Chapter 1

Green function

The Green function the fundamental response function of many body system. The single particle green function is defined as

$$G(t-t') = -\mathrm{i}\langle\phi \mid \mathcal{T}\psi(t)\psi^{\dagger}(t') \mid \phi\rangle \tag{1.1}$$

where ϕ is many body ground state . The operator $\psi(t)$ is defined on Heisenberg representation. We define the propagator as the Green function on the momentum space , namly

$$G(k,\omega) = -\mathrm{i}\int \frac{d^3k}{(2\pi)^2}\frac{d\omega}{2\pi}\langle\phi \mid \mathcal{T}\psi(x,t)\psi^{\dagger}(x',t') \mid \phi\rangle e^{\mathrm{i}k\cdot(x-x')-\omega(t-t')} \tag{1.2}$$

Figure 1.1: Single Fermion Propagator

1.1 Fermion Green function

Let's define th e ground state $\mid \phi\rangle$ as

$$\mid \phi\rangle = \prod_{|k|<k_f,\sigma} c^{\dagger}_{k\sigma} \mid 0\rangle \tag{1.3}$$

According to definition of Green function (1.1), the fermion green function could be calculated as

$$G(k,t-t') = -\mathrm{i}\langle\phi \mid \mathcal{T}c_{k\sigma}(t)c^{\dagger}_{k'\sigma'}(t') \mid \phi\rangle = -\mathrm{i}\theta(t-t')\delta_{kk'}(1-n_{k'\sigma'})e^{\mathrm{i}\omega_k(t'-t)} - \mathrm{i}\theta(t'-t)\delta_{kk'}n_{k\sigma}e^{\mathrm{i}\omega_k(t'-t)} \tag{1.4}$$

We consider the Fourier transformation to find the propagator

$$\begin{aligned}
G(k,\omega) &= \int_{-\infty}^{+\infty} dt - \mathrm{i}\left(\theta(t)(1-n_{k\sigma}) - \theta(-t)n_{k\sigma}\right)e^{\mathrm{i}(\omega-\varepsilon_k)t} \\
&= -\mathrm{i}\lim_{\varepsilon\to 0}\int_0^{\infty}\theta_{k-k_F}e^{\mathrm{i}(\omega-\omega_k+\mathrm{i}\varepsilon)t} + \mathrm{i}\lim_{\varepsilon\to 0}\int_{-\infty}^{0}\theta_{k_F-k}e^{\mathrm{i}(\omega-\omega_k+\mathrm{i}\varepsilon)t} \\
&= \lim_{\varepsilon\to 0}\frac{\theta_{k-k_F}}{\omega-\omega_k+\mathrm{i}\varepsilon} + \frac{\theta_{k_F-k}}{\omega-\omega_k+\mathrm{i}\varepsilon} \\
&= \frac{1}{\omega-\omega_k+\mathrm{i}\varepsilon}
\end{aligned} \tag{1.5}$$

The fermion green function consists two part. The first term stands for particle moving forwards in time. The second term stands for holes moving backwards in time. Hence, we define the free fermion Green function as

$$G(k,\omega) = \frac{1}{\omega - \omega_k + \mathrm{i}\varepsilon} = \xrightarrow{\quad k,\omega \quad} \tag{1.6}$$

Figure 1.2: Single Fermion Propagator

1.2 Boson Green function

By the smae way , the bosonic green function can be calculated with definition. The non-interacting bosonic gas caould be described by hamiltonian

$$H = \sum_q \omega_q \left(b_q^\dagger b_q + \frac{1}{2} \right) \tag{1.7}$$

The ground state of hamiltonian (1.7) is just vacuum $\mid 0\rangle$. The physical field is defined as [1]

$$\phi_q = \sqrt{\frac{\hbar}{2m\omega_q}} \left(b_q + b_{-q}^\dagger \right) \tag{1.8}$$

The Green function for bosons could be defined by field ϕ_q .

$$\begin{aligned} G(q,t,t') = -\mathrm{i}\langle \phi \mid \mathcal{T}\phi_q(t)\phi_{q'}^\dagger(t') \mid \phi\rangle &= -\mathrm{i}\frac{\hbar}{2m\omega_q}\langle \phi \mid \mathcal{T} b_q(t) b_q^\dagger(t') \mid \phi\rangle - \mathrm{i}\frac{\hbar}{2m\omega_q}\langle \phi \mid \mathcal{T} b_{-q}^\dagger(t) b_{-q}(t') \mid \phi\rangle \\ &= -\mathrm{i}\frac{\hbar}{2m\omega_q} e^{\mathrm{i}\omega_q(t'-t)}\theta(t-t') - \mathrm{i}\frac{\hbar}{2m\omega_q} e^{-\mathrm{i}\omega_q(t'-t)}\theta(t'-t) \end{aligned} \tag{1.9}$$

We calculate the propagator from Eq(1.9).

$$\begin{aligned} G(q,\omega) &= -\mathrm{i}\frac{\hbar}{2m\omega_q}\int_{-\infty}^{+\infty} dt\theta(t)e^{\mathrm{i}(\omega-\omega_q)t} + \theta(-t)e^{\mathrm{ii}(\omega+\omega_q)t} \\ &= -\mathrm{i}\frac{\hbar}{2m\omega_q}\left(\int_0^{+\infty} dt e^{\mathrm{i}(\omega-\omega_q+\mathrm{i}\varepsilon)t} + \int_{-\infty}^{0} dt e^{\mathrm{i}(\omega+\omega_q-\mathrm{i}\varepsilon)t} \right) \\ &= \frac{\hbar}{2m\omega_q}\left(\frac{1}{\omega-\omega_q+\mathrm{i}\varepsilon} - \frac{1}{\omega+\omega_q+\mathrm{i}\varepsilon} \right) \\ &= \frac{\hbar}{2m\omega_q}\frac{2\omega_q}{\omega^2 - (\omega_q+\mathrm{i}\varepsilon)^2} \end{aligned} \tag{1.10}$$

Note:-

The bosonic green function contains two part. The first part involves forward boson emitting process. The second part involves boson absorbing process .

$$G(q,\omega) = = \frac{\hbar}{2m\omega_q}\left(\frac{1}{\omega-(\omega_q+\mathrm{i}\varepsilon)} - \frac{1}{\omega+(\omega_q+\mathrm{i}\varepsilon)} \right) \tag{1.11}$$

At static limit $\omega \to 0$, then $G(q,\omega) = -\frac{\hbar}{2m\omega^2}$, it will induce effective atraction interaction.

[1]You can refer to scalar field quantization in field theory theory

Chapter 2

Imaginary-time Green function

In this section , we discuss imaniary time Green function. The imaginary Green function is defined as

$$G_{\lambda\lambda'}(\tau-\tau') = -\langle \mathcal{T}\psi_\lambda(\tau)\psi^\dagger_{\lambda'}(\tau')\rangle = -\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta()}\psi_\lambda(\tau)\psi^\dagger_{\lambda'}(\tau')\right] \tag{2.1}$$

For a non-interacting system, the expectaton is

$$\langle \psi^\dagger_{\lambda'}\psi_\lambda\rangle = \delta_{\lambda'\lambda}\begin{cases} n(\varepsilon_\lambda) & \text{Bosons} \\ f(\varepsilon_\lambda) & \text{Fermions}\end{cases} \tag{2.2}$$

where $f(\varepsilon_\lambda), n(\varepsilon_\lambda)$ is the bosonic/fermionic distribution.

$$\begin{cases} f(\varepsilon_\lambda) = \dfrac{1}{e^{\beta\varepsilon_\lambda}-1} \\ n(\varepsilon_\lambda) = \dfrac{1}{e^{\beta\varepsilon_\lambda}+1}\end{cases} \tag{2.3}$$

The green function (2.1) could be written into

$$G_{\lambda\lambda'}(\tau-\tau') = -e^{\varepsilon_\lambda(\tau'-\tau)}\begin{cases} [\theta(\tau-\tau')(1+n(\varepsilon_\lambda)) + \theta(\tau'-\tau)n(\varepsilon_\lambda)] & \text{Bosons} \\ [\theta(\tau-\tau')(1-f(\varepsilon_\lambda)) - \theta(\tau'-\tau)f(\varepsilon_\lambda)] & \text{Fermions}\end{cases} \tag{2.4}$$

The most eminent property of imagnary-time Green function is the periodicty for bosons and anti-periodicity for fermions. We take $-\beta < \tau < 0$, the Green function (2.1) expands into

$$\begin{aligned} G_{\lambda\lambda'}(\tau) &= -\langle \mathcal{T}c_\lambda(\tau)c^\dagger_{\lambda'}(0)\rangle = -\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}c^\dagger_{\lambda'}(0)c_\lambda(\tau)\right] \\ &= -\eta\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}c^\dagger_{\lambda'}(0)e^{H\tau}c_\lambda e^{-H\tau}\right] \quad \text{Trace identity} \\ &= -\eta\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}e^{H(\tau+\beta)}c_\lambda e^{-H(\tau+\beta)}c^\dagger_{\lambda'}(0)\right] \\ &= \eta G(\tau+\beta) \end{aligned} \tag{2.5}$$

Let's consider the identity for Green function

$$\begin{aligned} G(\tau) &= \frac{1}{\beta}\int_0^\beta G(\tau')\delta(\tau-\tau')d\tau' = \frac{1}{\beta}\int_0^\beta G(\tau')\beta\sum_{n=-\infty}^{\infty}e^{-\mathrm{i}\omega_n(\tau-\tau')} \\ &= \sum_{n=-\infty}^{+\infty}\left(\int_0^\beta G(\tau')e^{\mathrm{i}\omega_n\tau'}\right)e^{-\mathrm{i}\omega_n\tau} \\ &= \sum_{\omega_n} G(\mathrm{i}\omega_n)e^{\mathrm{i}\omega_n\tau} \end{aligned} \tag{2.6}$$

The Eq(2.6) tells us Matsubara representation

$$\begin{cases} G(\tau) = \sum_{\omega_n} G(\mathrm{i}\omega_n) e^{\mathrm{i}\omega_n \tau} \\ G(\mathrm{i}\omega_n) = \int_0^\beta e^{\mathrm{i}\omega_n \tau} G(\tau) d\tau \end{cases} \tag{2.7}$$

Eq(2.7) is just the Matsubara representation of imagninary-time Green function . The Eq(**??**) should match with properties(2.5) . We introduce Matsubara frequency for bosons and fermions respectively.

$$\omega_n = \begin{cases} \frac{2\pi n}{\beta} & \text{Bosons} \\ \frac{(2n+1)\pi}{\beta} & \text{Fermions} \end{cases} \tag{2.8}$$

Claim 2.1 .

The imaginary-time Green function adimits poperties below

$$\int_0^\beta G(\tau) e^{\mathrm{i}\omega_n \tau} = \int_{-\beta}^0 G(\tau) e^{\mathrm{i}\omega_n \tau} \tag{2.9}$$

Proof.

$$\int_{-\beta}^0 G(\tau) e^{\mathrm{i}\omega_n \tau} = \int_0^\beta G(\tau+\beta) e^{\mathrm{i}(\tau+\beta)} d\tau = \int_0^\beta G(\tau) e^{\mathrm{i}\omega_n \tau} \tag{2.10}$$

□

2.0.1 Lehmann representation

In many body physis, the Lehmann representation is obtained by assumption of time-independent hamiltonian. Now, we expand the Green function into eigenstates.

$$\begin{aligned} G(\tau) &= -\frac{1}{Z} \operatorname{Tr}\left(e^{-\beta H} e^{H\tau} \psi_\lambda(0) e^{-H\tau} \psi^\dagger_{\lambda'}(0)\right) \\ &= -\frac{1}{Z} \sum_{n,n'} \langle n \mid e^{-\beta H} e^{H\tau} \psi_\lambda(0) e^{-H\tau} \mid n' \rangle \langle n' \mid \psi^\dagger_{\lambda'}(0) \mid n \rangle \\ &= -\frac{1}{Z} \sum_{n,n'} \langle n \mid \psi_\lambda(0) \mid n' \rangle \langle n' \mid \psi^\dagger_\lambda(0) \mid n \rangle e^{-\beta E_n} e^{(E'_n - E_n)\tau} \end{aligned} \tag{2.11}$$

We transform the Green function (2.11) into frequencey space by formula (2.7).

$$\begin{aligned} G(\mathrm{i}\omega_n) &= \int_0^\beta e^{\mathrm{i}\omega_n \tau} G(\tau) d\tau = \sum_{n,n'} \langle n \mid \psi_\lambda(0) \mid n' \rangle \langle n' \mid \psi^\dagger_\lambda(0) \mid n \rangle e^{-\beta E_n} \int_0^\beta e^{(E_n - E'_n + \mathrm{i}\omega_n)\tau} \\ &= -\frac{1}{Z} \sum_{n,n'} \frac{\langle n \mid \psi_\lambda(0) \mid n' \rangle \langle n' \mid \psi^\dagger_\lambda(0) \mid n \rangle}{E'_n - E_n + \mathrm{i}\omega_n} e^{-\beta E_n} \left(e^{(E_n - E'_n + \mathrm{i}\omega_n)\beta} - 1\right) \\ &= -\frac{1}{Z} \sum_{n,n'} \frac{\langle n \mid \psi_\lambda(0) \mid n' \rangle \langle n' \mid \psi^\dagger_\lambda(0) \mid n \rangle}{E'_n - E_n + \mathrm{i}\omega_n} \left(\eta e^{-\beta E n'} - e^{-\beta E_n}\right) \\ &= \frac{1}{Z} \sum_{n,n'} \frac{\langle n \mid \psi_\lambda(0) \mid n' \rangle \langle n' \mid \psi^\dagger_\lambda(0) \mid n \rangle}{E'_n - E_n + \mathrm{i}\omega_n} \left(e^{-\beta E n'} - \eta e^{-\beta E_n}\right) \end{aligned} \tag{2.12}$$

Question 1

If $-\beta < \tau < 0$. please derive Lehmann representation form of Green function.

2.0.2 Matsubara Green function for free fermion and free bosons

We obtain the fermionic green function for free fermion by using of (2.7)

$$G(\mathrm{i}\omega_n) = \int_0^\beta d\tau e^{\mathrm{i}\omega_n\tau} G(\tau) = -\int_0^\beta d\tau (1 - f(\varepsilon_\lambda)) e^{-(\varepsilon_\lambda - \mathrm{i}\omega_n)\tau} = \frac{1}{\varepsilon_\lambda - \mathrm{i}\omega_n} \frac{e^{-(\varepsilon_\lambda - \mathrm{i}\omega_n)\tau} - 1}{1 + e^{-\beta\varepsilon_\lambda}} = \frac{1}{\mathrm{i}\omega_n - \varepsilon_\lambda} \tag{2.13}$$

By the same way , we calculate the bosonic green function for free bosons.

$$G(\mathrm{i}\nu_n) = \int_0^\beta d\tau e^{\mathrm{i}\nu_n\tau} G(\tau) = \frac{1}{\mathrm{i}\nu_n - \varepsilon_\lambda} \tag{2.14}$$

Example 2.0.1 (.)

Calculate the finite temperature Green function for harmonics oscilators.

$$D(\tau) = -\langle \mathcal{T} x(\tau) x(0) \rangle \tag{2.15}$$

We expand the the Green function as

$$\begin{aligned} D(\tau) &= -\frac{\hbar}{2m\omega} \langle \mathcal{T}(b(\tau) + b^\dagger(\tau))(b(0) + b^\dagger(0)) \rangle \\ &= -\frac{\hbar}{2m\omega} \left(\langle \mathcal{T}(b(\tau) b^\dagger(0)) \rangle + \langle \mathcal{T}(b^\dagger(\tau) b(0)) \rangle \right) \end{aligned}$$

The first term could be given by Eq(2.4) . And the second term has such form

$$\begin{aligned} \langle \mathcal{T}(b^\dagger(\tau) b(0)) \rangle &= (\theta(\tau) n(\omega) + (n(\omega) + 1)\theta(-\tau)) = n(\omega) + \theta(-\tau) = -n(-\omega) - 1 - \theta(-\tau) \\ &= -\left(n(-\omega)\theta(-\tau) + (1 + n(-\omega))\theta(\tau) \right) \end{aligned}$$

Hence, the Green function can be founded by (2.13)

$$D(\tau) = \frac{\hbar}{2m\omega} \frac{2\omega}{(\mathrm{i}\nu_n)^2 - \omega^2} \tag{2.16}$$

2.1 Matsubara sum

We will often encounter summation for all Matsubara frequencies. In this section, we will develop complex contour integral method to deal with this problem. A very important example is the calculation of polarization bubble diagram

The suscetibility is given by bubble diagram Figure(2.1)

$$\begin{aligned} \chi(q, \mathrm{i}\nu_n) &= -2\mu_B^2 T \sum_{k,n} G(k+q, \mathrm{i}\omega_n + \mathrm{i}\nu_n) G(k, \mathrm{i}\omega_n) \\ &= -2\mu_B^2 T \sum_{k,n} \frac{1}{\mathrm{i}\omega_n + \mathrm{i}\nu_n - \varepsilon_{k+q}} \frac{1}{\mathrm{i}\omega_n - \varepsilon_k} \end{aligned} \tag{2.17}$$

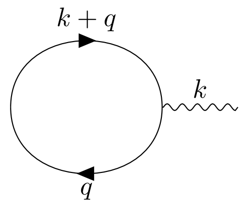

Figure 2.1: Polarization Bubble Feynman Diagram

The minus sign originates from Fermionic loop . We consider such contour integral

$$\int_{\mathcal{C}} f(z) \frac{1}{z+\mathrm{i}\nu_n-\varepsilon_{k+q}} \frac{1}{z-\varepsilon_k} = 2\pi\mathrm{i} \lim_{z\to\omega_n\cup\{\varepsilon_{k+1}-\mathrm{i}\nu_n,\varepsilon_k\}} \operatorname{Res}\left(f(z)\frac{1}{z+\mathrm{i}\nu_n-\varepsilon_{k+q}}\frac{1}{z-\varepsilon_k}\right) \tag{2.18}$$

Hence, we can derive the susceptibility as

$$\chi(q,\nu_n) = \sum_k \frac{f(\varepsilon_{k+q})-f(\varepsilon_k)}{\mathrm{i}\nu_n-(\varepsilon_{k+q}-\varepsilon_k)} \tag{2.19}$$

This result meets with Linhard response function.

Note:-

We notice that

$$\begin{aligned}\sum_{z\to z_n} \operatorname{Res}\left(f(z)\frac{1}{z+\mathrm{i}\nu_n-\varepsilon_{k+q}}\frac{1}{z-\varepsilon_k}\right) &= \sum_{z\to z_n}\left(\frac{z-\omega_n}{e^{\beta(z-\omega_n)}e^{\beta z_n}+1}\frac{1}{z+\mathrm{i}\nu_n-\varepsilon_{k+q}}\frac{1}{z-\varepsilon_k}\right)\\ &= -\frac{1}{\beta}\sum_n \frac{1}{\mathrm{i}\omega_n+\mathrm{i}\nu_n-\varepsilon_{k+q}}\frac{1}{\mathrm{i}\omega_n-\varepsilon_k}\end{aligned} \tag{2.20}$$

We substitute Eq(2.20) into (2.18), the result is obvious.