\dagger e^{-ik_w w} \left(\frac{c\Gamma^0 - i\Gamma^s}{2} \right) e^{iA_{n,n+\hat{s}}} e^{ik'_w w} \psi_{\vec{x}+\hat{s},k'_w} + \text{H.c.} \right] \\
&+ \frac{1}{N} m \sum_{n,k_w,k'_w} \psi_{\vec{x},k_w}^\dagger e^{-ik_w w + ik'_w w} \Gamma^0 \psi_{\vec{x},k'_w} \\
&+ \frac{1}{N} \sum_{n,k_w,k'_w} \left[\psi_{\vec{x},k_w}^\dagger e^{-ik_w w + ik'_w (w+1)} \left(\frac{c\Gamma^0 - i\Gamma^4}{2} \right) e^{iA_{n,n+\hat{w}}} \psi_{\vec{x},k'_w} + \text{H.c.} \right] \\
=& \sum_{k_w,\vec{x},s} \left[\psi_{\vec{x},k_w}^\dagger \left(\frac{c\Gamma^0 - i\Gamma^s}{2} \right) e^{iA_{\vec{x},\vec{x}+\hat{s}} \psi_{\vec{x}+\hat{s},k_w}} + \text{H.c.} \right] \\
&+ \sum_{k_w,\vec{x},s} \psi_{\vec{x},k_w}^\dagger \left\{ \sin\left(k_w + A_{\vec{x}4}\right) \Gamma^4 + \left[m + c\cos\left(k_w + A_{\vec{x}4}\right)\right] \Gamma^0 \right\} \psi_{\vec{x},k_w}
\end{aligned} \tag{233}$$

where $s = 1, 2, 3$ and $A_{\vec{x}4} = A_{\vec{x},\vec{x}+\hat{w}}$. We can only pick one k_w term and obtain a three dimensional Hamiltonian with an additional parameter.

$$H_{\text{3D}}[A,\theta] = \sum_{\vec{x},s} \left[\psi_x^\dagger \left(\frac{c\Gamma^0 - i\Gamma^s}{2} \right) e^{iA_{\vec{x},\vec{x}+\hat{s}}} \psi_{\vec{x}+\hat{s}} + \text{H.c.} \right] + \sum_{\vec{x},s} \psi_{\vec{x}}^\dagger \left[\sin\theta_{\vec{x}} \Gamma^4 + \left(m + c\cos\theta_{\vec{x}}\right) \Gamma^0 \right] \psi_{\vec{x}} \tag{234}$$

where $\theta_{\vec{x}} = k_w + A_{\vec{x}4}$ is the field parameter. Likewise the effective action can be defined as

$$\exp^{iS_{\text{3D}}[A,\theta]} = \int D[\psi] D[\bar{\psi}] \exp \left\{ i \int dt \left[\sum_{\vec{x}} \bar{\psi}_{\vec{x}} \left(i\partial_\tau - A_{\vec{x}0} \right) \psi_{\vec{x}} - H[A,\theta] \right] \right\} \tag{235}$$

The (3+1)-dimensional action is thus

$$S_{\text{3D}} = \frac{G_3\left(\theta_0\right)}{4\pi} \int d^3x dt \epsilon^{\mu\nu\sigma\tau} A_\mu \partial_\nu \delta\theta \partial_\sigma A_\tau \tag{236}$$

The factor can be direnctly obtained from the C_2:

$$\begin{aligned}
G_3(\theta_0) =& -\frac{\pi^2}{15} \epsilon^{\mu\nu\rho\sigma 4} \int \frac{d^3k d\omega}{(2\pi)^5} \text{Tr} \left[\left(G \frac{\partial G^{-1}}{\partial q^\mu} \right) \left(G \frac{\partial G^{-1}}{\partial q^\nu} \right) \left(G \frac{\partial G^{-1}}{\partial q^\rho} \right) \left(G \frac{\partial G^{-1}}{\partial q^\sigma} \right) \left(G \frac{\partial G^{-1}}{\partial q^4} \right) \right] \times 5 \\
=& -\frac{\pi}{6} \epsilon^{\mu\nu\rho\sigma} \int \frac{d^3k d\omega}{(2\pi)^4} \text{Tr} \left[\left(G \frac{\partial G^{-1}}{\partial q^\mu} \right) \left(G \frac{\partial G^{-1}}{\partial q^\nu} \right) \left(G \frac{\partial G^{-1}}{\partial q^\rho} \right) \left(G \frac{\partial G^{-1}}{\partial q^\sigma} \right) \left(G \frac{\partial G^{-1}}{\partial \theta_0} \right) \right]
\end{aligned} \tag{237}$$

Hence it must satisfies

$$\int d\theta_0 G_3(\theta_0) = C_2 \in \mathbb{Z} \tag{238}$$

Likewise it can also be represented by Berry curvature

$$\begin{aligned} G_3(\theta_0) &= \frac{1}{32\pi^2}\int d^3k \epsilon^{ijk4}\,\mathrm{Tr}\left[f_{ij}f_{k\theta}\right] \times 4 = \frac{1}{8\pi^2}\int d^3k\epsilon^{ijk}\,\mathrm{Tr}\left[f_{k\theta}f_{ij}\right] \\ &= \frac{1}{8\pi^2}\int d^3k\epsilon^{ijk}\,\mathrm{Tr}\left[f_{k\theta}f_{ij}\right] = -\frac{1}{8\pi^2}\int d^3k\epsilon^{ijk}\,\mathrm{Tr}\left[f_{\theta k}f_{ij}\right] \end{aligned} \tag{239}$$

We then introduce the non-Abelian Chern Simons term

$$\mathcal{K}^A = \frac{1}{16\pi^2}\epsilon^{ABCD}\,\mathrm{Tr}\left[\left(f_{BC} - \frac{i}{3}\left[a_B, a_C\right]\right)\cdot a_D\right] \tag{240}$$

Then

$$\begin{aligned} \partial_A\mathcal{K}^A =& \partial_A\frac{1}{16\pi^2}\epsilon^{ABCD}\,\mathrm{Tr}\left[\left(f_{BC} - \frac{i}{3}[a_B, a_c]\right)\cdot a_D\right] \\ =& \partial_A\frac{1}{16\pi^2}\epsilon^{ABCD}\left\{\left[f_{BC}^{\alpha\beta} - \frac{i}{3}\left(a_B^{\alpha\gamma}a_C^{\gamma\beta} - a_C^{\alpha\gamma}a_B^{\gamma\beta}\right)\right]a_D^{\beta\alpha}\right\} \\ =& \frac{1}{16\pi^2}\epsilon^{ABCD}\left\{\left[\frac{2i}{3}\left(\partial_A a_B^{\alpha\gamma}a_C^{\gamma\beta} + a_B^{\alpha\gamma}\partial_A a_C^{\gamma\beta} - \partial_A a_C^{\alpha\gamma}a_B^{\gamma\beta} - a_C^{\alpha\gamma}\partial_A a_B^{\gamma\beta}\right)a_D^{\beta\alpha}\right]\right\} \\ &+ \frac{1}{16\pi^2}\epsilon^{ABCD}\left\{\left[\partial_B a_C^{\alpha\beta} - \partial_C a_B^{\alpha\beta} + \frac{2i}{3}\left(a_B^{\alpha\gamma}a_C^{\gamma\beta} - a_C^{\alpha\gamma}a_B^{\gamma\beta}\right)\right]\partial_A a_D^{\beta\alpha}\right\} \\ =& \frac{1}{16\pi^2}\epsilon^{ABCD}\left[\frac{2i}{3}\left(a_A^{\alpha\gamma}a_B^{\gamma\beta} - a_B^{\alpha\gamma}a_A^{\gamma\beta}\right)\partial_C a_D^{\beta\alpha}\right] - \frac{1}{16\pi^2}\epsilon^{CADB}\left[\frac{i}{3}\left(a_A^{\alpha\gamma}a_B^{\gamma\beta} - a_B^{\alpha\gamma}a_A^{\gamma\beta}\right)\partial_C a_D^{\beta\alpha}\right] \\ &+ \frac{1}{16\pi^2}\epsilon^{ABCD}\left[\frac{i}{3}\partial_A a_B^{\alpha\gamma}a_C^{\gamma\beta}a_D^{\beta\alpha} - \partial_A a_B^{\gamma\beta}a_D^{\beta\alpha}a_C^{\alpha\gamma}\right] + \frac{1}{16\pi^2}\epsilon^{CABD}\left[\left(\partial_A a_B^{\alpha\beta} - \partial_B a_A^{\alpha\beta}\right)\partial_C a_D^{\beta\alpha}\right] \\ &+ \frac{1}{16\pi^2}\epsilon^{ABCD}\left[\frac{2i}{3}\partial_A a_B^{\alpha\gamma}a_C^{\gamma\beta}a_D^{\beta\alpha} - \partial_A a_B^{\gamma\beta}a_D^{\beta\alpha}a_C^{\alpha\gamma}\right] \\ =& \frac{1}{32\pi^2}\epsilon^{ABCD}\left[\left(\partial_A a_B^{\alpha\beta} - \partial_B a_A^{\alpha\beta}\right)\left(\partial_C a_D^{\beta\alpha} - \partial_D a_C^{\beta\alpha} + i\left[a_C, a_D\right]^{\beta\alpha}\right)\right] \\ &+ \frac{1}{32\pi^2}\epsilon^{ABCD}\left[i[a_A, a_B]^{\alpha\beta}\left(\partial_C a_D^{\beta\alpha} - \partial_D a_C^{\beta\alpha}\right)\right] \\ =& \frac{1}{32\pi^2}\epsilon^{ABCD}\,\mathrm{Tr}\left[f_{AB}f_{CD}\right] \end{aligned} \tag{241}$$

where in the last line we have used the fact that

$$\epsilon^{ABCD}\,\mathrm{Tr}[a_A a_B a_C a_D] = 0 \tag{242}$$

Hence

$$\frac{\partial P_3(\theta_0)}{\partial\theta_0} \equiv G_3(\theta_0) = \int d^3k\partial_A\mathcal{K}^A \tag{243}$$

Since we can choose an appropriate gauge such that $\mathcal{K}^i$ $(i = 1, 2, 3)$ is single valued, the above equation can be simplified as

$$\frac{\partial P_3(\theta_0)}{\partial \theta_0} \equiv G_3(\theta_0) = \int d^3k \partial_\theta \mathcal{K}^\theta \tag{244}$$

Hence

$$\begin{aligned} P_3(\theta_0) &= \int d^3k \mathcal{K}^\theta \\ &= \frac{1}{16\pi^2} \int d^3k \epsilon^{\theta ijk} \operatorname{Tr}\left[\left(f_{ij} - \frac{i}{3}\left[a_i, a_j\right]\right) \cdot a_k\right] \end{aligned} \tag{245}$$

Since under gauge invariance, the three dimensional integration of Chern Simons terms is invariant modulo a integer, consider the following gauge tranformation:

$$a_i \longrightarrow u^{-1} a_i u - i u^{-1} \partial_i u \tag{246}$$

$$\begin{aligned} A &= \int d^3k \epsilon^{\theta ijk} \operatorname{Tr}\left[\left(\partial_i a_j' - \partial_j a_i'\right) a_k'\right] \\ &= \int d^3k \epsilon^{\theta ijk} \operatorname{Tr}\left[\left(2\partial_i a_j'\right) a_k'\right] \\ &= \int d^3k \epsilon^{\theta ijk} \operatorname{Tr}\left[\partial_i(u^{-1}a_j u - iu^{-1}\partial_j u)(u^{-1}a_k u - iu^{-1}\partial_k u)\right] \\ &= \int d^3k \epsilon^{\theta ijk} \operatorname{Tr} 2\left[\partial_i u^{-1} a_j u + u^{-1}\partial_i a_j u + u^{-1} a_j \partial_i u - i\partial_i u^{-1}\partial_j u\right](u^{-1}a_k u - iu^{-1}\partial_k u) \\ &= \int d^3k \epsilon^{\theta ijk} \operatorname{Tr} 2\left[2\partial_i u^{-1} a_j a_k u - 2i\partial_i u \partial_j u^{-1} a_k + (u^{-1}\partial_i u)(u^{-1}\partial_j u)(u^{-1}\partial_k u) + \partial_i a_j a_k\right] \end{aligned} \tag{247}$$

$$\begin{aligned} B &= \int d^3k \epsilon^{\theta ijk} \frac{2i}{3} \operatorname{Tr}\left[a_i', a_j'\right] a_k' \\ &= \int d^3k \epsilon^{\theta ijk} \frac{4i}{3} \operatorname{Tr}\left[a_i' a_j' a_k'\right] \\ &= \int d^3k \epsilon^{\theta ijk} \frac{4i}{3} \operatorname{Tr}\left[(u^{-1}a_i u - iu^{-1}\partial_i u)(u^{-1}a_j u - iu^{-1}\partial_j u)(u^{-1}a_k u - iu^{-1}\partial_k u)\right] \\ &= \int d^3k \epsilon^{\theta ijk} \frac{4i}{3} \operatorname{Tr}[a_i a_j a_k] + \int d^3k \epsilon^{\theta ijk} \frac{4i}{3} \operatorname{Tr}\left[i(u^{-1}\partial_i u)(u^{-1}\partial_j u)(u^{-1}\partial_k u)\right] \\ &\quad + \int d^3k \epsilon^{\theta ijk} \frac{4i}{3} \operatorname{Tr}\left[-iu^{-1}a_i a_j \partial_k u - i\partial_i u u^{-1} a_j a_k - iu^{-1} a_k a_i \partial_j u\right] \\ &\quad + \int d^3k \epsilon^{\theta ijk} \frac{4i}{3} \operatorname{Tr}\left[-u^{-1}a_i \partial_j u u^{-1}\partial_k u - u^{-1}a_j \partial_k u u^{-1}\partial_i u - u^{-1}a_k \partial_i u u^{-1}\partial_j u\right] \\ &= \int d^3k \epsilon^{\theta ijk} \frac{4i}{3} \operatorname{Tr}[a_i a_j a_k] + \int d^3k \epsilon^{\theta ijk} \frac{4i}{3} \operatorname{Tr}\left[i(u^{-1}\partial_i u)(u^{-1}\partial_j u)(u^{-1}\partial_k u)\right] \\ &\quad - \int d^3k \epsilon^{\theta ijk} \operatorname{Tr} 4\left[\partial_i u^{-1} a_j a_k u - i\partial_i u \partial_j u^{-1} a_k\right] \end{aligned} \tag{248}$$

Hence

$$
\begin{aligned}
\Delta P_3 &= \frac{1}{16\pi^2}A + \frac{1}{16\pi^2}B - P_3 \\
&= \frac{1}{16\pi^2}\int d^3k\epsilon^{\theta ijk}\frac{2}{3}\,\mathrm{Tr}\left[(u^{-1}\partial_i u)(u^{-1}\partial_j u)(u^{-1}\partial_k u)\right] \\
&= \frac{1}{24\pi^2}\int d^3k\epsilon^{\theta ijk}\,\mathrm{Tr}\left[(u^{-1}\partial_i u)(u^{-1}\partial_j u)(u^{-1}\partial_k u)\right]
\end{aligned}
\tag{249}
$$

We can also simplify S_{3D} by inserting $G_3 = \partial P_3/\partial\theta$:

$$
\begin{aligned}
S_{3D} &= -\frac{G_3\left(\theta_0\right)}{4\pi}\int d^3xdt\epsilon^{\mu\nu\sigma\tau}\partial_\mu\delta\theta A_\nu\partial_\sigma A_\tau \\
&= \frac{G_3\left(\theta_0\right)}{4\pi}\int d^3xdt\epsilon^{\mu\nu\sigma\tau}A_\mu\partial_\nu\delta\theta\partial_\sigma A_\tau \\
&= \frac{1}{4\pi}\int d^3xdt\epsilon^{\mu\nu\sigma\tau}A_\mu(\partial P_3/\partial\theta)\partial_\nu\delta\theta\partial_\sigma A_\tau \\
&= \frac{1}{4\pi}\int d^3xdt\epsilon^{\mu\nu\sigma\tau}A_\mu\partial_\nu P_3\partial_\sigma A_\tau \\
&= \frac{1}{4\pi}\int d^3xdt\epsilon^{\mu\nu\sigma\tau}P_3\partial_\mu A_\nu\partial_\sigma A_\tau
\end{aligned}
\tag{250}
$$

3.2 Physical Consequencies of the Effective Action $\mathbf{S_{3D}}$

The response function is

$$
j^\mu = \frac{\delta S_{3D}}{\delta A_\mu}
\tag{251}
$$

Since

$$
\begin{aligned}
\delta S_{3D} &= \frac{1}{4\pi}\int d^3xdt\epsilon^{\eta\nu\sigma\tau}P_3\partial_\eta\delta A_\nu\partial_\sigma A_\tau + \frac{1}{4\pi}\int d^3xdt\epsilon^{\eta\nu\sigma\tau}P_3\partial_\eta A_\nu\partial_\sigma\delta A_\tau \\
&= -\frac{1}{4\pi}\int d^3xdt\epsilon^{\eta\nu\sigma\tau}\delta A_\nu\partial_\eta P_3\partial_\sigma A_\tau - \frac{1}{4\pi}\int d^3xdt\epsilon^{\eta\nu\sigma\tau}\delta A_\tau\partial_\sigma P_3\partial_\eta A_\nu \\
&= \frac{1}{2\pi}\int d^3xdt\epsilon^{\eta\nu\sigma\tau}\delta A_\eta\partial_\nu P_3\partial_\sigma A_\tau
\end{aligned}
\tag{252}
$$

we can see that

$$
j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\sigma\tau}\partial_\nu P_3\partial_\sigma A_\tau
\tag{253}
$$

3.2.1 Hall Effect Induced by Spatial Gradient of $\mathbf{P_3}$

Consider a system in which P_3 depends only on z. Hence

$$
j^\mu = \frac{\partial_z P_3}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho, \quad \mu,\nu,\rho = t,x,y
\tag{254}
$$

Consider a uniform electric field E_x in the x direction. Then

$$j^y = \frac{\partial_z P_3}{2\pi}(\partial_0 A_x - \partial_x A_0) = \frac{\partial_z P_3}{2\pi} E_x \tag{255}$$

Integrate this result in a finite range of z

$$J_y^{2D} = \int_{z_1}^{z_2} dz \frac{\partial_z P_3}{2\pi} E_x = \frac{1}{2\pi}\left(\int_{z_1}^{z_2} dP_3\right) E_x \tag{256}$$

The net Hall conductance in the region $z_1 \leq z \leq z_2$ is

$$\sigma_{xy}^{2D} = \int_{z_1}^{z_2} dP_3/2\pi \tag{257}$$

3.2.2 TME induced by the temporal gradient of $\mathbf{P_3}$

Consider a system in which $P_3 = P_3(t)$ is spatially uniform but time dependent. Then

$$j^i = -\frac{\partial_t P_3}{2\pi}\epsilon^{ijk}\partial_j A_k, \quad i,j,k = x,y,z \tag{258}$$

Hence

$$\vec{j} = -\frac{\partial_t P_3}{2\pi}\vec{B} \tag{259}$$

Let θ adiabatically evolve from 0 to 2π. The charge pumped along the z direction is (assume that the magnetic field is along z direction and is constant.)

$$\Delta Q = \int dt \int dxdy\vec{j} = -\frac{1}{2\pi}\int_0^{2\pi} d\theta P_3 B_z L_x L_y = -\frac{B_z L_x L_y}{2\pi} C_2 \tag{260}$$

Likewise we can define the charge polarization as

$$\vec{j} = \partial_t \vec{P} \tag{261}$$

In a static uniform $\vec{B}$ field

$$\partial_t \vec{P} = -\frac{\partial_t P_3}{2\pi}\vec{B} \tag{262}$$

Hence

$$\vec{P} = -\frac{\vec{B}}{2\pi}(P_3 + \text{const}) \tag{263}$$

From the above equation, we learned that the charge polarization is induced by a magnetic field. This phenomenon is called the magnetoelectric effect.

Similar to what we have done before, we can replace the periodic boundary condition along the z direction by the open boundary condition and calculate the Landau levels. When $-4c < m < -2c$, let θ evolve from 0 to 2π. There are N degenerate states on the bottom surface fall below the Fermi energy and accordingly N degenerate states rise above the Fermi energy on the top surface. Hence

$$\Delta Q = -NC_2 = -\frac{B_z L_x L_y}{2\pi} C_2 \tag{264}$$

From above we can also see that

$$\nabla \cdot \vec{j} = -\frac{\partial_t P_3}{2\pi} \nabla \cdot \vec{B} \tag{265}$$

Hence, in the presence of magnetic monopoles: $\rho_m = \nabla \cdot \vec{B}/2\pi$.

$$\partial_t \rho_e = (\partial_t P_3)\, \rho_m \tag{266}$$

Let P_3 evolve from 0 to $\frac{\Theta}{2\pi}$, the monopole will acquire a charge of $Q_e = \frac{\Theta}{2\pi} Q_m$.

3.3 $\mathbf{Z_2}$ Topological Classification of Time-Reversal Invariant Insulators

The time-reversal invariance requires that

$$T^\dagger h(-\vec{k}) T = h^T(\vec{k}) \tag{267}$$

where the time-reversal matrices satisfy

$$T^\dagger T = \mathbb{I} \quad T^* T = \mathbb{I} \tag{268}$$

For two TRI band insulators, one with the Hamiltonian $h_1(\vec{k})$, the other with $h_2(\vec{k})$. Define the interplation function such that

$$\begin{aligned} h(\vec{k}, 0) = h_1(\vec{k}), \quad h(\vec{k}, \pi) = h_2(\vec{k}) \\ T^\dagger h(-\vec{k}, -\theta) T = h^T(\vec{k}, \theta) \end{aligned} \tag{269}$$

where $h(\vec{k}, \theta)$ remains gapped for any $\theta \in [0, 2\pi]$.

Claim For any two of such interpolations, $C_2[h(\vec{k}, \theta)] - C_2[h'(\vec{k}, \theta)] = 0 \ mod \ 2$.

Proof Similarly define two new interpolations

$$\begin{aligned} g_1(k, \theta) &= \begin{cases} h(k, \theta), & \theta \in [0, \pi] \\ h'(k, 2\pi - \theta), & \theta \in [\pi, 2\pi] \end{cases} \\ g_2(k, \theta) &= \begin{cases} h'(k, 2\pi - \theta), & \theta \in [0, \pi] \\ h(k, \theta), & \theta \in [\pi, 2\pi] \end{cases} \end{aligned} \tag{270}$$

Then

$$C_2[h] - C_2[h'] = C_2[g_1] + C_2[g_2] \tag{271}$$

From the definition and the TRI constraint

$$T^\dagger g_1(-\vec{k},-\theta)T = g_2^T(\vec{k},\theta) \leftrightarrow g_2^T(-\vec{k},-\theta)T^\dagger = T^\dagger g_1(\vec{k},\theta) \tag{272}$$

consider an eigenstate $|\vec{k},\theta;\alpha\rangle$ of $g_1(\vec{k},\theta)$ with eigenvalue $E_\alpha(\vec{k},\theta)$.

$$\begin{aligned} g_2^T(-\vec{k},-\theta)T^\dagger|\vec{k},\theta;\alpha\rangle_1 &= T^\dagger g_1(\vec{k},\theta)|\vec{k},\theta;\alpha\rangle_1 \\ &= E_\alpha(\vec{k},\theta)T^\dagger|\vec{k},\theta;\alpha\rangle_1 \end{aligned} \tag{273}$$

Since g_2 is in fact a Hamiltonian, $g_2^\dagger = g_2$. Then we can see that $g_2^T = g_2^*$ Hence

$$\begin{aligned} &g_2^*(-\vec{k},-\theta)T^\dagger|\vec{k},\theta;\alpha\rangle_1 = E_\alpha(\vec{k},\theta)T^\dagger|\vec{k},\theta;\alpha\rangle_1 \Rightarrow \\ &g_2(-\vec{k},-\theta)T^T(|\vec{k},\theta;\alpha\rangle_1)^* = E_\alpha(\vec{k},\theta)T^T(|\vec{k},\theta;\alpha\rangle_1)^* \end{aligned} \tag{274}$$

Thus $T^T(|\vec{k},\theta;\alpha\rangle_1)^*$ is an eigenvector of g_2 with the eigenvalue E_α. This eigenvector can be represented as a linear combination of the basis of eigenvectors of g_2.

$$T^T\left(|\vec{k},\theta;\alpha\rangle_1\right)^* = \sum_\beta U_{\alpha\beta}(\vec{k},\theta)|-\vec{k},-\theta;\beta\rangle_2 \tag{275}$$

Hence

$$\begin{aligned} |\vec{k},\theta;\alpha\rangle_1 &= T\left(\sum_\beta U_{\alpha\beta}(\vec{k},\theta)|-\vec{k},-\theta;\beta\rangle_2\right)^* \\ \langle\vec{k},\theta;\alpha|_1 &= \left(\sum_\beta U^*_{\alpha\beta}(\vec{k},\theta)\langle-\vec{k},-\theta;\beta|_2\right)^* T^\dagger \end{aligned} \tag{276}$$

Consider the Berry phase gauge vector

$$\begin{aligned} a_{1j}^{\alpha\beta}(\vec{k},\theta) &= -i\left\langle \vec{k},\theta;\alpha\,|\partial_j|\,\vec{k},\theta;\beta\right\rangle_1 \\ &= -i\left[\sum_{\gamma,\delta} U^*_{\alpha\gamma}\langle-\vec{k},-\theta;\gamma|\partial_j\left(U_{\beta\delta}|-\vec{k},-\theta;\delta\rangle_2\right)\right]^* \\ &= i\sum_{\gamma,\delta} U_{\alpha\gamma}\langle-\vec{k},-\theta;\gamma|\partial_{-j}|-\vec{k},-\theta;\delta\rangle_2 U^\dagger_{\delta\beta} - i\sum_\gamma U_{\alpha\gamma}(\vec{k},\theta)\partial_j U^*_{\beta\gamma}(\vec{k},\theta) \\ &= \sum_{\gamma,\delta} U_{\alpha\gamma} a_{2j}^{\gamma\delta *}(-\vec{k},-\theta)\left(U^\dagger\right)_{\delta\beta} - i\sum_\gamma U_{\alpha\gamma}(\vec{k},\theta)\partial_j U^*_{\beta\gamma}(\vec{k},\theta) \end{aligned} \tag{277}$$

As for the Berry curvature

$$\begin{aligned} \partial_i a_{1j} - \partial_j a_{1i} =& \partial_i U a^*_{2j}U^\dagger + U\partial_i a^*_{2j}U^\dagger + Ua^*_{2j}\partial_i U^\dagger - i\partial_i U\partial_j U^\dagger - iU\partial_i\partial_j U^\dagger \\ &- \partial_j U a^*_{2i}U^\dagger - U\partial_j a^*_{2i}U^\dagger - Ua^*_{2i}\partial_j U^\dagger + i\partial_j U\partial_i U + iU\partial_j\partial_i U^\dagger \\ ia_i a_j - ia_j a_i =& iUa^*_{2i}a^*_{2j}U^\dagger + Ua^*_{2i}\partial_j U^\dagger + U\partial_i U^\dagger U a^*_{2j}U^\dagger - iU\partial_i U^\dagger U\partial_j U^\dagger \\ &- iUa^*_{2j}a^*_{2i}U^\dagger - Ua^*_{2j}\partial_i U^\dagger - U\partial_j U^\dagger U a^*_{2i}U^\dagger + iU\partial_j U^\dagger U\partial_i U^\dagger \\ =& iU[a_{2i},a_{2j}]^*U^\dagger + Ua^*_{2i}\partial_j U^\dagger - \partial_i U a^*_{2j}U^\dagger + i\partial_i U\partial_j U^\dagger \\ &- Ua^*_{2j}\partial_i U^\dagger + \partial_j U a^*_{2i}U^\dagger - i\partial_j U\partial_i U^\dagger \end{aligned} \tag{278}$$

Hence

$$f_{1ij}(\vec{k},\theta) = U f^*_{2ij}(-\vec{k},-\theta)U^\dagger \tag{279}$$

We then see that

$$C_2[g_1] = C_2[g_2] \tag{280}$$

Hence

$$C_2[h] - C_2[h'] = 2C_2[g_1] = 0 \ mod \ 2 \tag{281}$$

We can then define the ralative second Chern parity

$$N_3\left[h_1(\vec{k}), h_2(\vec{k})\right] = (-1)^{C_2[h(\vec{k},\theta)]} \tag{282}$$

As usual we set the vacuum Hamiltonian $h_0(\vec{k}) = h_0$ as a reference. Then if a Hamiltonian h satisfies $N[h_0, h] = -1$, this Hamiltonian is called a $\mathbb{Z}_2$ nontrivial Hamiltonian. Otherwise it is a $\mathbb{Z}_2$ trivial Hamiltonian.

There is also a less rigorous way to define the classification. For a TRI band insulator, from what we have learned, it should also satisfy

$$a_i(\vec{k}) = U a^*_i(-\vec{k})U^\dagger - iU\partial_i U^\dagger \tag{283}$$

Notice that

$$\begin{aligned}(a_i^{\alpha\beta})* &= (-i\left\langle \vec{k},\theta_0;\alpha\,|(\partial/\partial k_i)|\,\vec{k},\theta_0;\beta\right\rangle)^* \\ &= i\frac{\partial}{\partial k_i}\langle\vec{k},\theta_0;\beta|\vec{k},\theta_0;\alpha\rangle = -i\langle\vec{k},\theta_0;\beta|\frac{\partial}{\partial k_i}|\vec{k},\theta_0;\alpha\rangle \\ &= a_i^{\beta\alpha}\end{aligned} \tag{284}$$

Consider the P_{3*} with $a^*(-\vec{k})$ as the Berry gauge vector.

$$P_{3*} == \frac{1}{16\pi^2}\int d^3k\epsilon^{\theta ijk}\,\mathrm{Tr}\left[\left(f^*_{ij} - \frac{i}{3}\left[a^*_i, a^*_j\right]\right)\cdot a^*_k\right] \tag{285}$$

We only need to consider two terms. The first one is

$$\begin{aligned}\int d^3k\epsilon^{\theta ijk}\,\mathrm{Tr}\left[\left(\partial_i a^*_j(-\vec{k})a^*_k(-\vec{k})\right)\right] &= \int d^3k\epsilon^{\theta ijk}\partial_i(a_j^{\alpha\beta}(-\vec{k}))^*(a_k^{\beta\alpha}(-\vec{k}))^* \\ &= \int d^3k\epsilon^{\theta ijk}\partial_i a_j^{\beta\alpha}(-\vec{k})a_k^{\alpha\beta}(-\vec{k}) \\ &= -\int d^3k\epsilon^{\theta ijk}\partial_{-i}a_k^{\alpha\beta}(-\vec{k})a_j^{\beta\alpha}(-\vec{k}) \\ &= -\int d^3k\epsilon^{\theta ijk}\partial_i a_k^{\alpha\beta}(\vec{k})a_j^{\beta\alpha}(\vec{k})\end{aligned} \tag{286}$$

The second term is

$$
\begin{aligned}
\int d^3k\epsilon^{\theta ijk}\,\mathrm{Tr}\left[a_i^* a_j^* a_k^*\right] &= \int d^3k\epsilon^{\theta ijk}(a_i^{\alpha\beta})^*(a_j^{\beta\gamma})^*(a_k^{\gamma\alpha})^* \\
&= \int d^3k\epsilon^{\theta ijk}a_i^{\beta\alpha}a_j^{\gamma\beta}a_k^{\alpha\gamma} \\
&= \int d^3k\epsilon^{\theta ijk}a_k^{\alpha\gamma}a_j^{\gamma\beta}a_i^{\beta\alpha} \\
&= -\int d^3k\epsilon^{\theta ijk}a_i^{\alpha\gamma}a_j^{\gamma\beta}a_k^{\beta\alpha} \\
&= -\int d^3k\epsilon^{\theta ijk}\,\mathrm{Tr}\left[a_i a_j a_k\right]
\end{aligned}
\tag{287}
$$

Hence

$$
P_{3*} = -P_3 \tag{288}
$$

From the gauge transformation we can learn that

$$
2P_3 = \Delta P_3 = \frac{i}{24\pi^2}\int d^3k\epsilon^{ijk}\,\mathrm{Tr}\left[\left(U\partial_i U^\dagger\right)\left(U\partial_j U^\dagger\right)\left(U\partial_k U^\dagger\right)\right] \in \mathbb{Z} \tag{289}
$$

Hence $P_3 = 0$ *mod* 2 or $P_3 = 1/2$ *mod* 2. For two TRI Hamiltonians h_1 and h_2, as we have shown before, $C[h(\vec{k},\theta)] = 2(P_3(h_1) - P_3(h_2)$ *mod* 2. Since $P_3(h_0) = 0$, Hamiltonians with $P_3(h) = 1/2$ is $\mathbb{Z}_2$ nontrivial.

3.4 Physical Properties of $\mathbf{Z_2}$-nontrivial Insulators

For a $\mathbb{Z}_2$ nontrivial insulator, $P_3 = n + \frac{1}{2}$ where $n \in \mathbb{Z}$.

$$
\begin{aligned}
S_{\mathrm{3D}} &= \frac{1}{4\pi}\int d^3xdt\epsilon^{\mu\nu\sigma\tau}P_3(x,t)\partial_\mu A_\nu\partial_\sigma A_\tau \\
&= \frac{2n+1}{8\pi}\int d^3xdt\epsilon^{\mu\nu\sigma\tau}\partial_\mu A_\nu\partial_\sigma A_\tau
\end{aligned}
\tag{290}
$$

If the system has a closed boundary the above integral can be quantized and hence can be time riversal invariant. However, for an open boundary system, since $\epsilon^{\mu\nu\sigma\tau}\partial_\mu A_\nu\partial_\sigma A_\tau = 2\mathbf{E}\cdot\mathbf{B}$, the effective action is no longer time reversal invariant. Consider a concrete case where the nontrivial insulator lies at $z < 0$ and the vacuum lies at $z > 0$. The effective action is

$$
S_{\mathrm{3D}} = \frac{1}{4\pi}\int d^3xdt\epsilon^{\mu\nu\sigma\tau}A_\mu\partial_\nu P_3\partial_\sigma A_\tau \tag{291}
$$

where $P_3 = n + 1/2$ at $z < 0$ and $P_3 = 0$ at $z > 0$. Hence

$$
\partial_z P_3 = -(n+1/2)\delta(z) \tag{292}
$$

The above action then becomes

$$
\begin{aligned}
S_{\mathrm{3D}} &= -\frac{2n+1}{8\pi}\int d^3xdt\epsilon^{\mu 3\sigma\tau}A_\mu(n+\frac{1}{2})\delta(z)\partial_\sigma A_\tau \\
&= \frac{2n+1}{8\pi}\int dxdydt\epsilon^{3\mu\nu\rho}A_\mu\partial_\nu A_\rho = S_{\mathrm{surf}}
\end{aligned}
\tag{293}
$$

From what we have learned, the Hall conductance of the surface is

$$\sigma_H = (n + \frac{1}{2})\frac{1}{2\pi} \tag{294}$$

Since on a 2D surface, a nontrivial insulator has odd number of $((2n+1))$ gapless Dirac cones, if they are endowed with a mass (in this way they become gapped), they can carry Hall conductance. Consider a Dirac fermion with a mass m, the Hall conductance it carries is

$$\sigma_H = \frac{1}{4\pi}\,\mathrm{sgn}(m)\left(= \frac{e^2}{2h}\,\mathrm{sgn}(m)\right) \tag{295}$$

Consider a perturbative term on the surface that disrupts the time reversal invariant symmetry and assign each Dirac fermion a mass term m_i. The net Hall conductance would be

$$\sigma_H = \sum_{i}^{2n+1} sgn(m_i)\frac{1}{4\pi} \tag{296}$$

The coefficient is an odd number which agrees with the result obtained from the surface theory. Consider a concrete example generated from the $H_[3D]$ with the field configuration set as

$$\theta(\vec{x}) = \theta(z) = \frac{\pi}{2}[1 - \tanh(z/4\xi)] \tag{297}$$

With a periodic boundary condition along x and y direction and an open boundary condition along z direction, the Hamiltonian can be written as

$$\begin{aligned}
H_{3\mathrm{D}}[A,\theta] =& \sum_{\vec{x},s}\left[\psi^\dagger_{\vec{x}}\left(\frac{c\Gamma^0 - i\Gamma^s}{2}\right)\psi_{\vec{x}+\hat{s}} + \mathrm{H.c.}\right] + \sum_{\vec{x},s}\psi^\dagger_{\vec{x}}\left[\sin\theta_x\Gamma^4 + (m + c\cos\theta_{\vec{x}})\,\Gamma^0\right]\psi_{\vec{x}} \\
=& \sum_{z,k_xk_y}\left[\psi^\dagger_{k_xk_y}(z)\left(\frac{c\Gamma^0 - i\Gamma^3}{2}\right)\psi_{k_xk_y}(z+1) + \mathrm{H.c.}\right] \\
&+ \sum_{z,k_x,k_y}\psi^\dagger_{k_xk_y}(z)\left[(m + c\cos\theta(z) + c\cos k_x + c\cos k_y)\,\Gamma^0 + \sin k_x\Gamma^1 + \sin k_y\Gamma^2\right]\psi_{k_xk_y}(z) \\
&+ \sum_{z,k_x,k_y}\psi^\dagger_{k_xk_y}(z)\sin\theta(z)\Gamma^4\psi_{k_xk_y}(z)
\end{aligned} \tag{298}$$

Under time reversal tranformation, k_x and k_y are odd; Γ^0 is even; $\Gamma^{1,2,3,4}$ is odd. Hence from above we see that only the last term is TRB, we call the last term as H_1 and the rest terms as H_0 which is TRI. Fortunately, from the field confiuration, $\theta(z)$ soon coverge to 0 when $z \to \infty$ and to π when $z \to \infty$. Hence due to the $\sin\theta(z)$ term in H_1, it is localized to $z = 0$. Hence the TRB term is only at the z=0 boundary and the bulk is still TRI.

$H_0(\theta)$ thus is a well defined interpolation between $\theta = 0$ and $\theta = \pi$. Consider the $-4c < m < -2c$ situation where $C_2[H_0] = 1$. Hence the relative parity between $\theta = 0$ and $\theta = \pi$ is -1. However, we should notice that at $\theta = 0$ the old Hamiltonian becomes

$$H_{3\mathrm{D}}[A,\theta] = \sum_{\vec{x},s}\left[\psi^\dagger_{\vec{x}}\left(\frac{c\Gamma^0 - i\Gamma^s}{2}\right)\psi_{\vec{x}+\hat{s}} + \mathrm{H.c.}\right] + \sum_{\vec{x},s}\psi^\dagger_{\vec{x}}\left[(m + c)\,\Gamma^0\right]\psi_{\vec{x}} \tag{299}$$

which is adiabatically connected to the $m \rightarrow -\infty$ situation. Hence $H_0(\theta=0)$ is a trivial insulator and $H_0(\theta=\pi)$ is a non trivial insulator which means there are an odd number of gapless Diarc fermions at $z=0$ surface. Hence near one of the Dirac cone at (k_x, k_y), the effective Hamiltonian can be written as

$$h_{\text{surf}} = \delta k_x \sigma_x + \delta k_y \sigma_y \tag{300}$$

Now ,we consider the H_1 term. Since $\{\Gamma^4, h_0\} = 0$, we find that near the Dirac cone, $\{\Gamma^4, h_{\text{surf}}\} = 0$. Hence the h_1 term is actually $m\sigma_z$. From the original Hamiltonian we can obtain the element of the sigma matrices by

$$\sigma^i_{\alpha\beta} = \left\langle k, \alpha \left| \frac{\partial h_0}{\partial k_i} \right|_k |k, \beta \right\rangle, \quad i = x, y \tag{301}$$

and likewise we can obtain the mass term. The detailed calculation is not given here. The result is

$$m = \frac{1}{2} \sum_{\alpha\beta} \sigma^z_{\alpha\beta} \left\langle k=0, \beta \left| h_1 \right| k=0, \alpha \right\rangle \tag{302}$$

Consider a parametrized Hamiltonian given by $h = h_0 + \lambda h_1$ from above we learn that the mass term is proportional to λ and positive when $\lambda < 0$. Hence the surface Hall conductance is given by

$$\sigma_H = \frac{sgn(m)}{4\pi} = -\frac{sgn(\lambda)}{4\pi} \tag{303}$$

Alternatively, we can obtain the Hall conductance by examine the variance of P_3. When $\lambda = 1$,

$$\sigma_H = \int_{-\infty}^{\infty} dP_3 \frac{1}{2\pi} = \int_{\pi}^{0} dP_3 \frac{1}{2\pi} = -\frac{C_2}{4\pi} = -\frac{1}{4\pi} \tag{304}$$

When $\lambda = -1$, the effct on Hamiltonian is the same as letting $\theta(z) \rightarrow -\theta(z)$. Hence the Hall conductance is

$$\sigma_H = \int_{-\infty}^{\infty} dP_3 \frac{1}{2\pi} = \int_{-\pi}^{0} dP_3 \frac{1}{2\pi} = \frac{C_2}{4\pi} = \frac{1}{4\pi} \tag{305}$$

The above results agree well with each other.

Consider a TRI breaking field $\mathbf{M}$ coupled to a TRI insulator. For a nontrivial insulator occupying a region $\mathcal{V}$ with a boundary $\partial\mathcal{V}$. Assume that the gradient of P_3 is given by

$$\nabla P_3(\vec{x}) = \left(g[M(\vec{x})] + \frac{1}{2} \right) \int_{\partial\mathcal{V}} d\hat{\mathbf{n}}(\vec{y}) \delta^3(\vec{x} - \vec{y}) \tag{306}$$

where $g[M(\vec{x})] \in \mathbb{Z}$ and $\hat{\mathbf{n}}$ is the unit vector normal to the boundary. Form this we can obtain the effective action.

$$S_{\text{surf}} = -\frac{1}{4\pi} \int_{\partial\mathcal{V}} d\hat{n}_\mu \left(g[M(\vec{x})] + \frac{1}{2} \right) \epsilon^{\mu\upsilon\sigma\tau} A_\nu \partial_\sigma A_\tau \tag{307}$$

3.4.1 TME

Consider a electric field parallel to the cylinder. At the interface, due to the gaps opened by magnetic field, we can obtain a circulating current

$$j = \sigma_H E = (n + \frac{1}{2})\frac{e^2}{h}E \tag{308}$$

This circulating current can also be generated by a bulk magnetization

$$\mathbf{M}_t = -\left(n + \frac{1}{2}\right)\frac{e^2}{hc}\mathbf{E} \tag{309}$$

Hence we can combine this term with a conventional magnetization term to obtain the effective total magnetization.

$$\mathbf{M} = \mathbf{M}_c + \mathbf{M}_t \tag{310}$$

Then

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}_c + (2n+1)\frac{e^2}{\hbar c}\mathbf{E} \tag{311}$$

Likewise, if we introduce a adiabatically varing magnetic field parallel the cylinder from 0 to $\mathbf{B}$, we would induce a circulating current which will induce a current parallel to the cylinder due to the Hall conductance. The Hall current is

$$j = \sigma_H \partial_t B \tag{312}$$

Hence the final result after the magnetic field stops varying and stays at $\mathbf{B}$, is cummulated charges at the top and the bottom of the cylinder.

$$\mathbf{P}_t = \left(n + \frac{1}{2}\right)\frac{e^2}{hc}\mathbf{B} \tag{313}$$

Combine this term with conventional polarization vector, we obtain

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}_c - (2n+1)\frac{e^2}{\hbar c}\mathbf{B} \tag{314}$$

Consider another approach. We only introduce the conventional terms at first.

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}_c \quad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}_c \tag{315}$$

The total action is the combination of the original action and the topological action

$$\begin{aligned}
S_{\text{tot}} =& S_{\text{Maxwell}} + S_{\text{topo}} \\
=& \int d^3x dt \left[-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}F_{\mu\nu}\mathcal{P}^{\mu\nu} - \frac{1}{c}j^\mu A_\mu\right] + \frac{\alpha}{4\pi}\int d^3x dt \epsilon^{\mu\nu\sigma\tau}P_3\partial_\mu A_\nu \partial_\sigma A_\tau \\
=& \int d^3x dt \left[-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}F_{\mu\nu}\mathcal{P}^{\mu\nu} - \frac{1}{c}j^\mu A_\mu\right] + \frac{\alpha}{8\pi}\int d^3x dt \epsilon^{\mu\nu\sigma\tau}P_3(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial_\sigma A_\tau \\
=& \int d^3x dt \left[-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}F_{\mu\nu}\mathcal{P}^{\mu\nu} - \frac{1}{c}j^\mu A_\mu\right] + \frac{\alpha}{8\pi}\int d^3x dt \epsilon^{\mu\nu\sigma\tau}P_3 F_{\mu\nu}\partial_\sigma A_\tau \\
=& \int d^3x dt \left[-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}F_{\mu\nu}\mathcal{P}^{\mu\nu} - \frac{1}{c}j^\mu A_\mu\right] + \frac{\alpha}{16\pi}\int d^3x dt P_3 \epsilon^{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}
\end{aligned} \tag{316}$$

The variation of the action over A_μ is

$$\begin{aligned}
\delta S &= \int d^3xdt \left[-\frac{1}{16\pi} 2(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) F^{\mu\nu} + \frac{1}{2}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)\mathcal{P}^{\mu\nu} - \frac{1}{c} j^\mu \delta A_\mu \right] \\
&\quad + \frac{\alpha}{16\pi} \int d^3xdt P_3 \epsilon^{\mu\nu\sigma\tau} 2(\partial_\mu \delta A_\nu - \partial_{nu} \delta A_\mu) F_{\sigma\tau} \\
&= \int d^3xdt \left[-\frac{1}{16\pi} 4(\partial_\mu \delta A_\nu) F^{\mu\nu} + \frac{1}{2} 2(\partial_\mu \delta A_\nu)\mathcal{P}^{\mu\nu} - \frac{1}{c} j^\mu \delta A_\mu \right] \\
&\quad + \frac{\alpha}{16\pi} \int d^3xdt P_3 \epsilon^{\mu\nu\sigma\tau} 4(\partial_\mu \delta A_\nu) F_{\sigma\tau} \\
&= \int d^3xdt \left[\frac{1}{4\pi} \delta A_\nu \partial_\mu F^{\mu\nu} - \delta A_\nu \partial_\mu \mathcal{P}^{\mu\nu} - \frac{1}{c} j^\mu \delta A_\mu \right] \\
&\quad - \frac{\alpha}{4\pi} \int d^3xdt \epsilon^{\mu\nu\sigma\tau} \delta A_\nu \partial_\mu (P_3 F_{\sigma\tau})
\end{aligned} \tag{317}$$

Hence

$$\begin{aligned}
\frac{1}{4\pi}\partial_\mu F^{\mu\nu} - \partial_\mu \mathcal{P}^{\mu\nu} - \frac{\alpha}{4\pi}\epsilon^{\mu\nu\sigma\tau}\partial_\mu (P_3 F_{\sigma\tau}) = \frac{j^\nu}{c} \quad \Rightarrow \\
-\frac{1}{4\pi}\partial_\nu F^{\mu\nu} + \partial_\nu \mathcal{P}^{\mu\nu} + \frac{\alpha}{4\pi}\epsilon^{\mu\nu\sigma\tau}\partial_\nu \left(P_3 F_{\sigma\tau}\right) = \frac{1}{c} j^\mu
\end{aligned} \tag{318}$$

When $\mu = 0$

$$\begin{aligned}
\frac{1}{4\pi}\partial_i E^i + \partial_i \mathcal{P}^i + \frac{\alpha}{4\pi}\epsilon^{ijk}\partial_i \left(P_3 F_{jk}\right) = \rho \\
\frac{1}{4\pi}\partial_i E^i + \partial_i \mathcal{P}^i + \frac{\alpha}{4\pi}(\partial_i P_3)\epsilon^{ijk} F_{jk} = \rho \\
\frac{1}{4\pi}\partial_i E^i + \partial_i \mathcal{P}^i + \frac{\alpha}{4\pi}(\partial_i P_3)(-2B_i) = \rho
\end{aligned} \tag{319}$$

Hence

$$\nabla \cdot \mathbf{D} = 4\pi\rho + 2\alpha \left(\nabla P_3 \cdot \mathbf{B}\right) \tag{320}$$

When $\mu = i$.

$$\begin{aligned}
&-\frac{1}{4\pi}\frac{\partial E_i}{c\partial t} - \frac{1}{4\pi}\partial_j F^{ij} - \frac{\partial P^i}{c\partial t} + \partial_j \mathcal{P}^{ij} - \frac{\alpha}{4\pi}\frac{\partial P_3}{c\partial t}\epsilon^{ijk}F_{jk} + \frac{\alpha}{4\pi}\partial_j P_3 \epsilon^{ij\sigma\tau} F_{\sigma\tau} = \frac{j_i}{c} \Rightarrow \\
&\partial_j \epsilon^{ijk} B_k - 4\pi \partial_j \epsilon^{ijk} M_k - \frac{\partial(E_i + 4\pi P_i)}{c\partial t} = \frac{4\pi j^i}{c} - \alpha \frac{\partial P_3}{c\partial t} 2B_i - \alpha \epsilon^{ijk}\partial_j P_3 2E_k \Rightarrow \\
&\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c}\mathbf{j} - 2\alpha \left(\left(\nabla P_3 \times \mathbf{E}\right) + \frac{1}{c}\left(\partial_t P_3\right)\mathbf{B} \right)
\end{aligned} \tag{321}$$

Combine the above results with the homogeneous Maxwell equations, we arrive at

$$\begin{aligned}\nabla \cdot \mathbf{D} &= 4\pi\rho + 2\alpha\left(\nabla P_3 \cdot \mathbf{B}\right)\\ \nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} &= \frac{4\pi}{c}\mathbf{j} - 2\alpha\left(\left(\nabla P_3 \times \mathbf{E}\right) + \frac{1}{c}\left(\partial_t P_3\right)\mathbf{B}\right)\\ \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0\\ \nabla \cdot \mathbf{B} &= 0\end{aligned} \tag{322}$$

4 Dimensional Reduction to (2+1) Dimensions

4.1 Effective Action of (2+1)-dimensional Insulators

Similar to what we have done in the dimensional reduction process from (4+1) dimensions to (3+1) dimensions, we require that the gauge vector is translationally invariant along w and z direction:

$$A_{n,n+\hat{i}} = A_{n+\hat{w},n+\hat{w}+\hat{i}} = A_{n+\hat{z},n+\hat{z}+\hat{i}} \tag{323}$$

Likewise, the Hamiltonian is

$$\begin{aligned}H[A] = &\sum_{k_z,k_w,\mathbf{x}}\sum_{s=1,2}\left[\psi^{\dagger}_{\mathbf{x};k_z,k_w}\left(\frac{c\Gamma^0 - i\Gamma^s}{2}\right)e^{iA_{x,x+\hat{s}}}\psi_{\mathbf{x}+\hat{s};k_z,k_w} + \text{H.c.}\right]\\ &+ \sum_{k_z,k_w,\mathbf{x}}\sum_{s=1,2}\psi^{\dagger}_{\mathbf{x};k_z,k_w}\cdot\left\{\sin\left(k_z + A_{\mathbf{x}3}\right)\Gamma^3 + \sin\left(k_w + A_{\mathbf{x}4}\right)\Gamma^4\right\}\psi_{\mathbf{x};k_z,k_w}\\ &+ \sum_{k_z,k_w,\mathbf{x}}\sum_{s=1,2}\psi^{\dagger}_{\mathbf{x};k_z,k_w}\cdot\left\{\left[m + c\cos\left(k_z + A_{\mathbf{x}3}\right) + c\cos\left(k_w + A_{\mathbf{x}4}\right)\right]\Gamma^0\right\}\psi_{\mathbf{x};k_z,k_w}\end{aligned} \tag{324}$$

We then obtain the (2+1)-dimensional effective theory by choosing only one slice of the above and assiging $k_z + A_{\mathbf{x}3} \to \theta_{\mathbf{x}}$ and $k_w + A_{\mathbf{x}4} \to \varphi_{\mathbf{x}}$.

$$\begin{aligned}H_{2D}[A,\theta,\varphi] = &\sum_{s=1,2}\left[\psi^{\dagger}_{\mathbf{x}}\left(\frac{c\Gamma^0 - i\Gamma^s}{2}\right)e^{iA_{x,x+\hat{s}}}\psi_{\mathbf{x}} + \text{H.c.}\right]\\ &+ \sum_{s=1,2}\psi^{\dagger}_{\mathbf{x}}\cdot\left\{\sin\theta_{\mathbf{x}}\Gamma^3 + \sin\varphi_{\mathbf{x}}\Gamma^4 + \left(m + \cos\theta_{\mathbf{x}} + \cos\varphi_{\mathbf{x}}\right)\right\}\psi_{\mathbf{x}}\end{aligned} \tag{325}$$

The effective action can be obtained

$$S_{\text{2D}} = \frac{G_2\left(\theta_0,\varphi_0\right)}{2\pi}\int d^2x dt \epsilon^{\mu\nu\rho}A_\mu\partial_\nu\delta\theta\partial_\rho\delta\varphi \tag{326}$$

where $G_2\left(\theta_0, \varphi_0\right)$ can be obtained from the old C_2 number without integrating over k_z and k_w:

$$\begin{aligned}
G_2(\theta_0,\varphi_0) &= (-2\pi)\frac{-\pi^2}{15}\int\frac{d^2kd\omega}{(2\pi^5)}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[\left(G\frac{\partial G^{-1}}{\partial q^\mu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\nu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\rho}\right)\left(G\frac{\partial G^{-1}}{\partial \theta_0}\right)\left(G\frac{\partial G^{-1}}{\partial \varphi_0}\right)\right]\times(5\times 4)\\
&= \frac{2\pi}{3}\int\frac{d^2kd\omega}{(2\pi^5)}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[\left(G\frac{\partial G^{-1}}{\partial q^\mu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\nu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\rho}\right)\left(G\frac{\partial G^{-1}}{\partial \theta_0}\right)\left(G\frac{\partial G^{-1}}{\partial \varphi_0}\right)\right]\\
&= (-2\pi)\frac{1}{32\pi^2}\int d^2k\times\epsilon^{ijk\varphi}\,\mathrm{tr}[f_{ij}f_{k\varphi}]\quad\text{(where one of i,j,k has to be }\theta)\\
&= (-2\pi)\frac{1}{32\pi^2}\int d^2k4\times\left\{2\epsilon^{i\theta k\varphi}\,\mathrm{tr}[f_{i\theta}f_{k\varphi}]+\epsilon^{ij\theta\varphi}\,\mathrm{tr}[f_{ij}f_{\theta\varphi}]\right\}\\
&= \frac{1}{4\pi}\int d^2k\epsilon^{ij}\,\mathrm{tr}[+2f_{i\theta}f_{k\varphi}-f_{ij}f_{\theta\varphi}]
\end{aligned} \tag{327}$$

Hence

$$\int G_2 d\theta d\varphi = -2\pi C_2 \tag{328}$$

As we have proved

$$\partial_A\mathcal{K}^A = \frac{1}{32\pi^2}\epsilon^{ABCD}\,\mathrm{Tr}[f_{AB}f_{CD}] \tag{329}$$

we can see that

$$G_2(\theta_0,\varphi_0) = -2\pi\int d^2k(\partial_x\mathcal{K}^x+\partial_y\mathcal{K}^y+\partial_\theta\mathcal{K}^\theta+\partial_\varphi\mathcal{K}^\varphi) \tag{330}$$

If $\mathcal{K}^x$ and $\mathcal{K}^y$ are sinle valued, the above equation can be simplified as

$$G_2(\theta_0,\varphi_0) = -2\pi\int d^2k(\partial_\theta\mathcal{K}^\theta+\partial_\varphi\mathcal{K}^\varphi) = \partial_\theta\Omega_\varphi-\partial_\varphi\Omega_\theta \tag{331}$$

where

$$\Omega_\varphi = -2\pi\int d^2k\mathcal{K}^\theta\quad \Omega_\theta = 2\pi\int d^2k\mathcal{K}^\varphi \tag{332}$$

We can then define the gauge vector potential

$$\Omega_\mu \equiv \Omega_\theta\partial_\mu\delta\theta+\Omega_\varphi\partial_\mu\delta\varphi \tag{333}$$

Hence the gauge curvature is

$$\begin{aligned}
\partial_\mu\Omega_\nu-\partial_\nu\Omega_\mu =&\partial_\mu(\Omega_\theta\partial_\nu\delta\theta+\Omega_\varphi\partial_\nu\delta\varphi)-\partial_\nu(\Omega_\theta\partial_\mu\delta\theta+\Omega_\varphi\partial_\mu\delta\varphi)\\
=&\partial_\mu\Omega_\theta\partial_\nu\delta\theta+\partial_\mu\Omega_\varphi\partial_\nu\delta\varphi-\partial_\nu\Omega_\theta\partial_\mu\delta\theta-\partial_\nu\Omega_\varphi\partial_\mu\delta\varphi\\
=&\partial_\varphi\Omega_\theta\partial_\mu\delta\varphi\partial_\nu\delta\theta+\partial_\theta\Omega_\varphi\partial_\mu\delta\theta\partial_\nu\delta\varphi-\partial_\varphi\Omega_\theta\partial_\nu\delta\varphi\partial_\mu\delta\theta-\partial_\theta\Omega_\varphi\partial_\nu\delta\theta\partial_\mu\delta\varphi\\
=&\partial_\theta\Omega_\varphi(\partial_\mu\delta\theta\partial_\nu\delta\varphi-\partial_\nu\delta\theta\partial_\mu\delta\varphi)-\partial_\varphi\Omega_\theta(\partial_\mu\delta\theta\partial_\nu\delta\varphi-\partial_\nu\delta\theta\partial_\mu\delta\varphi)\\
=&(\partial_\theta\Omega_\varphi-\partial_\varphi\Omega_\theta)(\partial_\mu\delta\theta\partial_\nu\delta\varphi-\partial_\nu\delta\theta\partial_\mu\delta\varphi)\\
=&G_2(\theta_0,\varphi_0)(\partial_\mu\delta\theta\partial_\nu\delta\varphi-\partial_\nu\delta\theta\partial_\mu\delta\varphi)
\end{aligned} \tag{334}$$

We can then write the effective action in a simpler form

$$
\begin{aligned}
S_{\mathrm{2D}} &= \frac{G_2(\theta_0,\varphi_0)}{2\pi}\int d^2xdt\epsilon^{\mu\nu\rho}A_\mu\partial_\nu\delta\theta\partial_\rho\delta\varphi \\
&= \frac{G_2(\theta_0,\varphi_0)}{2\pi}\frac{1}{2}\int d^2xdt\epsilon^{\mu\nu\rho}A_\mu(\partial_\nu\delta\theta\partial_\rho\delta\varphi-\partial_\rho\delta\theta\partial_\nu\delta\varphi) \\
&= \frac{1}{2\pi}\frac{1}{2}\int d^2xdt\epsilon^{\mu\nu\rho}A_\mu(\partial_\nu\Omega_\rho-\partial_\rho\Omega_\nu) \\
&= \frac{1}{2\pi}\int d^2xdt\epsilon^{\mu\nu\tau}A_\mu\partial_\nu\Omega_\tau
\end{aligned}
\tag{335}
$$

Then

$$
\delta S_{2D} = \frac{1}{2\pi}\int d^2xdt\epsilon^{\mu\nu\tau}\delta A_\mu\partial_\nu\Omega_\tau \tag{336}
$$

Hence the response function is

$$
j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu\Omega_\rho \tag{337}
$$

For a concrete example, consider the Dirac model $h(\mathbf{k}) = \sum_a d_a(\mathbf{k})\Gamma^a$ that we have discussed where

$$
C_2 = \frac{3}{8\pi^2}\int d^4k\epsilon^{abcde}\hat{d}_a\partial_x\hat{d}_b\partial_y\hat{d}_c\partial_z\hat{d}_d\partial_w\hat{d}_e \tag{338}
$$

and

$$
d_a(\mathbf{k}) = \left[\left(m+c\sum_i\cos k_i\right),\sin k_x,\sin k_y,\sin k_z,\sin k_w\right] \tag{339}
$$

Here $d_a(\mathbf{k}) \to d_a(\mathbf{k},\theta,\varphi)$. Hence

$$
G_2(\theta,\varphi) = -2\pi C_2 = -\frac{3}{4\pi}\int d^2k\epsilon^{abcde}\frac{d_a\partial_{k_x}d_b\partial_{k_y}d_c\partial_\theta d_d\partial_\varphi d_e}{|\mathbf{d}(\mathbf{k},\theta,\varphi)|^5} \tag{340}
$$

Hence the gauge curvature is

$$
\begin{aligned}
\partial_\mu\Omega_\nu-\partial_\nu\Omega_\mu =& -\frac{3}{4\pi}\int d^2k\epsilon^{abcde}\frac{d_a\partial_{k_x}d_b\partial_{k_y}d_c\partial_\theta d_d\partial_\varphi d_e}{|\mathbf{d}(\mathbf{k},\theta,\varphi)|^5}(\partial_\mu\delta\theta\partial_\nu\delta\varphi-\partial_\nu\delta\theta\partial_\mu\delta\varphi) \\
=& -\frac{3}{4\pi}\int d^2k\epsilon^{abcde}\frac{d_a\partial_{k_x}d_b\partial_{k_y}d_c\partial_\mu d_d\partial_\nu d_e}{|\mathbf{d}(\mathbf{k},\theta,\varphi)|^5}+\frac{3}{4\pi}\int d^2k\epsilon^{abcde}\frac{d_a\partial_{k_x}d_b\partial_{k_y}d_c\partial_\nu d_d\partial_\mu d_e}{|\mathbf{d}(\mathbf{k},\theta,\varphi)|^5} \\
=& -3\epsilon^{abcde}\int\frac{d^2k}{2\pi}\frac{d_a\partial_{k_x}d_b\partial_{k_y}d_c\partial_\mu d_d\partial_\nu d_e}{|\mathbf{d}(\mathbf{k},\theta,\varphi)|^5}
\end{aligned}
\tag{341}
$$

Consider a different Dirac model

$$
h(\mathbf{k},\mathbf{n}) = \sin k_x\Gamma^1+\sin k_y\Gamma^2+(\cos k_x+\cos k_y-2)\Gamma^0+m\sum_{a=0,3,4}\hat{n}_a\Gamma^a \tag{342}
$$

For such a model the **d** vector can be rewritten as

$$\mathbf{d}(\mathbf{k},\theta,\varphi)=\mathbf{d}_0(\mathbf{k})+\begin{pmatrix}0\\0\\m\hat{\mathbf{n}}\end{pmatrix} \tag{343}$$

where

$$\mathbf{d}_0(\mathbf{k})=(\sin k_x,\sin k_y,0,0,\cos k_x+\cos k_y-2) \tag{344}$$

In the $m \ll 2$ limit, the article states that (Although personally, I disagree with this limit.)

$$h(\mathbf{k},\hat{\mathbf{n}})\simeq \Sigma_{a=1,2}k_a\Gamma^a+\Sigma_{b=0,3,4}m\hat{n}_b\Gamma^b \tag{345}$$

In this case, the gauge curvature is

$$\begin{aligned}
\partial_\mu\Omega_\nu-\partial_\nu\Omega_\mu=&-3\epsilon^{abcde}\int\frac{d^2k}{2\pi}\frac{d_a\partial_{k_x}d_b\partial_{k_y}d_c\partial_\mu d_d\partial_\nu d_e}{|\mathbf{d}(\mathbf{k},\theta,\varphi)|^5}\\
=&-3\epsilon^{12ade}\int\frac{d^2k}{2\pi}\frac{d_a\partial_\mu d_d\partial_\nu d_e}{|\mathbf{d}(\mathbf{k},\theta,\varphi)|^5}\\
=&-3\epsilon^{12ade}\int\frac{d^2k}{2\pi}\frac{d_a\partial_\mu d_d\partial_\nu d_e}{\left(\sqrt{k_x^2+k_y^2+m^2}\right)^5}\\
=&-3\epsilon^{12ade}\frac{1}{2\pi}d_a\partial_\mu d_d\partial_\nu d_e\frac{\pi\left(\frac{8\pi m}{(m^2+4\pi^2)\sqrt{m^2+8\pi^2}}+\arctan\frac{4\pi^2}{\pi\left(m\sqrt{m^2+8\pi^2}\right)}\right)}{3m^3}\\
=&-\frac{\pi}{4}\epsilon^{ade}\hat{n}_a\partial_\mu\hat{n}_d\partial_\nu\hat{n}_e\\
=&-\frac{\pi}{4}\hat{\mathbf{n}}\cdot\partial_\mu\hat{\mathbf{n}}\times\partial_\nu\hat{\mathbf{n}}
\end{aligned} \tag{346}$$

Hence

$$j^\mu=\frac{1}{4\pi}\epsilon^{\mu\nu\rho}(\partial_\nu\Omega_\rho-\partial_\rho\Omega_\nu)=-\frac{1}{16}\epsilon^{\mu\nu\rho}\hat{\mathbf{n}}\cdot\partial_\nu\hat{\mathbf{n}}\times\partial_\rho\hat{\mathbf{n}} \tag{347}$$

Lets just go back to see the physical consequnces of the current term. Consider the following parameter field configuration:

$$\varphi(t)=2\pi t/T \quad \theta(\vec{x})=\theta(y)=\frac{\pi}{2}\left[1+\tanh\left(\frac{y}{\xi}\right)\right] \tag{348}$$

The current along the x direction is

$$j_x=\frac{1}{2\pi}\left(\partial_y\Omega_t-\partial_t\Omega_y\right) \tag{349}$$

Let φ evolve from 0 to 2π. The charge flowed through $x = 0$ is

$$\begin{aligned}
\Delta Q &= \int dt I_x = \int dt dy \left(\partial_y \Omega_t - \partial_t \Omega_y\right) \frac{1}{2\pi} \\
&= \int dt dy \frac{1}{2\pi} \left(\partial_\theta \Omega_\varphi - \partial_\varphi \Omega_\theta\right) \left(\partial_y \delta\theta \partial_t \delta\varphi - \partial_t \delta\theta \partial_y \delta\varphi\right) \\
&= \int_0^\pi d\theta \int_0^{2\pi} d\varphi \frac{1}{2\pi} \left(\partial_\theta \Omega_\varphi - \partial_\varphi \Omega_\theta\right) \\
&= \int_0^\pi d\theta \frac{1}{2\pi} \int_0^{2\pi} d\varphi \partial_\theta \Omega_\varphi = \int_0^\pi d\theta \partial_\theta \left(\int_0^{2\pi} d\varphi \Omega_\varphi \frac{1}{2\pi} \right) \\
&= \int_0^\pi d\theta \partial_\theta \left(- \int d^2 k d\varphi \mathcal{K}^\theta \right) \\
&= - \int_0^\pi dP_3(\theta)
\end{aligned} \tag{350}$$

Since the Dirac Hamiltonian is TRI, when $-4c \leq m \leq -2c$:

$$\int_0^\pi dP_3(\theta) = \frac{1}{2} C_2 = \frac{1}{2} \tag{351}$$

The pumped charge is

$$\Delta Q = -\frac{1}{2} \tag{352}$$

4.2 $\mathbf{Z_2}$ Classification of TRI Insulators

For any two (2+1)-dimensional TRI insulators $h_1(\mathbf{k})$ and $h_2(\mathbf{k})$, define a interpolation $h(\mathbf{k}, \theta)$ between them such that

$$\begin{aligned}
&h(\mathbf{k}, 0) = h_1, \quad h(\mathbf{k}, \pi) = h_2 \\
&T^\dagger h(-\mathbf{k}, -\theta) T = h^T(\mathbf{k}, \theta)
\end{aligned} \tag{353}$$

hence $h(\mathbf{k}, \theta)$ is a (3+1)-dimensional TRI insulator with $N_3[h(\mathbf{k}, \theta)] = \pm 1$. Next, we shall prove that $N_3[h(\mathbf{k}, \theta)]$ is indepedent of the choice of the interpolation $h(\mathbf{k}, \theta)$.

Consider two interpolations $h(\mathbf{k}, \theta)$ and $h'(\mathbf{k}, \theta)$. Define the interpolation between the interpolations as $g(\mathbf{k}, \theta, \varphi)$ such that

$$\begin{aligned}
g(\mathbf{k}, \theta, 0) = h(\mathbf{k}, \theta), &\quad g(\mathbf{k}, \theta, \pi) = h'(\mathbf{k}, \theta) \\
g(\mathbf{k}, 0, \varphi) = h_1(\mathbf{k}), &\quad g(\mathbf{k}, \pi, \varphi) = h_2(\mathbf{k}) \\
g^T(\mathbf{k}, \theta, \varphi) &= T^\dagger g(-\mathbf{k}, -\theta, -\varphi) T
\end{aligned} \tag{354}$$

Firstly, the interpolation can be viewed as an interpolation between $h(\mathbf{k}, \theta)$ and $h'(\mathbf{k}, \theta)$. Hence

$$(-1)^{C_2[g]} = N_3[h] N_3[h'] \tag{355}$$

Secondly the interpolation can be view as an interpolation between $h_1(\mathbf{k})$ and $h_2(\mathbf{k})$. Since the above two Hamiltonians are independent of φ, we can define another (4+1)-dimensional interpolation between them such that the (4+1)-dimensional interpolation is also independent of φ. Then the gauge vector $a_\varphi = 0$ which means the Chern number of this interpolation is zero. Due to the equivalence between the two (4+1)-dimensional interpolations, we learn that

$$(-1)^{C_2[g]} = 1 \tag{356}$$

Consequently,

$$N_3[h] = N_3[h'] \tag{357}$$

We thus porved that the $N_3[h(\mathbf{k}, \theta)]$ is indenpendent of the choice of the interpolation. Hence the Z_2 classification in (2+1)-dimensional TRI insulators is well defined.

4.3 Physical Properties of the $\mathbf{Z_2}$ Nontrivial Insulators

Consider the interface between a nontrivial TRI insulator $h(\mathbf{k})$ and a trivial TRI insulator h_0. Define an interpolation between them $h(\mathbf{k}, \theta)$ such that $h(\mathbf{k}, 0) = h_0$ and that $h(\mathbf{k}, \pi) = h_1(\mathbf{k})$. Given a field configuration

$$\theta(x, y) = \frac{\pi}{2}\left[1 - \tanh\left(\frac{y}{\xi}\right)\right] \tag{358}$$

We can see that $h[\mathbf{k}, \theta(y)]$ and $h[\mathbf{k}, -\theta(y)]$ both creats domain walls between h_0 and $h_1(\mathbf{k})$, as $\theta = \pi (y \to -\infty)$ and $\theta = 0 (y \to \infty)$. Consider the following Hamiltonian

$$h(\mathbf{k}, \mathbf{x}) = \begin{cases} h[\mathbf{k}, \theta(y)], & x < 0 \\ h[\mathbf{k}, -\theta(y)], & x > 0 \end{cases} \tag{359}$$

Consider a rectangular-shpaed loop surrounding the $x = 0, y = 0$ point. The localized charge is

$$Q = \int_A d^2x\rho = \frac{1}{2\pi}\int_A d^2x \left(\partial_x\Omega_y - \partial_y\Omega_x\right) = \frac{1}{2\pi}\oint_C \Omega \cdot d\mathbf{l} \tag{360}$$

Since

$$\Omega_\mu \equiv \Omega_\theta \partial_\mu \delta\theta + \Omega_\varphi \partial_\mu \delta\varphi = \Omega_\theta \partial_\mu \delta\theta \quad \delta\varphi = 0 \text{ here} \tag{361}$$

we can see that

$$\omega \cdot d\mathbf{l} = \Omega_\theta d\theta \tag{362}$$

If we let the loop to be much larger than the interface width ξ, then θ varies from $-\pi$ to π hence

$$Q = \frac{1}{2\pi}\oint \Omega_\theta d\theta = \int d^2k d\theta \mathcal{K}^\varphi = P_3[h(\mathbf{k}), \theta] = (n + \frac{1}{2}) \tag{363}$$

Integrate the ρ and j^x in the y direction

$$\begin{aligned} \rho_{1d}(x) &= \frac{1}{2\pi}\int_{-L}^{L} dy \left(\partial_x\Omega_y - \partial_y\Omega_x\right) \\ j_{1d}(x) &= \frac{1}{2\pi}\int_{-L}^{L} dy \left(\partial_y\Omega_t - \partial_t\Omega_y\right) \end{aligned} \tag{364}$$

where $L \gg \xi$. Since in this case $\delta\varphi = 0$ and that in the deep vacuum zone and the deep nontrivial insulator zone, where $|y| \gg \xi$, $\delta\theta = 0$, we can see that $\Omega \to 0$ when $|y| \to \infty$ Hence the above equations can be simplified as

$$\begin{aligned} \rho_{1d}(x) &= \frac{1}{2\pi}\int_{-L}^{L} dy\,(\partial_x\Omega_y) = \partial_x P_3(x,t) \\ j_{1d}(x) &= \frac{1}{2\pi}\int_{-L}^{L} dy\,(-\partial_t\Omega_y) = -\partial_t P_3(x,t) \end{aligned} \tag{365}$$

where

$$P_3(x,t) = \int_{-L}^{L} dy\Omega_y(x,y,t)/2\pi \tag{366}$$

is the magnetoelectric polarization.

Consider a time dependent Hamiltonian $h(\mathbf{k},\mathbf{x},t)$ that satisfies

$$\begin{aligned} h(\mathbf{k},\mathbf{x},t=0) &= h[\mathbf{k},\theta(y)] \\ h(\mathbf{k},\mathbf{x},t=T) &= h[\mathbf{k},-\theta(y)] \end{aligned} \tag{367}$$

The pumped charge is

$$\begin{aligned} Q_{\text{pump}} &= \int_0^T dt j_{1d}(x) = -[P_3(x,T) - P_3(x,0)] \\ &= -\left[\int_{-\pi}^{0} d\theta\Omega_\theta \frac{1}{2\pi} - \int_{\pi}^{0} d\theta\Omega_\theta \frac{1}{2\pi}\right] \\ &= -\int_{-\pi}^{\pi} d\theta \int d^2k K^\varphi = -P_3 = -\left(n + \frac{1}{2}\right) \end{aligned} \tag{368}$$

5 Unified Theory of Topological Insulators

5.1 Phase Space Chern Simon Theory

Rewrite the effective theory of the IQHE:

$$\begin{aligned} S_{\text{eff}} &= \frac{1}{2\pi}\int dk_x \int dk_y f_{xy}(\mathbf{k}) \frac{1}{4\pi}\int d^2x \int dt A_\mu \epsilon^{\mu\nu\tau}\partial_\nu A_\tau \\ &= \frac{1}{4\pi}\int \frac{d^2k}{2\pi}\epsilon^{ij}\partial_i a_j \int dt d^2x \epsilon^{\mu\nu\tau} A_\mu \partial_\nu A_\tau \\ &= \frac{1}{4\pi}\int \frac{d^2k}{2\pi}\epsilon^{ij}\,\mathrm{tr}[\partial_i a_j] \int dt d^2x \epsilon^{\mu\nu\tau} A_\mu \partial_\nu A_\tau \end{aligned} \tag{369}$$

We define the phase space coordinate as $\mathbf{q} = (t, x, y, k_x, k_y)$ and the phase space gauge vectors $\mathbf{A} = (A_0, A_1, A_2, 0, 0)$ and $\mathbf{a} = (0, 0, 0, a_1, a_2)$. Then the above can be written as

$$S_{2+1} = \frac{1}{8\pi^2}\int d^5q \epsilon^{ABCDE} A_A \partial_B A_C \,\mathrm{Tr}\left[\partial_D a_E\right] \tag{370}$$

Consider the dimensional reduction process:

Step 1. Replace $k_y + A_y$ by a parameter $\theta(x,t)$. (One should notice that A_y is replaced by $\theta(x,t) - \theta_0$ and that k_y is replaced by θ_0).

Step 2. Replace ∂_{k_y} with $\partial\theta$. Hence the Berry gauge field $a_{k_y}^{\alpha\beta} = -i \left\langle \alpha; \mathbf{k} \left| \partial_{k_y} \right| \beta; \mathbf{k} \right\rangle$ is replaced by $a_\theta^{\alpha\beta} = -i \left\langle \alpha; k_x, \theta \left| \partial_\theta \right| \beta; k_x, \theta \right\rangle$.

Step 3. Remove the $\int dy$ and $\int \frac{dk_y}{2\pi}$ terms.

We should keep in mind that after the dimensional reduction process, all things are no longer dependent on the already reduced dimension. Hence

$$
\begin{aligned}
S_{1+1} =& \frac{1}{4\pi} \int dk \epsilon^{ij} \operatorname{Tr}\left[\partial_i a_j\right] \int dt dx \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \\
=& \frac{1}{4\pi} \int dk \operatorname{Tr}\left[\partial_{k_x} a_\theta - \partial_\theta a_{k_x}\right] \int dt dx \left(-\theta \partial_x A_t + A_t \partial_x \theta + \theta \partial_t A_x - A_x \partial_t \theta\right) \\
=& \frac{1}{2\pi} \int dk \operatorname{Tr}\left[\partial_{k_x} a_\theta - \partial_\theta a_{k_x}\right] \int dt dx \left(A_t \partial_x \theta - A_x \partial_t \theta\right)
\end{aligned}
\tag{371}
$$

We should notice that the Barry phase gauge field can gain real space components:

$$
a_\theta^{\alpha\beta} \partial_\mu \theta = \langle \alpha, k_x, \theta(x,t) | \partial_\mu \theta \partial_\theta | \beta, k_x, \theta(x,t) \rangle = \langle \alpha, k_x, \theta(x,t) | \partial_\mu | \beta, k_x, \theta(x,t) \rangle \equiv a_\mu^{\alpha\beta} \tag{372}
$$

Also since the spatial dependence of the Barry phase gauge field is introduced solely by $\theta(x,t)$

$$
\partial_\mu a_{k_x} = \partial_\theta a_{k_x} \partial_\mu \theta \tag{373}
$$

Hence

$$
\begin{aligned}
S_{1+1} =& \frac{1}{2\pi} \int dk dt dx \left\{ A_t \operatorname{Tr}[\partial_{k_x} a_x - \partial_x a_{k_x}] - A_x \operatorname{Tr}[\partial_{k_x} a_t - \partial_t a_{k_x}] \right\} \\
=& \frac{1}{2\pi} \int d^3 q \epsilon^{ABC} A_A \operatorname{Tr}\left[\partial_B a_C\right]
\end{aligned}
\tag{374}
$$

where the phase space is $(t, x, 0)$ and the phase space vector is $(A_t, A_x, 0)$. Reduce the dimensions one more time. This time we should keep in mind that the gauge field is already time dependent due to the last dimesional reduction procedure. In higher dimensions, the gauge field can be spatial and temporal dependent due to prior dimensional reductions. Hence

$$
\begin{aligned}
S_{0+1} &= \int dt - (\phi - \phi_0) \operatorname{Tr}\left[\partial \phi_0 a_t - \partial t a_\phi\right] \\
&= - \int dt \operatorname{Tr}\left[\partial_t (\phi - \phi_0) a_\phi - \partial \phi_0 (\phi - \phi_0) a_t\right] \\
&= - \int dt \operatorname{Tr}\left[\partial_t \phi a_\phi + a_t\right] \\
&= - \int dt \operatorname{Tr}\left[\partial_t \phi \tilde{a}_\phi + \tilde{a}_t\right]
\end{aligned}
\tag{375}
$$

In the last line, we should keep in minde that

$$\tilde{a}_t^{\alpha\beta} = -i \left\langle \alpha; t, \phi(t) \left| \left(\frac{\partial}{\partial t} \right)_\phi \right| \beta; t, \phi(t) \right\rangle \tag{376}$$

and that

$$\tilde{a}_\phi^{\alpha\beta} = -i \left\langle \alpha; t, \phi(t) \left| \left(\frac{\partial}{\partial \phi} \right)_t \right| \beta; t, \phi(t) \right\rangle \tag{377}$$

Hence the real a_t is

$$\begin{aligned} a_t^{\alpha\beta} &= -i \left\langle \alpha; t, \phi(t) \left| \partial_t \right| \beta; t, \phi(t) \right\rangle \\ &= -i \langle \alpha; t, \phi | \left(\frac{\partial}{\partial t} \right)_\phi + \frac{\partial \phi}{\partial t} \left(\frac{\partial}{\partial \phi} \right)_t \bigg] | \beta; t, \phi \rangle = \tilde{a}_t^{\alpha\beta} + \partial_t \phi \tilde{a}_\phi^{\alpha\beta} \end{aligned} \tag{378}$$

Hence the action can be written as

$$S_{0+1} = - \int dt \, \mathrm{Tr} \left[a_t \right] \tag{379}$$

Next, we will consider the (4+1)-dimensional case.

$$S_{eff} = \frac{1}{24\pi^2} \frac{1}{32\pi^2} \int d^4 x dt \, \epsilon^{\mu\nu\rho\sigma\tau} A_\mu \partial_\nu A_\rho \partial_\sigma A_\tau \int d^4 k \epsilon^{ijkl} \, \mathrm{tr} \left[f_{ij} f_{kl} \right] \tag{380}$$

Define $D_i = \partial_i + a_i$. As

$$f_{ij} = \partial_i a_j - \partial_j a_i + i \left[a_i, a_j \right] \tag{381}$$

we can see that

$$\begin{aligned} D_i a_j - D_j a_i &= \left(\partial_i + i a_i \right) a_j - \left(\partial_j + i a_j \right) a_i \\ &= \partial_i a_j - \partial_j a_i + i \left[a_i, a_j \right] = f_{ij} \end{aligned} \tag{382}$$

Hence

$$\begin{aligned} \epsilon^{ijkl} \, \mathrm{tr} \left[f_{ij} f_{kl} \right] &= \epsilon^{ijkl} \, \mathrm{tr} \left[\left(D_i a_j - D_j a_i \right) \left(D_k a_l - D_l a_k \right) \right] \\ &= 4 \, \epsilon^{ijkl} \, \mathrm{tr} \left[D_i a_j D_k a_l \right] \end{aligned} \tag{383}$$

Hence

$$\begin{aligned} S_{4+1} &= \frac{1}{192\pi^4} \int d^4 x dt \epsilon^{\mu\nu\rho\sigma\tau} A_\mu \partial_\nu A_\rho \partial_\sigma A_\tau \int d^4 k \epsilon^{ijkl} \, \mathrm{tr} \left[D_i a_j D_k a_l \right] \\ &= \frac{1}{192\pi^4} \int d^9 q \epsilon^{ABCDEFGHl} A_A \partial_B A_C \partial_D A_E \, \mathrm{Tr} \left[D_F a_G D_H a_l \right] \end{aligned} \tag{384}$$

Dimensional reduction to (3+1) dimensions

$$
\begin{aligned}
S_{3+1} =&\frac{1}{96\pi^3}\int d^3xdt\epsilon^{\mu\nu\rho\sigma\tau}A_\mu\partial_\nu A_\rho\partial_\sigma A_\tau\int d^3k\epsilon^{ijkl}\,\mathrm{tr}\,[D_ia_jD_ka_l]\\
=&\frac{1}{96\pi^3}\int d^7q\epsilon^{\nu\rho\sigma\tau}\theta\partial_\nu A_\rho\partial_\sigma A_\tau\int d^3k\epsilon^{ijkl}\,\mathrm{tr}\,[D_ia_jD_ka_l]\\
&+\frac{1}{96\pi^3}\int d^7q\epsilon^{\mu\nu\sigma\tau}A_\mu\partial_\nu\theta\partial_\sigma A_\tau\int d^3k\epsilon^{ijkl}\,\mathrm{tr}\,[D_ia_jD_ka_l]\\
&+\frac{1}{96\pi^3}\int d^7q\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu A_\rho\partial_\sigma\theta\int d^3k\epsilon^{ijkl}\,\mathrm{tr}\,[D_ia_jD_ka_l]\\
=&\frac{3}{96\pi^3}\int d^7q\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu A_\rho\partial_\sigma\theta\int d^3k\epsilon^{ijkl}\,\mathrm{tr}\,[D_ia_jD_ka_l]\\
=&\frac{3}{96\pi^3}\int d^7q\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu A_\rho\partial_\sigma\theta\int d^3k\epsilon^{\theta ijk}\,\mathrm{tr}\,[D_\theta a_iD_ja_k]\\
&+\frac{3}{96\pi^3}\int d^7q\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu A_\rho\partial_\sigma\theta\int d^3k\epsilon^{i\theta jk}\,\mathrm{tr}\,[D_ia_\theta D_ja_k]\\
&+\frac{3}{96\pi^3}\int d^7q\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu A_\rho\partial_\sigma\theta\int d^3k\epsilon^{ij\theta k}\,\mathrm{tr}\,[D_ia_jD_\theta a_k]\\
&+\frac{3}{96\pi^3}\int d^7q\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu A_\rho\partial_\sigma\theta\int d^3k\epsilon^{ijk\theta}\,\mathrm{tr}\,[D_ia_jD_ka_\theta]\\
=&\frac{3}{96\pi^3}\int d^7q\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu A_\rho\int d^3k\epsilon^{\theta ijk}\,\mathrm{tr}\,[D_\sigma a_iD_ja_k]\\
&+\frac{3}{96\pi^3}\int d^7q\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu A_\rho\int d^3k\epsilon^{i\theta jk}\,\mathrm{tr}\,[D_ia_\sigma D_ja_k]\\
&+\frac{3}{96\pi^3}\int d^7q\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu A_\rho\int d^3k\epsilon^{ij\theta k}\,\mathrm{tr}\,[D_ia_jD_\sigma a_k]\\
&+\frac{3}{96\pi^3}\int d^7q\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu A_\rho\int d^3k\epsilon^{ijk\theta}\,\mathrm{tr}\,[D_ia_jD_ka_\sigma]\\
=&\int\frac{d^7q}{32\pi^3}\epsilon^{AB\cdots G}A_A\partial_BA_C\,\mathrm{Tr}\,[D_Da_ED_Fa_G]
\end{aligned}
\tag{385}
$$

We can then obtain the general rule of dimensional reduction

Step 1. Remove a $\partial_A A_B$ term and replace the (2n+1)-dimensional totally antisymmetric tensor with a (2n-1)-dimensional one.

Step 2. Remove the integrals on the reduced dimension.(Remove the $\int\frac{dx_d dk_d}{2\pi}$.)

Step 3. Multiply the action by n which is the number of external gauge fields in the original action.

We can then immediately obtain the (2+1)-dimensional action

$$
\begin{aligned}
S_{2+1} &= \int \frac{d^5 q}{32\pi^3} \epsilon^{ABCDE} A_A \operatorname{Tr}\left[D_B a_C D_D a_E\right] \times (2 \times 2\pi) \\
&= \int \frac{d^5 q}{8\pi^2} \epsilon^{ABCDE} A_A \operatorname{Tr}\left[D_B a_C D_D a_E\right]
\end{aligned}
\tag{386}
$$

More generally, the effective theory for a (2n+1)-dimensional topological insulator is

$$
S_{2n+1} = \frac{C_n}{(n+1)!(2\pi)^n} \int d^{2n+1} x \epsilon^{\mu_1 \mu_2 \cdots \mu_{2n+1}} \times A_{\mu_1} \partial_{\mu_2} A_{\mu_3} \cdots \partial_{\mu_{2n}} A_{\mu_{2n+1}} \tag{387}
$$

where

$$
C_n = \frac{1}{n! 2^n (2\pi)^n} \int d^{2n} k \epsilon^{i_1 i_2 \cdots i_{2n}} \operatorname{Tr}\left[f_{i_1 i_2} f_{i_3 i_4} \cdots f_{i_{2n-1} i_{2n}}\right] \tag{388}
$$

Hence the action can also be written as

$$
\begin{aligned}
S_{2n+1} =& \frac{1}{(n+1)!(2\pi)^n} \frac{1}{n! 2^n (2\pi)^n} \int d^{2n+1} x \epsilon^{\mu_1 \mu_2 \cdots \mu_{2n+1}} \times A_{\mu_1} \partial_{\mu_2} A_{\mu_3} \cdots \partial_{\mu_{2n}} A_{\mu_{2n+1}} \\
&\times \int d^{2n} k \epsilon^{i_1 i_2 \cdots i_{2n}} \operatorname{Tr}\left[D_{i_1} a_{i_2} D_{i_3} a_{i_4} \cdots D_{i_{2n-1}} a_{i_{2n}}\right] \times 2^n \\
=& \frac{1}{n!(n+1)!(2\pi)^{2n}} \int d^{4n+1} q \epsilon^{A_1 A_2 \cdots A_{4n+1}} A_{A_1} \partial_{A_2} A_{A_3} \cdots \partial_{A_{2n}} A_{A_{2n+1}} \operatorname{Tr}\left[D_{A_{2n+2}} a_{A_{2n+3}} \cdots D_{A_{4n}} a_{A_{4n+1}}\right]
\end{aligned}
\tag{389}
$$

In the original action, there are only $(n+1)$ terms of external gauge fields. Thus a (2n+1)-dimensional action can at most have (n+1) decendants. Its mth decendant is

$$
\begin{aligned}
S^{(m)}_{2n+1-m} =& \frac{(n+1)\cdots(n-m+2)}{n!(n+1)!(2\pi)^{2n-m}} \int d^{4n+1-2m} q \epsilon^{A_1 A_2 \cdots A_{4n+1-2m}} A_{A_1} \partial_{A_2} A_{A_3} \cdots \partial_{A_{2n-2m}} A_{A_{2n-2m+1}} \\
&\times \operatorname{Tr}\left[D_{A_{2n-2m+2}} a_{A_{2n-2m+3}} \cdots D_{A_{4n-2m}} a_{A_{4n-2m+1}}\right] \\
=& \frac{1}{n!(n-m+1)!(2\pi)^{2n-m}} \int d^{4n+1-2m} q \epsilon^{A_1 A_2 \cdots A_{4n+1-2m}} A_{A_1} \partial_{A_2} A_{A_3} \cdots \partial_{A_{2n-2m}} A_{A_{2n-2m+1}} \\
&\times \operatorname{Tr}\left[D_{A_{2n-2m+2}} a_{A_{2n-2m+3}} \cdots D_{A_{4n-2m}} a_{A_{4n-2m+1}}\right] \\
=& \frac{(2n+1+m)\cdots(n-m+2)}{n!(2n-m+1)!(2\pi)^{2n-m}} \int d^{4n+1-2m} q \epsilon^{A_1 A_2 \cdots A_{4n+1-2m}} A_{A_1} \partial_{A_2} A_{A_3} \cdots \partial_{A_{2n-2m}} A_{A_{2n-2m+1}} \\
&\times \operatorname{Tr}\left[D_{A_{2n-2m+2}} a_{A_{2n-2m+3}} \cdots D_{A_{4n-2m}} a_{A_{4n-2m+1}}\right] \\
=& \frac{\begin{pmatrix} 2n+1-m \\ n \end{pmatrix}}{(2n-m+1)!} \int \frac{d^{4n+1-2m} q}{(2\pi)^{2n-m}} \epsilon^{A_1 A_2 \cdots A_{4n+1-2m}} A_{A_1} \partial_{A_2} A_{A_3} \cdots \partial_{A_{2n-2m}} A_{A_{2n-2m+1}} \\
&\times \operatorname{Tr}\left[D_{A_{2n-2m+2}} a_{A_{2n-2m+3}} \cdots D_{A_{4n-2m}} a_{A_{4n-2m+1}}\right] \\
\equiv& \mathrm{CS}^{n-m+1}_{4n-2m+1}
\end{aligned}
\tag{390}
$$

where CS_s^t is a mixed Chern Simons term. We should notice that t indicates the number of the external gauge field terms and s is the dimension of the phase space. Consider a (d+1)-dimensional space. Immediately we learn that $s = 2d+1$. Since there are t external fields and $(t-1)$ derivatives acting on the external gauge fields. Hence if we want non zero CS_s terms, we should require that

$$2t - 1 \leq d + 1 \tag{391}$$

Hence $0 \leq t \leq [d/2] + 1$:

$$\mathrm{CS}_{2d+1}^t, t = 0, 1, \ldots, [d/2] + 1 \tag{392}$$

5.2 $\mathbf{Z_2}$ Topological Insulator in Generic Dimensions

We will first consider the charge-hole transformation and time-reversal transformation.

$$C : A_\mu \to -A_\mu, \quad T : A_\mu \to \begin{cases} A_0 \\ -A_i \end{cases} \tag{393}$$

For each transformation, the change of the momentum operator $-i\partial_\mu$ is the same as the change of the external gauge fields. As both of the transformations are antiunitary, we can see that

$$C : \partial_\mu \to \partial_\mu, \quad T : \partial_\mu \to \begin{cases} -\partial_0 \\ \partial_i \end{cases} \tag{394}$$

Hence, under particle-hole transformation:

$$C : S_{2n+1}^{CS} \to (-1)^{n+1} S_{2n+1}^{CS} \quad T : S_{2n+1}^{CS} \to (-1)^n S_{2n+1}^{CS} \tag{395}$$

Hence S_{4n+1}^{CS} is C odd but T even, while S_{4n-1}^{CS} is C even but T odd. For different dimensions we should choose their symmetries accordingly.

Next, we will try to put the Z_2 classification on the (1+1)-dimensional TRI insulators and find out that it does not work. Consider following Hamiltonian

$$h_{2\mathrm{D}}(\mathbf{k}) = \Gamma^1 \sin k_x + \Gamma^2 \sin k_y + \Gamma^0 \left[m + c\left(\cos k_x + \cos k_y\right)\right] \tag{396}$$

From our privious study, we can see that when $-2c \leq m \leq 2c$, this Hamiltonian is not trivial. Consider this Hamiltonian as an interpolation between $h_1(k) = h_{2\mathrm{D}}(k, 0)$ and $h_2(k) = h_{2\mathrm{D}}(k, \pi)$. In this way, the two (1+1)-dimensional Hamiltonian cannot be continuously deformed to each other without breaking the time reversal symmetry. However we can actually construct a such interpolation:

$$h_0(k, \theta) = \Gamma^1 \sin k + \Gamma^{02} \sin^2 \theta + \Gamma^0 \left(m + c \cos k_x + c \cos \theta\right) \tag{397}$$

where $\Gamma^{02} = i\Gamma^0\Gamma^2$ is also time reversal invariant. Hence in this Hamiltonian, for each θ the one dimensional $h_0(k) = h_0(k, \theta)$ is always TRI. Thus in this way the h_1 and h_2 can actually be adiabatically deformed into each other which is in contradiction to the nontriviality of the h_{2D} interpolation.

If we want a Z_2 classification to work we should require that the base Hamiltonian space is simply connected which means each two Hamiltonians is path connected and the paths (interpolations) between them can be adiabatically deformed into each other. Equivalently, any closed loops in this space can be continuously deformed into one point. However, we can see that the closed loop formed by the above mentioned two Hamiltonians are unable to be deformed to one point.

$$g(k,\theta) = \begin{cases} h_{2\mathrm{D}}(k,\theta), & \theta \in [0,\pi] \\ h_0(k, 2\pi - \theta), & \theta \in [\pi, 2\pi] \end{cases} \tag{398}$$

Since the second one is trivial and the upper one is nontrivial, the winding number (Chern number) of this loop is nonzero.

Generally, the Z_2 classification of the descendants of $(2n+1)$-dimensional insulators only works for $[(2n-1)+1]$-dimensional and $[(2n-2)+]$-dimensional descendants. This phenomena can also be viewed from another perspective. Back to the case we are studying, the edge theory of the (4+1)-dimensional insulators with $C_2 = 1$ is described by

$$H_{\partial(4+1)} = v\vec{\sigma} \cdot \overrightarrow{\mathbf{p}} \tag{399}$$

While in its (3+1)-dimensional nontrivial descendant, the edge theory is

$$H_{\partial(3+1)} = v\left(\sigma_x p_x + \sigma_y p_y\right) \tag{400}$$

which is stable due to the prevention of mass term by TRI. The edge teory of a (2+1)-dimensional nontrivial descendant is

$$H_{\partial(2+1)} = vx\sigma_z p_z \tag{401}$$

Now the edge theory of a (1+1)-dimensional nontrivial descendant is

$$H_{\partial(0+1)} = 0 \tag{402}$$

It does not have a nontrivial structure. Hence we can decide whether Z_2 classification can be performed by studying whether the edge state of the system is stable.

6 Epilogue

I managed to cover most of the topics in the article, however, due to limited time, there are some imperfections. I will keep furnishing this paper.

7 References

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