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Chapter 1

Solid theory

1.1 Path integral

1.1.1 Propagator

Generally speaking, we could define tome revolution operaor $U(t_a, t_b) = e^{-\mathrm{i}H(t_a - t_b)}$ for a given hamiltonian $H = \frac{p^2}{2m} + V(x)$. We define propagator as

Definition 1.1: .

The propagator is defined as

$$iG(x_a, t_a; x_b, t_b) = \langle x_a \mid U(t_a.t_b) \mid x_b \rangle \tag{1.1}$$

The propagator satisfied to Scrodinger equation. We will prove this argument below

$$i\partial_t G(x, t; x_0, t_0) = \langle x \mid \mathrm{i}\partial_t e^{-\mathrm{i}H(t-t_0)} \mid x_0 \rangle = \langle x \mid HU(t, t_0) \mid x_0 \rangle = H\langle x \mid U(t, t_0) \mid x_0 \rangle \tag{1.2}$$

The Eq(11.21) is equivalent to

$$i\partial_t G(x, t; x_0, t_0) = HG(x, t; x_0, t_0) \tag{1.3}$$

Example 1.1.1 (.)

The free particle propagator sis given as

$$G(x_, t; x_0, t_0) = -\mathrm{i}\sqrt{\frac{m}{2\pi i t}} \exp\left(\frac{\mathrm{i}m(x - x_0)^2}{2t}\right) \tag{1.4}$$

The propagator (11.4) could be derived by solving Eq(11.3). We write down it into Fourier space

$$\mathrm{i}\hbar \frac{dG(t, k)}{dt} = \frac{\hbar^2 k^2}{2m} G(t, k) \implies \frac{dG(t, k)}{dt} = -\mathrm{i}\frac{\hbar k^2}{2m} G(t, k) \tag{1.5}$$

The solution of Eq (11.5) is given by

$$G(t, k) = Ce^{-\mathrm{i}\frac{\hbar k^2}{2m}t} \tag{1.6}$$

where C is renormalization factor. We transform the solution (11.16) into real space

$$G(t,x) = C\int_{-\infty}^{+\infty}\frac{dk}{2\pi}e^{\mathrm{i}(kx-\frac{\hbar k^2}{2m}t)}dk = Ce^{\mathrm{i}\frac{mx^2}{2\hbar t}}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}e^{-\mathrm{i}\frac{\hbar t}{2m}(k-\frac{mx}{\hbar t})^2} = C\sqrt{\frac{m}{2\hbar\mathrm{i}\pi\hbar t}}e^{\mathrm{i}\frac{mx^2}{2\hbar t}} \tag{1.7}$$

If we take $C = 1$ and $x \to x - x_0$, it will turn into Eq(11.4).

1.2 Path integral formalism of propagator

We consider the propagator at time interval $[t_i, t_f]$, then we could introduce N slices to decompose propagator. We denote $x_i = x_0, x_f = x_N; t_i = t_0, t_f = t_N$, then the propagator could be expressed into path integral formalism.

$$\begin{aligned}&\mathrm{i}G(x_i,t_0;x_f,x_N) = \mathrm{i}\langle x_f \mid U(t_i,t_f) \mid x_i\rangle = \int dx_{N-1}\cdots dx_1\langle x_f \mid e^{-\mathrm{i}H(t_f-t_{N-1})} \mid x_{N-1}\rangle\langle x_{N-1} \mid e^{-\mathrm{i}H(t_{N-2}-t_{N-2})} \mid x_{N-2}\rangle \\ &\cdots\langle x_1 \mid e^{-\mathrm{i}H(t_1-t_0)} \mid x_0\rangle \\ &= \int dx_{N-1}\cdots dx_1 G(x_f,t_f;x_{N-1},t_{N-1})\cdots G(x_1,t_1;x_i,t_i)\end{aligned} \tag{1.8}$$

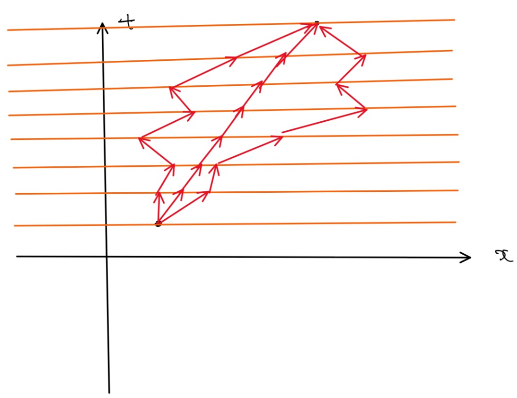

Fig 1.1: The propagator is the summation of all possible paths .

We also need to transform the (1.8) into phase space to obatain the action functional.

$$\begin{aligned}G(x_n;t_N;x_{n-1},t_{n-1}) &= \langle x_n \mid e^{-\mathrm{i}H(t_n-t_{n-1})} \mid x_{n-1}\rangle = \langle x_n e^{-\mathrm{i}\frac{p^2}{2m}(t_n-t_{n-1})} \mid p_n\rangle\langle p_n \mid e^{-\mathrm{i}V(x_{n-1})(t_n-t_{n-1})} \mid x_{n-1}\rangle \\ &= \exp\left(\mathrm{i}\left(p_x(x_n-x_{n-1}) - \frac{p_n^2}{2m}(t_n-t_{n-1}) - V(x_{n-1})(t_n-t_{n-1})\right)\right) \\ &= e^{\mathrm{i}L(t_n)}\end{aligned} \tag{1.9}$$

The propgator (1.8) is written into action functional

$$\mathrm{ii}G(x_i,t_0;x_f,x_N) = \int \mathcal{D}[x]e^{\mathrm{i}\int_{t_i}^{t_f}L(t)dt} \tag{1.10}$$

where the integral measurement is defined as

$$\int\mathcal{D}[x] = \int\prod_{n=1}^{N-1}\frac{dx_i dp_i}{2\pi\hbar} \tag{1.11}$$

We use saddle expansion to calculate the path integral for free particle

Example 1.2.1 (.)

Saddle expansion for free particle
The classical path reads as

$$x = x_a + \frac{x_b - x_a}{t_b - t_a}(t - t_a) \tag{1.12}$$

then we have

$$(x_i - x_{i-1})^2 = (x_c^i - x_c^{i-1} + \delta x_i - \delta x_{i-1}) = (x_c^i - x_c^{i-1})^2 + 2(x_c^i - x_c^{i-1})(\delta x_i - \delta x_{i-1}) + (\delta x_i - \delta x_{i-1})^2 \tag{1.13}$$

The calssical action is given as

$$S_c = \int_{t_a}^{t_b} L(t) = \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_a} \tag{1.14}$$

The fluctuations arise from second order term , then the path integral

1.3 Linear response theory

Consider external perturbation on system , then the system could be described as

$$H + f(t)O_1 \tag{1.15}$$

We asuume peturbation is slowly added , where $f(\infty) = 0$. The eigenstates $\mid \psi_n\rangle$ evolution could be written as

$$\mid \psi_n(t)\rangle = \mathcal{T}\left[e^{-\int_{-\infty}^{t} dt'(H+f(t)O_1)} \mid \psi_n\rangle\right] \tag{1.16}$$

We expand the time order evolution operator into first order

$$\begin{aligned}
\mid \psi_n(t)\rangle &= \mathcal{T}\left[e^{-\int_{-\infty}^{t} dt'(H+f(t)O_1)}\right] \mid \psi_n\rangle \\
&= \mathcal{T}\left[e^{-\mathrm{i}\int_{-\infty}^{t} dt' H}(1 + -\mathrm{i}\int_{-\infty}^{t} dt'' f(t'')O_1)\right] \mid \psi_n\rangle \\
&= e^{-\mathrm{i}H(t-t_{-\infty})} - \mathrm{i}\int_{-\infty}^{t} dt' e^{-\mathrm{i}\int_{t''}^{t} H dt'} f(t'')O_1 e^{-\mathrm{i}\int_{-\infty}^{t''} dt' H} \mid \psi_n\rangle \\
&= e^{-\mathrm{i}H(t-t_{-\infty})} - \mathrm{i}\int_{-\infty}^{t} dt' e^{-\mathrm{i}H(t-t')}[f(t')O_1]e^{-\mathrm{i}H(t'-t_{-\infty})} \mid \psi_n\rangle
\end{aligned} \tag{1.17}$$

We call the second term as $\mid \delta\psi_n(t)\rangle$, which means wavefunction variations.

$$\begin{aligned}
\mid \delta\psi_n(t)\rangle &= -\mathrm{i}\int_{-\infty}^{t} dt' e^{-\mathrm{i}H(t-t')}[f(t')O_1]e^{-\mathrm{i}H(t'-t_{-\infty})} \mid \psi_n\rangle \\
&= -\mathrm{i}\int_{-\infty}^{t} dt' f(t') e^{-\mathrm{i}H(t-t_{-\infty})} \underbrace{e^{\mathrm{i}H(t'-t_{-\infty})}O_1 e^{-\mathrm{i}H(t'-t_{-\infty})}}_{O_1(t')} dt' \mid \psi_n\rangle
\end{aligned} \tag{1.18}$$

We define the $O_1(t)$ at Heisenberg picture. Hence, the operator response $\delta O_2(t)$ could be expressed into

$$\begin{aligned}\delta O_2(t) &= \langle \psi_n(t) \mid O_2 \mid \psi_n(t)\rangle - \langle \psi_n \mid e^{\mathrm{i}H(t-t_{-\infty})} O_2 e^{-\mathrm{i}H(t-t_{-\infty})} \mid \psi_n\rangle \\ &= \mathrm{i}\int_0^t dt' f(t') \left[\langle \psi_n \mid O_1(t') e^{\mathrm{i}H(t-t_{-\infty})} O_2 e^{-\mathrm{i}H(t-t_{-\infty})} \mid \psi_n\rangle - \langle \psi_n \mid e^{\mathrm{i}H(t-t_{-\infty})} O_1 e^{-\mathrm{i}H(t-t_{-\infty})} O_1(t') \mid \psi_n\rangle\right] \\ &- \mathrm{i}\int_{-\infty}^t \langle \psi_n \mid [O_1(t), O_2(t')] \mid \psi_n\rangle f(t') dt' \end{aligned} \tag{1.19}$$

We define the retarded Green function as the integral kernel of Eq(1.19).

Definition 1.2: .

Retarded Green function

$$\chi(t-t') = -\mathrm{i}\int_{-\infty}^t \langle \psi_n \mid [O_2(t), O_1(t')] \mid \psi_n\rangle dt' \Theta(t-t') \tag{1.20}$$

We use a simple example to explain the retarded Green function.

Example 1.3.1 (.)

Harmonic oscillator in a external E field

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \qquad H_1 = -eEx \tag{1.21}$$

Please calculate polarizability

Firstly, we calculate the first order perturbation wavefunction

$$\mid \psi_0' \rangle = \mid 0\rangle + \sum_{n\neq 0} \frac{\langle n \mid H_1 \mid 0\rangle}{E_0 - E_1} \mid 1\rangle \tag{1.22}$$

The polarizability for ground state is given by

$$P = \langle \psi_0' \mid ex \mid \psi_0' \rangle = 2\sum_{n\neq 0} \frac{\langle 0 \mid ex \mid n\rangle\langle n \mid ex \mid 0\rangle}{E_0 - E_n} = 2\times \frac{e^2 \frac{\hbar}{m\omega}}{\hbar\omega} = \frac{2e^2}{m\omega^2} \tag{1.23}$$

We will use Green function formalism to solve this problem , we calculate the retard Green function

$$\begin{aligned}\chi(t-t') &= -\mathrm{i}\Theta(t-t')\langle 0 \mid [x(t), x(t')] \mid 0\rangle \\ &= -\mathrm{i}\Theta(t-t')\left[\langle 0 \mid e^{\mathrm{i}Ht} x e^{-\mathrm{i}H(t-t')} x e^{-\mathrm{i}Ht'} \mid 0\rangle - \langle 0 \mid e^{\mathrm{i}Ht'} x e^{-\mathrm{i}H(t'-t)} x e^{-\mathrm{i}Ht} \mid 0\rangle\right] \\ &= -\mathrm{i}\Theta(t-t') \mid \langle 0 \mid x \mid 1\rangle \mid^2 \left(e^{-\mathrm{i}(E_1-E_0)(t-t')} - e^{\mathrm{i}(E_1-E_0)(t-t')}\right)\end{aligned} \tag{1.24}$$

then we calculate frequency response

$$\begin{aligned}\chi(\omega) &= -\mathrm{i} \mid \langle 0 \mid x \mid 1\rangle \mid^2 \int_{-\infty}^{\infty} \Theta(t) \left[e^{\mathrm{i}(\omega-(E_1-E_0)+\mathrm{i}\eta t)} - e^{\mathrm{i}(\omega-(E_1-E_0)+\mathrm{i}\eta t)}\right] \\ &= \mid \langle 0 \mid x \mid 1\rangle \mid^2 \left[\frac{1}{\omega-\omega_0+\mathrm{i}\eta} - \frac{1}{\omega+\omega_0+\mathrm{i}\eta}\right]\end{aligned} \tag{1.25}$$

The static limit of Eq(1.25) meets with perturbation results (1.23)

1.4 Path integral and restard Green functon

We start from retard Green function (1.20). It could be written as

$$\langle 0 \mid O_2(t)O_2(t') \mid 0\rangle = \langle 0 \mid U^\dagger(t,-\infty)O_2U(t,t')O_1U(t',-\infty) \mid 0\rangle = \langle 0 \mid U^\dagger(+\infty,-\infty)U(+\infty,t)O_2U(t,t')O_2U(t',-\infty) \mid 0\rangle \tag{1.26}$$

Assuming that the perturbaton is adibatic process, then $\langle 0 \mid U^\dagger(+\infty,-\infty) \mid 0\rangle$ is up to a phase.

$$\langle 0 \mid O_2(t)O_2(t) \mid 0\rangle = \frac{\langle 0 \mid U(+\infty,t)O_2U(t,t')O_2U(t',-\infty) \mid 0\rangle}{\langle 0 \mid U(+\infty,-\infty) \mid 0\rangle} \tag{1.27}$$

We consider adopt coordinate eigenstates, then the numerator could be written into

$$\langle 0 \mid U(+\infty,-\infty) \mid 0\rangle = \int \mathcal{D}[x]e^{\mathrm{i}\int_{-\infty}^{+\infty} L(x,\dot{x})dt} \tag{1.28}$$

By the same way , the denominator is written into

$$\begin{aligned}\langle 0 \mid U(+\infty,t)O_2U(t,t')O_2U(t',-\infty) \mid 0\rangle &= \int \mathcal{D}[x]e^{\mathrm{i}\int_t^{+\infty} dtL(x,\dot{x})}O_2(x(t))e^{\mathrm{i}\int_{t'}^{t} dtL(x,\dot{x})}O_1(x(t'))e^{\mathrm{i}\int_{-\infty}^{t'} dtL(x,\dot{x})} \\ &= \int \mathcal{D}[x]O_2(x(t))O_1(x(t'))e^{\mathrm{i}\int_{-\infty}^{+\infty} dtL(x,\dot{x})}\end{aligned} \tag{1.29}$$

The $O_1(x(t'))$ is function about $x(t)$. The key idea is to make sure operator is time ordered. We combine the cases $t > t'$ and $t' > t$, then we have

$$\frac{\int \mathcal{D}[x]O_2(x(t))O_1(x(t'))e^{\mathrm{i}\int_{-\infty}^{+\infty} dtL(x,\dot{x})}}{\int \mathcal{D}[x]e^{\mathrm{i}\int_{-\infty}^{+\infty} L(x,\dot{x})dt}} = \mathcal{T}\langle 0 \mid O_2(t)O_2(t') \mid 0\rangle \tag{1.30}$$

We will continue to use harmonic oscilator to explain Eq(1.30).

Example 1.4.1 (.)

The harmonic oscilator

We will use Eq(1.30) to calculate Green function , which has been calculated in Example(1.3). The Green function could be calculated with path integral

$$\mathrm{i}G(t_2,t_1) = \frac{\int \mathcal{D}[x]x(t_2)x(t_1)e^{\mathrm{i}\int_{-\infty}^{+\infty} dtL(x,\dot{x})}}{\int \mathcal{D}[x]e^{\mathrm{i}\int_{-\infty}^{+\infty} L(x,\dot{x})dt}} \tag{1.31}$$

where $L = \frac{p^2}{2m} - \frac{1}{2}m\omega_0^2x^2(t)$. The Lagragian could be written into frequency domain

$$\begin{aligned}S &= \int_{-\infty}^{+\infty} dt\left(\frac{p^2}{2m} - \frac{1}{2}m\omega x^2(t)\right) = \int_{-\infty}^{+\infty} dt\left(\frac{1}{2}m\dot{x}^2(t) - \frac{1}{2}m\omega^2(t)\right) \\ &= \int_{-\infty}^{+\infty} dt\int \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\left[-\frac{1}{2}m\omega\omega' x(\omega)x(\omega')e^{-\mathrm{i}(\omega+\omega')t} - \frac{1}{2}m\omega_0^2x(\omega)x(\omega')e^{-\mathrm{i}(\omega+\omega')t}\right] \\ &= \int \frac{d\omega}{2\pi}\frac{1}{2}m\left(\omega^2-\omega_0^2\right)x(\omega)x(-\omega)\end{aligned} \tag{1.32}$$

Hence, the Green function on frequency domain is given as

$$G(\omega) = \frac{1}{m}\frac{1}{\omega^2 - \omega_0^2 + \mathrm{i}\varepsilon} \tag{1.33}$$

The Green function on time domain could be deived as

$$G(t_2, t_1) = \int \frac{d\omega}{2\pi}\frac{1}{m}\frac{1}{\omega^2 - \omega_0^2 + \mathrm{i}\varepsilon} e^{-\mathrm{i}\omega(t_2 - t_1)} \tag{1.34}$$

1.5 Matsubara Green function

The time order Green function could be calculated with path integral . However, the time order Green function couldn't serve physical observale quantity. The retarded Green function is physical observable quantiy but it's hard to calculate. Both of them can be connected by the imaginary Green function,or Matsubara Green function. The Matsubara Green function is defined as

$$\mathcal{G}(\tau) = -\langle \mathcal{T}_\tau [O_2(\tau) O_1(\tau)] \rangle \tag{1.35}$$

where $O(\tau) = e^{H\tau} O e^{-H\tau}$. The Matsubara green function of fermions and bosons satisfies to periodic condition and anti-periodic condition relatively .

$$\mathcal{G}(\tau) = \tag{1.36}$$

Chapter 2

Green function

The Green function the fundamental response function of many body system. The single particle green function is defined as

$$G(t-t') = -\mathrm{i}\langle \phi \mid \mathcal{T}\psi(t)\psi^\dagger(t') \mid \phi\rangle \tag{2.1}$$

where ϕ is many body ground state . The operator $\psi(t)$ is defined on Heisenberg representation. We define the propagator as the Green function on the momentum space , namly

$$G(k,\omega) = -\mathrm{i}\int \frac{d^3k}{(2\pi)^2}\frac{d\omega}{2\pi}\langle \phi \mid \mathcal{T}\psi(x,t)\psi^\dagger(x',t') \mid \phi\rangle e^{\mathrm{i}k\cdot(x-x')-\omega(t-t')} \tag{2.2}$$

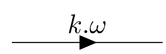

Figure 2.1: Single Fermion Propagator

2.1 Fermion Green function

Let's define th e ground state $\mid \phi\rangle$ as

$$\mid \phi\rangle = \prod_{|k|<k_f,\sigma} c^\dagger_{k\sigma} \mid 0\rangle \tag{2.3}$$

According to definition of Green function (2.1), the fermion green function could be calculated as

$$G(k,t-t') = -\mathrm{i}\langle \phi \mid \mathcal{T}c_{k\sigma}(t)c^\dagger_{k'\sigma'}(t') \mid \phi\rangle = -\mathrm{i}\theta(t-t')\delta_{kk'}(1-n_{k'\sigma'})e^{\mathrm{i}\omega_k(t'-t)} - \mathrm{i}\theta(t'-t)\delta_{kk'}n_{k\sigma}e^{\mathrm{i}\omega_k(t'-t)} \tag{2.4}$$

We consider the Fourier transformation to find the propagator

$$\begin{aligned}
G(k,\omega) &= \int_{-\infty}^{+\infty} dt - \mathrm{i}\left(\theta(t)(1-n_{k\sigma}) - \theta(-t)n_{k\sigma}\right) e^{\mathrm{i}(\omega-\varepsilon_k)t} \\
&= -\mathrm{i}\lim_{\varepsilon\to 0}\int_0^\infty \theta_{k-k_F} e^{\mathrm{i}(\omega-\omega_k+\mathrm{i}\varepsilon)t} + \mathrm{i}\lim_{\varepsilon\to 0}\int_{-\infty}^0 \theta_{k_F-k} e^{\mathrm{i}(\omega-\omega_k+\mathrm{i}\varepsilon)t} \\
&= \lim_{\varepsilon\to 0}\frac{\theta_{k-k_F}}{\omega-\omega_k+\mathrm{i}\varepsilon} + \frac{\theta_{k_F-k}}{\omega-\omega_k+\mathrm{i}\varepsilon} \\
&= \frac{1}{\omega-\omega_k+\mathrm{i}\varepsilon}
\end{aligned} \tag{2.5}$$

The fermion green function consists two part. The first term stands for particle moving forwards in time. The second term stands for holes moving backwards in time. Hence, we define the free fermion Green function as

$$G(k,\omega) = \frac{1}{\omega - \omega_k + \mathrm{i}\varepsilon} = \xrightarrow{\quad k,\omega \quad} \tag{2.6}$$

Figure 2.2: Single Fermion Propagator

2.2 Boson Green function

By the smae way , the bosonic green function can be calculated with definition. The non-interacting bosonic gas caould be described by hamiltonian

$$H = \sum_q \omega_q \left(b_q^\dagger b_q + \frac{1}{2} \right) \tag{2.7}$$

The ground state of hamiltonian (2.7) is just vacuum $\mid 0\rangle$. The physical field is defined as [1]

$$\phi_q = \sqrt{\frac{\hbar}{2m\omega_q}} \left(b_q + b_{-q}^\dagger \right) \tag{2.8}$$

The Green function for bosons could be defined by field ϕ_q .

$$\begin{aligned} G(q,t,t') = -\mathrm{i}\langle \phi \mid \mathcal{T}\phi_q(t)\phi_{q'}^\dagger(t') \mid \phi\rangle &= -\mathrm{i}\frac{\hbar}{2m\omega_q}\langle \phi \mid \mathcal{T} b_q(t) b_q^\dagger(t') \mid \phi\rangle - \mathrm{i}\frac{\hbar}{2m\omega_q}\langle \phi \mid \mathcal{T} b_{-q}^\dagger(t) b_{-q}(t') \mid \phi\rangle \\ &= -\mathrm{i}\frac{\hbar}{2m\omega_q} e^{\mathrm{i}\omega_q(t'-t)}\theta(t-t') - \mathrm{i}\frac{\hbar}{2m\omega_q} e^{-\mathrm{i}\omega_q(t'-t)}\theta(t'-t) \end{aligned} \tag{2.9}$$

We calculate the propagator from Eq(2.9).

$$\begin{aligned} G(q,\omega) &= -\mathrm{i}\frac{\hbar}{2m\omega_q}\int_{-\infty}^{+\infty} dt\theta(t) e^{\mathrm{i}(\omega-\omega_q)t} + \theta(-t) e^{\mathrm{ii}(\omega+\omega_q)t} \\ &= -\mathrm{i}\frac{\hbar}{2m\omega_q}\left(\int_0^{+\infty} dt e^{\mathrm{i}(\omega-\omega_q+\mathrm{i}\varepsilon)t} + \int_{-\infty}^0 dt e^{\mathrm{i}(\omega+\omega_q-\mathrm{i}\varepsilon)t} \right) \\ &= \frac{\hbar}{2m\omega_q}\left(\frac{1}{\omega-\omega_q+\mathrm{i}\varepsilon} - \frac{1}{\omega+\omega_q+\mathrm{i}\varepsilon} \right) \\ &= \frac{\hbar}{2m\omega_q}\frac{2\omega_q}{\omega^2 - (\omega_q+\mathrm{i}\varepsilon)^2} \end{aligned} \tag{2.10}$$

Note:-

The bosonic green function contains two part. The first part involves forward boson emitting process. The second part involves boson absorbing process .

$$G(q,\omega) = = \frac{\hbar}{2m\omega_q}\left(\frac{1}{\omega-(\omega_q+\mathrm{i}\varepsilon)} - \frac{1}{\omega+(\omega_q+\mathrm{i}\varepsilon)} \right) \tag{2.11}$$

At static limit $\omega \to 0$, then $G(q,\omega) = -\frac{\hbar}{2m\omega^2}$, it will induce effective atraction interaction.

[1] You can refer to scalar field quantization in field theory theory

2.3 Imaginary-time Green function

In this section , we discuss imaniary time Green function. The imaginary Green function is defined as

$$G_{\lambda\lambda'}(\tau-\tau') = -\langle \mathcal{T}\psi_\lambda(\tau)\psi^\dagger_{\lambda'}(\tau')\rangle = -\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta()}\psi_\lambda(\tau)\psi^\dagger_{\lambda'}(\tau')\right] \tag{2.12}$$

For a non-interacting system, the expectaton is

$$\langle\psi^\dagger_{\lambda'}\psi_\lambda\rangle = \delta_{\lambda'\lambda}\begin{cases} n(\varepsilon_\lambda) & \text{Bosons} \\ f(\varepsilon_\lambda) & \text{Fermions}\end{cases} \tag{2.13}$$

where $f(\varepsilon_\lambda), n(\varepsilon_\lambda)$ is the bosonic/fermionic distribution.

$$\begin{cases} f(\varepsilon_\lambda) = \dfrac{1}{e^{\beta\varepsilon_\lambda}-1} \\ n(\varepsilon_\lambda) = \dfrac{1}{e^{\beta\varepsilon_\lambda}+1}\end{cases} \tag{2.14}$$

The green function (2.12) could be written into

$$G_{\lambda\lambda'}(\tau-\tau') = -e^{\varepsilon_\lambda(\tau'-\tau)}\begin{cases} [\theta(\tau-\tau')(1+n(\varepsilon_\lambda))+\theta(\tau'-\tau)n(\varepsilon_\lambda)] & \text{Bosons} \\ [\theta(\tau-\tau')(1-f(\varepsilon_\lambda))-\theta(\tau'-\tau)f(\varepsilon_\lambda)] & \text{Fermions}\end{cases} \tag{2.15}$$

The most eminent property of imagnary-time Green function is the periodicty for bosons and anti-periodicity for fermions. We take $-\beta<\tau<0$, the Green function (2.12) expands into

$$\begin{aligned} G_{\lambda\lambda'}(\tau) &= -\langle\mathcal{T}c_\lambda(\tau)c^\dagger_{\lambda'}(0)\rangle = -\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}c^\dagger_{\lambda'}(0)c_\lambda(\tau)\right] \\ &= -\eta\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}c^\dagger_{\lambda'}(0)e^{H\tau}c_\lambda e^{-H\tau}\right] \quad \text{Trace identity} \\ &= -\eta\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}e^{H(\tau+\beta)}c_\lambda e^{-H(\tau+\beta)}c^\dagger_{\lambda'}(0)\right] \\ &= \eta G(\tau+\beta) \end{aligned} \tag{2.16}$$

Let's consider the identity for Green function

$$\begin{aligned} G(\tau) &= \frac{1}{\beta}\int_0^\beta G(\tau')\delta(\tau-\tau')d\tau' = \frac{1}{\beta}\int_0^\beta G(\tau')\beta\sum_{n=-\infty}^{\infty}e^{-\mathrm{i}\omega_n(\tau-\tau')} \\ &= \sum_{n=-\infty}^{+\infty}\left(\int_0^\beta G(\tau')e^{\mathrm{i}\omega_n\tau'}\right)e^{-\mathrm{i}\omega_n\tau} \\ &= \sum_{\omega_n}G(\mathrm{i}\omega_n)e^{\mathrm{i}\omega_n\tau} \end{aligned} \tag{2.17}$$

The Eq(2.17) tells us Matsubara representation

$$\boxed{\begin{cases} G(\tau) = \sum_{\omega_n}G(\mathrm{i}\omega_n)e^{\mathrm{i}\omega_n\tau} \\ G(\mathrm{i}\omega_n) = \int_0^\beta e^{\mathrm{i}\omega_n\tau}G(\tau)d\tau \end{cases}} \tag{2.18}$$

Eq(2.18) is just the Matsubara representation of imagninary-time Green function . The Eq(**??**) should match with properties(2.16) . We introduce Matsubara frequency for bosons and fermions respectively.

$$\omega_n = \begin{cases} \frac{2\pi n}{\beta} & \text{Bosons} \\ \frac{(2n+1)\pi}{\beta} & \text{Fermions} \end{cases} \tag{2.19}$$

Claim 2.1 .
The imaginary-time Green function adimits poperties below

$$\int_0^\beta G(\tau)e^{\mathrm{i}\omega_n\tau} = \int_{-\beta}^0 G(\tau)e^{\mathrm{i}\omega_n\tau} \tag{2.20}$$

Proof.

$$\int_{-\beta}^0 G(\tau)e^{\mathrm{i}\omega_n\tau} = \int_0^\beta G(\tau+\beta)e^{\mathrm{i}(\tau+\beta)}d\tau = \int_0^\beta G(\tau)e^{\mathrm{i}\omega_n\tau} \tag{2.21}$$

□

2.3.1 Lehmann representation

In many body physis, the Lehmann representation is obtained by assumption of time-independent hamiltonian. Now, we expand the Green function into eigenstates.

$$\begin{aligned} G(\tau) &= -\frac{1}{Z}\mathrm{Tr}\left(e^{-\beta H}e^{H\tau}\psi_\lambda(0)e^{-H\tau}\psi^\dagger_{\lambda'}(0)\right) \\ &= -\frac{1}{Z}\sum_{n,n'}\langle n \mid e^{-\beta H}e^{H\tau}\psi_\lambda(0)e^{-H\tau} \mid n'\rangle\langle n' \mid \psi^\dagger_{\lambda'}(0) \mid n\rangle \\ &= -\frac{1}{Z}\sum_{n,n'}\langle n \mid \psi_\lambda(0) \mid n'\rangle\langle n' \mid \psi^\dagger_\lambda(0) \mid n\rangle e^{-\beta E_n}e^{(E'_n - E_n)\tau} \end{aligned} \tag{2.22}$$

We transform the Green function (2.22) into frequencey space by formula (2.18).

$$\begin{aligned} G(\mathrm{i}\omega_n) &= \int_0^\beta e^{\mathrm{i}\omega_n\tau}G(\tau)d\tau = \sum_{n,n'}\langle n \mid \psi_\lambda(0) \mid n'\rangle\langle n' \mid \psi^\dagger_\lambda(0) \mid n\rangle e^{-\beta E_n}\int_0^\beta e^{(E_n - E'_n + \mathrm{i}\omega_n)\tau} \\ &= -\frac{1}{Z}\sum_{n,n'}\frac{\langle n \mid \psi_\lambda(0) \mid n'\rangle\langle n' \mid \psi^\dagger_\lambda(0) \mid n\rangle}{E'_n - E_n + \mathrm{i}\omega_n}e^{-\beta E_n}\left(e^{(E_n - E'_n + \mathrm{i}\omega_n)\beta} - 1\right) \\ &= -\frac{1}{Z}\sum_{n,n'}\frac{\langle n \mid \psi_\lambda(0) \mid n'\rangle\langle n' \mid \psi^\dagger_\lambda(0) \mid n\rangle}{E'_n - E_n + \mathrm{i}\omega_n}\left(\eta e^{-\beta En'} - e^{-\beta E_n}\right) \\ &= \frac{1}{Z}\sum_{n,n'}\frac{\langle n \mid \psi_\lambda(0) \mid n'\rangle\langle n' \mid \psi^\dagger_\lambda(0) \mid n\rangle}{E'_n - E_n + \mathrm{i}\omega_n}\left(e^{-\beta En'} - \eta e^{-\beta E_n}\right) \end{aligned} \tag{2.23}$$

Question 1

If $-\beta < \tau < 0$. please derive Lehmann representation form of Green function.

2.3.2 Matsubara Green function for free fermion and free bosons

We obtain the fermionic green function for free fermion by using of (2.18)

$$G(\mathrm{i}\omega_n) = \int_0^\beta d\tau e^{\mathrm{i}\omega_n\tau} G(\tau) = -\int_0^\beta d\tau (1 - f(\varepsilon_\lambda)) e^{-(\varepsilon_\lambda - \mathrm{i}\omega_n)\tau} = \frac{1}{\varepsilon_\lambda - \mathrm{i}\omega_n} \frac{e^{-(\varepsilon_\lambda - \mathrm{i}\omega_n)\tau} - 1}{1 + e^{-\beta\varepsilon_\lambda}} = \frac{1}{\mathrm{i}\omega_n - \varepsilon_\lambda} \tag{2.24}$$

By the same way , we calculate the bosonic green function for free bosons.

$$G(\mathrm{i}\nu_n) = \int_0^\beta d\tau e^{\mathrm{i}\nu_n\tau} G(\tau) = \frac{1}{\mathrm{i}\nu_n - \varepsilon_\lambda} \tag{2.25}$$

Example 2.3.1 (.)
Calculate the finite temperature Green function for harmonics oscilators.

$$D(\tau) = -\langle \mathcal{T} x(\tau) x(0) \rangle \tag{2.26}$$

We expand the the Green function as

$$\begin{aligned} D(\tau) &= -\frac{\hbar}{2m\omega} \langle \mathcal{T}(b(\tau) + b^\dagger(\tau))(b(0) + b^\dagger(0)) \rangle \\ &= -\frac{\hbar}{2m\omega} \left(\langle \mathcal{T}(b(\tau) b^\dagger(0)) \rangle + \langle \mathcal{T}(b^\dagger(\tau) b(0)) \rangle \right) \end{aligned}$$

The first term could be given by Eq(2.15) . And the second term has such form

$$\begin{aligned} \langle \mathcal{T}(b^\dagger(\tau) b(0)) \rangle &= (\theta(\tau) n(\omega) + (n(\omega) + 1)\theta(-\tau)) = n(\omega) + \theta(-\tau) = -n(-\omega) - 1 - \theta(-\tau) \\ &= -\left(n(-\omega)\theta(-\tau) + (1 + n(-\omega))\theta(\tau) \right) \end{aligned}$$

Hence, the Green function can be founded by (2.24)

$$D(\tau) = \frac{\hbar}{2m\omega} \frac{2\omega}{(\mathrm{i}\nu_n)^2 - \omega^2} \tag{2.27}$$

2.4 Matsubara sum

We will often encounter summation for all Matsubara frequencies. In this section, we will develop complex contour integral method to deal with this problem. A very important example is the calculation of polarization bubble diagram

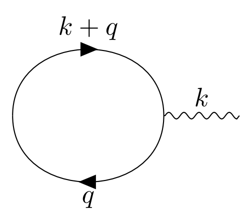

Figure 2.3: Polarization Bubble Feynman Diagram

The suscetibility is given by bubble diagram Figure(2.3)

$$\chi(q, \mathrm{i}\nu_n) = -2\mu_B^2 T \sum_{k,n} G(k+q, \mathrm{i}\omega_n + \mathrm{i}\nu_n) G(k, \mathrm{i}\omega_n)$$
$$= -2\mu_B^2 T \sum_{k,n} \frac{1}{\mathrm{i}\omega_n + \mathrm{i}\nu_n - \varepsilon_{k+q}} \frac{1}{\mathrm{i}\omega_n - \varepsilon_k} \tag{2.28}$$

The minus sign originates from Fermionic loop . We consider such contour integral

$$\int_{\mathcal{C}} f(z) \frac{1}{z + \mathrm{i}\nu_n - \varepsilon_{k+q}} \frac{1}{z - \varepsilon_k} = 2\pi \mathrm{i} \lim_{z \to \omega_n \cup \{\varepsilon_{k+1} - \mathrm{i}\nu_n, \varepsilon_k\}} \mathrm{Res}\left(f(z) \frac{1}{z + \mathrm{i}\nu_n - \varepsilon_{k+q}} \frac{1}{z - \varepsilon_k} \right) \tag{2.29}$$

Hence, we can derive the susceptibility as

$$\chi(q, \nu_n) = \sum_k \frac{f(\varepsilon_{k+q}) - f(\varepsilon_k)}{\mathrm{i}\nu_n - (\varepsilon_{k+q} - \varepsilon_k)} \tag{2.30}$$

This result meets with Linhard response function.

Note:-

We notice that

$$\sum_{z \to z_n} \mathrm{Res}\left(f(z) \frac{1}{z + \mathrm{i}\nu_n - \varepsilon_{k+q}} \frac{1}{z - \varepsilon_k} \right) = \sum_{z \to z_n} \left(\frac{z - \omega_n}{e^{\beta(z-\omega_n)} e^{\beta z_n} + 1} \frac{1}{z + \mathrm{i}\nu_n - \varepsilon_{k+q}} \frac{1}{z - \varepsilon_k} \right)$$
$$= -\frac{1}{\beta} \sum_n \frac{1}{\mathrm{i}\omega_n + \mathrm{i}\nu_n - \varepsilon_{k+q}} \frac{1}{\mathrm{i}\omega_n - \varepsilon_k} \tag{2.31}$$

We substitute Eq(2.31) into (2.29), the result is obvious.

2.5 Path integral

2.5.1 Coherent state

2.5.2 Bosonic path integral

In this section , we will derive bosonic path integral fromalism. Our start point is partition function . We consider write partition function into coherent state

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right) = \int \frac{dz d\bar{z}}{2\pi \mathrm{i}} e^{-z\bar{z}} \langle z \mid e^{-\beta H} \mid z \rangle \tag{2.32}$$

We consider divide Bo;tzmann factor $e^{-\beta H}$ into time slices $e^{-\beta H} = \left(e^{-\Delta\tau H}\right)^N$. By using of completenes relation of coherent state, the partition function (6.1) could be written into

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right) = \int \prod_{i=0}^{N-1} \frac{dz_i d\bar{z}_i}{2\pi \mathrm{i}} e^{-z_i \bar{z}_i} \langle z_N \mid e^{-\Delta\tau H} \mid z_{N-1} \rangle \langle z_{N-1} \mid e^{-\Delta\tau H} \mid z_{N-2} \rangle \cdots \langle z_1 \mid e^{-\Delta\tau H} \mid z_0 \rangle \tag{2.33}$$

The hamiltonian H is the polynomials about operator $b, b^\dagger$ with normal order. However, the Boltmann factor is not normal order . In other words, we cann't replace Boltmann factor by c number equivalents.

$$\langle z_k \mid e^{-\Delta\tau H} \mid z_{k+1} \rangle = \langle z_k \mid 1 - \Delta\tau H[b^\dagger, b] \mid z_{k-1} \rangle = \langle z_k \mid 1 - \Delta\tau H[\bar{z}_k, z_{k-1}] \mid z_{k+1} \rangle = e^{\bar{z}_k z_{k-1}} \left(1 - \Delta\tau H[\bar{z}_k, z_{k-1}]\right)$$
$$= e^{\bar{z}_k z_{k-1} - \Delta\tau H[\bar{z}_k, z_{k-1}]} \tag{2.34}$$

We substitute (2.34) into (6.1) . The partition function becomes [2]

$$Z = \int \prod_{k=1}^{N-1} \frac{dz_k d\bar{z}_k}{2\pi \mathrm{i}} e^{(\bar{z}_{i+1}-\bar{z}_i)z_i - \Delta\tau H[\bar{z}_k, z_{k-1}]} \tag{2.35}$$

Furthermore, we take limit $N \to$ to continumm limits.

$$\sum_{k=0}^{N} (\bar{z}_{i+1}-\bar{z}_i)z_i - \Delta\tau H[\bar{z}_k, z_{k-1}] = \sum_{k=0}^{N} \Delta\tau \frac{(\bar{z}_{i+1}-\bar{z}_i)}{\Delta\tau} z_i - \Delta\tau H[\bar{z}_k, z_{k-1}] \to -\int_0^\beta d\tau \left(\bar{z}\partial z + H[\bar{z}, z]\right) \tag{2.36}$$

We define functional measurement as

$$\mathcal{D}[\bar{z}, z] = \int \prod_{k=1}^{N-1} \frac{dz_k d\bar{z}_k}{2\pi \mathrm{i}} \tag{2.37}$$

The partion function could be written into

$$\boxed{Z = \int \mathcal{D}[\bar{z}, z] e^{-S} \qquad S = \int_0^\beta d\tau \left(\bar{z}\partial z + H[\bar{z}, z]\right)} \tag{2.38}$$

2.5.3 Gaussian path integral

A most important path integral is Gaussian path integral . We start from action S_E

$$S_E = \int_0^\beta d\tau \bar{z}_\alpha(\tau)(\partial_\tau + h_{\alpha\beta})z_\beta(\tau) \tag{2.39}$$

The action (2.39) could be written into Matsubara representation .

$$\begin{aligned} S_E &= \int_0^\beta d\tau \bar{z}_\alpha(\tau)(\partial_\tau + h_{\alpha\beta})z_\beta(\tau')\delta(\tau-\tau') \\ &= \frac{1}{\beta} \sum_{\mathrm{i}\nu_n, \mathrm{i}\nu_n'} \bar{z}_\beta(\mathrm{i}\nu_n') \left(\mathrm{i}\nu_n + h_{\alpha\beta}\right) z_\beta(\mathrm{i}\nu_n) \int_0^\beta d\tau e^{\mathrm{i}(\nu_n - \nu_n')\tau} \\ &= \sum_{\mathrm{i}\nu_n} \bar{z}_\beta(\mathrm{i}\nu_n) \left(\mathrm{i}\nu_n + h_{\alpha\beta}\right) z_\beta(\mathrm{i}\nu_n) \end{aligned} \tag{2.40}$$

Claim 2.2 ,
We have such identity for coherent state integral

$$\int \prod_{k=1}^{n} \frac{dz_k d\bar{z}_k}{2\pi \mathrm{i}} e^{-\bar{z}_\alpha M_{\alpha\beta} z_\beta} = \frac{1}{\det(M)} \tag{2.41}$$

Proof. Let's consider the unitary transformation for quadratic form $\bar{z}_\alpha M_{\alpha\beta} z_\beta$.

$$\bar{z}_\alpha M_{\alpha\beta} z_\beta = z_\alpha (U^\dagger U M_\mu U^\dagger U)_{\alpha\beta} z_\beta = z_\alpha' (U M U^\dagger)_{\alpha\beta} z_\beta' = \sum_{n=1}^{\infty} \lambda_n \bar{z}_k' z_k' \tag{2.42}$$

[2] $Z_N = Z_0$

Hence, the integral (2.41) could be calculated as

$$\int \prod_{k=1}^{n} \frac{dz_k d\bar{z}_k}{2\pi \mathrm{i}} e^{-\bar{z}_\alpha M_{\alpha\beta} z_\beta} = \int \prod_{k=1}^{n} \frac{dz'_k d\bar{z}'_k}{2\pi \mathrm{i}} e^{-\sum\limits_{k=1}^{n} \lambda_k \bar{z}'_k z_k} = \frac{1}{\det(M)} \tag{2.43}$$

□

Hence, we can write down the partion function as

$$Z = \frac{1}{\det(\partial_\tau + h)} \tag{2.44}$$

2.5.4 Fermonic path integral

We introduce the Grassmann coherent state to formulate fermioinc path integral. By the same way, we can write down the fermionic path integral .

We introduce the Grassmann variables to formulate fermionic path integral . The Grassmann varibales satisfies to anti-commutating propeties

$$\{\xi_i, \xi_j\} = 0 \quad \{\bar{\xi}_i, \bar{\xi}_j\} = 0 \quad \{\xi_i, \bar{\xi}_j\} = 0 \tag{2.45}$$

Furthermore, the Grassmann varibales also sati-commutate with fermioonic operators

$$\{\xi_i, c_j\} = \{\xi_i, c_j^\dagger\} = \{\bar{\xi}_i, c_j\} = \{\bar{\xi}_i, c_j^\dagger\} = 0 \tag{2.46}$$

Definition 2.5.1: .

Fermionic coherent state $\mid \xi\rangle$ is defined as

$$\mid \xi\rangle = e^{-\xi c^\dagger} \mid 0\rangle = (1 - \xi c^\dagger) \mid 0\rangle \tag{2.47}$$

The fermionic coherent state definition (2.5.4) could be generalized into many particles

$$\mid \xi_1, \xi_2, \cdots \xi_n\rangle = e^{-\sum\limits_{i}^{n} \xi_i c_i^\dagger} \mid 0\rangle \tag{2.48}$$

If the function could be expanded into power series with Grassmann variables, then we call this function is Grassmann anlytical function.

$$\psi(\xi) = \psi_0 + \psi_1 \xi + \psi_2 \xi^2 + \cdots \tag{2.49}$$

We give the diffrentiation and integration rules for Grassmann variables .

$$\partial_\xi(\xi) = 1 \quad \partial_\xi(\bar{\xi}\xi) = -\bar{\xi} \tag{2.50}$$

The Eq(2.50) tells us anti-commutation relation between ∂_ξ and $\partial_{\bar{\xi}}$.

$$\{\partial_\xi, \partial_{\bar{\xi}}\} = \partial_\xi \partial_{\bar{\xi}} + \partial_{\bar{\xi}} \partial_\xi = 0 \tag{2.51}$$

Eq(2.50) suggests suh definition

$$\int d\xi 1=\int d\xi\partial_\xi\xi=0\quad\int d\xi\xi=1 \tag{2.52}$$

Thus, integration and differentiation is same for Grassmann varibales

$$\partial_\xi\equiv\int d\xi \tag{2.53}$$

Now we try to find resolution identity

$$I=\int d\xi d\bar{\xi}\mu(\xi)\mid\xi\rangle\langle\xi\mid\qquad\mu(\xi)=e^{-\bar{\xi}\xi} \tag{2.54}$$

The inner product could be defined as

$$\langle f\mid g\rangle=\int d\bar{\xi}d\xi e^{-\bar{\xi}\xi}\bar{f}(\xi)g(\xi)=\int d\bar{\xi}d\xi e^{-\bar{\xi}\xi}\mid(\bar{f}_0+\bar{f}_1\bar{\xi})(\bar{g}_0+\bar{g}_1\bar{\xi})=\bar{f}_0f_0+\bar{g}_0g_0 \tag{2.55}$$

Cases

We give some cases about resolution identity of fermionic coherent states

$$\mid\psi\rangle=\int\prod_{i=1}^N d\bar{\xi}d\xi e^{-\sum\limits_{i=1}^N\bar{\xi}_i\xi_i}\psi(\bar{\xi}_1,\bar{\xi}_2,\cdot\bar{\xi}_n)\mid\xi_1,\xi_2,\cdots\xi_n\rangle \tag{2.56}$$

we consider annihilating fermion on coherent states

$$\begin{cases}c\mid\xi\rangle=c(\mid 0\rangle-\xi c^\dagger\mid 0\rangle)=\xi\mid 0\rangle=\xi_i\mid\xi\rangle\implies\langle\xi\mid c^\dagger=\bar{\xi}\langle\xi\mid\\ c^\dagger\mid\xi\rangle=c^\dagger(1-\xi c^\dagger)\mid 0\rangle=c^\dagger\mid 0\rangle=-\partial_\xi\mid\xi\rangle\implies\langle\xi\mid c=\partial_\xi\langle\xi\mid=\end{cases} \tag{2.57}$$

We consider matrix element on the coherent states

$$\begin{cases}\langle\xi\mid c\mid\psi\rangle=\int d\bar{\xi}d\xi e^{-\bar{\xi}\xi}\langle\xi\mid c\mid\xi\rangle\langle\xi\mid\psi\rangle=\int d\bar{\xi}d\xi\xi\langle\xi\mid\psi\rangle=\int d\bar{\xi}\langle\xi\mid\psi\rangle=\partial_{\bar{\xi}}\psi(\bar{\xi})\\ \langle\xi\mid c^\dagger\mid\psi\rangle=\int d\bar{\xi}d\xi e^{-\bar{\xi}\xi}\langle\xi\mid c^\dagger\mid\xi\rangle\langle\xi\mid\psi\rangle=\int d\bar{\xi}d\xi\xi\langle\xi\mid\psi\rangle=\int d\bar{\xi}d\bar{\xi}(1-\bar{\xi}\xi)\bar{\xi}\langle\xi\mid\psi\rangle=\bar{\xi}\partial_{\bar{\xi}}\psi(\bar{\xi})\end{cases}$$

The evaluation pf density operator N could be wriiten as

$$\begin{aligned}&\frac{\langle\xi_1,\xi_2,\cdots\xi_n\mid\hat{N}\mid\xi_1,\xi_2,\cdots\xi_n\rangle}{\langle\xi_1,\xi_2,\cdots\xi_n\mid\xi_1,\xi_2,\cdots\xi_n\rangle}=\sum_{i=1}^N\frac{\langle\xi_1,\xi_2,\cdots\xi_n\mid c_i^\dagger c_i\mid\xi_1,\xi_2,\cdots\xi_n\rangle}{\langle\xi_1,\xi_2,\cdots\xi_n\mid\xi_1,\xi_2,\cdots\xi_n\rangle}=\sum_{i=1}^N\bar{\xi}_i\frac{\langle\xi_1,\xi_2,\cdots\xi_n\mid c_i\mid\xi_1,\xi_2,\cdots\xi_n\rangle}{\langle\xi_1,\xi_2,\cdots\xi_n\mid\xi_1,\xi_2,\cdots\xi_n\rangle}\\&=\sum_{i=1}^N\bar{\xi}_i\frac{\langle\xi_1,\xi_2,\cdots\xi_n\mid c_i\mid\xi_1,\xi_2,\cdots\xi_n\rangle}{\langle\xi_1,\xi_2,\cdots\xi_n\mid\xi_1,\xi_2,\cdots\xi_n\rangle}\\&=\sum_i\bar{\xi}_i\xi\end{aligned} \tag{2.58}$$

Cases

We consider general form of hamiltonian

$$H=\sum_i\varepsilon_ic_i^\dagger c_i+V\sum_{i,j}c_i^\dagger c_k^\dagger c_kc_i \tag{2.59}$$

$$\langle H\rangle = \sum_i \varepsilon_k \bar{\xi}_i \xi_i + V\sum_{i,k} \mid \xi_i \mid^2 \mid \xi_k \mid^2 \tag{2.60}$$

In most cases , we need to evaluate the Grassmann Gaussian integal .

Theorem 2.1 .

$$\mathcal{Z}[\bar{\zeta},\zeta] = \int \prod_{i=1}^{N} d\bar{\xi}_i d\xi_i e^{-\sum\limits_{i,j}\bar{\xi}_i M_{ij}\xi + \bar{\xi}_i\zeta_i + \bar{\zeta}_i\xi_i} \tag{2.61}$$

Proof. Firstly , we consider more simple form , namly

$$\begin{aligned}
\int \prod_{i=1}^{N} d\bar{\xi}_i d\xi_i e^{-\sum\limits_{i,j}\bar{\xi}_i M_{ij}\xi} &= \int \prod_{i=1}^{N} d\bar{\xi}_i d\xi_i (1 - M_{\sigma(1),1}\bar{\xi}_{\sigma(1)}\xi_1)\cdots(1 - M_{\sigma(n),n}\bar{\xi}_{\sigma(n)}\xi_n) \\
&= \int \prod_{i=1}^{N} d\bar{\xi}_i d\xi_i (-M_{\sigma(1),1}\bar{\xi}_{\sigma(1)}\xi_1)(-M_{\sigma(2),2}\bar{\xi}_{\sigma(2)}\xi_2)\cdots(-M_{\sigma(n),n}\bar{\xi}_{\sigma(n)}\xi_n) \\
&= \int \prod_{i=2}^{N} d\bar{\xi}_i d\xi_i d\bar{\xi}_1 M_{\sigma(1),1}\bar{\xi}_{\sigma(1)}(-M_{\sigma(2),2}\bar{\xi}_{\sigma(2)}\xi_2)\cdots(-M_{\sigma(n),n}\bar{\xi}_{\sigma(n)}\xi_n) \\
&\quad\vdots \\
&= \int\int \prod_{i=2}^{N} d\bar{\xi}_i \bar{\xi}_{\sigma(1)}\bar{\xi}_{\sigma(2)}\cdots\bar{\xi}_{\sigma(n)} M_{\sigma(1),1}M_{\sigma(2),2}\cdots M_{\sigma(n),n} \\
&= \det(M)
\end{aligned} \tag{2.62}$$

Now we consider the shft the varibales ξ_i

$$\begin{aligned}
&\sum_{ij}(\bar{\eta}_i + \bar{\alpha}_i)M_{ij}(\eta_j + \alpha_j) + \sum_i \bar{\zeta}_i(\eta_i + \alpha_i) + (\bar{\eta}_i + \bar{\alpha}_i)_i\zeta_i = \sum_{ij}\bar{\eta}_i M_{ij}\eta_j + \sum_i \eta_i(\zeta_i + \sum_j M_{ij}\alpha_j) \\
&\sum_i(\bar{\zeta}_i + \sum_j \bar{\alpha}_j M_{ji})\eta_i + \sum_{ij}\bar{\alpha}_i M_{ij}\alpha_j + \sum_i \bar{\zeta}_i\alpha_i + \bar{\alpha}_i\zeta_i
\end{aligned} \tag{2.63}$$

We let $\zeta_i + M_{ij}\alpha_j = 0$, then (2.63) turns into

$$\sum_{ij}(\bar{\eta}_i + \bar{\alpha}_i)M_{ij} = \sum_{ij}\bar{\eta}_i M_{ij}\eta_j + \sum_{ij}\bar{\zeta}_i M^{-1}_{ij}\zeta_j - \sum_i \bar{\zeta}_i M^{-1}_{ij}\zeta_j - \sum_i \bar{\zeta}_i M^{-1}_{ij}\zeta_j \tag{2.64}$$

□

Hence, the Eq(2.61) could be derived with Eq(2.62)

$$Z = \int \mathcal{D}[\bar{\xi},\xi]e^{-S} \qquad S = \int_0^\beta d\tau\left(\bar{\xi}\partial\xi + H[\bar{\xi},\xi]\right) \tag{2.65}$$

By the same way , the fermionic path integral could be derived as

Question 2

Please review the process on section (2.5.2) to derive Eq(2.65)

2.6 Fluctuatio -dissipatio theorem

2.6.1 Kramers-Kronig theorem

The Kramers-Kronig theorem related the real part and imgnary part of susceptibility. The susceptibility is the frequency space has such form

$$\chi(\omega) = \mathcal{F}\left(-\mathcal{T}\langle[A(t), A(t')]\rangle\theta(t-t')\right) = \mathcal{F}(-\mathcal{T}\langle[A(t), A(t')]\rangle) * \mathcal{F}(\theta(t-t')) \tag{2.66}$$

Note:-

$$\mathcal{F}(t)) = \lim_{\varepsilon\to 0}\int_0^\infty dte^{t(-\varepsilon+\mathrm{i}\omega)}t = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon - \mathrm{i}\omega} = \mathrm{i}\left(\frac{1}{\omega} - \mathrm{i}\pi\delta(\omega)\right) \tag{2.67}$$

We substitute (2.67) into (2.66)

$$\chi(\omega) = \mathcal{F}(\chi''(t)) * F(\theta(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \mathrm{i}\left(\frac{1}{\omega-\omega'} - \mathrm{i}\delta(\omega-\omega')\right)\chi''(\omega') \tag{2.68}$$

Accrodin to (2.68), we can derive

$$\begin{aligned}
\chi(\omega) = \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{\chi''(\omega')}{\omega-\omega'} &= \frac{\mathrm{i}}{\pi}\int_0^\infty \frac{\chi''(\omega')}{\omega-\omega'} + \frac{\mathrm{i}}{\pi}\int_0^\infty \frac{\chi''(-\omega')}{\omega+\omega'} \\
&= \frac{\mathrm{i}}{\omega}\int_0^\infty \frac{(\omega+\omega')\chi''(\omega) + (\omega-\omega')\chi^{*\prime\prime}(\omega)}{\omega^2-\omega'^2} \\
&= \frac{\mathrm{i}}{\pi}\int_0^\infty \frac{2\omega\Re\chi''(\omega')}{\omega^2-\omega'^2}d\omega' - \frac{1}{\pi}\int_0^\infty \frac{2\omega'\Im\chi''(\omega')}{\omega^2-\omega'^2}d\omega'
\end{aligned} \tag{2.69}$$

Hence, we could derive Kramers-Kronig relation

$$\begin{cases}
\Re\chi(\omega) = -\dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{2\omega'\Im\chi''(\omega')}{\omega^2-\omega'^2}d\omega' \\
\Im\chi(\omega) = \dfrac{\mathrm{i}}{\pi}\displaystyle\int_0^\infty \frac{2\omega\Re\chi''(\omega')}{\omega^2-\omega'^2}d\omega'
\end{cases} \tag{2.70}$$

Chapter 3

Density functional theory

3.1 Kohn-Heisenberg theorem

We consider a system with Halmiltonian

$$H = T + V + U \qquad \begin{cases} T = \int d^3r \psi_\sigma^\dagger \left(-\frac{\hbar^2}{2m}\right) \nabla^2 \psi_\sigma \\ V = \int d^3r \psi_\sigma^\dagger V(r) \psi_\sigma \\ u = \frac{e^2}{2} \int d^3r d^3r' \frac{\psi_\sigma^\dagger(r)\psi_{\sigma'}^\dagger(r')\psi_{\sigma'}(r')\psi_\sigma(r)}{\mid r - r' \mid} \end{cases} \tag{3.1}$$

The hamiltonian is determined by potential $V(r)$. In the language of field theory, H is the functional of $V(r)$. Furthermore, the ground state $\mid \psi_a \rangle$ and the ground density $\rho(r)$ are also functionals of $V(r)$.

$$V(r) \rightarrow \mid \psi_a \rangle \rightarrow \rho(r) \tag{3.2}$$

The variables $\rho(r)$ and $V(r)$ are conjugate variables. We could also describe system in terms of $\rho(r)$.

First, we denote ground state for $H = T + U + V$ as $\psi(r_1, \cdots r_n)$ and ground state for $H' = T + U + V'$ as $\psi'(r_1, \cdots r_n)$. It's obvious that the state $\psi'(r_1, \cdots r_n)$ isn't ground state of H. Otherwise , we will deduce that $V(r)$ and $V(r')$ are same potential.

$$\begin{cases} (T + U + V)\psi(r_1, \cdots r_n) = E\psi(r_1, \cdots r_n) \\ (T + U + V')\psi(r_1, \cdots r_n) = E\psi(r_1, \cdots r_n) \end{cases} \implies (V - V')\psi(r_1, \cdots r_n) = (E - E')\psi(r_1, \cdots r_n) \tag{3.3}$$

Hence, the state $\psi'(r_1, \cdots r_n)$ isn't ground state for H if $\psi(r_1, \cdots r_n)$ is ground state for H. We could have

$$\langle \psi' \mid H \mid \psi' \rangle > \langle \psi \mid H \mid \psi \rangle \iff \langle \psi' \mid H + V' - V \mid \psi' \rangle \langle > \langle \psi \mid H \mid \psi \rangle \tag{3.4}$$

By the same way,

$$\langle \psi \mid H + V - V' \mid \psi \rangle > \langle \psi' \mid H \mid \psi' \rangle \tag{3.5}$$

Adding Eq(11.4,11.5),

$$\int d^3r \left(\rho'(r) - \rho(r)\right)\left(V(r) - V'(r)\right) > 0 \tag{3.6}$$

The Eq(11.16) tells that diffrent $V(r)$ gives different $\rho(r)$. This is just Khon -Heisenberg theorem , where the ground state of any interacting many particle systems with a given fixed interparticle interaction is a unique functional of the electron.

Now we discuss the benefits of theory in terms of $\rho(r)$. We defined the ground state energy functional

$$\begin{cases} E_G([\rho(r)] = F[\rho(r)] + \int d^3 r V(r)\rho(r) \\ F[\rho(r)] = \langle \psi_G \mid T + U \mid \psi_G \rangle \end{cases} \tag{3.7}$$

The ground state energy is separated into two parts , where F only depends on density $\rho(r)$ and the other term depends on lattice potential $V(r)$.

3.2 LDA

We can use local approximation such that we use the results from uniform electron gas at different values of ρ to approximate $F[\rho(r)]$. We express the ground state energy into

$$E_G[\rho(r)] = T_o[\rho(r)] + V_H[\rho(r)] + E_{xc}[\rho(r)] + \int d^3 r V(r)\rho(r) \tag{3.8}$$

The ground state could be approximated with Slater determinant $\phi_{i\sigma}(r)$,which is also determined by $\rho(r)$.

- $$T_o[\rho(r)] = \int d^3 r \sum_{i,\sigma} \phi^*_{i\sigma}(r) \left(-\frac{\hbar^2}{2m}\nabla^2\right) \phi_{i\sigma}(r) \tag{3.9}$$

  This is not real ground state of system. The slater determinent will distorted by interaction.

- $$V_H(\rho) = \frac{1}{2}\int d^3 r d' r \rho(r) \frac{e^2}{\mid r - \prime \mid} \rho(r) \tag{3.10}$$

- $$E_{xc}[\rho(r)] = F[\rho(r)] - T_o[\rho(r)] - V_H[\rho(r)] \tag{3.11}$$

  We spproximate $E_{xc}[\rho(r)]$ with function of density $\rho[\rho(r)]$ in a local way . The energy $E_{xc}[\rho(r)]$ doesn;t depend on $\nabla\rho$, which implies that it doesn't depend on fluctuations of density $\rho(r)$. We minimize energy functional under constraint of $\int d^3 r \phi^*_{i\sigma}\phi_{i\sigma} = 1$.

We write down the energy functional explicitly

$$E_G\rho(r) = \sum_{i\sigma}\int d^3 r \psi^*_{i\sigma}\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r) +\right)\psi_{i\sigma} + V_H + \int d^3 r E_{xc}[\rho(r)] - \sum_{i\sigma}\int d^3 r \lambda_{i\sigma}\left(\phi^*_{i\sigma}\phi_{i\sigma} - 1\right) \tag{3.12}$$

We take variation with respect to $\phi^*_{i\sigma}(r)$ to obtain motion equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi_{i\sigma} + \int d^3 r \frac{e^2}{\mid r - r' \mid}\rho(r') + \frac{\delta E_{xc}(\rho)}{\delta\rho(r)}\psi_{i\sigma} = \lambda_{i\sigma}\psi_{i\sigma} \tag{3.13}$$

The Eq(3.13) is analogous to single particle Schrodinger equation. We define effective potential as

$$V_{\text{eff}}(r) = V(r) + V'(r) + \frac{\delta E_{xc}(r)}{\delta\rho(r)} \tag{3.14}$$

We need to assume an initial density contribution $\rho(r)$, then we have V_{eff}. We can solve the band structure and obtain revisited $\rho(r)$. This process is interated continuely untill convergence. The unknown $\frac{\delta E_{xc}(\rho)}{\delta\rho(r)}\psi_{i\sigma}$ can be obtained through homogeneous electron gas at density ρ.

3.2.1 Exchange energy E_{xc}

We solve the exchange energy for electron with $k < k_F$ with Hatree-Fock theory.

$$\varepsilon_{Fock}(k) = -\frac{2e^2}{\pi} k_F F(x) \qquad \text{where} \quad F(x) = \frac{1}{2} + \frac{1-x^2}{4x} \log \left| \frac{1+x}{1-x} \right| \tag{3.15}$$

The total energy could be founded as

$$\sum_k \varepsilon_{\text{Fock}}(k) = \frac{V}{(2\pi)^3} \int -\frac{2e^2}{\pi} k_F F(x) = -N \left(\frac{3e^2}{4\pi} \right) k_F \tag{3.16}$$

Note:-

$$\begin{aligned}
\int_0^1 x^2 F(x) dx &= \int_0^1 x^2 \left(\frac{1}{2} + \frac{1-x^2}{4x} \log \left| \frac{1+x}{1-x} \right| \right) dx \\
&= \frac{1}{6} + \frac{1}{4} \int_0^1 (1-x^2) x \log \left| \frac{1+x}{1-x} \right| dx \\
&= \frac{1}{6} + \frac{1}{2} \int_0^1 x(1-x^2) \sum_{n=0}^{+\infty} \frac{1}{2n+1} x^{2n+1} dx \\
&= \frac{1}{6} + \frac{1}{2} \sum_{n=0}^{+\infty} \frac{1}{(2n+1)(2n+3)} - \frac{1}{(2n+1)(2n+5)} \\
&= \frac{1}{6} + \sum_{n=0}^{+\infty} \frac{1}{(2n+1)(2n+3)(2n+5)} \\
&= \frac{1}{6} + \frac{1}{2} \sum_{n=0}^{\infty} B(n+\frac{1}{2}, 3) \\
&= \frac{1}{6} + \frac{1}{2} \int_0^1 \sum_{n=0}^{\infty} x^{n-\frac{1}{2}} (1-x)^2 \\
&= \frac{1}{4}
\end{aligned}$$

Hence, the Eq(3.16) reads

$$\sum_k \varepsilon_{\text{Fock}}(k) = -\frac{V}{(2\pi)^3} \cdot \frac{2e^2 k_F^4}{\pi} \int d\Omega \int_0^{+\infty} x^2 F(x) dx = -\frac{V e^2 k_F^4}{4\pi^3} = -\frac{3e^2}{4\pi} k_F N \qquad k_F^3 = \frac{3\pi^2 N}{V} \tag{3.17}$$

With the definition of density ρ

$$\rho \frac{4\pi}{3} (r_s a_o)^3 = 1 \qquad r_s = \left(\frac{3\rho}{4\pi} \right)^{\frac{1}{3}} \frac{1}{a_0} \tag{3.18}$$

The correlation energy could be written as

$$\varepsilon_x = -\frac{e^2}{2a_0} \frac{3^{\frac{4}{3}}}{2\pi^{\frac{1}{3}}} \rho^{\frac{4}{3}} a_0 \tag{3.19}$$

3.3 Thomas-Fermi approximation

The energy functional could be written as

$$E[\rho(r)] = \int d^3rT[\rho(r)] + \frac{e^2}{2}\int \frac{\rho(r)\rho(r')}{\mid r - r' \mid} d^3rd^3r' + \int d^3rE_{xc}[\rho(r)] + \int d^3rV(r)\rho(r) - \mu \int d^3(\rho(r) - N) \quad (3.20)$$

We make variation with respect to $\rho(r)$.

$$\int d^3\delta\rho(r)\left[\frac{\delta T(\rho)}{\delta\rho} + V(r) + e^2\int d^3r'\frac{\rho(r')}{\mid r - r' \mid} + \frac{\delta E_{xc}(\rho)}{\delta\rho(r)} - \mu\right] = 0 \quad (3.21)$$

The kinetic energy could be expressed into density

$$\int d^3rT[\rho(r)] = \int d^3r\rho(r)\frac{3}{5}\frac{\hbar^2k_F^2}{2m} = \frac{3}{10}\frac{\hbar^2}{m}(3\pi^2)^{\frac{2}{3}}\int \rho^{\frac{5}{3}}(r)d^3r \implies \frac{\delta T[\rho(r)]}{\delta\rho(r)} = \frac{\hbar^2}{2m}(3\pi^2\rho)^{\frac{2}{3}} \quad (3.22)$$

We can derive motion equation

$$\frac{\hbar^2}{2m}(3\pi^2\rho)^{\frac{2}{3}} + V(r) + V_i(r) = \mu \quad (3.23)$$

Chapter 4

4.1 Caroli-de-Gennes-Matricon Vortex core state

We start from Bdg equation , which reads as

$$\begin{pmatrix} H_0(r) & \Delta(r) \\ \Delta^*(r) & -H_0(r) \end{pmatrix} \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} = E \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} \tag{4.1}$$

where $H_0 = \frac{1}{2m}\left(-\mathrm{i}\hbar\nabla - e\vec{A}\right)^2 - \mu$, $\Delta(r) = \Delta(r)e^{\mathrm{i}\varphi}$. Now we make ansatz that the solution is free at z direction .

$$\begin{pmatrix} u(r) \\ v(r) \end{pmatrix} = e^{\mathrm{i}k_z z} e^{\mathrm{i}m\varphi} \begin{pmatrix} f(r) \\ g(r) \end{pmatrix} \tag{4.2}$$

In this system, it have $SO(2)$ symmetry around z axis. Hence, we could define conserve quantity L_z , which is the summation of spin and orbit angular momentum.

$$L_z = -\mathrm{i}\hbar\frac{\partial}{\partial\varphi} - \frac{\hbar}{2}\sigma_z \tag{4.3}$$

Obviously, the state (4.2) is the eigenstate of operator (4.3) .

$$L_z \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} = (n + \frac{1}{2})\hbar \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} \tag{4.4}$$

Now, we substitute (4.2) into (4.1) to derive the simplified equation

$$\begin{cases} \left(-\dfrac{\hbar^2}{2m}\dfrac{\partial}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} - \dfrac{(m+1)^2}{r^2} + k_{\parallel}^2\right) f(r) + \Delta(r)g(r) = Ef(r) \\ \Delta(r)f(r) + \left(-\dfrac{\hbar^2}{2m}\dfrac{\partial}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} - \dfrac{m^2}{r^2} + k_{\parallel}^2\right) f(r) = Eg(r) \end{cases} \tag{4.5}$$

where $k_{\parallel}^2 = k_f^2 - k_z^2$. The chemical potential is equal to Fermi energy $\frac{\hbar^2 k_f^2}{2m}$. The kinetiic energy along z diretion contributes term $\frac{\hbar^2 k_z^2}{2m}$. The Bdg equation have particle hole symmetry

$$\mathrm{i}\sigma_y K \begin{pmatrix} H_0(r) & \Delta(r) \\ \Delta^*(r) & -H_0(r) \end{pmatrix} K\mathrm{i}\sigma_y = \begin{pmatrix} H_0(r) & \Delta(r) \\ \Delta^*(r) & -H_0(r) \end{pmatrix} \tag{4.6}$$

Hence, we could find the negative energy solution with particle hole transformation

$$\mathrm{i}\sigma_y K \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} = e^{-\mathrm{i}k_z z} e^{-\mathrm{i}m\phi} \begin{pmatrix} -g(r) \\ f(r)e^{-\mathrm{i}\phi} \end{pmatrix} \tag{4.7}$$

We consider the solution at the vertex core . The gap function vanishes, then the $f(r)$ and $g(r)$ are discoupled . We could write down two equations as

$$\begin{cases} \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+1)^2}{r^2} k_+^2\right] f(r) = 0 \\ \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} k_-^2\right] g(r) = 0 \end{cases} \tag{4.8}$$

The solution is given Bessel function , namly $f(r) = A_+ f(k_+ r), g(r) = A_- g(k_- r)$. The wavevector $k_\pm$ is defined as

$$k_\pm = \sqrt{k_\parallel^2 \pm \frac{2mE}{\hbar^2}} = k_\parallel \pm \frac{mE}{\hbar^2 k_\parallel} \tag{4.9}$$

In order to find finite solution at $r = 0$, we only consider the first class Bessel function. The Bdg equation could be rewritten into

$$[\Delta(r)\sigma_1 + h_l(r)\sigma_3] \begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = \left(E - \frac{\hbar^2}{4mr^2}(2n+1)\right) \begin{pmatrix} f(r) \\ g(r) \end{pmatrix} \tag{4.10}$$

where $l^2 = \frac{(m+1)^2 + m^2}{2} = m^2 + m + \frac{1}{2}$. We use Hankel function $H_l^1(k_\parallel r)$ to expand solution $f(r)$

$$\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = H_l^1(k_\parallel r) \begin{pmatrix} \tilde{f}(r) \\ \tilde{g}(r) \end{pmatrix} + h.c \tag{4.11}$$

Note:-

$$h_l(r)f(r) = \big(h_l(r)H_l(k_\parallel r)\big)\tilde{f}(r) + 2\frac{d}{dr}H_l(k_\parallel r)\frac{d}{dr}\tilde{f}(r) + H_l(k_\parallel r)\frac{d^2}{dr^2}\tilde{f}(r) + H_l(k_\parallel r)\frac{1}{r}\frac{d}{dr}\tilde{f}(r) \tag{4.12}$$

$(\tilde{f}, \tilde{g})$ ia the slow varing function. Hence, we neglect the last term on the (4.12)

The Eq(4.10) could be simplified into

$$\left[-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + 2\frac{\log H_l(k_\parallel r)}{dr}\right)\sigma_s + \Delta(r)\sigma_1\right] \begin{pmatrix} \tilde{f}(r) \\ \tilde{g}(r) \end{pmatrix} = \left(E - \frac{\hbar^2(2n+1)}{4mr^2}\right) \begin{pmatrix} \tilde{f}(r) \\ \tilde{g}(r) \end{pmatrix} \tag{4.13}$$

The equation (4.13) is difficult to solve about $\Delta(r)$. However, we could derive approximation solution under limit $k_F r \gg 1$. The Eq((4.13)) will be turns into

$$\left[-\mathrm{i}\hbar v_\parallel \sigma_3 \frac{d}{dr} + \Delta(r)\sigma_1\right] \begin{pmatrix} \tilde{f}(r) \\ \tilde{g}(r) \end{pmatrix} = \left[E - \frac{\hbar^2(2m+1)}{4mr^2}\right] \begin{pmatrix} \tilde{f}(r) \\ \tilde{g}(r) \end{pmatrix} \tag{4.14}$$

The centrigugal term on the right side is small quantity in relative to energy gap Δ

$$\frac{\hbar^2(m+\frac{1}{2})}{2mr^2\Delta} = \frac{\pi\xi}{\hbar f}\frac{\hbar^2(m+\frac{1}{2})}{2mr^2} \sim \frac{\xi}{r}\frac{1}{k_f r} \ll 1 \tag{4.15}$$

We write down the Eq(4.14)

$$\begin{cases} -\mathrm{i}\hbar v_\parallel \frac{d}{dr}\tilde{f}(r) + \Delta(r)\tilde{g}(r) = \left(E - \frac{\hbar^2(2m+1)}{4mr^2}\right)\tilde{f}(r) \\ \mathrm{i}\hbar v_\parallel \frac{d}{dr}\tilde{g}(r) + \Delta(r)\tilde{f}(r) = \left(E - \frac{\hbar^2(2m+1)}{4mr^2}\right)\tilde{g}(r) \end{cases} \tag{4.16}$$

Firstly, we find the solution for homogenous part of Eq(4.16) . We could take ansatz that $\tilde{f}(r) = -\mathrm{i}\tilde{g}(r)$ .The homogenepus equation reads as

$$\frac{d\tilde{f}(r)}{dr} + \frac{\Delta(r)}{\hbar v_{\parallel}}\tilde{f}(r) = 0 \implies \tilde{f}(r) = e^{-\int_0^r \Delta(r')dr'} \tag{4.17}$$

We consider the solution for the inhomogeous equation (4.16) . We could take ansatz , namly $\tilde{f}(r) = e^{\mathrm{i}\frac{\psi(r)}{2}}e^{-\int_0^r \Delta(r')dr'}, \tilde{g}(r) = -\mathrm{i}e^{-\mathrm{i}\frac{\psi(r)}{2}}e^{-\int_0^r \Delta(r')dr'}$, where $\psi(r)$ is small quantity . We expand the equation into first order

$$\frac{1}{2}\frac{d\psi(r)}{dr} - \frac{1}{2}\frac{\Delta(r)}{\hbar v_{\parallel}}\psi(r) = \frac{1}{\hbar v_{\parallel}}\left(E - \frac{\hbar^2(2m+1)}{4mr^2}\right) \tag{4.18}$$

We could solve from (4.18)

$$\psi(r) = -\int_r^{\infty} e^{K_1(r)}\frac{2}{\hbar v_{\parallel}}\left(E - \frac{\hbar^2(2m+1)}{4mr^2}\right) \tag{4.19}$$

Now we begin to find energy E with boundary condition. We consider the phase at r_c^-

$$\begin{cases} f(k_+ r) \sim \cos\left(k_+ r_c + \frac{(n+1)^2}{2k_+ r_c} - \frac{n+1}{2}\pi - \frac{\pi}{4}\right) \\ f(k_- r) \sim \cos\left(k_- r_c + \frac{n^2}{2k_- r_c} - \frac{n}{2}\pi - \frac{\pi}{4}\right) \end{cases} \tag{4.20}$$

The phase difference is given as

$$\lim_{r\to r_c} \delta\phi = 2k_0 r_c + \frac{2n+1}{2k_{\parallel r_c}} - \frac{\pi}{2} + \pi \tag{4.21}$$

The minus sign on the Eq(4.19) would contribute π phase on the (4.21). Hence, we could write down phase condition $2k_0 r_c + \frac{2n+1}{2k_{\parallel r_c}} = \psi(r_c)$

Chapter 5

Bosonization and Luttinger liquid

In this secton, we will discuss Bosonization. We focus particle-hole excitation in the vivinity of Fermi surface $x = \pm k_F$. Then, the fermion annihilation operator f_n reads as

$$f_n = R(x_n)e^{\mathrm{i}k_F x_n} + L(x_n)e^{-\mathrm{i}k_F x_n} \tag{5.1}$$

Hence, the term $f_n^\dagger f_n$ is given as

$$f_n^\dagger f_n = R^\dagger(x_n)R(x_n) + L^\dagger(x_n)L(x_n) + e^{\mathrm{i}2k_F x_n}\left(R^\dagger(x_n)L^\dagger(x_n) + L^\dagger(x_n)R^\dagger(x_n)\right) \tag{5.2}$$

The right and left mover density could be described as

$$\rho_R(x) =: R^\dagger(x)R(x) : \rho_L(x) =: L^\dagger(x)L(x) : \tag{5.3}$$

The principal idea of bosonization is to describe the fermionic system in terms of the density fluctuation operator in (5.3). The density $\rho(x)$ and current $j(x)$ could be written as

$$\rho(x) = \bar{\psi}\psi = \rho_R(x) + \rho_L(x) \qquad j(x) = \bar{\psi}\gamma_5\psi = \rho_R(x) - \rho_L(x) \tag{5.4}$$

We discuss the properties of operator $\rho_{R/L}(x)$. We consider the commutator of $\rho_R(-p)$ and $\rho_R(p')$,

$$[\rho_R(-p), \rho(p')] = \frac{1}{L}\sum_{k,k'}[R^\dagger(k+p)R(k), R^\dagger(k')R(k'+p')] = \frac{1}{L}\sum_k \left(R^\dagger(k+p)R(k'+p') - R(k+p-p')R(k)\right) \tag{5.5}$$

We use the expression of normal ordering

$$R^\dagger(k+p)R(k+p') =: R^\dagger(k+p)R(k+p') : +\delta_{p,p'}n^R_{k+p} \tag{5.6}$$

The result turns into

$$[\rho_R(-p), \rho_R(p)] = \delta_{pp'}\frac{1}{L}\sum_k (n^R_{k+p} - n^R_k) = -\frac{p}{2L} \tag{5.7}$$

By the same way, we could derive

$$[\rho_L(-p), \rho_L(p')] = \frac{p}{2\pi}\delta_{pp'} \tag{5.8}$$

We consider real space commutator

$$[\rho_R(x),\rho_R(x')] = \frac{1}{L}\sum_{p,p'} e^{\mathrm{i}(p'x'-px)}[\rho_R(-p),\rho_R(p')] = \frac{1}{L}\sum_{p} -\frac{p}{2\pi}e^{\mathrm{i}(p'x'-px)} = -\frac{\mathrm{i}}{2\pi}\partial_x\delta(x-x') \tag{5.9}$$

and

$$[\rho_R(x),\rho_R(x')] = \frac{\mathrm{i}}{2\pi}\partial_x\delta(x-x') \tag{5.10}$$

Now we introduce the phase field $\phi_{R/L}(x)$ [1], which reads as

$$\begin{cases} \phi_R(x) = \dfrac{2\pi}{\sqrt{L}}\displaystyle\sum_{p>0} e^{-\alpha p/2}\frac{1}{\mathrm{i}p}\left(e^{\mathrm{i}px}\rho_R(p) - e^{-\mathrm{i}px}\rho_R(-p)\right) = \phi_R^+(x)+\phi_R^-(x) \\ \phi_L(x) = \dfrac{2\pi}{\sqrt{L}}\displaystyle\sum_{p>0} e^{-\alpha p/2}\frac{1}{\mathrm{i}p}\left(e^{\mathrm{i}px}\rho_L(p) - e^{-\mathrm{i}px}\rho_L(-p)\right) = \phi_L^+(x)+\phi_L^-(x) \end{cases} \tag{5.11}$$

Using $\phi_R(x)$ and $\phi_L(x)$, we define

$$\begin{cases} \theta_+ = \phi_R + \phi_L \\ \theta_- = \phi_R - \phi_L \end{cases} \tag{5.12}$$

Hence, the Eq(5.4) could be expressed into

$$\rho(x) = \frac{1}{2\pi}\partial_x\theta_+(x) \quad j(x) = \frac{1}{2\pi}\partial_x\theta_-(x) \tag{5.13}$$

Here, we consider the commutator for $\phi_R(x)$.

$$\begin{aligned} -[\phi_R(x),\phi_R(x')] &= \frac{4\pi^2}{L}\sum_{p,p'>0} e^{-\alpha(p+p')/2}\frac{-1}{pp'}[e^{\mathrm{i}px}\rho_R(p) - e^{-\mathrm{i}px}\rho_R(-p), e^{\mathrm{i}p'x'}\rho_R(p') - e^{-\mathrm{i}p'x'}\rho_R(-p')] \\ &= -\frac{4\pi^2}{L}\sum_{p,p'>0} e^{-\alpha(p+p')/2}\frac{-1}{pp'}\frac{p}{2\pi}\left[e^{\mathrm{i}(px+p'x')}\delta_{p,-p'} + e^{\mathrm{i}(px-p'x')}\delta_{p,p'} - e^{-\mathrm{i}(px-p'x')}\delta_{p,p'} - e^{-\mathrm{i}(px+p'x')}\delta_{p,-p'}\right] \\ &= -\frac{2\pi}{L}\sum_{p>0} e^{-\alpha p}\frac{1}{p}[-e^{\mathrm{i}p(x-x')} + e^{-\mathrm{i}p(x-x')}] \\ &= 2\mathrm{i}\int_0^\infty \frac{e^{-\alpha p}\sin p(x-x')}{p} \\ &= \mathrm{i}\pi\mathrm{sgn}(x-x') \end{aligned} \tag{5.14}$$

Note:-

The last line on the (7.41) could be derived with Feynmann technology

$$\frac{dF(\alpha)}{d\alpha} = -\int_0^\infty e^{-\alpha p}\sin p(x-x') = -\mathrm{Im}\left(\frac{1}{\alpha - \mathrm{i}(x-x')}\right) = -\pi\delta(x-x') \tag{5.15}$$

Hence, we could derive $F(\alpha)$

$$F(\alpha) = -\frac{\pi}{2}\mathrm{sgn}(x-x') \tag{5.16}$$

[1] Why introduce ϕ field

The Eq(7.41) tells us $\theta_\pm$ are conjugate field to each other.

$$\begin{cases} [\theta_\pm(x), \theta_\pm(x')] = [\phi_R(x) \pm \phi_L(x), \phi_R(x') \pm \phi_L(x')] = [\phi_R(x), \phi_R(x')] + [\phi_L(x), \phi_L(x')] = 0 \\ [\theta_\pm(x), \theta_\mp(x')] = [\phi_R(x) \pm \phi_L(x), \phi_R(x') \pm \phi_L(x')] = [\phi_R(x), \phi_R(x')] - [\phi_L(x), \phi_L(x')] = 2\pi \mathrm{isgn}(x - x') \end{cases} \tag{5.17}$$

We define the momentum operator $\pi(x)$ as

$$\pi(x) = -\frac{1}{4\pi}\partial_x \theta_-(x) \tag{5.18}$$

then we have canonical commutation relation

$$[\theta_+(x), \pi(x')] = \mathrm{i}\delta(x - x') \tag{5.19}$$

The fermionic operator $R(x), L(x)$ could be written as

$$\begin{cases} R(x) = \frac{1}{\sqrt{2\pi\alpha}}\eta_1 e^{\mathrm{i}\phi_R(x)} \\ L(x) = \frac{1}{\sqrt{2\pi\alpha}}\eta_2 e^{\mathrm{i}\phi_L(x)} \end{cases} \implies \begin{cases} R(x) = \frac{1}{\sqrt{2\pi\alpha}}\eta_1 e^{\mathrm{i}(\theta_+(x)+\theta_-(x))} \\ L(x) = \frac{1}{\sqrt{2\pi\alpha}}\eta_2 e^{\mathrm{i}(-\theta_+(x)+\theta_-(x))} \end{cases} \tag{5.20}$$

We could derive other quantities

$$\bar{\psi}\psi = R^\dagger(x)L(x) + L^\dagger(x)R(x) = \frac{1}{2\pi\alpha}\left[e^{\mathrm{i}(\phi_L(x)-\phi_R(x))} + e^{-\mathrm{i}(\phi_L(x)-\phi_R(x))}\right] = \frac{1}{\pi\alpha}\cos\theta_-(x) \tag{5.21}$$

$$\mathrm{i}\bar{\psi}\gamma_5\psi = \mathrm{i}\left(R^\dagger(x)L(x) - L^\dagger(x)R(x)\right) = \frac{\mathrm{i}}{2\pi\alpha}\left[e^{\mathrm{i}(\phi_L(x)-\phi_R(x))} - e^{-\mathrm{i}(\phi_L(x)-\phi_R(x))}\right] = \frac{1}{\pi\alpha}\sin\theta_-(x) \tag{5.22}$$

We summarize Bosonization rule as below

$$\begin{cases} R(x) = \frac{1}{\sqrt{2\pi\alpha}}\eta_1 e^{\mathrm{i}(\theta_+(x)+\theta_-(x))} \\ L(x) = \frac{1}{\sqrt{2\pi\alpha}}\eta_2 e^{\mathrm{i}(-\theta_+(x)+\theta_-(x))} \\ \bar{\psi}\psi = R^\dagger(x)L(x) + L^\dagger(x)R(x) = \frac{1}{2\pi\alpha}\left[e^{\mathrm{i}(\phi_L(x)-\phi_R(x))} + e^{-\mathrm{i}(\phi_L(x)-\phi_R(x))}\right] = \frac{1}{\pi\alpha}\cos\theta_-(x) \\ \mathrm{i}\bar{\psi}\gamma_5\psi = \mathrm{i}\left(R^\dagger(x)L(x) - L^\dagger(x)R(x)\right) = \frac{\mathrm{i}}{2\pi\alpha}\left[e^{\mathrm{i}(\phi_L(x)-\phi_R(x))} - e^{-\mathrm{i}(\phi_L(x)-\phi_R(x))}\right] = \frac{1}{\pi\alpha}\sin\theta_-(x) \end{cases} \tag{5.23}$$

5.1 Examples

The first example is XY model on one-dimensional. The hamiltonian is reads as

$$H_{XY} = -\frac{J_\perp}{2}\sum_i f_{i+1}^\dagger f_i + f_i^\dagger f_{i+1} \tag{5.24}$$

We expand the hamiltonian around the Fermi surface

$$H_{XY} = -J_\perp \sum_k k R^\dagger(k)R(k) + (R \to L) \tag{5.25}$$

Let's consider the commutation relation between H_{XY}

$$[\rho_R(p), H_{XY}] = -\sum_{k,k'}[R^\dagger(k)R(k+p), J_\perp k' R^\dagger(k')R'(k')] = -\sum_{k,k'} J_\perp k' \left[-R^\dagger(k')R(k+p)\delta_{kk'} + \delta_{k+p,k'}R^\dagger(k)R(k')\right]$$
$$= -J_\perp p \rho_R(p) \tag{5.26}$$

By the same way

$$[\rho_L(p), H_{XY}] = -J_\perp p \rho_L(p) \tag{5.27}$$

We could construct effective hamiltonian $\tilde{H}_{XY}$ satisfies to commutation relation (5.25,5.26)

$$\tilde{H}_{XY} = J_\perp \sum_p \left[\rho_R(p)\rho_R(-p) + \rho_L(-p)\rho_L(p)\right] \tag{5.28}$$

We use the Bosonization rule (**??**) to express (5.28) into $\theta_\pm$

$$\tilde{H}_{XY} = \frac{J_\perp}{8\pi} \int dx \left[(\partial_x\theta_+(x))^2 + \partial_x\theta_-(x))^2\right] \tag{5.29}$$

The tranverse field part is given by

$$H_z = \sum_i (f_n f_{n+1} - \frac{1}{2})(f_n f_{n+1} - \frac{1}{2}) = \sum_n \left[\rho_R(x_n) + \rho_L(x_n) + e^{\mathrm{i}2k_F x_n} \frac{1}{2\pi\alpha}(e^{\mathrm{i}(\phi_R+\phi_L)} + e^{-\mathrm{i}(\phi_R+\phi_L)})\right] \tag{5.30}$$

5.2 Luttinger model

5.3 Abelian bosonization of Luttinger liquid

The free boson hamiltonian reads as

$$\mathcal{H}_0 = \frac{v_F}{2} \sum_\sigma \left((\partial_x\phi_\sigma)^2 + \Pi_\sigma^2\right) \tag{5.31}$$

We define charge Bose field and spin Bose field as below

$$\begin{cases} \phi_c = \dfrac{1}{\sqrt{2}}(\phi_\uparrow + \phi_\downarrow) \\ \phi_s = \dfrac{1}{\sqrt{2}}(\phi_\uparrow - \phi_\downarrow) \end{cases} \tag{5.32}$$

The fee Bose hamiltonian $\mathcal{H}_0$ is splitted into charge and spin parts

$$\mathcal{H}_0 = \frac{v_F}{2}\left((\partial_x\phi_c)^2 + \Pi_c^2\right) + \frac{v_F}{2}\left((\partial_x\phi_s)^2 + \Pi_s^2\right) \tag{5.33}$$

The spin-1/2 Luttinger model could be written into

$$\mathcal{H}_0 = \frac{v_c}{2}\left(\frac{1}{K_c}\Pi_c^2 + K_c(\partial_x\phi_c)^2\right) + \frac{v_s}{2}\left(\frac{1}{K_s}\Pi_c^2 + K_s(\partial_x\phi_s)^2\right) + V_c\cos(\sqrt{4\pi}\phi_c) + V_s\cos(\sqrt{4\pi}\phi_s) \tag{5.34}$$

where charge and spin velocities are

$$
\begin{cases} v_c = \sqrt{\left(v_F + \frac{g_4}{\pi}\right)^2 - \left(\frac{g_{1,\parallel}}{\pi} - \frac{2g_2}{\pi}\right)^2} \\ v_s = \sqrt{\left(v_F - \frac{g_4}{\pi}\right)^2 - \left(\frac{g_{1,\parallel}}{\pi}\right)^2} \end{cases} \tag{5.35}
$$

The charge Luttinger parameter and spin Luttinger paramete K_c and K_s is given as

$$
k_c = \sqrt{\frac{\pi v_F + g_4 + 2g_2 - g_{1,\parallel}}{\pi v_F + g_4 - 2g_2 + g_{1,\parallel}}} \quad k_s = \sqrt{\frac{\pi v_F - g_4 - g_{1,\parallel}}{\pi v_F - g_4 + g_{1,\parallel}}} \tag{5.36}
$$

The coupling coefficients V_c and V_s is given as

$$
V_c = \frac{g_3}{2\pi^2} \quad V_s = \frac{g_{1,\perp}}{2\pi^2} \tag{5.37}
$$

Note:-

- Forward scattering term

$$
\begin{aligned}
g_4 \sum_{s,\sigma} \psi^\dagger_{s,\sigma} \psi^\dagger_{s,-\sigma} \psi_{s,-\sigma} \psi_{s,\sigma} = g_4 \sum_{s,\sigma} \rho_{s\sigma}\rho_{s,-\sigma} &= 2g_4 \left(\rho_{R,\uparrow}\rho_{R,\downarrow} + \rho_{L,\uparrow}\rho_{L,\downarrow}\right) \\
&= \frac{2g_4}{\pi} \left((\partial_x \phi_{R,\uparrow})(\partial_x \phi_{R,\downarrow}) + (\partial_x \phi_{L,\uparrow})(\partial_x \phi_{L,\downarrow})\right) \\
&= \frac{g_4}{2\pi} \left[(\partial_x \phi_{R,c})^2 - (\partial_x \phi_{R,s})^2 + (\partial_x \phi_{L,c})^2 - (\partial_x \phi_{L,s})^2\right] \\
&= \frac{g_4}{2\pi} \left[(\partial_x \phi_c)^2 + (\partial_x \vartheta_c)^2 - (\partial_x \phi_s)^2 - (\partial_x \vartheta_s)^2\right]
\end{aligned} \tag{5.38}
$$

- Forward process on opposite branches

$$
g_2 \sum_{\sigma',\sigma} \psi^\dagger_{1,\sigma} \psi^\dagger_{-1,\sigma'} \psi_{-1,\sigma'} \psi_{1,\sigma} = g_2 \sum_{\sigma',\sigma} \rho_{R,\sigma}\rho_{L,\sigma'} = g_2 \partial_x \phi_{Rc} \partial_x \phi_{Lc} = \frac{g_2}{\pi} \left[(\partial_x \phi_c)^2 - (\partial_x \vartheta_c)^2\right] \tag{5.39}
$$

- Backscattering process without spin flip

$$
\begin{aligned}
g_{1,\parallel} \sum_{\sigma} \psi^\dagger_{1,\sigma} \psi^\dagger_{-1,\sigma} \psi_{1,\sigma} \psi_{-1,\sigma} = -g_{1,\parallel} \sum_{\sigma} \rho_{R\sigma}\rho_{L\sigma} &= -g_{1,\parallel} \left(\rho_{R\uparrow}\rho_{L\uparrow} + \rho_{R\downarrow}\rho_{L\downarrow}\right) \\
&= -\frac{g_{1,\parallel}}{2\pi} \left[\partial_x(\phi_{Rc} + \phi_{Rs})\partial_x(\phi_{Lc} + \phi_{Ls}) + \partial_x(\phi_{Rc} - \phi_{Rs})\partial_x(\phi_{Lc} - \phi_{Ls})\right] \\
&= -\frac{g_{1,\parallel}}{\pi} \left[(\partial_x \phi_{Rc})(\partial_x \phi_{Lc}) + (\partial_x \phi_{Rs})(\partial_x \phi_{Rs})\right] \\
&= -\frac{g_{1,\parallel}}{2\pi} \left[(\partial_x \phi_c)^2 - (\partial_x \vartheta_c)^2 + (\partial_x \phi_s)^2 - (\partial_x \vartheta_s)^2\right]
\end{aligned} \tag{5.40}
$$

- Scattering process on opposite branches with spin flip

$$
g_{1,\perp} \sum_{\sigma} \psi^\dagger_{1,\sigma} \psi^\dagger_{-1,-\sigma} \psi_{1,-\sigma} \psi_{-1,\sigma} = \frac{g_{1,\perp}}{4\pi^2} \left[e^{-\sqrt{4\pi}\mathrm{i}(\phi_{R,\uparrow} - \phi_{L,\downarrow} - \phi_{R,\downarrow} + \phi_{L,\uparrow})} + h.c\right] = \frac{g_{1,\perp}}{2\pi^2} \cos \phi_s \tag{5.41}
$$

- Umklapp scattering

$$H_u = g_3 e^{\mathrm{i}(4p_F - G)x}\psi^\dagger_{-1,\uparrow}\psi^\dagger_{-1,\downarrow}\psi_{1,\downarrow}\psi_{1,\uparrow} + h.c = \frac{g_3}{4\pi^2}e^{\mathrm{i}\sqrt{4\pi}(\phi_{R,\uparrow}+\phi_{L,\downarrow}+\phi_{R,\downarrow}+\phi_{L,\uparrow})} + h.c = \frac{g_3}{2\pi^2}\cos\sqrt{4\pi}\phi_c \tag{5.42}$$

The Luttinger model could be collected as

$$\begin{aligned}\mathcal{H}_0 = &\frac{1}{2}\left((v_F + \frac{g_4}{\pi} + \frac{2g_2}{\pi} - \frac{g_\parallel}{\pi})(\partial_x\phi_c)^2 + (v_F + \frac{g_4}{\pi} - \frac{2g_2}{\pi} + \frac{g_\parallel}{\pi})(\partial_x\vartheta_c)^2\right)\\ &+ \frac{1}{2}\left((v_F - \frac{g_4}{\pi} - \frac{g_\parallel}{\pi})(\partial_x\phi_s)^2 + (v_F - \frac{g_4}{\pi} + \frac{g_\parallel}{\pi})(\partial_x\vartheta_s)^2\right) + \frac{g_{1,\perp}}{2\pi^2}\cos\phi_s + \frac{g_3}{2\pi^2}\cos\sqrt{4\pi}\phi_c\end{aligned} \tag{5.43}$$

We will summarize properties about Tomonaga-Lutinger model (5.34) below .

- Spin and charge freedom doesn't couple: spin-charge separation
- The charge veclocities is larger that spin if system lies on repulsive interaction.
- The fermionic field ψ_s is expressed as

$$\psi_\alpha = \frac{1}{\sqrt{2\pi}}\eta_{\alpha,s}e^{-\alpha\mathrm{i}\sqrt{2\pi}(\phi_{\alpha,c}+\sigma\phi_{\alpha,s})} \tag{5.44}$$

- *SU*(2) *chiral currents* are given by

$$\begin{cases} J^3_R = \psi^\dagger_{R,\uparrow}\psi_{R,\uparrow} - \psi^\dagger_{R,\downarrow}\psi_{R,\downarrow} = \frac{1}{2\sqrt{\pi}}\partial_x\phi_{R,s}\\ J^3_L = \psi^\dagger_{L,\uparrow}\psi_{L,\uparrow} - \psi^\dagger_{L,\downarrow}\psi_{L,\downarrow} = \frac{1}{2\sqrt{\pi}}\partial_x\phi_{L,s}\\ J^\pm_R = \psi^\dagger_{R,\pm}\psi_{R,\mp} = e^{\mp\sqrt{4\pi}(\phi_{R,\uparrow}-\phi_{R,\downarrow})} = e^{\mp 2\sqrt{2\pi}(\phi_{R,\uparrow}-\phi_{R,\downarrow})}\\ J^\pm_L = \psi^\dagger_{L,\pm}\psi_{L,\mp} = e^{\pm\sqrt{4\pi}(\phi_{L,\uparrow}-\phi_{L,\downarrow})} = e^{\pm 2\sqrt{2\pi}(\phi_{L,\uparrow}-\phi_{L,\downarrow})}\end{cases} \tag{5.45}$$

- *Charge density wave*

$$\mathcal{O}_{\mathrm{CDW}} = e^{-2p_F x}\sum_\sigma \psi^\dagger_{1,\sigma}\psi_{-1,\sigma} = \frac{e^{-2p_F x}}{\pi}(e^{-\mathrm{i}\sqrt{4\pi}(\phi_{R\uparrow}+\phi_{L\uparrow})} + e^{-\mathrm{i}\sqrt{4\pi}(\phi_{R\downarrow}+\phi_{L\downarrow})}) = \frac{e^{-2p_F x}}{\pi}e^{-\mathrm{i}\sqrt{2\pi}\phi_c}\cos\sqrt{2\pi}\phi_s \tag{5.46}$$

- *Spin density wave*

$$\mathcal{O}^3_{SDW} = e^{-\mathrm{i}2p_F x}\sum_{\alpha,\beta}\psi^\dagger_{R,\alpha}\sigma^3_{\alpha\beta}\psi_{L,\beta} = \frac{e^{-\mathrm{i}2p_F x}}{\pi}(e^{-\mathrm{i}\sqrt{4\pi}(\phi_{R\uparrow}+\phi_{L\uparrow})} - e^{-\mathrm{i}\sqrt{4\pi}(\phi_{R\downarrow}+\phi_{L\downarrow})}) = -\frac{e^{-\mathrm{i}2p_F x}}{\pi}e^{-\mathrm{i}\sqrt{2\pi}\phi_c}\sin\sqrt{2\pi}\phi_s \tag{5.47}$$

- *Singlet superconductivity*

$$\begin{aligned}\mathcal{O}_{SS} &= \frac{1}{\sqrt{2}}(\psi^\dagger_{R,\uparrow}\psi^\dagger_{L,\downarrow} - \psi^\dagger_{R,\downarrow}\psi^\dagger_{L,\uparrow}) = \frac{1}{\sqrt{2\pi^2}}\left(e^{-\mathrm{i}\sqrt{4\pi}(\phi_{R,\uparrow}-\vartheta_{L,\downarrow})} - e^{-\mathrm{i}\sqrt{8\pi}(\phi_{R,\downarrow}-\vartheta_{L,\uparrow})}\right)\\ &= \frac{1}{\sqrt{8\pi^2}}\left(e^{-\mathrm{i}\sqrt{2\pi}(\vartheta_c+\phi_s)} - e^{-\mathrm{i}\sqrt{2\pi}(\vartheta_c-\phi_s)}\right)\\ &= -\mathrm{i}\sqrt{\frac{1}{2\pi^2}}e^{-\mathrm{i}\sqrt{2\pi}\vartheta_c}\sin\sqrt{2\pi}\phi_s\end{aligned} \tag{5.48}$$

- *Triplet superconductivity*

$$
\begin{cases}
\mathcal{O}^1_{\text{TS}} = \psi^\dagger_{R,\uparrow}\psi^\dagger_{L,\uparrow} = \dfrac{1}{2\pi}e^{-\mathrm{i}\sqrt{4\pi}(\phi_{R\uparrow}-\phi_{L\uparrow})} = \dfrac{1}{2\pi}e^{-\mathrm{i}\sqrt{2\pi}(\vartheta_c+\vartheta_s)} \\
\mathcal{O}^0_{\text{TS}} = \sqrt{\dfrac{1}{2\pi^2}}e^{-\mathrm{i}\sqrt{2\pi}\vartheta_c}\cos\sqrt{2\pi}\phi_s \\
\mathcal{O}^{-1}_{\text{TS}} = \psi^\dagger_{R,\downarrow}\psi^\dagger_{L,\downarrow} = \dfrac{1}{2\pi}e^{-\mathrm{i}\sqrt{4\pi}(\phi_{R\downarrow}-\phi_{L\downarrow})} = \dfrac{1}{2\pi}e^{-\mathrm{i}\sqrt{2\pi}(\vartheta_c-\vartheta_s)}
\end{cases}
\tag{5.49}
$$

5.4 Correlation function of the Luttinger model

The hamiltonian of Lutiinger liquid reads as

$$
H = (\pi v_F + g_4)(\rho_L^2 + \rho_R^2) + 2g_2\rho_R\rho_L \tag{5.50}
$$

The density operator ρ_R, ρ_L are bosonic operator. Hence, we us bosonic Bogliubov transformation to diagonalize hamiltonian

$$
\rho_R = \cosh\theta\tilde{\rho}_R + \sinh\theta\tilde{\rho}_L \tag{5.51}
$$

The hamiltonian (5.50) could be written into

$$
H = (\tilde{\rho}_R, \tilde{\rho}_L)\begin{pmatrix}\cosh\theta & \sinh\theta\\ \sinh\theta & \cosh\theta\end{pmatrix}\begin{pmatrix}\pi v_F + g_4 & g_2\\ g_2 & \pi v_F + g_4\end{pmatrix}\begin{pmatrix}\cosh\theta & \sinh\theta\\ \sinh\theta & \cosh\theta\end{pmatrix}\begin{pmatrix}\tilde{\rho}_R\\ \tilde{\rho}_L\end{pmatrix} \tag{5.52}
$$

We choose θ as

$$
\tanh 2\theta = -\frac{g_2}{\pi v_F + g_4} \tag{5.53}
$$

Hence, the hamiltonian could be diagonalized as

$$
H = \pi v(\tilde{\rho}_R^2 + \tilde{\rho}_L^2) = \frac{v}{2}\left((\partial_x\tilde{\theta})^2 + (\partial_x\tilde{\phi})^2\right) \tag{5.54}
$$

The velocity v is

$$
\pi v = \sqrt{(\pi v_F + g_4)^2 - g_2^2} \tag{5.55}
$$

We express the θ with Luttinger parameter. We consider operator $\tilde{\rho}_R$ in the form of bose field ϕ

$$
\begin{cases}
\tilde{\rho}_R = \cosh\theta\dfrac{1}{\sqrt{\pi}}\partial_x\phi_R + \sinh\theta\dfrac{1}{\sqrt{\pi}}\partial_x\phi_L = \dfrac{\cosh\theta+\sinh\theta}{2\sqrt{\pi}}\partial_x\phi + \dfrac{\cosh\theta-\sinh\theta}{2\sqrt{\pi}}\partial_x\vartheta \\
\tilde{\rho}_L = \cosh\theta\dfrac{1}{\sqrt{\pi}}\partial_x\phi_L + \sinh\theta\dfrac{1}{\sqrt{\pi}}\partial_x\phi_R = \dfrac{\cosh\theta+\sinh\theta}{2\sqrt{\pi}}\partial_x\phi - \dfrac{\cosh\theta-\sinh\theta}{2\sqrt{\pi}}\partial_x\vartheta
\end{cases}
\tag{5.56}
$$

Now we could written down relation with Luttinger parameter K

$$
K = \frac{\cosh\theta+\sinh\theta}{\cosh\theta-\sinh\theta} \implies \tanh\theta = \frac{K-1}{K+1} \quad \cosh\theta = \frac{K+1}{2\sqrt{K}} \quad \sinh\theta = \frac{K-1}{2\sqrt{K}} \tag{5.57}
$$

Let's consider the fermionic propagator

$$
\begin{aligned}
\langle \mathcal{T}\psi_R(x,t)\psi_R(x',t')\rangle &= \frac{1}{2\pi}\mathcal{T}\langle e^{\mathrm{i}\sqrt{4\pi}\phi(x,t)}e^{-\mathrm{i}\sqrt{4\pi}\phi(x',t')}\rangle \sim \frac{1}{2\pi}e^{2\pi\mathcal{T}\langle\phi(x,t)\phi(x',t')\rangle} \\
&\sim \frac{1}{2\pi}\left(\frac{1}{(x-x')-v(t-t')+\mathrm{i}\varepsilon}\right)^{\frac{(K+1)^2}{4K}}\left(\frac{1}{(x-x')-v(t-t')+\mathrm{i}\varepsilon}\right)^{\frac{(K-1)^2}{4K}} \qquad (5.58)
\end{aligned}
$$

The correlation for left -moving fermion is

$$
\langle \mathcal{T}\psi_R(x,t)\psi_R(x',t')\rangle \sim \frac{1}{2\pi}\left(\frac{1}{(x-x')+v(t-t')+\mathrm{i}\varepsilon}\right)^{\frac{(K+1)^2}{4K}}\left(\frac{1}{(x-x')-v(t-t')-\mathrm{i}\varepsilon}\right)^{\frac{(K-1)^2}{4K}} \qquad (5.59)
$$

Let's consider correlations for parameter.

- *Charge density wave*

  The CDW parameter reads as

$$
\mathcal{O}_{\mathrm{CDW}} = e^{\mathrm{i}2p_F x}\left(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L\right) = e^{\mathrm{i}2p_F x}\left(e^{\mathrm{i}\sqrt{4\pi}\phi} + e^{-\mathrm{i}\sqrt{4\pi}\phi}\right) \qquad (5.60)
$$

  Hence, the correlator is founded to be

$$
\langle \mathcal{T}(\mathcal{O}_{\mathrm{CDW}}\mathcal{O}_{\mathrm{CDW}}^\dagger)\rangle \sim \frac{1}{(2\pi)^2}\mathcal{T}\langle e^{\mathrm{i}\sqrt{4\pi}\phi}e^{-\mathrm{i}\sqrt{4\pi}\phi}\rangle \sim \frac{1}{(2\pi)^2}\left(\frac{1}{(x-x')^2-v^2(t-t')^2+\mathrm{i}\varepsilon}\right)^{1/K} \qquad (5.61)
$$

  which could be derive with (5.59) .

- *Superconducting parameter*

  The superconducting parameter reads as

$$
\mathcal{O}_S = \psi_R^\dagger\psi_L^\dagger \qquad (5.62)
$$

  The correlation for superconducting parameter is given by

$$
\langle \mathcal{T}(\mathcal{O}_{\mathrm{S}}\mathcal{O}_{\mathrm{S}}^\dagger)\rangle \sim \frac{1}{(2\pi)^2}\langle e^{\mathrm{i}\sqrt{4\pi}\vartheta}e^{-\mathrm{i}\sqrt{4\pi}\vartheta}\rangle \sim \frac{1}{(2\pi)^2}\left(\frac{1}{(x-x')^2-v^2(t-t')^2+\mathrm{i}\varepsilon}\right)^{K} \qquad (5.63)
$$

5.5 Problem

Question 3: .

Please find $\langle F \mid R^\dagger(x)R(x) \mid F\rangle$

Proof.

$$
\langle F \mid R^\dagger(x)R(x) \mid F\rangle = \frac{1}{L}\sum_{k<0}e^{-\mathrm{i}k(x-x')}\langle F \mid R^\dagger(k)R(k) \mid F\rangle = \frac{1}{2\pi}\int_{-\infty}^{0}dk e^{[\varepsilon-\mathrm{i}(x-x')]k} = \frac{1}{2\pi}\frac{1}{\varepsilon-\mathrm{i}(x-x')} \qquad (5.64)
$$

We describe it with bosons

$$\langle F \mid R^\dagger(x)R(x) \mid F\rangle = \frac{1}{2\pi\alpha}\langle 0 \mid e^{-\mathrm{i}\phi_R(x)}e^{\mathrm{i}\phi_R(x')} \mid 0\rangle \tag{5.65}$$

Using the BCH formula

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} = e^B e^A e^{\frac{1}{2}[A,B]} \tag{5.66}$$

We would calculate Eq(5.65) with Eq(5.66)

$$\begin{aligned} e^{-\mathrm{i}\phi_R(x)}e^{\mathrm{i}\phi_R(x')} &= e^{\mathrm{i}[-\phi_R(x)+\phi_R(x')]}e^{\frac{1}{2}[\phi_R(x),\phi_R(x')]} = e^{\mathrm{i}[-\phi_R^+(x)-\phi_R^-(x)+\phi_R^-(x')+\phi_R^+(x')]}e^{\frac{1}{2}[\phi_R(x),\phi_R(x')]} \\ &= e^{\mathrm{i}[-\phi_R^+(x)+\phi_R^+(x')]}e^{\mathrm{i}[-\phi_R^-(x)+\phi_R^-(x')]}e^{\frac{1}{2}[-\phi_R^+(x)+\phi_R^+(x'),-\phi_R^-(x)+\phi_R^-(x')]}e^{\frac{1}{2}[\phi_R(x),\phi_R(x')]} \\ &= e^{\mathrm{i}[-\phi_R^+(x)+\phi_R^+(x')]}e^{\mathrm{i}[-\phi_R^-(x)+\phi_R^-(x')]}e^{[\phi_R^+(x),\phi_R^-(x)]-[\phi_R^+(x'),\phi_R^-(x)]} \end{aligned} \tag{5.67}$$

Hence, we obtain

$$\begin{aligned} \langle F \mid R^\dagger(x)R(x) \mid F\rangle &= \frac{1}{2\pi\alpha}\langle 0 \mid e^{-\mathrm{i}\phi_R(x)}e^{\mathrm{i}\phi_R(x')} \mid 0\rangle = \frac{1}{2\pi\alpha}e^{[\phi_R^+(x),\phi_R^-(x)]-[\phi_R^+(x'),\phi_R^-(x)]} \\ &= \frac{1}{2\pi\alpha}\exp\left(-\frac{2\pi}{L}\sum_{p>0}e^{-\alpha p}\frac{1-e^{\mathrm{i}p(x-x')}}{p}\right) \end{aligned} \tag{5.68}$$

We take $L \to +\infty$, then the sum will turn into integral

$$\begin{aligned} \frac{2\pi}{L}\sum_{p>0}e^{-\alpha p}\frac{1-e^{\mathrm{i}p(x-x')}}{p} &= \int_0^{+\infty} e^{-\alpha p}\frac{1-e^{\mathrm{i}p(x-x')}}{p} = -\int_0^\infty e^{-\alpha p}\sum_{n=1}^\infty \frac{(\mathrm{i}p(x-x'))^n}{n!} \\ &= -\int_0^\infty e^{-\alpha p}\sum_{n=1}^\infty (x-x')^n\frac{\mathrm{i}p^{n-1}}{n!} \\ &= -\sum_{n=1}^\infty \frac{1}{n}\frac{(x-x')^n}{n} \\ &= -\log\left(1-\frac{\mathrm{i}(x-x')}{a}\right) \\ &= \frac{1}{2\pi}\frac{1}{\alpha-\mathrm{i}(x-x')} \end{aligned} \tag{5.69}$$

□

Question 4: .

Please prove $\{R(x), R(x')\} = 0$

Proof.

$$R(x), R(x') = \frac{1}{2\pi\alpha}e^{\mathrm{i}\phi_R(x)}e^{\mathrm{i}\phi_R(x')} = \frac{1}{2\pi\alpha}e^{\mathrm{i}\phi_R(x')}e^{\mathrm{i}\phi_R(x)}e^{[\phi_R(x),\phi_R(x')]} = -\frac{1}{2\pi\alpha}e^{\mathrm{i}\phi_R(x')}e^{\mathrm{i}\phi_R(x)} \tag{5.70}$$

□

Chapter 6

Plasmon and Lindhard function

6.1 Random phase approximation

Electrons on the positive charge background could be described by hamiltonian

$$H = -\frac{\hbar^2}{2m}\sum_{i=1} N\frac{\hbar^2\nabla_i^2}{2m} + \frac{1}{2}\sum_{i\neq j}\frac{e^2}{\mid r_i - r_j \mid} + H_{\text{positive charge background}} \tag{6.1}$$

The Columb interaction could be expanded into momentum space by Fourier transformation

$$\begin{aligned}\frac{1}{2}\sum_{i\neq j}\frac{e^2}{\mid r_i - r_j \mid} = \frac{1}{2}\sum_q v(q)e^{\mathrm{i}\vec{q}\cdot(\vec{r}_i - \vec{r}_j)} = \frac{1}{2}\sum_q v(q)(\sum_{i=1}^{N}\sum_{j=1}^{N} v(q)e^{\mathrm{i}\vec{q}\cdot(\vec{r}_i - \vec{r}_j)} - N) \\ = \sum_q v(q)\left(\rho_q^\dagger\rho_q - N\right)\end{aligned} \tag{6.2}$$

where $\rho_q = \sum_j e^{-\mathrm{i}\vec{q}\cdot\vec{r}_j}, \rho(r) = \sum_{i=1}^{N}\delta(r - r_i)$. The background charge corresponds to $q = 0$ components, namlly $\frac{V(0)}{2}(N^2 - N)$. We should remove it .

The hamiltonian (6.1) written into second quantization form

$$H = \sum_{k\sigma}(\varepsilon_{k\sigma} - \mu)c_{k\sigma}^\dagger c_{k\sigma} + \frac{1}{2V}\sum_{k,k^{prime}}\sum_{q\neq 0}\frac{4\pi e^2}{q^2}c_{k+q,\sigma}^\dagger c_{k'-q,\sigma'}^\dagger c_{k'-q,\sigma'}c_{k+q,\sigma} \tag{6.3}$$

We remove the background charge contribution. We define the density operator ρ_q as

$$\rho_q = \sum_k c_{k+q}^\dagger c_k \quad \sum \rho_q^\dagger = \sum_k c_{k^\dagger}c_{k+q} = \rho_{-q} \tag{6.4}$$

Note:-

- $$\begin{aligned}[H_0, \rho_q] = \sum_{k\sigma}\varepsilon_k[c_{k\sigma}^\dagger c_{k\sigma}, c_{k+q\sigma}^\dagger c_{k\sigma}] &= \sum_{k\sigma}\varepsilon_k' c_{k'\sigma}^\dagger[c_{k'\sigma}, c_{k+q\sigma}^\dagger c_{k\sigma}] + \varepsilon_k'[c_{k'\sigma}^\dagger, c_{k+q\sigma}^\dagger c_{k\sigma}]c_{k'\sigma} \\ &= \sum_k(\varepsilon_{k+q} - \varepsilon_k)c_{k+q\sigma}^\dagger c_{k\sigma} \\ &= \sum_k \hbar\omega_{kq}\rho_{kq}\end{aligned} \tag{6.5}$$

- $$\begin{aligned}[\rho_{q_1}, \rho_{q_2}] = \sum_{k_1,k_2} [c^\dagger_{k_1+q_1} c_{k_1}, c^\dagger_{k_2+q_2} c_{k_2}] &= \sum_{k_1,k_2} c^\dagger_{k_1+q_1} [c_{k_1}, c^\dagger_{k_2+q_2} c_{k_2}] + [c^\dagger_{k_1+q_1}, c^\dagger_{k_2+q_2} c_{k_2}] c_{k_1} \\ &= \sum_{k_1,k_2} c^\dagger_{k_1+q_1} \{c_{k_1}, c^\dagger_{k_2+q_2}\} c_{k_2} - c^\dagger_{k_2+q_2} \{c^\dagger_{k_1+q_1}, c_{k_2}\} c_{k_1} \\ &= \sum_k c^\dagger_{k+q_1+q_2} c_k - c^\dagger_{k+q_2} c_{k-q_1} \end{aligned} \tag{6.6}$$

- $$\begin{aligned}\frac{1}{2}\sum_{q\neq 0} v(q')[\rho_{p'}\rho_{-q'}, \rho_q] &= \frac{1}{2}\sum_{q\neq 0} v(q')[\rho_{q'}, \rho_q]\rho_{-q'} + v(q')\rho_{q'}[\rho_{-q'}, \rho_q] \\ &= \frac{1}{2}\sum_{k,q\neq 0} v(q')(c^\dagger_{k+q'+q\sigma} c_{k\sigma} - c^\dagger_{k+q\sigma} c_{k-q'\sigma})\rho_{-q'} + (c^\dagger_{k-q'+q\sigma} c_{k\sigma} - c^\dagger_{k+q\sigma} c_{k+q'\sigma})\rho_{q'} \end{aligned} \tag{6.7}$$
$$\tag{6.8}$$

The $\rho_q = \sum_i e^{-\mathrm{i}\vec{q}\cdot\vec{r}_i}$ is a summation of phase , which is random at high densities. Hence, the RPA could be applied.

$$\rho_{q_1}\rho_{q_2-q_1} = \sum_{k_1,k_2} e^{-\mathrm{i}\vec{q}_1\cdot\vec{r}_i} \cdot e^{-\mathrm{i}(\vec{q}_2-\vec{q}_1)\cdot\vec{r}_i} \approx N\rho_{q_2} \tag{6.9}$$

Hence , the commutator becomes

$$[H, \rho_{k\sigma}] = \hbar\omega_{kq}\rho_{kq\sigma} + v(q)(n_k - n_{k+q})\rho_q \tag{6.10}$$

Let's consider the eigenequation , which reads

$$[H, \sum_k a_k\rho_k] = \hbar\omega \sum_k a_k\rho_k \tag{6.11}$$

which is equivalent to

$$[H, \sum_k a_k\rho_k] = \sum_k a_k\left(\hbar\omega_{kq}\rho_{k\sigma} + v(q)(n_k - n_{k+q})\rho_q\right) = \hbar\omega\sum_k a_k\rho_{kq} \tag{6.12}$$

Note:-

Let's erive consistent equation . We can derive from Eq(6.12)

$$\begin{aligned} &\hbar\omega_{kq}a_k + \sum_{k'\sigma'} v(q)(n_{k'} - n_{k'+q})a_{k'\sigma'} = \hbar\omega a_k \\ &\implies a_k = \frac{v(q)}{\hbar(\omega-\omega_{kq})}\sum_{k',\sigma'}(n_{k'} - n_{k'+q})a_{k'} \\ &\implies \sum_k (n_k - n_{k+q})a_k = \sum_{k',\sigma}\frac{v(q)(n_{k'} - n_{k'+q})}{\hbar(\omega-\omega_{kq})}\sum_k (n_k - n_{k+q})a_k \\ &\implies \sum_{k',\sigma}\frac{v(q)(n_{k'-n_{k'+q}})}{\hbar(\omega-\omega_{kq})} = 1 \end{aligned}$$

6.2 Dieletric functon

Suppose we add an external potential ,

$$H_e(t) = \sum_i V_e(r_i) e^{-\mathrm{i}\omega t + \eta t} = \frac{1}{V} \sum_q V_e(q) e^{-\mathrm{i}\omega t + \eta t} \rho_q \tag{6.13}$$

The motion equation reads as

$$-\mathrm{i}\hbar \dot{\rho}_k = [H, \rho_k] + [H_e(t), \rho_k] \tag{6.14}$$

Note:-

With RPA approximation,

$$[H_e(t), \rho_{kq}] = \sum_{q'} V_e(q) e^{-\mathrm{i}\omega t + \eta t} [\rho_q, \rho_{kq}] = \frac{2}{V} V_e(q)(n_{k+q} - n_k) e^{-\mathrm{i}\omega t + \eta t}$$

Hence, we can derive

$$\begin{aligned} & -\omega \langle \rho_{kq} \rangle = \omega_{kq} \langle \rho_{kq} \rangle + \frac{2}{V} V_e(q)(n_k - n_{k-q}) + \frac{2}{V}(n_{k+q} - n_k)\langle \rho_q \rangle \\ & \implies \langle \rho_{kq} \rangle = -\sum_k \frac{2}{V} \frac{n_{k+q} - n_k}{\hbar\omega - \omega_{kq}} \underbrace{(V_e(q) + V(q)\langle \rho_q \rangle)}_{V_{tot}} \end{aligned} \tag{6.15}$$

We define the vacuum polarization as

$$\chi(q, \omega) = \frac{2}{V} \sum_k \frac{n_{k+q} - n_k}{\hbar\omega - \omega_{kq}} \tag{6.16}$$

The total potential V_{tot} can be expressed as

$$V_{tot} = V_e(q, t) + \frac{4\pi e^2}{q^2} \langle \rho_q \rangle = V_e - V(q)\chi(q, \omega) V_{tot} \implies V_{tot} = \frac{V_e}{(1 + v(q)\chi(q, \omega))} \tag{6.17}$$

Note:-

From the classical Laplace equation

$$-\nabla^2 V = 4\pi(-e)^2 \langle \rho \rangle \implies V_{tot} = \frac{4\pi e^2}{q^2} \langle \rho_q \rangle \tag{6.18}$$

We define dielectric function with Lindhard function

$$\varepsilon(q, \omega) = 1 + v(q)\chi(q, \omega) = 1 + \frac{2v(q)}{V} \sum_k \frac{n_{k+q} - n_k}{\hbar\omega - \omega_{kq}} \tag{6.19}$$

Let's consider the real part and imaginary part of dielectric function.

$$\begin{cases} \varepsilon_1(q, \omega) = 1 + v(q)\chi(q, \omega) = 1 + \dfrac{2v(q)}{V} \displaystyle\sum_k \mathcal{P}\left(\frac{n_{k+q} - n_k}{\hbar\omega - \omega_{kq}} \right) \\ \varepsilon_2(q, \omega) = \dfrac{2\pi v(q)}{\hbar V} \displaystyle\sum_k n_k \left[\delta(\omega - (\omega_{k+q} - \omega_k)) + \delta(\omega + (\omega_{k+q} - \omega_k)) \right] \end{cases} \tag{6.20}$$

6.3 Lindhard function

Let's discuss some typical behaviour of Lindhard function.

- $\frac{q}{k_F} \to 0$, $n_{k+q} - n_k = \frac{\partial n}{\partial \varepsilon} \nabla_q \varepsilon = -\delta(\varepsilon - \varepsilon_F) q v_F \cos\theta$

$$\begin{aligned}
\chi_0(q,\omega) = \frac{2}{V}\sum_k \frac{n_{k+q} - n_k}{\hbar\omega - \omega_{kq}} &= 2\int \frac{d^3k}{(2\pi)^3}\frac{-\delta(\varepsilon-\varepsilon_F)v_F q\cos\theta}{\omega - qv_F\cos\theta + \mathrm{i}\eta} \\
&= 2N(0)\int \frac{d\cos\theta d\varphi}{4\pi}\frac{-qv_F\cos\theta}{\omega - qv_F\cos\theta + \mathrm{i}\eta} \\
&= N(0)\int_{-1}^{1} dx \frac{-x}{s - x + \mathrm{i}\eta} \\
&= N(0)\int_{-1}^{1} dx \left[\mathcal{P}\left(\frac{x}{x-s}\right) + \mathrm{i}x\pi\delta(x-s)\right] \\
&= 2N(0)\left(1 - \frac{s}{2}\log\left|\frac{1+s}{1-x}\right|\right) + \mathrm{i}N(0)\pi s\Theta(\mid s^2 - 1\mid)
\end{aligned} \tag{6.21}$$

Note:-

$$N = \left(\frac{4\pi k^3}{3}\right)\cdot\left(\frac{L}{2\pi}\right)^3 \implies \log N = \frac{3}{2}\log\varepsilon \implies D(\varepsilon) = \frac{dN}{d\varepsilon} = \frac{3N}{2\varepsilon}$$

$$\frac{1}{(2\pi)^3}k^2dk = \frac{1}{(2\pi)^3}\cdot\frac{1}{4\pi}\frac{d\frac{4\pi}{3}k^3}{d\varepsilon}\cdot d\varepsilon = \frac{N(\varepsilon)}{4\pi}d\varepsilon$$

We consider two different limits

$$\chi_0(q,\omega) = \begin{cases} 2N(0)\left(1 - s^2 + \mathrm{i}\frac{\pi}{2}\right) & (s \ll 1) \\ 2N(0)\left(-\frac{1}{3s^2} - \frac{1}{5s^4}\right) & (s \gg 1)\end{cases} \tag{6.22}$$

In the plasmon region, the imaginary part of Lindhard vanishes.

$$\varepsilon(q,\omega) = 1 + \frac{4\pi e^2}{q^2}\cdot 2N(0)\left(-\frac{1}{3s^2} - \frac{1}{5s^4}\right) \tag{6.23}$$

Chapter 7

Fermi Liquid

7.1 Quasi-particles and Landau interaction parameters

7.2 Renormalization to physical properties

Let's consider a simple classical example. The object is connected by string . The other end of string is fixed on wall.

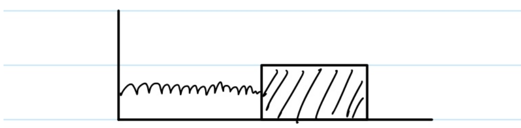

Fig 7.1

The input of system is force F and response is displacement s. The susceptibility is defined as

$$x = -\frac{x}{F} = \frac{1}{k} \tag{7.1}$$

The energy of sytem is described as

$$E = E_{\text{Ela}} - kx = \frac{1}{2}kx^2 - Fx \to E = -\frac{1}{2}\chi F^2 \tag{7.2}$$

The susceptibility also can be defined from energy

$$\chi = -\frac{\partial^2 E}{\partial F^2} \tag{7.3}$$

Example 7.2.1 (.)
We consider magnetism system . where the external field H will response to magnetization M . The energy increment is gien by

$$dE = HdM \tag{7.4}$$

Hence , the total energy is given by

$$E = E_M - HM \to \chi = -\frac{\partial^2 E}{\partial H^2} \tag{7.5}$$

Now we consider a more complex system, where object is connected with two springs . By the same way , the susceptibility is given by

$$\chi = \frac{1}{k_0 + k'} = \frac{\chi_0}{1 + \frac{k'}{k_0}} \tag{7.6}$$

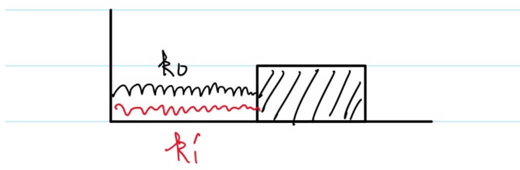

Fig 7.2

We understood this process with close loop process as shown on ().

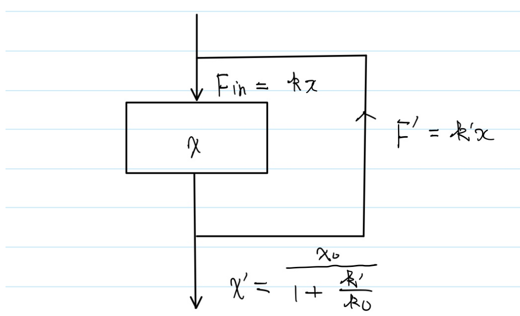

Fig 7.3

The feedback process will change the susceptibility. We expand the renoramlized susceptibility into Taylor series

$$\chi' = \chi_0 + \chi_1 + \chi_2 \cdots \text{where} \quad \chi_n = \chi_0 \left(-\frac{k'}{k}\right)^n \tag{7.7}$$

Question 5: .

If the k' is negative, what interesting things will be happend?

7.2.1 Magnetic susceptibility

We consider energy fnctional with second order . The spin index is polarized at z axis. Hnece, the desnsity variation are diagonal.

$$f^a(p,p')\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta}\delta n_{\beta\alpha}(p)\cdot\delta n_{\delta\gamma}(p')\to f^a(p,p')\vec{\sigma}\cdot\vec{\sigma}\delta n_{p\sigma}\delta n_{p'\sigma'} \tag{7.8}$$

Hence, we can derive that

$$\delta\varepsilon^{(2)}=\frac{1}{2N_0V}F_0^a\sum\sigma\cdot\sigma'\delta n_{p\sigma}\delta_{p'\sigma'}=\frac{V}{2N_0}F_0^a(S_z)^2 \tag{7.9}$$

We introduce molecular field h_{mol} , which induces energy increment ΔV

$$\Delta V=--V\int h_{\mathrm{mol}}\cdot dS_z\implies h_{\mathrm{mol}}=-\frac{1}{V}\frac{\partial E}{\partial S}=-\frac{1}{N_0}F_0^aS_z \tag{7.10}$$

The total magnetic field is given by

$$h_{\mathrm{tot}}=h_{\mathrm{ex}}+h_{\mathrm{mol}} \tag{7.11}$$

The total magnetization S_z and total field can be related with susceptibility χ_0

$$S_z=\chi_0h_{\mathrm{tot}}=\chi(0)\left(h_{ex}-N_0^{-1}F_0^aS_z\right)\implies\chi=\frac{\chi_0}{1+\chi_0N_0^{-1}F_0^a} \tag{7.12}$$

7.2.2 Compressiblity

In this subsection, we will discuss another quantity , namly compressibility.

Note:-

The defnition of compressibility is

$$\chi_{\mathrm{comp}}=-\frac{1}{V}\frac{\partial V}{\partial p} \tag{7.13}$$

We can use observale quantity n and μ to express the compressiblity.

$$V=\frac{N}{n}\qquad PV=N\mu \tag{7.14}$$

The compressiblity can be written as

$$\chi=-\frac{1}{N}\frac{d}{d\mu}\left(\frac{N}{n}\right)=\frac{1}{n^2}\frac{dn}{d\mu} \tag{7.15}$$

The density variation functional could be written as

$$\delta\varepsilon^{(2)}=\frac{1}{2N_0V}F_0^s\sum_{p,p'}\delta n_p\delta n_{p'}=\frac{V}{2N_0}F_0^s\sum(\delta n)^2 \tag{7.16}$$

By the same way, we can define molecular field

$$h_{\mathrm{mol}}=-\frac{F_0^s\delta n}{N_0} \tag{7.17}$$

which means that

$$\chi=\frac{\chi_0}{1+F_0^s} \tag{7.18}$$

7.2.3 Effective mass

The effective mass is renormalized with p wave channel . We will derive the explicit form below.The density on the real space could be written as

$$n(r,t)=\sum_\sigma\int\frac{d^3p}{(2\pi)^3}n_p(r,t) \tag{7.19}$$

The current is given by

$$\vec{j}(r,t)=\sum_\sigma\int\frac{d^3p}{(2\pi)^3}\nabla_p\varepsilon(p,\sigma)n_{p\sigma}(r,t) \tag{7.20}$$

We take linear order approximation

$$\begin{cases}\varepsilon_{p\sigma}(r,t)=\varepsilon_p^0+\int\frac{d^3p'}{(2\pi)^3}f_{\sigma\sigma'}(p,p')\delta n_{p'\sigma'}\\ n_{p\sigma}=n_{p,\sigma}^0+\delta n_{p\sigma}(r,t)\end{cases} \tag{7.21}$$

We substitute the EQ(7.21) into (7.22)

$$\vec{j}(r,t)=\sum_\sigma\int\frac{d^3p}{(2\pi)^3}\nabla_p\left(\varepsilon_p^0+\int\frac{d^3p'}{(2\pi)^3}f_{\sigma\sigma'}(p,p')\delta n_{p'\sigma'}\right)\left(n_{p,\sigma}^0+\delta n_{p\sigma}(r,t)\right) \tag{7.22}$$

We remove the background current, then we have

$$\begin{aligned}\vec{j}(r,t)&=\sum_\sigma\int\frac{d^3p}{(2\pi)^3}\left[\nabla_p\varepsilon_p^0\delta n_{p\sigma}(r,t)+\nabla_p\int\frac{d^3p'}{(2\pi)^3}f_{\sigma\sigma'}(p,p')\delta n_{p'\sigma'}n_{p,\sigma}^0\right]\\&=\int\frac{d^3p}{(2\pi)^3}\left[\nabla_p\varepsilon_p^0\delta n_{p\sigma}(r,t)-\int\frac{d^3p'}{(2\pi)^3}f_{\sigma\sigma'}(p,p')\delta n_{p'\sigma'}\nabla_p n_{p,\sigma}^0\right]\\&=\int\frac{d^3p}{(2\pi)^3}\nabla_p\varepsilon_p^0\delta n_{p\sigma}(r,t)-\int\frac{d^3p}{(2\pi)^3}\frac{\partial n_{p,\sigma}^0}{\partial\varepsilon}\vec{v}_F\int\frac{d^3p'}{(2\pi)^3}f_{\sigma\sigma'}(p,p')\delta n_{p'\sigma'}\\&=\int\frac{d^3p}{(2\pi)^3}\nabla_p\varepsilon_p^0\delta n_{p\sigma}(r,t)-\int\frac{d^3p}{(2\pi)^3}\frac{\partial n_{p,\sigma}^0}{\partial\varepsilon}v_F\frac{4\pi}{2l+1}\sum_l P_l(\cos\theta)P_l(\cos\theta')\int\frac{d^3p'}{(2\pi)^3}\delta n_{p'\sigma'}\\&=\int\frac{d^3p}{(2\pi)^3}\vec{v}_F\left(1+\frac{F_0^s}{3}\right)\implies\frac{1}{m^*}=\frac{1}{m}\int\frac{d^3p}{(2\pi)^3}\end{aligned} \tag{7.23}$$

7.2.4 Arbitrary channel constribution

We define the radial density by integrate out momentum p

$$\delta n=V\int\frac{d^3p}{(2\pi)^3}\delta n(p)=V\int\frac{p^2dp}{(2\pi)^3}\delta n(p)\int d\Omega=V\int d\Omega\delta n(\Omega) \tag{7.24}$$

The angular density distribution could be expand into normal modes

$$\delta n(\Omega)=\sum_{l,m}n_{l,m}Y_{l,m}(\Omega) \tag{7.25}$$

The kinetic increment could be decomposed into normal modes

$$
\begin{aligned}
\Delta E^2 &= \frac{1}{2V}\int \frac{dp^3}{(2\pi)^3} f_{\sigma,\sigma'}(p,p')\delta n_{p\sigma}\delta n_{p'\sigma'} \\
&= \frac{V}{2}\int d\Omega_p d\Omega_{p'} f_{\sigma,\sigma'}(p,p')\delta n(\Omega_p)\delta n(\Omega_p') \\
&= \frac{V}{2}N^{-1}(0)\int d\Omega_p d\Omega_{p'}\left(\sum_{l_1}\sum_{m_1=-l_1}^{l_1} F_l^s Y_{l_1m_1}(\Omega_p)\bar{Y}_{l_1m_1}(\Omega_p')\right)\left(\sum_{l_2}\sum_{m_2=-l_2}^{l_2} n_{l_2m_2}Y_{l_2m_2}(\Omega_p)\right) \\
&\left(\sum_{l_2}\sum_{m_3=-l_3}^{l_3} n_{l_3m_3}Y_{l_3m_3}(\Omega_p')\right) + (s\leftrightarrow a) \\
&= \frac{V}{2}N^{-1}(0)\sum_l^{\infty}\frac{4\pi}{2l+1}\sum_{m=-l}^{l} F_l^s \mid \delta n_{lm}\mid^2 + (s\leftrightarrow a)
\end{aligned} \tag{7.26}
$$

We consider the first order increment

$$
\delta E^{(0)} = \sum_p \varepsilon_p \delta n_p = \int \frac{d^3p}{(2\pi)^3}\varepsilon_p\delta n_p \tag{7.27}
$$

7.2.5 Pomeranchuk instability

7.3 The Boltzmann equation and zero sound

7.3.1 Boltzmann equation

We start from Boltzmann equation. The particle density on the phase space is described by distribution function $f(\mathbf{r},\mathbf{p},t)$. In the other words ,

$$
f(\mathbf{r},\mathbf{p},t)d\mathbf{r}^3 d\mathbf{p}^3 \tag{7.28}
$$

Due to occurence of collisions, the particle number lying on the phase space will change . We consider the particle variation on the phase space. The Liouville theorem tells us phase space volumme conservation.

$$
\Delta N_{\text{Collision}} = \left(f(\mathbf{r}+\Delta\mathbf{r},\mathbf{p}+\Delta p,t) - f(\mathbf{r},\mathbf{p},t)\right)d\mathbf{r}^3 d\mathbf{p}^3 \tag{7.29}
$$

The total differential of f is

$$
df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\frac{\partial p}{\partial t}dt + \frac{\partial f}{\partial p}\frac{\partial p}{\partial t}dt = (\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla_{\mathbf{r}}f + \vec{F}\cdot\nabla_{\mathbf{p}}f)dt \tag{7.30}
$$

We can see that the particle number flow comprise real space flow and momentum space flow. The total flow is equal to collison section. The collison section consists of forward process and reverse process.

$$
I = \int d^3\mathbf{r}d^3\mathbf{p}I(\Omega)(f(\mathbf{r}_1',\mathbf{p}_1',t)f(\mathbf{r}_1',\mathbf{p}_1',t) - f(\mathbf{r}_1,\mathbf{p}_1,t)f(\mathbf{r}_1,\mathbf{p}_1,t)) \tag{7.31}
$$

The $I(g,\Omega)$ is scattering section , which can be determined by Fermin golden rule. We consider the fermion system, the density distribution has very strong limit in virtue of Pauli principle.

$$
I = \frac{1}{V^2}\sum \mid \langle 3,4\mid V\mid 1,2\rangle\mid^2 \delta_{p_1+p_2=p_3+p_4}\delta_{\sigma_1+\sigma_2=\sigma_3+\sigma_4}\delta_{\varepsilon_1+\varepsilon_2=\varepsilon_3+\varepsilon_4}\left(n_1n_2(1-n_3)(n_4) - (1-n_2)(1-n_2)n_3n_4\right) \tag{7.32}
$$

Note:-

If we consider random approximation , then the collision section will turns into

$$\langle I \rangle = -\frac{\delta N}{\tau} \tag{7.33}$$

Hence, the solution will recover into equibrilium distrition gradually.

$$N(T) = N_0(1 - e^{-\frac{T}{\tau}}) \tag{7.34}$$

7.4 Zero sound

We consider collisonless cases . Hence, the Boltzman equation could be written as

$$\frac{\partial n(\mathbf{r},\mathbf{p},t)}{\partial t} + \nabla_{\mathbf{p}}\varepsilon(\mathbf{p},t)\nabla_{\mathbf{r}}n(\mathbf{r},\mathbf{p},t) - \nabla_{\mathbf{r}}\varepsilon(\mathbf{p},t)\nabla_{\mathbf{p}}n(\mathbf{r},\mathbf{p},t) = 0 \tag{7.35}$$

It's self-evident that Eq(10.8) is a nonlinear equation . We expand the density distribution and energy functional.

$$\begin{cases} \varepsilon(\mathbf{r},\mathbf{p},t) = \varepsilon_0(p) + \dfrac{1}{V}\displaystyle\sum_{p'} f^s(\mathbf{p},\mathbf{p}')\delta n_{\mathbf{p}'}(\mathbf{r},t) \\ n(\mathbf{r},\mathbf{p}) = n_0(\mathbf{r},\mathbf{p}) + \delta n(\mathbf{r},\mathbf{p},t) \end{cases} \tag{7.36}$$

At the equibrillium state, the density distrbution and energy functional don't rely on space position $\mathbf{r}$. The Boltzmann equation can be expanded into first order in relative with $\delta(\mathbf{r},\mathbf{p},t)$.

$$\frac{\partial \delta n(\mathbf{r},\mathbf{p},t)}{\partial t} + \vec{v}_{\mathbf{p}} \cdot \nabla_{\mathbf{r}}\delta n(\mathbf{r},\mathbf{p},t) - \nabla_{\mathbf{p}}\varepsilon \cdot \frac{1}{V}\sum_{p'} f^s(p.p')\nabla_{\mathbf{r}}\delta n(\mathbf{r},\mathbf{p}',t) = 0 \tag{7.37}$$

We insert relation $\nabla_{\mathbf{p}}\varepsilon = \dfrac{\partial \varepsilon}{\partial \varepsilon}\nabla_{\mathbf{p}}\varepsilon$ into Eq(**??**). The Boltzmann equation will reduce into

$$\frac{\partial \delta n(\mathbf{r},\mathbf{p},t)}{\partial t} + \vec{v}_{\mathbf{p}}\left(\cdot\nabla_{\mathbf{r}}\delta n(\mathbf{r},\mathbf{p},t) - \frac{\partial n}{\partial \varepsilon} \cdot \frac{1}{V}\sum_{p'} f^s(p.p')\nabla_{\mathbf{r}}\delta n(\mathbf{r},\mathbf{p}',t)\right) = 0 \tag{7.38}$$

We expand the density fluactuation into Fourier modes, namly

$$\delta n(\mathbf{r},\mathbf{p},t) = \sum_q \delta n(\mathbf{p}) e^{\mathrm{i}(\vec{q}\cdot\vec{\mathbf{r}} - \omega t)} \tag{7.39}$$

where wavevector $\vec{q}$ is small . We insert Fourier mode into Eq(7.39)

$$(\omega - \vec{q}\cdot\vec{v}_F)\delta n(\mathbf{p}) + (\vec{v}_F \cdot \vec{q})\frac{\partial n}{\partial \varepsilon}\frac{1}{V}\sum_{p'} f^s(\mathbf{p},\mathbf{p}')\delta n(\mathbf{p}) = 0 \tag{7.40}$$

We define the dimensionless quantty $s = \dfrac{\omega}{q v_F}$ and choose direction of $\vec{v}_F$ as z axis. Now we integrate out the radial part by $\displaystyle\int \frac{p^2 dp}{2\pi}$

$$(s-\cos\theta)\delta\hat{n}(\Omega)-\cos\theta\underbrace{\int\frac{p^2dp}{2\pi}\frac{\partial n}{\partial\varepsilon}}_{\frac{N(0)}{4\pi}}\frac{1}{V}\sum_{p'}f^s(\mathbf{p},\mathbf{p}')\delta n(\mathbf{p})=0 \tag{7.41}$$

The osscilation on the Fermi surface is tesor wave. The Fermi surface oscilation is transfered by Landau ineraction. Hence, we consider decomposing the desity $\hat{n}(\Omega)$ into $SO(3)$ irreucible tensors.

$$\frac{1}{V}\sum_{p'}f^s(\mathbf{p},\mathbf{p}')\delta n(\mathbf{p})=\int d\Omega'\left(F_l^s\sum_l\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}(\Omega)\bar{Y}_{lm}(\Omega')\right)\left(\sum_{l'}\sum_{m=-l'}^{l'}u_l'Y_{l'm'}(\Omega')\right) \tag{7.42}$$

$$=\sum_{l'}\frac{4\pi}{2l'+1}u_{l'}F_{l'}^sY_{l'm}(\Omega) \tag{7.43}$$

where Ω is the solid angle expanded by momentum p and Ω' expanded by Ω'. We insert Eq(7.43) into Eq(7.41) to derive such identity

$$\sum_{l'}\sum_{m=-l'}^{l'}n_{l'}Y_{l'm}-\sum_{l'}\sum_{m=-l'}^{l'}\frac{1}{2l'+1}\frac{\cos\theta}{s-\cos\theta}u_{l'}F_{l'}^sY_{l'm}(\Omega)=0 \tag{7.44}$$

We can see from Eq(7.44) that the angular momentum m can be viewed as internal gauge. Hence, We fix m to zero. Now, we can derive from Eq(7.44)

$$\frac{u_l}{\sqrt{(2l+1)}}-\sum_{l'}\frac{1}{\sqrt{(2l'+1)(2l+1)}}\int d\Omega\frac{\cos\theta}{s-\cos\theta}Y_{l0}(\Omega)Y_{l'0}(\Omega)F_{l'}^s\frac{u_{l'}}{\sqrt{2l'+1}}=0 \tag{7.45}$$

We define the integral $Omega_{ll'}$ as below.

$$\Omega_{ll'}=-\frac{1}{\sqrt{(2l'+1)(2l+1)}}\int d\Omega\frac{\cos\theta}{s-\cos\theta}Y_{l0}(\Omega)Y_{l'0}(\Omega)F_{l'}^s \tag{7.46}$$

We consider the zero order of Eq(7.45).

$$u_0+\Omega_{00}F_0^s=0\implies\frac{1}{F_o^s}=-\Omega_{00} \tag{7.47}$$

Note:-

We give some details about calculation of Ω_{00}

$$\begin{aligned}\Omega_{00}&=-\int\frac{d\Omega}{4\pi}\frac{\cos\theta}{s-\cos\theta}=-\frac{1}{2}\int_{-1}^{1}\frac{xdx}{s-x}=-\frac{1}{2}\int_{-1}^{1}\left[x\mathcal{P}(\frac{1}{s-x})+\mathrm{i}\pi\delta(s-x)\right]dx\\&=-\frac{1}{2}\int_{-1}^{1}\left[\frac{s}{s-x}-1+\mathrm{i}\pi\delta(s-x)\right]dx\\&=-\frac{s}{2}\log\frac{s+1}{s-1}+1-\mathrm{i}\frac{\pi}{2}\Theta(\mid s^2-1\mid)\end{aligned} \tag{7.48}$$

The solution of the Eq(7.47) could be depicted in Fig (7.4)

Note:-

We consider two limits

- $s \to 1^+$

$$-\frac{s}{2}\log\left|\frac{s+1}{s-1}\right| + 1 \approx 1 + \log\left|\frac{s-1}{2}\right| = \frac{!}{F_o^s} \implies s \approx 1 + e^{-\frac{2}{F_o^s}} \tag{7.49}$$

- $s \to \infty$

$$\frac{s}{2}\log\left|\frac{s-1}{s+1}\right| = \frac{s}{2}\left(-\frac{1}{s} - \frac{1}{2s^2} - \frac{1}{3s^3} - \frac{1}{s} + \frac{s^2}{2} - \frac{1}{3s^3}\right) + 1 = -\frac{1}{3s^2} \implies s = \sqrt{\frac{F_0^s}{3}} \tag{7.50}$$

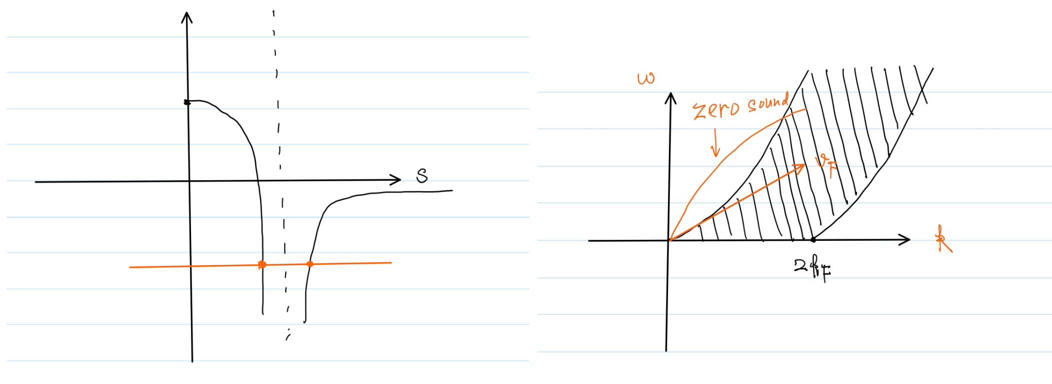

Fig 7.4

The real physcial solution lies on region $s > 1$. Otherwise, it will collapse into particle-hole continnum.

Chapter 8

Bethe ansatz

8.1 Heisenberg model

We will begin with classical picture. Considering two body collision problem, the momentum will be exchanged according to energy-momentum conservation.

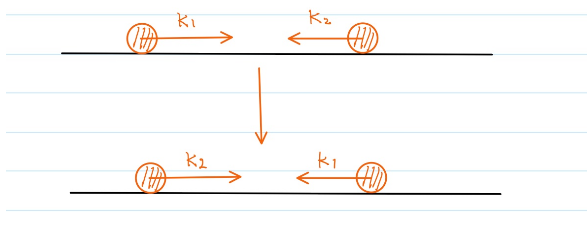

Fig 8.1: Two ball will ecahnge momentum after collision

Now we use quantum mechanical language to express this proccess. The initial wavefunction is $\phi_{\text{ini}} = e^{\mathrm{i}(k_1 x_1 + k_2 x_2)}$, the final wavefunction woulbe be $\phi_{\text{ini}} = e^{\mathrm{i}(k_2 x_2 + k_2 1 x_2)}$.

If we consider many body collision problem on one dimension, the momentum $\{k_1, \cdots k_n\}$ will be permutated. Bethe estabalish many body wavefunction ansatz as

$$\phi(x_1, x_2, \cdots x_n) = \sum_{P \in S_N} A_p e^{\mathrm{i}(k_{P(1)} x_1 + \cdots k_{P(n)} x_n)} \tag{8.1}$$

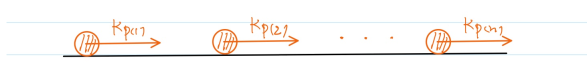

Fig 8.2: Many body state after collision

The one dimensional Heisenberg model hamiltonian reads as

$$H = \frac{1}{2} \sum_{n=1}^{N} J S_n^+ S_{n+1}^- + J S_n^- S_{n+1}^+ + 2\Delta (S_n^z S_{n+1}^z - \frac{1}{4}) \tag{8.2}$$

The Bethe ansatz wavefunction for Heisenberg model is

$$\Psi_M = \sum_{\{x_i\}} \phi(x_1, x_2, \cdots x_M) S_1^- \cdots S_M^- \mid \Psi_0\rangle \tag{8.3}$$

where $\Psi_0 = \mid\uparrow, \uparrow, \cdots \uparrow\rangle$. The scattering amplitude $\phi(x_1, x_2, \cdots x_n)$ is given by (8.1)
We consider single magnon cases , where the wavefunction reads as

$$\Psi_1 = \sum_{\{x_i\}} \phi(x_i) S_i^- \mid \Psi_0\rangle \tag{8.4}$$

The wavefuntion (8.4) is the eigenstate of Schrodinger equation [1]

$$H\Psi = E\Psi \implies \sum_{\{x_i\}} \phi(x_i) S_{i+1}^- \mid \Psi_0\rangle + \sum_{\{x_i\}} \phi(x_i) S_{i-1}^- \mid \Psi_0\rangle + (E_0 - \Delta) \sum_{\{x_i\}} \phi(x_i) S_i^- \mid \Psi_0\rangle = E \sum_{\{x_i\}} \phi(x_i) S_i^- \mid \Psi_0\rangle \tag{8.5}$$

The Eq(5.5) is equivalent to

$$\frac{J}{2}[\phi(x_{i-1}) + \phi(x_{i+1})] = (E - E_0 + \Delta)\phi(x_i) \tag{8.6}$$

The wavefunction also satisfies to periodic condition

$$\phi(x_i) = \phi(x_{i+N}) = \phi(x_i) e^{\mathrm{i}kN} \implies k = \frac{2\pi m}{N} \tag{8.7}$$

Hence, the dispersion is given by

$$E = E_0 - \Delta + J \cos k \tag{8.8}$$

8.1.1 Two magnon cases

In this cases , it invloves two body scattering process. The wavefunction is nontrivial

$$\phi(x_1, x_2) = A_1 e^{\mathrm{i}(k_1 x_1 + k_2 x_2)} + A_2 e^{\mathrm{i}(k_2 x_1 + k_1 x_2)} \tag{8.9}$$

The wavefunction Ψ_2 should be satisfied to Schrodinger equation

$$\begin{cases} \frac{J}{2}\left(\phi(x_1 \pm 1, x_2) + \phi(x_1, x_2 \pm 1)\right) = (E - E_0 + 2\Delta)\phi(x_1, x_2) & \mid x_1 - x_2 \mid > 1 \\ \frac{J}{2}\left(\phi(x_1 - 1, x_2) + \phi(x_1, x_2 + 1)\right) = (E - E_0 + \Delta)\phi(x_1, x_2) & \mid x_1 - x_2 \mid = 1 \end{cases} \tag{8.10}$$

The sufficient and neccessary condition meets with Eq(8.10) is

$$\frac{J}{2}[\phi(x_1, x_2) + \phi(x_2, x_2)] = \Delta\phi(x_1, x_2) \implies \frac{J}{2}(A_1 + A_2)[e^{\mathrm{i}(k_1+k_2)x_1} + e^{\mathrm{i}(k_1+k_2)x_2}] = \Delta \left[A_1 e^{\mathrm{i}(k_1 x_1 + k_2 x_2)} + A_2 e^{\mathrm{i}(k_2 x_1 + k_1 x_2)}\right] \tag{8.11}$$

We give the relation between scattering amplitudes

$$\frac{A_1}{A_2} = -\frac{e^{\mathrm{i}(k_1+k_2)} + 1 - 2\Delta/J e^{\mathrm{i}k_1}}{e^{\mathrm{i}(k_1+k_2)} + 1 - 2\Delta/J e^{\mathrm{i}k_2}} = -e^{\mathrm{i}\Theta(k_1,k_2)} \tag{8.12}$$

The phase $\theta(k_1, k_2)$ satisifies to $\Theta(k_1, k_2) = -\Theta(k_2, k_1)$. We will introduce rapadity to simplify (8.12) .

[1] It will be cost $-\frac{1}{2}\Delta$ energy when flip a spin

Note:-

$$
\begin{aligned}
e^{\mathrm{i}(k_1+k_2)} + 1 - 2\Delta/J e^{\mathrm{i}k_1} &= e^{\mathrm{i}\frac{1}{2}(k_1+k_2)}\left[e^{\mathrm{i}\frac{1}{2}(k_1+k_2)} + e^{-\mathrm{i}\frac{1}{2}(k_1+k_2)} - 2\Delta/J e^{\frac{i}{2}(k_1-k_2)}\right] \\
&= e^{\mathrm{i}\frac{1}{2}(k_1+k_2)}\left[2\cos\frac{k_1+k_2}{2} - 2\Delta/J\cos\frac{k_1-k_2}{2} - \mathrm{i}2\Delta/J\sin\frac{k_1-k_2}{2}\right] \\
&= e^{\mathrm{i}\frac{1}{2}(k_1+k_2)}\left[\frac{2\cos\frac{k_1+k_2}{2} - 2\Delta/J\cos\frac{k_1-k_2}{2}}{4\Delta/J\sin\frac{k_1-k_2}{2}} - \frac{\mathrm{i}}{2}\right]
\end{aligned}
\tag{8.13}
$$

We take $\Delta = J$, then we will have

$$
\frac{2\cos\frac{k_1+k_2}{2} - 2\Delta/J\cos\frac{k_1-k_2}{2}}{4\Delta/J\sin\frac{k_1-k_2}{2}} = \frac{-\sin\frac{k_1}{2}\sin\frac{k_2}{2}}{\sin\frac{k_1}{2}\cos\frac{k_2}{2} - \sin\frac{k_2}{2}\cos\frac{k_1}{2}} = \frac{1}{\cot\frac{k_1}{2} - \cot\frac{k_2}{2}} \tag{8.14}
$$

We substitute it into (8.12), then we have

$$
\frac{A_1}{A_2} = -\frac{1 - \frac{\mathrm{i}}{2}(\cot\frac{k_1}{2} - \cot\frac{k_2}{2})}{1 + \frac{\mathrm{i}}{2}(\cot\frac{k_1}{2} - \cot\frac{k_2}{2})} = \frac{\frac{1}{2}(\cot\frac{k_1}{2} - \cot\frac{k_2}{2}) + \mathrm{i}}{\frac{1}{2}(\cot\frac{k_1}{2} - \cot\frac{k_2}{2}) - \mathrm{i}} \tag{8.15}
$$

We call $\lambda_i = \frac{1}{2}\cot\frac{k_i}{2}$ as *rapidity*. Let's give the explicit relation

$$
e^{\mathrm{i}k_i} = \frac{\lambda_i + \frac{1}{2}\mathrm{i}}{\lambda_i - \frac{1}{2}\mathrm{i}} \tag{8.16}
$$

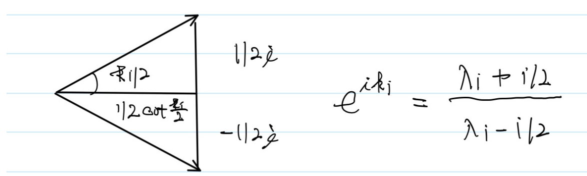

Fig 8.3: Relation about *rapidity*

The $e^{\mathrm{i}\Theta(k_1,k_2)}$ coule be expressed as

$$
e^{\mathrm{i}\Theta(k_1,k_2)} = -\frac{\lambda_1 - \lambda_2 + \mathrm{i}}{\lambda_1 - \lambda_2 - \mathrm{i}} \tag{8.17}
$$

The wavefunction $\phi(x_1, x_2)$ could be written as

$$
\phi(x_1, x_2) = e^{\mathrm{i}(k_1x_1 + k_2x_2 + \frac{1}{2}\Theta(k_1,k_2))} + e^{\mathrm{i}(k_2x_1 + k_1x_2 + \frac{1}{2}\Theta(k_1,k_2))} \tag{8.18}
$$

Now we use periodic condition to solve wavevector k_1, k_2 , , namly $\phi(x_1, x_2) = \phi(x_2, x_1 + N)$

$$
\phi(x_1, x_2) = e^{\mathrm{i}(k_1x_1 + k_2x_2 + \frac{1}{2}\Theta(k_1,k_2))} + e^{\mathrm{i}(k_2x_1 + k_1x_2 + \frac{1}{2}\Theta(k_2,k_1))} = e^{\mathrm{i}(k_1x_2 + k_2x_1 + \frac{1}{2}\Theta(k_1,k_2))}e^{\mathrm{i}k_2N} + e^{\mathrm{i}(k_1x_1 + k_2x_2 + \frac{1}{2}\Theta(k_2,k_1))}e^{\mathrm{i}k_1N} \tag{8.19}
$$

We could derive

$$
e^{\mathrm{i}(k_1N - \Theta(k_1,k_2))} = 1 \qquad e^{\mathrm{i}(k_2N - \Theta(k_2,k_1))} = 1 \tag{8.20}
$$

The Eq(11.19) is equivalent to

$$\begin{cases} k_1 N - \Theta(k_1 + k_2) = 2\pi m_1 \\ k_2 N - \Theta(k_2 + k_1) = 2\pi m_2 \end{cases} \tag{8.21}$$

The phase $\Theta(k_1, k_2)$ on periodic condition (11.23) oringinates from interaction.

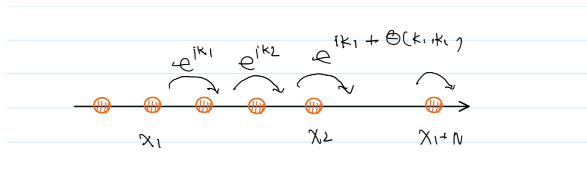

Fig 8.4: The scattering phase shift will contribute to periodic condition

The equivalent description of Eq(11.19) is

$$\left(\frac{\lambda_1 + \frac{1}{2}\mathrm{i}}{\lambda_1 - \frac{1}{2}\mathrm{i}}\right)^N = \frac{\lambda_1 - \lambda_2 + \mathrm{i}}{\lambda_1 - \lambda_2 + \mathrm{i}} \quad \left(\frac{\lambda_2 + \frac{1}{2}\mathrm{i}}{\lambda_2 - \frac{1}{2}\mathrm{i}}\right)^N = \frac{\lambda_2 - \lambda_1 + \mathrm{i}}{\lambda_2 - \lambda_1 + \mathrm{i}} \tag{8.22}$$

We seek bound state solution , which implies $\left(\frac{\lambda_{1,2}+\frac{\mathrm{i}}{2}}{\lambda_{1,2}-\frac{\mathrm{i}}{2}}\right)^N \to 0, \infty$ .Hence, $\lambda_1 - \lambda_2 = \pm\mathrm{i}$. The energy is given as

$$\begin{aligned} E = J(\cos k_1 + \cos k_2 - 2) &= \frac{J}{2}\left(e^{\mathrm{i}k_1} + e^{-\mathrm{i}k_1} + e^{\mathrm{i}k_2} + e^{-\mathrm{i}k_2} - 4\right) \\ &= \frac{J}{2}\left(\frac{\lambda_1 + \frac{\mathrm{i}}{2}}{\lambda_1 - \frac{\mathrm{i}}{2}} + \frac{\lambda_1 - \frac{\mathrm{i}}{2}}{\lambda_1 + \frac{\mathrm{i}}{2}} + \frac{\lambda_2 + \frac{\mathrm{i}}{2}}{\lambda_2 - \frac{\mathrm{i}}{2}} + \frac{\lambda_2 - \frac{\mathrm{i}}{2}}{\lambda_2 + \frac{\mathrm{i}}{2}} - 4\right) \\ &= -\frac{J}{2}\left(\frac{1}{\lambda_1^2 + 1/4} + \frac{1}{\lambda_2^2 + 1/4}\right) \end{aligned} \tag{8.23}$$

We could set $\lambda_1 = x + \frac{\mathrm{i}}{2}, \lambda_2 = x - \frac{\mathrm{i}}{2}$, then the energy is given as

$$E = -\frac{J}{2}\frac{1}{x^2 + 1} \tag{8.24}$$

If the energy is real , then x only pure real or pure imaginary. If the x is pure real, then the energy will be determined with mass momentum.

$$e^{\mathrm{i}(k_1+k_2)} = \frac{x + \mathrm{i}}{x - \mathrm{i}} \cdot \frac{x}{x} = \frac{x + \mathrm{i}}{x - \mathrm{i}} \tag{8.25}$$

Hence, the energy coul be expressed as

$$E = \frac{J}{2}\left(\cos(k_1 + k_2) - 1\right) \tag{8.26}$$

8.2 Bethe ansatz solution

Chapter 9

Heisenberg model

9.1 Exchange interaction

Heisenberg proposed the nearest exchange interaction to explain the spontaneous magnetization . The hydrogen molecular model provide key clue to this problem. As shown in Fig1.1, two electrons labeled by 1,2 located on the 1s orbit. The spatial wavefunction can be written as

$$\begin{cases} \phi_S = \frac{1}{\sqrt{2}} \left(\phi_a(r_1)\phi_b(r_2) + \phi_a(r_2)\phi_b(r_1)\right) \\ \phi_A = \frac{1}{\sqrt{2}} \left(\phi_a(r_1)\phi_b(r_2) - \phi_a(r_2)\phi_b(r_1)\right) \end{cases}$$

which corresponds to the energy

$$\begin{cases} E_S = 2\varepsilon_0 + K + J_e \\ E_A = 2\varepsilon_0 + K - J_e \end{cases}$$

The ε_0 is the on-site energy for $1s$ orbit electron . K is the Coulomb interaction

$$K = \int V_{ab} \mid \psi_a(r_1) \mid^2 \mid \psi_b(r_2) \mid^2$$

The V_{ab} is the Coulomb interaction

$$V_{ab} = e^2 \int \left(\frac{1}{r_{12}} + \frac{1}{r_{ab}} - \frac{1}{r_{a2}} - \frac{1}{r_{b1}} \right)$$

The J is the exchange energy

$$J = \int V(ab)\psi_a(r_1)\psi_a(r_2)^*\psi_b(r_1)\psi_b(r_2)^*$$

Pauli principle restrict that the product of the spatial wavefunction and spin wavefunction must be anti-symmetry. In the other words, the coherent energy of hydrogen atom depend on the spin orientation. Hence, Heisenberg written out the hamiltonian based on the exchange interaction.

$$H = -\sum_{\langle i,j \rangle} J_{ij} S_i \cdot S_j$$

The exchange integral oringinates from Pauli principle.

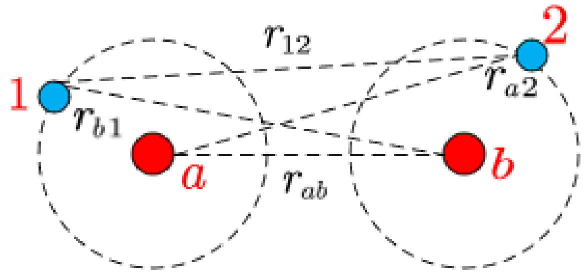

Fig 9.1: Hydrogen molecule exchange interaction

9.2 Heisenberg ferromagnetism

The Holstein-Primakoff transfermation swap the angular momentum operator into bosonic operator.

$$\begin{cases} \hat{S}^z = S - \hat{b}^\dagger \hat{b} \\ \hat{S}^\dagger = \sqrt{2S}\left(1 - \frac{\hat{b}^\dagger \hat{b}}{2S}\right)\hat{b} \\ \hat{S}^- = \sqrt{2S}\hat{b}^\dagger\left(1 - \frac{\hat{b}^\dagger \hat{b}}{2S}\right) \end{cases}$$

In the limit that expectations of $\hat{S}_z$ on each site are near to S , there are very few excitations in the bosonic representation, this mapping may be mapped into

$$\begin{cases} S^z = S - \hat{b}^\dagger \hat{b} \\ S^\dagger = \sqrt{2S}\hat{b} \\ \hat{S}^- = \sqrt{2S}\hat{b}^\dagger \end{cases}$$

Consider the exchange energy J is larger than zero, where the spins on the sites tend to align in the same direction. The ground state of ferromagnetism can be written as

$$\mid \text{Ground}\rangle = \bigotimes_i \mid S, S\rangle$$

Using the Holstein-Primakoff transfermation to reduce the heisenberg hamiltonian into

$$H = -J \sum_{\langle i,j \rangle} S^2 - JS \sum_{\langle i,j \rangle} (\hat{b}_i^\dagger \hat{b}_j + \hat{b}_j^\dagger \hat{b}_i - \hat{b}_i^\dagger \hat{b}_i - \hat{b}_j^\dagger \hat{b}_j)$$

This system is translationally invariant, which can be taken Fourier transformation . The result is

$$H = -\frac{1}{2} J S^2 N Z + \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} \qquad \varepsilon_{\mathbf{k}} = 2J \sum_i (1 - \cos k_i)$$

The decription of the excitations can be reduce into harmonic oscillator like modes .Because the ferromagnet is an eigenstate of Hamiltonian, there no zero-point fluctuations. At the lower energy sector, the dispersion is propotional to the k^2

$$\varepsilon_{\mathbf{k}} \sim \mid k^2 \mid$$

These excitation modes are Goldstein mode which arise from the spin rotational symmetry broken. According to the Ginzburg -Landau theory, the symmetry broken can be described by an parameter -maggnetization .We will study the magnetization

$$M = \frac{1}{N}\sum_i \langle S - \hat{b}_i^\dagger \hat{b}_i\rangle = S - \frac{1}{N}\sum_{\mathbf{k}} \langle \hat{b}_{\mathbf{K}}^\dagger \hat{b}_{\mathbf{k}}\rangle$$

The fluctuation of magnetization is just the occupation number for the bosonic modes .

$$\begin{aligned}\Delta M &= \frac{1}{N}\sum_{\mathbf{k}} \frac{1}{e^{\beta\varepsilon_{\mathbf{k}}}-1} = \frac{1}{V}\int_{\mathrm{BZ}} d^d\mathbf{k}\frac{1}{e^{\beta\varepsilon_{\mathbf{k}}}-1} \\ &= T\int_0^{\sqrt{T/SJ}} \frac{k^{d-1}dk}{SJk^2} \\ &\sim T^{1.5}(d=3)\end{aligned}$$

The magnetization fluctuation is divergent in the $d \leq 2$, where the Mermin-Wagner theorem tells us that there is no continue symmetry broken at finite temperature. The three dimension is just the Bloch $T^{1.5}$ law ,that the magnetization fluctuation is propotional to $T^{1.5}$.

9.3 Heisenberg anti-ferromagnet

The Hamiltonian minimized the energy when the spin on the nearset site is opposite. This is also know as the Neel state shown in Fig 1.2

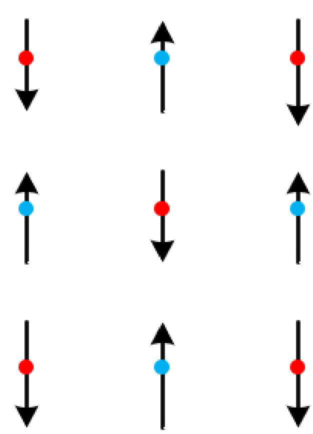

Fig 9.2: Neel state

We decompose the lattice into two sublattice A, B . For A sublattice, all spins on the site are oriented up . The Holstein-Primakoff for the A sublattice is the same as the ferromagnet .

However, We should change the forms of Holstein-Primakoff for the A sublattice.

Now we use this to rewrite the Heiseberg hamiltonian into such forms

$$H = -J\sum_{i,j} S^2 + JS\sum_{\langle i,j\rangle}\left[\hat{a}_i^\dagger\hat{a}_i + \hat{a}_j^\dagger\hat{a}_j + \hat{a}_i^\dagger\hat{a}_j^\dagger + \hat{a}_i\hat{a}_j\right]$$

We introduce the new coordinate x, d , where x is the coordinate for the all sites and the d is the nearest vector running over all nearest sites.

$$\begin{cases} S^z = S - \hat{b}^\dagger \hat{b} \\ \hat{S}^\dagger = \sqrt{2S}\,\hat{b} \\ \hat{S}^- = \sqrt{2S}\,\hat{b}^\dagger \end{cases} \qquad \begin{array}{ll} |S,S\rangle & |0\rangle \\ |S,S-1\rangle & |1\rangle \\ |S,S-2\rangle & |2\rangle \\ & \vdots \end{array}$$

Fig 9.3: Holstein-Primakoff for A sublattice

$$\begin{cases} S^z = -S + \hat{b}^\dagger \hat{b} \\ \hat{S}^\dagger = \sqrt{2S}\,\hat{b}^\dagger \\ \hat{S}^- = \sqrt{2S}\,\hat{b} \end{cases} \qquad \begin{array}{ll} & \vdots \\ |S,-S+2\rangle & |2\rangle \\ |S,-S+1\rangle & |1\rangle \\ |S,-S\rangle & |0\rangle \end{array}$$

Fig 9.4: Holstein-Primakoff for B sublattice

$$H = -\frac{JS^2Nz}{2} + \frac{JSz}{2}\sum_{x,d}\left[2\hat{a}_x^\dagger \hat{a}_x + \hat{a}_z^\dagger \hat{a}_{x+d}^\dagger + \hat{a}_z \hat{a}_{x+d}\right]$$

which transforms into momentum representation

$$H = -\frac{JS(S+1)Nz}{2} + \frac{JSz}{2}\sum_{\mathbf{k}}\begin{bmatrix}\hat{a}_{\mathbf{k}}^\dagger & \hat{a}_{-\mathbf{k}}\end{bmatrix}\begin{bmatrix}1 & \gamma_{\mathbf{k}} \\ \gamma_{\mathbf{k}} & 1\end{bmatrix}\begin{bmatrix}\hat{a}_{\mathbf{k}} \\ \hat{a}_{-\mathbf{k}}^\dagger\end{bmatrix} \qquad \gamma_{\mathbf{k}} = \sum_i \cos k_i$$

This Hamiltonian is the same as the BCS superconductor hamiltonian which can be diagonalized by Bogoliubov transformation.

$$\begin{bmatrix}\hat{a}_{\mathbf{k}} \\ \hat{a}_{-\mathbf{k}}^\dagger\end{bmatrix} = \begin{bmatrix}\cosh\theta_{\mathbf{k}} & -\sinh\theta_{\mathbf{k}} \\ -\sinh\theta_{\mathbf{k}} & \cosh\theta_{\mathbf{k}}\end{bmatrix}\begin{bmatrix}\hat{\alpha}_{\mathbf{k}} \\ \hat{\alpha}_{-\mathbf{k}}^\dagger\end{bmatrix}$$

The result is given by

$$H = -\frac{JS(S+1)Nz}{2} + \sum_{\mathbf{k}}\varepsilon_{\mathbf{k}}\left(\hat{\alpha}_{\mathbf{k}}^\dagger \hat{\alpha}_{\mathbf{k}} + \frac{1}{2}\right)$$

where

$$\varepsilon_{\mathbf{k}} = Jsz\sqrt{1-\gamma_{\mathbf{k}}} \sim Jsz \mid k \mid$$

The energy spectrum exists the zero point energy, which leads to the fluctuation even at the zero temperature. The magnetization fluctuation is given by

$$\begin{aligned}\Delta M &= \frac{1}{N}\sum_{\mathbf{k}}\langle(u_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}}^\dagger + v_{\mathbf{k}}\hat{\alpha}_{-\mathbf{k}})(u_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}} + v_{\mathbf{k}}\hat{\alpha}_{-\mathbf{k}}^\dagger)\rangle \\ &= \frac{1}{N}\sum_{\mathbf{k}}\left(u_{\mathbf{k}}^2\langle\hat{\alpha}_{\mathbf{k}}^\dagger\hat{\alpha}_{\mathbf{k}}\rangle + \mathbf{v}_{\mathbf{k}}^2\langle\hat{\alpha}_{-\mathbf{k}}\hat{\alpha}_{-\mathbf{k}}^\dagger\rangle\right) \\ &= \frac{1}{N}\sum_{\mathbf{k}}\left(u_{\mathbf{k}}^2\langle\hat{\alpha}_{\mathbf{k}}^\dagger\hat{\alpha}_{\mathbf{k}}\rangle + \mathbf{v}_{\mathbf{k}}^2\langle\hat{\alpha}_{-\mathbf{k}}^\dagger\hat{\alpha}_{-\mathbf{k}}\rangle + v_{\mathbf{k}}^2\right)\end{aligned}$$

At the zero temperature , the bosonic occupation number vanishes . We only consider zero point term

$$\begin{aligned}
\Delta M &= \frac{1}{N}\sum_{\mathbf{k}} v_{\mathbf{k}}^2 \\
&= \frac{1}{2V}\int_{\mathrm{BZ}}\left[\frac{1}{\sqrt{1-\gamma_{\mathbf{k}}}}-1\right] \\
&= \frac{1}{4zV}\int dk k^{d-2}
\end{aligned}$$

This integral is divergent in the 1d and convergent in $d \geq 2$.

9.4 Bethe ansatz

The Heisenberg model has the $SU(2)$ symmetry , but we only consider the spin rotational symmetry around the z axis. The toal spin $S_z = \sum_n s_n^z$ is a conserved quantity , which leads to a good quantum number $S_z = \frac{N}{2} - M$. The state is the ferromagntic ground state with eigenvalue $E_0 = -\frac{JN}{4}$ when $M = 0$.

$$| \psi_0\rangle = |\uparrow, \uparrow, \cdots, \uparrow\rangle$$

We consider the sector$M = 1$ where the eigenstate is the superposition of the n basis. The basis label by n which can be constructed by the lower operator

$$| n\rangle = S_n^- \ |\uparrow, \uparrow, \cdots, \uparrow\rangle$$

We write the eigenstate $| \psi_1\rangle$ in the sector $M = 1$ as

$$| \psi_1\rangle = \sum_{n=1}^{N} f(n) \mid n\rangle$$

Define the translational operator T as

$$T \mid n\rangle = | \ (n+1) \pmod{\mathrm{n}}\rangle$$

Consider the translational invariance on the Heisenberg model

$$T \mid \psi_i\rangle = \sum_{n=1}^{N} f(n) \mid n+1\rangle$$

which implies that

$$f(n+1) = \mu f(n)$$

We use the periodic condition to find out μ

$$T^N \mid \psi_0\rangle = \mu^n \sum_{n=1}^{N} f(n) \mid n\rangle \implies \mu^N = 1$$

Now the eigenstate can be written as

$$| \psi_1\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{ikn} \mid n\rangle$$

The eigenvalue for the $\mid \psi_1\rangle$ is the

$$E = E_0 + 2J(1 - \cos k) \qquad k = \frac{2\pi}{N}$$

Let's write down the generic eigenstate for the sector $M = 2$

$$\mid \psi_2\rangle = \sum_{1\le n_2\le n_2\le N} f(n_1, n_2) \mid n_1, n_2\rangle$$

where

$$\mid n_1, n_2\rangle = S^-_{n_1} S^-_{n_2} \left|\uparrow, \uparrow, \cdots, \uparrow\right\rangle$$

Bethe made an ansatz that the coefficient is determined by

$$f(n_1, n_2) = Ae^{\mathrm{i}(k_1 n_1 + k_2 + n_2)} + A' e^{\mathrm{i}(k_1 n_2 + k_2 n_1)}$$

We substitute this state into eigenequation

$$2(E - E_0) f(n_1, n_2) = J(4f(n_1, n_2) - f(n_1 \pm 1, n_2) - f(n_1, n_2 \pm 1))$$

which implies the eigenvalue

$$E = E_0 + 2 \sum_{i=1,2} (1 - \cos k_i)$$

On the other hand, the we can derive another equation if $n_2 = n_2 + 1$

$$2(E - E_0) f(n_1, n_2) = J(2f(n_1, n_2) - f(n_1 - 1, n_2) - f(n_1, n_2 + 1))$$

This equation leads to the relation of coefficient A, A'

$$\frac{A}{A'} = e^{\mathrm{i}\theta} = -\frac{e^{\mathrm{i}(k_1+k_2)} + 1 - 2e^{\mathrm{i}k_1}}{e^{\mathrm{i}(k_1+k_2)} + 1 - 2e^{\mathrm{i}k_2}} \implies 2\cot\frac{\theta}{2} = \cot\frac{k_1}{2} - \cot\frac{k_2}{2}$$

The phase shift *theta* reflect the interaction between the magnons. And we use the periodic condition to determine the quasi momentum k_1, k_2

$$f(n_1, n_2) = f(n_2, n_1 + N) \implies Nk_1 = 2\pi n_1 + \theta \quad Nk_2 = 2\pi n_2 - \theta$$

Bethe generalize this idea to construct generic wavefunction

$$\mid \psi\rangle = \sum_{1\le n_2\le n_2\le N} f(n_1, n_2, \cdots, n_N) \mid n_1, n_2, \cdots, n_M\rangle$$

where

$$f(n_1, n_2, \cdots, n_N) = \sum_{P\in S_M} \exp\left(\mathrm{i}\sum_{j=1}^{M} k_{P(j)} n_j + \frac{1}{2}\sum_{l,l<j} \theta_{P(l)P(j)}\right)$$

The phase angle is given by

$$e^{\mathrm{i}\theta_{lj}} = -\frac{e^{\mathrm{i}(k_l+k_j)} + 1 - 2e^{\mathrm{i}k_l}}{e^{\mathrm{i}(k_l+k_j)} + 1 - 2e^{\mathrm{i}k_j}}$$

In virtue of translational symmetry , which means that $f(n_1, n_2, \cdots n_M) = f(n_2, \cdots n_M, n_1 + N)$

$$\sum_{j=1}^{M} k_{P(j)} n_j + \frac{1}{2} \sum_{l,l<j} \theta_{P(l)P(j)} = \sum_{j=2}^{M} k_{P'(j-1)} n_j + P'(j)$$

Chapter 10

Strong correlation theory

10.1 $t - J$ model

We use Schrieffer canonical transformation to derive $t - J$ model.

$$H_{\text{eff}} = e^S H e^{-S} = H_0 + [S, H_0] + \frac{1}{2}[S, [S, H_0]] + \cdots \tag{10.1}$$

The hopping process on Hubbard model could be splitted into four process

- Single occupancy hopping

$$H_t^0 = -t \sum_{\langle i,j \rangle, \sigma} \Big((1 - n_{i\bar{\sigma}}) c_{i\sigma}^\dagger c_{j\sigma} (1 - n_{j\bar{\sigma}}) + (1 - n_{j\bar{\sigma}}) c_{j\sigma}^\dagger c_{i\sigma} (1 - n_{i\bar{\sigma}}) \Big) \tag{10.2}$$

- Double occupancy moving

$$H_t^1 = -t \sum_{\langle i,j \rangle, \sigma} \Big(n_{i\bar{\sigma}} c_{i\sigma}^\dagger c_{j\sigma} n_{j\bar{\sigma}} + n_{j\bar{\sigma}} c_{j\sigma}^\dagger c_{i\sigma} n_{i\bar{\sigma}} \Big) \tag{10.3}$$

This fisrt process doen't create and destroy double occupancy . The second process creates a double accupancy and destroy a double occupancy, then the total number of double occupancy doesn't change.

- Double occupancy creation

$$H_d^+ = -t \sum_{\langle i,j \rangle, \sigma} \Big(n_{i\bar{\sigma}} c_{i\sigma}^\dagger c_{j\sigma} (1 - n_{j\bar{\sigma}}) + n_{j\bar{\sigma}} c_{j\sigma}^\dagger c_{i\sigma} (1 - n_{i\bar{\sigma}}) \Big) \tag{10.4}$$

- Double occupancy creation

$$H_d^- = -t \sum_{\langle i,j \rangle, \sigma} \Big((1 - n_{i\bar{\sigma}}) c_{i\sigma}^\dagger c_{j\sigma} n_{j\bar{\sigma}} + (1 - n_{j\bar{\sigma}}) c_{j\sigma}^\dagger c_{i\sigma} n_{i\bar{\sigma}} \Big) \tag{10.5}$$

We could expand the Schrieffer canonical transformation

$$H = H_t^0 + H_t^1 + H_d^+ + H_d^- + H_U + [S, H_t^0] + [S, H_t^1] + [S, H_d^+] + [S, H_d^-] + [S, H_U] + \frac{1}{2}[S, [S, H]] + \cdots \tag{10.6}$$

We notice the four hopping processes commutes relatively. The Schrieffer transformation operator doesn't change doubling. Hence, we guarantee the term $H_d^{\pm}$ vanishes on effective hamiltonian. We could impose condition for hamiltonian (10.11)

$$H_d^{\pm} + [S, H_U] = 0 \tag{10.7}$$

We could notice that

$$[H_d^{\pm}, H_U] = \pm H_U \tag{10.8}$$

Eq(10.8) means that double occupancy increases or dereases with one site.

Note:-

Claim 10.1 .

$$[c_{i\sigma}^{\dagger} c_{j\sigma}, H_U] = -c_{i\sigma}^{\dagger} c_j n_{i\bar{\sigma}} + c_{i\sigma}^{\dagger} c_{j\sigma} n_{j\bar{\sigma}} \tag{10.9}$$

We use the Claim(10.1), the Eq(10.8) is obvious.

$$\begin{cases} [H_d^+, H_U] = [-t \sum_{\langle i,j \rangle, \sigma} \Big(n_{i\bar{\sigma}} c_{i\sigma}^{\dagger} c_{j\sigma} (1 - n_{j\bar{\sigma}}) + n_{j\bar{\sigma}} c_{j\sigma}^{\dagger} c_{i\sigma} (1 - n_{i\bar{\sigma}}) \Big), U n_{i\uparrow} n_{j\downarrow}] = -U H_d^+ \\ [H_d^-, H_U] = [-t \sum_{\langle i,j \rangle, \sigma} \Big((1 - n_{i\bar{\sigma}}) c_{i\sigma}^{\dagger} c_{j\sigma} n_{j\bar{\sigma}} + (1 - n_{j\bar{\sigma}}) c_{j\sigma}^{\dagger} c_{i\sigma} n_{i\bar{\sigma}} \Big), U n_{i\uparrow} n_{j\downarrow}] = U H_d^- \end{cases}$$

Hence, we could take Schrieffer transformation operator S as

$$S = \frac{1}{U}(H_d^+ - H_d^-) \tag{10.10}$$

The Eq(10.11) will turn out

$$H = H_t^0 + H_t^1 + H_U + [S, H_t^0] + [S, H_t^1] + [S, H_d^+] + [S, H_d^-] + \frac{1}{2}[S, [S, H]] + \cdots \tag{10.11}$$

The first term would change the number of double occupancy. We expand hamiltonian (**??**) into order $(t/U)^2$.

$$\frac{1}{2}[S, [S, H_U]] = \frac{1}{2U}[S, H_d^+ + H_d^-] + \mathcal{O}(\frac{t^3}{U^3}) = \frac{1}{U}[H_d^+, H_d^-] + \mathcal{O}(\frac{t^3}{U^3}) \tag{10.12}$$

Note:-

$$[H_d^+, H_d^-] = t^2 \sum_{\langle i,j \rangle, \sigma, \sigma'} [n_{i\bar{\sigma}} c_{i\sigma}^{\dagger} c_{j\sigma} (1 - n_{j\bar{\sigma}}) + n_{j\bar{\sigma}} c_{j\sigma}^{\dagger} c_{i\sigma} (1 - n_{i\bar{\sigma}}), (1 - n_{i\bar{\sigma'}}) c_{i\sigma'}^{\dagger} c_{j\sigma'} n_{j\bar{\sigma'}} + (1 - n_{j\bar{\sigma'}}) c_{j\sigma'}^{\dagger} c_{i\sigma'} n_{i\bar{\sigma'}}] \tag{10.13}$$

The term could be decomposed such terms

- $$[n_{i\downarrow} c_{i\uparrow}^{\dagger} c_{j\uparrow} (1 - n_{j\downarrow}), (1 - n_{j\downarrow}) c_{j\uparrow}^{\dagger} c_{i\uparrow} n_{i\downarrow}] = n_{i\downarrow} (1 - n_{j\downarrow}) (n_{i\uparrow} - n_{j\uparrow}) \tag{10.14}$$

  This term is second order process. The double occupancy is destroyed with electron hopping, then the double occupancy is created with hopping back. We take summation for spin index

$$
\begin{aligned}
&[n_{i\downarrow}(1-n_{j\downarrow})(n_{i\uparrow}-n_{j\uparrow}) + i \leftrightarrow j] + (\uparrow\leftrightarrow\downarrow) = (n_{i\downarrow}n_{i\uparrow} - n_{i\downarrow}n_{j\uparrow} - n_{j\downarrow}n_{i\downarrow}n_{i\uparrow} + n_{j\downarrow}n_{i\downarrow}n_{j\uparrow}) \\
&+ n_{j\downarrow}n_{j\uparrow} - n_{j\downarrow}n_{i\uparrow} - n_{i\downarrow}n_{j\downarrow}n_{j\uparrow} + n_{i\downarrow}n_{j\downarrow}n_{j\uparrow} + (\uparrow\leftrightarrow\downarrow) \\
&= (n_{i\uparrow}-n_{i\downarrow})(n_{j\uparrow}-n_{j\downarrow}) - n_i n_j \\
&= 4S_z \cdot S_z - n_i \cdot n_j
\end{aligned}
$$

We don't consider double occupancy states. The term $n_{i\downarrow}n_{j\downarrow}(1-n_{\uparrow})$ has no contribution to final results.

- $$
\begin{aligned}
&[n_{i\downarrow}c^\dagger_{i\uparrow}c_{j\uparrow}(1-n_{j\downarrow}), (1-n_{j\uparrow})c^\dagger_{j\downarrow}c_{i\downarrow}n_{i\uparrow}] = n_{i\downarrow}(1-n_{j\downarrow})[c^\dagger_{i\uparrow}c_{j\uparrow}, (1-n_{j\uparrow})n_{i\uparrow}]c^\dagger_{j\downarrow}c_{i\downarrow} \\
&+ [n_{i\downarrow}(1-n_{j\downarrow}), c^\dagger_{j\downarrow}c_{i\downarrow}](1-n_{j\uparrow})n_{i\uparrow}c^\dagger_{i\uparrow}c_{j\uparrow} \\
&= n_{i\downarrow}(1-n_{j\downarrow})\left[-(1-n_{j\uparrow}) - n_{i\uparrow}\right] c^\dagger_{i\uparrow}c_{j\uparrow}c^\dagger_{j\downarrow}c_{i\downarrow} + (1-n_{j\uparrow})n_{i\uparrow}\left[-(1-n_{j\downarrow}) - n_{i\downarrow}\right] c^\dagger_{j\downarrow}c_{i\downarrow}c^\dagger_{i\uparrow}c_{j\uparrow}
\end{aligned}
$$

  This hopping proccess has an immediate process , where create double occupancy. Hence, this term could be reduce into

  $$
  [n_{i\downarrow}c^\dagger_{i\uparrow}c_{j\uparrow}(1-n_{j\downarrow}), (1-n_{j\uparrow})c^\dagger_{j\downarrow}c_{i\downarrow}n_{i\uparrow}] = c^\dagger_{i\uparrow}c_{i\downarrow}c^\dagger_{j\downarrow}c_{j\uparrow} = S^+_i S^-_j
  $$

- $$
\begin{aligned}
&[n_{i\downarrow}c^\dagger_{i\uparrow}c_{j\uparrow}(1-n_{j\downarrow}), (1-n_{j\uparrow})c^\dagger_{i\downarrow}c_{j\downarrow}n_{i\uparrow}] = n_{i\downarrow}(1-n_{j\downarrow})[c^\dagger_{i\uparrow}c_{j\uparrow}, (1-n_{j\uparrow})n_{i\uparrow}]c^\dagger_{i\downarrow}c_{j\downarrow} \\
&+ [n_{i\downarrow}(1-n_{j\downarrow}), c^\dagger_{i\downarrow}c_{j\downarrow}](1-n_{j\uparrow})n_{i\uparrow}c^\dagger_{i\uparrow}c_{j\uparrow} \\
&= n_{i\downarrow}(1-n_{j\downarrow})\left[-(1-n_{j\uparrow}) - n_{i\uparrow}\right] c^\dagger_{i\uparrow}c_{j\uparrow}c^\dagger_{i\downarrow}c_{j\downarrow} + (1-n_{j\uparrow})n_{i\uparrow}\left[(1-n_{j\downarrow}) + n_{i\downarrow}\right] c^\dagger_{i\downarrow}c_{j\downarrow}c^\dagger_{i\uparrow}c_{j\uparrow} \\
&= 0
\end{aligned}
$$

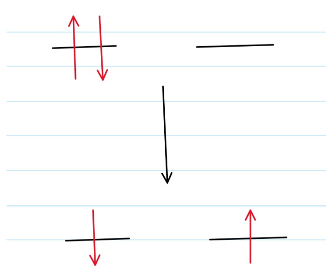

Fig 10.1: The propagator is the summation of all possible paths .

Hence, we could derive effectie model

$$
H = -t\sum_{\langle i,j\rangle,\sigma}\Big((1-n_{i\bar{\sigma}})c^\dagger_{i\sigma}c_{j\sigma}(1-n_{j\bar{\sigma}}) + (1-n_{j\bar{\sigma}})c^\dagger_{j\sigma}c_{i\sigma}(1-n_{i\bar{\sigma}})\Big) + \frac{4t^2}{U}\sum_{\langle i,j\rangle}(S_i\cdot S_j - \frac{1}{4}n_i n_j) \tag{10.15}
$$

10.2 Slave particle

Chapter 11

Superconductivity

In this chapter, we will focus on microscopic theory of superconductivity.

11.1 BCS theory

11.1.1 Cooper problem

Cooper consider two body problem with attractive interaction. Fermi gas has stableFfermi surface. The exsitence of Fermi surface will exert strong restriction to electron scattering. We consider the two electrons scattering process. This process demands momentum conservation , namly

$$\vec{k}_1' + \vec{k}_2' = \vec{k}_1 + \vec{k}_2 \tag{11.1}$$

In virtue of Pauli principle , the process shown on Fig(12.1) is nor permitted . However, we the electrons on the Fermi surface tend to form process shown on Fig(12.1).

The two electrons with opposite momentum has more scattering space. Hence, the system could be descried as

$$H = H_0 + H_I = \sum_{k,\sigma} \varepsilon_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma} - \frac{g}{V} \sum_{k,k'} c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger c_{-k'\downarrow} c_{k'\uparrow} \tag{11.2}$$

We consider the wavefunction could be expressed as

$$\mid \psi\rangle = \sum_k a(k) c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \mid \mathrm{FS}\rangle \tag{11.3}$$

Combing with Eq(11.21) and (11.3), we could be write down the eigenequation

$$H \mid \psi\rangle = E \mid \psi\rangle \implies (2\varepsilon_k + E_0) a(k) - \frac{g}{V} \sum_{k'} a(k') = E a(k) \tag{11.4}$$

We can slove the Δ from the consistent equation (11.4)

$$\Delta E = -2\hbar\omega_D \exp\left(-\frac{2}{N(0)g}\right) \tag{11.5}$$

Note:-

We make consistent equation for Eq(11.4)

$$\frac{a(k)}{\sum_{k'} a(k')} = \frac{g/V}{2\varepsilon_k + E_0 - E} \tag{11.6}$$

We make summation for k

$$1 = \sum_k \frac{g/V}{2\varepsilon_k + E_0 - E} \simeq \frac{g}{N(0)} \int_0^{\hbar\omega_D} d\varepsilon \frac{1}{2\varepsilon - \Delta E} = \frac{g}{2} N(0) \log\left(\frac{2\hbar\omega_D - \Delta E}{-\Delta E}\right) \tag{11.7}$$

We could summarize from the result (11.5) that it will form bounded state with lower energy than oringin fermi surface if we conisder two electron on the Fermi surface with attractive interaction. This phenomena is also call *Cooper instablity*.

11.1.2 BCS wavefunction

Schrieffer generalize the single Cooper pair to magny body wavefunction. Let's consider N Coopr pair, the single Cooper pair wavefunction could be described by $\psi(r_1, r_2, \sigma_2, \sigma_2)$. The BCS wavefunction could be written into

$$\Psi_{\mathrm{BCS}} = \mathcal{A}\left(\psi(r_1, r_2, \sigma_1, \sigma_2) \cdots \psi(r_{2N-1}, r_{2N}; \sigma_{2n-1}, \sigma_{2n})\right) \tag{11.8}$$

where $\mathcal{A}$ is the anti-symmetric operation. We write down the single Cooper pair function

$$\begin{aligned}
\psi(r_1, r_2; \sigma_1, \sigma_2) &= \phi(\mid r_1 - r_2 \mid) \otimes \frac{1}{\sqrt{2}} \left(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle\right) \\
&= \sum_k \chi(k) e^{\mathrm{i}k(r_1 - r_2)} \otimes \frac{1}{\sqrt{2}} \left(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle\right) \\
&= \sum_k \chi(k) c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \mid \mathrm{Vac}\rangle
\end{aligned} \tag{11.9}$$

Hence, the many body wavefunction could be written into

$$\mid \Psi_{\mathrm{BCS}}\rangle = \mathcal{N}^{-\frac{1}{2}} \left((\chi(k) c_{k\uparrow}) c_{-k\downarrow}\right)^{\frac{N}{2}} \mid \mathrm{vac}\rangle \tag{11.10}$$

The wavefunction (11.10) is written at canonical ensemble . We generalize the wavefunction into grand canonical ensemble

$$\mid \psi_{\mathrm{BCS}}\rangle = \exp\left(\chi(k) c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\right) \mid \mathrm{vac}\rangle = \prod_k (1 + \chi(k) c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}) \mid \mathrm{vac}\rangle \tag{11.11}$$

Hence, the many body wavefunction is decomposed into single particel wavefunction product state. Furthermore , we introduce the varitional parameter u_k, v_k to write the wavefunction into

$$\mid \psi_{BCS}(\phi)\rangle = \prod_k (\mid u_k \mid + \mid v_k \mid e^{\mathrm{i}\phi} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}) \mid \mathrm{vac}\rangle \tag{11.12}$$

The varitional parameter u_k and v_k control the component of superconductivity. The BCS wavefunction is also coherent state , which means coherence of wavefunction phase.

We could project out the N particle wavefunction from Eq(11.12)

$$\psi_{BCS}(N) = \int_0^{2\pi} e^{-\mathrm{i}\frac{N}{2}\phi} \psi_{\mathrm{BCS}}(\phi) \tag{11.13}$$

11.1.3 BCS wavefunction varitation

We have write down the BCS wavefunction on the previous section . The next step is to optimize BCS wavefunction . We substitute the BCS wavefunction into hamiltonian (11.21)

$$E = \langle \psi_{BCS} \mid H \mid \psi_{BCS} \rangle = 2\sum_k \varepsilon_k \mid v_k \mid^2 - \frac{g}{V} \sum_{k_1,k_2} g(k_1,k_2) u^*_{k_1} v_{k_1} u_{k_1} v^*_{k_2} \tag{11.14}$$

The Eq(**??**) shows that the varitional parameter on channel k_1, k_2 should be matched with phase to gaurantee real energy. Now we can introduce the θ_k to describe variational parameter, namly $u_k = \cos\theta_k, v_k = \sin\theta_k$.

$$E = 2\sum_k \varepsilon \sin^2\theta_k - \frac{1}{V}\sum_{k_1,k_2} g(k_1,k_2)\sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2 \tag{11.15}$$

We make variation for parameter θ_k

$$2\varepsilon_k \sin 2\theta_k - \frac{1}{V}\sum_{k'} g(k_1,k_2)\sin\theta_{k'}\cos\theta_{k'}\cos\theta_k = 0 \tag{11.16}$$

We solve the consitent equation (11.16) to derive gap function at zero temperature

$$\Delta = \hbar\omega \exp\left(-\frac{1}{gN(0)}\right) \tag{11.17}$$

Note:-

We construct consistent equation from Eq(11.16)

$$\tan 2\theta_k = \frac{\Delta}{\varepsilon_k} \tag{11.18}$$

where $\Delta(k_1) = \frac{1}{2V}\sum g(k_1,k_2)\sin 2\theta_{k_2}$. We sustitute the relation (11.18) into Δ_k

$$\Delta_k = \frac{1}{V}\sum_k \frac{\Delta_k}{2\xi_k} \simeq gN(0)\int_{-\hbar\omega_D}^{\hbar\Omega_D} \frac{1}{\sqrt{\Delta^2+\varepsilon^2}} d\varepsilon = 2\int_0^{\frac{\hbar\omega_D}{\Delta}} \frac{1}{\sqrt{1+x^2}} dx \tag{11.19}$$

11.1.4 Mean field theory

We use mean field theory to deal with hamiltonian (11.21) . We define supercondutor order parameter as $\Delta = \frac{g}{V}\sum\limits_k \langle c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\rangle$

$$\mathcal{O}_1\mathcal{O}_2 = \mathcal{O}_1\left(\mathcal{O}_2 - \langle\mathcal{O}_2\rangle + \langle\mathcal{O}_2\rangle\right) = \mathcal{O}_1\langle\mathcal{O}_2\rangle + \left(\mathcal{O}_1 - \langle\mathcal{O}_1\rangle + \langle\mathcal{O}_1\rangle\right)\left(\mathcal{O}_2 - \langle\mathcal{O}_2\rangle\right) \approx \mathcal{O}_1\langle\mathcal{O}_2\rangle + \langle\mathcal{O}_1\rangle\mathcal{O}_2 - \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle \tag{11.20}$$

Hence, we derive effective mean field hamiltonian

$$H = \sum_{k,\sigma} \varepsilon_{k\sigma} c^\dagger_{k\sigma} c_{k\sigma} - \sum_k \Delta c_{-k\downarrow} c_{k\uparrow} - \sum c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + \frac{g}{V}\sum_k \mid \Delta_k \mid^2 \tag{11.21}$$

The last term is called condensation energy. We introduce Numbu spinor $\psi = (c_{k\uparrow}, c^\dagger_{-k\downarrow})^{\mathbf{T}}$

$$H = \sum_k (c_{k\uparrow}, c^\dagger_{-k\downarrow}) \begin{pmatrix} \varepsilon_k & -\Delta_k \\ -\Delta^*_k & -\varepsilon_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} + E_0 + \frac{g}{V}\sum_k \mid \Delta_k \mid^2 \tag{11.22}$$

where $E_0 = \sum_k \varepsilon_k$. We use Bogliubov transformation to diagonalize hamiltonian (11.22) . This part is easy to do. I leave it for somple task.

$$\begin{pmatrix} \beta_{k\uparrow} \\ \beta^\dagger_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{k\downarrow} \end{pmatrix} \qquad \tanh 2\theta_k = \frac{\Delta}{\varepsilon_k} \qquad \cos^2\theta_k = \frac{1}{2}\left(1+\frac{\varepsilon_k}{E_k}\right) \qquad \sin^2\theta_k = \frac{1}{2}\left(1-\frac{\varepsilon_k}{E_k}\right) \tag{11.23}$$

The hamiltonian turns into

$$H = \sum_k E_k \left(\beta^\dagger_{k\uparrow}\beta_{k\uparrow} + \beta^\dagger_{-k\downarrow}\beta_{-k\downarrow} - 1\right) \qquad E_k = \sqrt{\varepsilon_k^2 + \Delta^2} \tag{11.24}$$

Hence, we could written the gap function into Bogliubov quasiparticle

$$\begin{aligned}
\Delta = \frac{g}{V}\sum_k c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow} &= \frac{g}{V}\sum_k (\cos\theta_k\beta_{k\uparrow} + \sin\theta_k\beta_{-k\downarrow})(\cos\theta_k\beta_{-k\downarrow} - \sin\theta_k\beta_{-k\uparrow}) \\
&= \frac{g}{V}\sum_k \sin\theta_k\cos\theta_k\langle\beta_{-k\downarrow}\beta^\dagger_{-k\downarrow} - \beta^\dagger_{k\uparrow}\beta_{k\uparrow}\rangle \\
&= \frac{gN(0)\Delta}{2}\int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{\tanh\frac{\beta\varepsilon_k}{2}}{\varepsilon} d\varepsilon \\
&= gN(0)\Delta \int_0^{\frac{\hbar\omega_D}{2k_BT_C}} \frac{\tanh x}{x} dx \\
&= gN(0)\Delta\left(\log\frac{\hbar\omega_D}{2k_BT_c}\tanh\frac{\hbar\omega_D}{2k_BT_c} - \int_0^{\frac{\hbar\omega_D}{2k_BT_c}} \log x \mathrm{sech}^2 x dx\right) \\
&= gN(0)\Delta\log\frac{2e^\gamma\hbar\omega_D}{\pi k_BT_c}
\end{aligned} \tag{11.25}$$

At the critical temperature, the gap is vanishing. We have that $k_BT_c = 1.13h\omega_D\exp\left(-\frac{1}{gN(o)}\right)$. We can derive the relation. between critical temperation and superconductor gap . We use the integral identiy (11.28) on the last step . We could derive the relation between critical temperature and gap function

$$\frac{\Delta}{k_KT_c} \approx 1.76 \tag{11.26}$$

The Eq(11.25) also tells us that the gap function is

$$\boxed{\Delta = gV\sum_k \frac{\Delta}{2E_k}\tanh\frac{\beta E_k}{2}} \tag{11.27}$$

Claim 11.1 .

$$\int_0^\infty \log x \mathrm{sech}^2 x dx = \log\frac{\pi}{4} - \gamma \tag{11.28}$$

We consider the integral below

Proof.

$$\int_0^{\infty} x^a \text{sech}^2 x dx = \int_0^{+\infty} dx \frac{4e^{-2x}}{(1+e^{-2x})^2} x^a = 4\sum_{n=0}^{\infty}\int_0^{\infty} x^a(-1)^{n-1}e^{-2nx}dx = \frac{2\Gamma(a+1)}{2^a}\eta(a) \tag{11.29}$$

The integral (11.28) is equal to

$$\frac{d}{da}\left(\frac{2\Gamma(a+1)}{2^a}\eta(a)\right)\Bigg|_{a=0} = (\Gamma'(1)\eta(0) + \eta'(0) - \log 2\eta(0)) = \log\frac{\pi}{4} - \gamma$$

□

11.2 Thermodynamic quantity

11.2.1 Condensation energy

We could find from hamiltonian (11.21) that

$$\begin{aligned} H &= \sum_{k,\sigma} \varepsilon_{k\sigma} c^{\dagger}_{k\sigma} c_{k\sigma} - \sum_k \Delta c_{-k\downarrow} c_{k\uparrow} - \sum c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} + \frac{g}{V}\sum_k \mid \Delta_k \mid^2 \\ &= \sum_k \xi_k \left(\beta^{\dagger}_k \beta_k + \beta^{\dagger}_{-k}\beta_{-k}\right) + (\varepsilon_k - \xi_k) + \frac{g}{V}\sum_k \mid \Delta_k \mid^2 \end{aligned} \tag{11.30}$$

The constant term on the (11.30) is just condensation energy

$$\begin{aligned} E_{\text{cond}} &= \sum_k (\varepsilon_k - \xi_k) + \frac{g}{V} \mid \Delta_k \mid^2 = \sum_k (\varepsilon_k - \sqrt{\varepsilon_k^2 + \Delta^2}) + \frac{\Delta^2}{2\xi_k} \\ &= V\frac{1}{V}\sum_k \left(\varepsilon_k - \frac{\varepsilon_k^2 + \Delta^2/2}{\sqrt{\varepsilon_k^2 + \Delta^2}}\right) \\ &= VN(0)\int_{-\hbar\omega_D}^{\hbar\omega_D} d\varepsilon \left(\varepsilon - \frac{\varepsilon^2 + \Delta^2/2}{\sqrt{\varepsilon^2+\Delta^2}}\right) \\ &= -VN(o)\Delta^2 \end{aligned} \tag{11.31}$$

We insert the gap function (11.19) on the first line of (11.31), the details about step are given below . The condensation energy relies on density states and superconductor gap .

Note:-

$$\begin{aligned} &\int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{\varepsilon^2 + \Delta^2/2}{\sqrt{\varepsilon^2+\Delta^2}} d\varepsilon = \Delta^2 \int_{-\frac{\hbar\omega_D}{\Delta}}^{\frac{\hbar\omega_D}{\Delta}} \frac{x^2 + \frac{1}{2}}{\sqrt{1+x^2}} dx = \Delta^2 \int_0^{\frac{\hbar\omega}{\Delta}} \frac{2x^2+1}{\sqrt{1+x^2}} dx = \Delta^2 \frac{\hbar\omega}{\Delta} \cdot \sqrt{1 + \left(\frac{\hbar\Omega_D}{\Delta}\right)} \\ &\approx 2(\hbar\omega_D)^2 \left(1 + \frac{1}{2}\left(\frac{\Delta}{\hbar\omega_D}\right)^2\right) \end{aligned} \tag{11.32}$$

11.2.2 Specfic heat

Firstly , Let us review the statistial mechanics. The free energy for fermionic system is given by

$$F = -\sum_k \frac{1}{\beta}\log(1 + e^{-\beta\varepsilon_k}) \tag{11.33}$$

The entropy could be obatained from free energy as

$$
\begin{aligned}
S = -\frac{\partial F}{\partial T} &= \sum_k \frac{\partial}{\partial T}\left(k_B T \log(1+e^{-\beta\varepsilon_k})\right) = -\sum_k \frac{\partial}{\partial T}\left(k_B T \log(1-f_k)\right) \\
&= -\sum_k k_B \left((1-f_k)\log(1-f_k) + f_k \log(1-f_k) - T\frac{1}{1-f_k}\frac{\partial f_k}{\partial T}\right) \\
&= -\sum_k k_B \left((1-f_k)\log(1-f_k) + f_k \log f_k\right)
\end{aligned} \tag{11.34}
$$

The capcity could be derived from entropy

$$
\begin{aligned}
C_V = T\frac{\partial S}{\partial T} = -\beta\frac{\partial S}{\partial \beta} &= -2\beta\left(\log\frac{f_k}{1-f_k}\frac{\partial f_k}{\partial \beta}\right) = -2\beta^2\sum_k \xi_k \frac{\partial f_k}{\partial \beta} \\
&= -2\beta^2 \sum_k \xi_k \frac{\partial f_k}{\partial(\beta\xi_k)}\left(\xi_k + \frac{\partial \xi_k}{\partial \beta}\right)
\end{aligned} \tag{11.35}
$$

We consider the sin freedom, then the entropy need to multiply by factor 2. The specific heat on the (11.35) consists of two part. The first part is just specific heat of normal metal when the temperature T is near critical temperature T_c

$$
c_n = 2\beta\sum_k\left(-\frac{\partial f_k}{\partial \xi_k}\right)\xi_k^2 = 2k_B^2 T N(0)\int_{\frac{h\omega_D}{k_B T}}^{\frac{h\omega_D}{k_B T}} \frac{x^2 e^x}{(1+e^x)^2}dx \approx 4k_B N(0) T\int_{-\infty}^{+\infty}\frac{x^2e^x}{(1+e^x)^2}dx \sim \frac{2\pi^2}{3}N(0)k_B^2 T \tag{11.36}
$$

Note:-

We calculate the integral on the (11.36)

$$
\int_0^{+\infty}\frac{x^2e^x}{(1+e^x)^2} = \int_0^{\infty}dx\frac{x^2e^{-x}}{(1+e^{-x})^2} = \sum_{n=1}^{\infty}\int_0^{\infty}x^2e^{-nx}dx == \Gamma(3)\eta(2) = \frac{\pi^2}{6} \tag{11.37}
$$

At the critical temperature, the second term gives the specific heat between superconductor state and normal state.

$$
c_n - c_s = -2\beta^2\sum_k \xi_k - \frac{\partial f_k}{\partial(\beta\xi_k)}\frac{\partial \xi_k}{\partial \beta} = -\beta^2 N(0)\frac{d\Delta^2}{\partial \beta}\bigg|_{T=T_c} = k_B N(0)\frac{d\Delta^2}{dT}\bigg|_{T=T_c} \tag{11.38}
$$

We can see that the specific heat is not continuous at T_c . Hence, this is second order phase transition. At the low temperature region $T \ll T_c$, We neglect the second term

$$
\begin{aligned}
c_{es} = 2k_B\beta\sum_k\left(-\frac{\partial f_k}{\partial \xi_k}\right)\xi_k^2 &= 2\beta N(0)k_B\int_{-\infty}^{+\infty}\xi^2 e^{-\beta\xi}d\xi = 2\frac{\Delta^2(0)}{T}N(0)e^{-\frac{\Delta(0)}{k_BT}}\int_{-\infty}^{\infty}e^{-\frac{\varepsilon^2}{2k_BT\Delta(0)}} \\
&= 2\frac{\Delta^2(0)}{T}N(0)\left(\frac{2\pi\Delta(0)}{k_BT}\right)^{0.5}e^{-\frac{\Delta(0)}{k_BT}}
\end{aligned} \tag{11.39}
$$

11.2.3 Gap function dependdence on temperature

We start gap function (11.27) directly

$$1 = N(0)V\int_0^{\hbar\omega_D}\frac{\tanh\frac{1}{2}\beta\xi}{\xi}d\varepsilon = \frac{2N(0)V}{\beta}\int_0^{\infty}\sum_{n=-\infty}^{n+\infty}\frac{1}{\omega_n^2+\xi^2}d\xi \approx \frac{2N(0)V}{\beta}\sum_{n=-\infty}^{n+\infty}\int_0^{\infty}\left(\frac{1}{\omega_n^2+\varepsilon^2}-\frac{\Delta(T)^2}{(\omega_n^2+\varepsilon^2)^2}+\cdots\right)$$

$$= N(0)V\left[\int_0^{\hbar\omega}\frac{\tanh\frac{1}{2}\beta h\varepsilon}{\varepsilon}d\varepsilon - \sum_{n=-\infty}^{n+\infty}\frac{2}{\beta}\int_0^{\hbar\omega}\frac{\Delta(T)^2}{(\omega_n^2+\varepsilon^2)^2}\right]$$

$$= N(0)V\left[\int_0^{\hbar\omega}\frac{\tanh\frac{1}{2}\beta h\varepsilon}{\varepsilon}d\varepsilon - \sum_{n=-\infty}^{n+\infty}\frac{2}{\beta}\int_0^{+\infty}\frac{\Delta(T)^2}{(\omega_n^2+\varepsilon^2)^2}\right]$$

$$= N(0)g\left[\log\frac{2e^{\gamma}\hbar\omega_D}{\pi k_B T} - \left(\frac{\Delta^2(T)}{\pi k_B^2 T^2}\right)^2\sum_{n=0}^{n+\infty}\frac{1}{(n+1)^2}\right] \tag{11.40}$$

We use the integral () on the last step . We substitute (11.25) into (11.40)

$$\Delta(T) = \pi k_B T\sqrt{\frac{T_c - T}{\eta(3)T_c}} \tag{11.41}$$

Note:-

The Fermionic distribution function could be expressed into Poisson summation ,namly

$$f(\varepsilon) = \sum_{n=-\infty}^{+\infty}\frac{1}{\beta}\frac{1}{\mathrm{i}\omega_n - \varepsilon} \tag{11.42}$$

where the ω_n is he Matsubara frequency (2.19) .

$$\tanh\frac{\beta\varepsilon}{2} = \frac{e^{\beta\varepsilon}-1}{e^{\beta\varepsilon}+1} = f(-\varepsilon) - f(\varepsilon) = \frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\frac{1}{\mathrm{i}\omega_n+\varepsilon} - \frac{1}{\mathrm{i}\omega_n-\varepsilon} = \frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\frac{2\varepsilon}{\omega_n^2+\varepsilon^2} \tag{11.43}$$

$$\int_0^{+\infty}\frac{1}{(1+x^2)^2}dx = \frac{1}{2}\int_0^{+\infty}\frac{x^{-0.5}}{(1+x)^2} = \frac{1}{2}B(0.5, 1.5) = \frac{\pi}{4} \tag{11.44}$$

11.3 Susceptibility

11.4 Single particle tunneling

If we consider two system connected each other, the system could be described by

$$H = H_R + H_L + H_T \tag{11.45}$$

The tunneling hamiltonian could be described as

$$H_T = \sum_k T_{kq}c_{kR}^{\dagger}c_{qL} + h.c \tag{11.46}$$

The current is defined as [1]

[1] Tou can refer to Claim (11.4)

$$I = -\frac{2e}{\hbar}\Im\left(\sum_{k,q} T_{kq} c^{\dagger}_{kR,\sigma} c_{qL,\sigma}\right) \tag{11.47}$$

Claim 11.2 .
The density operator on the tunneling process is defined as

$$I = -\frac{2e}{\hbar}\Im\left(\sum_{k,q} T_{kq} c^{\dagger}_{kR,\sigma} c_{qL,\sigma}\right)$$

We can use motion equation to write down the current operator

$$\begin{aligned} I = -e\frac{d\langle N_L\rangle}{dt} &= -2e\frac{1}{\mathrm{i}\hbar}[N_L, H] \\ &= -\frac{2e}{\hbar}\sum_{kq\sigma} T_{kq}[c^{\dagger}_{Lk\sigma}c_{Lk\sigma}, T_{qk}c^{\dagger}_{qR\sigma}c_{Lk\sigma} + T_{kq}c^{\dagger}_{Lk\sigma}c_{Rq\sigma}] \\ &= -\frac{2e}{\hbar}\Im\left(\sum_{k,q} T_{kq} c^{\dagger}_{kR,\sigma} c_{qL,\sigma}\right) \end{aligned} \tag{11.48}$$

According to fluctuation theorem , we need to evaluate the response function of current operator.

$$G(\tau) = -\langle \mathcal{T}_\tau A(\tau) A^{\dagger}(0)\rangle \tag{11.49}$$

Note:-
Accoring to Wick theorem

$$\begin{aligned} G(\tau) &= -\sum_{k,q;k',q'} T_{kq}T_{k'q'}\langle \mathcal{T} c^{\dagger}_{Rk}(\tau)c_{Lq}(\tau)c^{\dagger}_{Rk'}c_{Lq'} + c^{\dagger}_{Lk}(\tau)c_{Rq}(\tau)c^{\dagger}_{Lk'}c_{Rq'}\rangle \\ &= -\sum_{k,q} \mid T_{kq}\mid^2 \left[G_L(q,\tau)G_R(k,-\tau) + G_R(k,\tau)G_L(q,-\tau)\right] \end{aligned} \tag{11.50}$$

We transform it into Matsubra representation

$$\begin{aligned} G(\mathrm{i}\omega_n) = \int_0^{\beta} e^{\mathrm{i}\omega_n\tau}G(\tau)d\tau = -\sum_{k,q}\mid T_{kq}\mid^2 \int_0^{\beta} d\tau \left[G_L(q,\tau)G_R(k,-\tau) + L\to R\right]e^{\mathrm{i}\omega_n\tau} \\ -\sum_{k,q}\mid T_{kq}\mid^2\left[\sum_n G(q,p_n+\omega_n)G(k,p_n) + L\to R\right] \end{aligned} \tag{11.51}$$

11.5 p wave pairing and more

11.5.1 d vector formalism

The most celbrated example of p wave Cooper pairing is the ^3He . The p wave pairing has rich structuture. The p wave pairing order parameter have 18 components. There has three orbital freedom $p_z, p_x + \mathrm{i}p_y, p_x - \mathrm{i}p_y$ and spin triplet . The additional two freedom arises from complex order parameter , which could be splitted into real part and imagninary parts. We start from continuum model

$$H=\sum_{k,\sigma}(\varepsilon_k-\mu)c^\dagger_{k\sigma}c_{k\sigma}+\frac{1}{2V_d}\sum_{k,k',\alpha\beta}V(k,k')c^\dagger_{-k'\beta}c^\dagger_{k'\beta\alpha}c_{k\alpha}c_{-k\beta} \tag{11.52}$$

We use dipolar interaction $V(k,k')=-V_t\vec{k}\cdot\vec{k}'$. Let's define the order parameter as

Definition 11.1: .

The p wave pairing order parameter is defined as

$$\Delta^a_{\sigma\sigma'}=-\sum_{k'}V_t k'_a\langle c_{k'\sigma}c_{-k'\sigma'}\rangle \tag{11.53}$$

Under the definition (11.5.1), the parameter could be decomposed into three symmetric matrices channel [2] .

$$\Delta^a_{\sigma\sigma'}=\Delta_{\mu a}\cdot(\sigma_u\cdot \mathrm{i}\sigma_2)_{\sigma\sigma'} \tag{11.54}$$

In order to study the spin structure lying on the order parameter , we introduce the d vector to describe the pin texture. The definition (11.5.1) gives the order parameter along the a direction. We introduce the pairing matrix $\Delta_{\sigma\sigma'}$

$$\Delta_{\sigma\sigma'}=k_a\Delta^a_{\sigma\sigma'}=\Delta(k)d_\mu(k)(\sigma_\mu \mathrm{i}\sigma_2)_{\sigma\sigma'}=\Delta(k)\begin{pmatrix}-d_x(k)+\mathrm{i}d_y(k) & d_z(k)\\ d_z(k) & d_x(k)+\mathrm{i}d_y(k)\end{pmatrix} \tag{11.55}$$

The pairing matrix has properties below

- The pairing matrix is symmetric matrix , which describes triplet pairing
- The pairing matrix satisfies to

$$\Delta_{\alpha\beta}(k)=\Delta_{\beta\alpha}(-k) \tag{11.56}$$

The pairing wavefunction is the triplet superposition , namly

$$\varphi(k)=\phi_{\alpha\beta}\mid\alpha,\beta\rangle=\Delta^\uparrow\left|\uparrow\uparrow\right\rangle+\Delta_\downarrow\left|\downarrow\downarrow\right\rangle+\Delta_0(\left|\uparrow\downarrow\right\rangle+\left|\downarrow\uparrow\right\rangle) \tag{11.57}$$

The matrix ϕ is symmtric matrix which could be decomposed as

$$\phi=\begin{pmatrix}\Delta_\uparrow & \Delta_0\\ \Delta_0 & \Delta_\downarrow\end{pmatrix}=\frac{1}{2}(\Delta_\uparrow+\Delta_\downarrow)+\frac{1}{2}(\Delta_\uparrow-\Delta_\downarrow)\sigma_z+\Delta_0\sigma_1=-\frac{\mathrm{i}}{2}(\Delta_\uparrow+\Delta_\downarrow)(\sigma_2\mathrm{i}\sigma_2)-\frac{1}{2}(\Delta_\uparrow-\Delta_\downarrow)(\sigma_1\mathrm{i}\sigma_2)+\Delta_0(\sigma_3\mathrm{i}\sigma_2) \tag{11.58}$$

From (11.58), we known that $d_z=\Delta_0, d_y=-\frac{1}{2}(\Delta_\uparrow-\Delta_\downarrow), d_x=-\frac{\mathrm{i}}{2}(\Delta_\uparrow+\Delta_\downarrow)$. Hence , the d vector could be expressed with the matrix Δ

$$\vec{d}=\frac{1}{2}\left[-\Delta^\uparrow(\hat{k}_x+\mathrm{i}\hat{k}_y)+\Delta_\downarrow(\hat{k}_x-\mathrm{i}\hat{k}_y)+2\Delta_0\hat{k}_z\right] \tag{11.59}$$

[2] $(\mathrm{i}\sigma_2\sigma_\mu)=(\mathrm{i}\sigma_2\sigma_\mu)^{\mathbf{T}}$

Example 11.5.1 (.)
Rotation in spin space

$$\begin{aligned}
R_\Omega \mid \varphi\rangle &= \sum_{\alpha,\beta}\left[(\vec{d}\cdot\vec{\sigma})\mathrm{i}\sigma_2\right]_{\alpha\beta} R^1(\Omega)\otimes R^2(\Omega)\mid \alpha\beta\rangle \\
&= \sum_{\alpha,\beta,\gamma,\delta}\left[(\vec{d}\cdot\vec{\sigma})\mathrm{i}\sigma_2\right]_{\alpha\beta} R^1_{\gamma\alpha}(\Omega)\otimes R^2_{\delta\beta}(\Omega)\mid \gamma\delta\rangle \\
&= \sum_{\beta,\gamma,\delta}\left[R(\Omega)(\vec{d}\cdot\vec{\sigma})\mathrm{i}\sigma_2\right]_{\gamma\beta}\sigma_2 R_{\beta\delta}(-\Omega)\sigma_2\mid \gamma\delta\rangle \\
&= \sum_{k}\left[(\mathbf{R}(\vec{d})\cdot\vec{\sigma})\mathrm{i}\sigma_2\right]_{\alpha\beta}\mid \alpha\beta\rangle
\end{aligned} \tag{11.60}$$

Now we consider the infinisimal rotation $-\mathrm{i}\hat{n}\cdot\vec{S}$, where $\vec{S}$ is the total momentum acting on spin space $\vec{S}=\vec{S}_1\otimes\mathbf{1}+\mathbf{1}\otimes\vec{S}_1$.

$$\begin{aligned}
-\mathrm{i}\hat{n}\cdot\vec{S}\mid\varphi\rangle &= \sum_{\alpha,\beta}\left[(\vec{d}\cdot\vec{\sigma})\sigma_2\right]_{\alpha\beta}\left((\hat{n}\cdot\vec{\sigma})_{\gamma\alpha}\mid\gamma\beta\rangle+(\hat{n}\cdot\vec{\sigma})_{\delta\beta}\mid\alpha\delta\rangle\right) \\
&= \sum_{\alpha\beta}\left[(\hat{n}\cdot\vec{\sigma})(\vec{d}\cdot\vec{\sigma})\sigma_2\right]_{\alpha\beta}(\mid\alpha\beta\rangle+\mid\beta\alpha\rangle) \\
&= \sum_{\alpha,\beta}\left[(\hat{n}\cdot\vec{d})\sigma_2+\mathrm{i}\left(\hat{n}\times\vec{d}\right)\cdot\vec{\sigma}\sigma_2\right]_{\alpha\beta}(\mid\alpha\beta\rangle+\mid\beta\alpha\rangle) \\
&= \varphi(\hat{n}\times\vec{d})
\end{aligned} \tag{11.61}$$

We use the symmetrix matrix properties on the second step on Eq(11.61). We prove a properties of rotation matrix

$$\sigma_2 R(\Omega)\sigma_2 = R^*(-\Omega) \implies (\sigma_2 R(-\Omega)\sigma_2)^* \tag{11.62}$$

Example 11.5.2 (.)
Rotation in real space

$$\begin{aligned}
-\mathrm{i}\hat{L}\mid\varphi\rangle &= -\sum_{\alpha\beta}\vec{r}\times\nabla_r\int dk e^{\mathrm{i}\vec{k}\cdot\vec{r}}D_{\alpha\beta}(k)\mid\alpha,\beta\rangle = \sum_{\alpha\beta}\varepsilon_{mnl}r_m\nabla_{r_m}\int dk e^{\mathrm{i}\vec{k}\cdot\vec{r}}D_{\alpha\beta}(k)\mid\alpha,\beta\rangle\hat{e}_l \\
&= \sum_{\alpha\beta}\varepsilon_{mnl}r_m\int dk\mathrm{i}k_n e^{\mathrm{i}\vec{k}\cdot\vec{r}}D_{\alpha\beta}(k)\mid\alpha,\beta\rangle\hat{e}_l \\
&= \sum_{\alpha\beta}\varepsilon_{mnl}-\int dk k_m e^{\mathrm{i}\vec{k}\cdot\vec{r}}\frac{\partial}{\partial k_n}D_{\alpha\beta}(k)\mid\alpha,\beta\rangle\hat{e}_l \\
&= \varphi(Ld(k))
\end{aligned} \tag{11.63}$$

We show that spin direction is orthogonal to d vector . The Eq(11.61) tells us

$$(\vec{S}\cdot\vec{d})\mid\varphi\rangle = \sum_{\alpha,\beta}\left[(\hat{n}\cdot\vec{d})\sigma_2+\mathrm{i}\left(\hat{d}\times\vec{d}\right)\cdot\vec{\sigma}\sigma_2\right]_{\alpha\beta}(\mid\alpha\beta\rangle+\mid\beta\alpha\rangle)=0 \tag{11.64}$$

Futher more, we could derive vector $\vec{S}$ as

$$\frac{\langle\varphi \mid \vec{S} \mid \varphi\rangle}{\langle\psi \mid \psi\rangle} = \mathrm{i}\vec{d} \times \vec{d}^{*} \tag{11.65}$$

Note:-

We give some thing details about Eq(11.65)

$$\begin{aligned}
\frac{\langle\varphi \mid \vec{S} \mid \varphi\rangle}{\langle\psi \mid \psi\rangle} &= \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \Big(\sigma_2(\vec{d}\cdot\vec{\sigma})\Big)^{*}_{\gamma\delta} \Big(\vec{\sigma}(\vec{d}\cdot\vec{\sigma})\sigma_2\Big)^{*}_{\alpha\beta} \left(\langle\gamma\delta \mid \alpha\beta\rangle + \langle\gamma\delta \mid \beta\alpha\rangle\right) \\
&= \frac{1}{2}\mathrm{Tr}\Big(\sigma_2(\vec{d}\cdot\vec{\sigma})\vec{\sigma}(\vec{d}\cdot\vec{\sigma})\sigma_2\Big) \\
&= \frac{1}{2}\mathrm{tr}\Big((\vec{d}\cdot\vec{\sigma})\mathrm{i}(\vec{d}\times\vec{\sigma})\Big) \\
&= \mathrm{i}\vec{d}\times\vec{d}^{*}
\end{aligned} \tag{11.66}$$

We use Eq(11.67) above.

$$\frac{\vec{\sigma}(\vec{d}\cdot\vec{\sigma})}{\langle\varphi \mid \varphi\rangle} = d_n\sigma_m\sigma_n = d_n(\delta_{mn} + \mathrm{i}\varepsilon_{mnk}\sigma_k) = \vec{d} + \mathrm{i}(\vec{d}\times\vec{\sigma}) \tag{11.67}$$

The direction of $\vec{d}$ is not the real spin direction. For example, if $d \parallel \hat{z}$, then Cooper pairing will reads as $\Delta^0 = \Delta_{\uparrow\downarrow} + \Delta_{\downarrow\uparrow}$, which lies on $x - y$ plane. The d vector will gives infinite angular momentum Cooper pairing if $\vec{d}$ vector is complex .

We know that the order parameter is invariant under transformation $\Delta(k) \to e^{\mathrm{i}\pi}\Delta(k), \hat{d}(k) \to -\hat{d}(k)$. Hence , the d vector lies on $RP^2 = S^2/Z_2$. The fundamental on the S^2 is trivial. Hoever , the fundamental group on the RP^2 is $\pi_1(RP^2) = Z_2$.

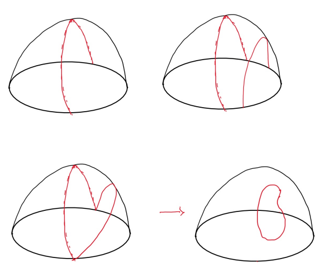

Figure 11.1: The first picture shows the non contractable loop on the RP^2. The second picture shows two noncontractable loop on the SP^2. If this two loopa are touched, then they turn into single contractable loop.

11.5.2 ^3He phase

In this section , we use the d vector representation to discuss ^3He phase . We start from hamiltonian (11.52) with mean field approximation

$$H = \sum_{k,\sigma}(\varepsilon_k - \mu)c^{\dagger}_{k\sigma}c_{k\sigma} + \frac{1}{2V_d}\sum_{k,\alpha\beta} c^{\dagger}_{-k\beta}c^{\dagger}_{k\alpha}\Delta_{\alpha\beta}(k) + \Delta_{\beta\alpha}(-k)c_{k\alpha}c_{k\beta} \tag{11.68}$$

We introduce Nambu spinor to reduce hamiltonian (11.68) , namly $\psi_k = (c_{k\uparrow}, c_{k\downarrow}, c^{\dagger}_{-k\uparrow}, c^{\dagger}_{-k\downarrow})^{\mathbf{T}}$

$$H = (c^\dagger_{k\uparrow}, c^\dagger_{k\downarrow}, c_{-k\uparrow}, c_{-k\downarrow}) \begin{pmatrix} \varepsilon_k - \mu & \Delta(k) \\ \Delta^8(*) & -(\varepsilon_k - \mu) \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \\ c^\dagger_{-k\uparrow}, \\ c^\dagger_{-k\downarrow} \end{pmatrix} \tag{11.69}$$

The Bloch hamiltonian of Eq(11.69) could be written as

$$H(k) = (\varepsilon_k - \mu)\,\Gamma_4 + d_z(k)\Gamma_1 - d_x(k)\Gamma_3 + d_y(k)\Gamma_0 \tag{11.70}$$

where

$$\Gamma_0 = -\mathrm{i}\sigma_2 \otimes 1 \quad \Gamma_1 = \sigma_1 \otimes \tau_1 \quad \Gamma_2 = \sigma_1 \otimes \tau_2 \quad \Gamma_3 = \sigma_1 \otimes \sigma_z \quad \Gamma_4 = \sigma_z \otimes 1 \tag{11.71}$$

The nergy spectrum of (11.70) is given as

$$E_k = \pm\sqrt{(\varepsilon_k - \mu)^2 + \mid \Delta(k) \mid^2} \tag{11.72}$$

- ^3He B phase

  We start from d vector representation

$$\Delta(k)d_\mu(k) = \Delta_{\mu a}k_a \tag{11.73}$$

The matrix element $\Delta_{\mu\alpha}$ maps the vector k on the momentum space to spin space . The most simple cases is that $\vec{d}$ vector is parallel to $\hat{k}$. We could see from (11.72) that the B phase is fully gapped. The wavefunction $\varphi(\vec{k})$ is charaterized as

$$\varphi(k) = \Big[(-\hat{k}_x + \mathrm{i}\hat{k}_y)\,|\uparrow\uparrow\rangle + (\hat{k}_x + \mathrm{i}\hat{k}_y)\,|\downarrow\downarrow\rangle + \hat{k}_z(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\Big] \tag{11.74}$$

In the B phase, the d vector vector is real vector, which means no Cooper pairing spin angular momentum from (11.65) . The orbital angular momentum could be from (11.63) ., namly $\langle L\rangle = 0$. In this phase , we require colinear spin -orbital coupling ($p-p$) channel. The Glodstein mode will appear with $SO_L(3) \otimes SO_s(3)/SO_J(3)$ symmetry.

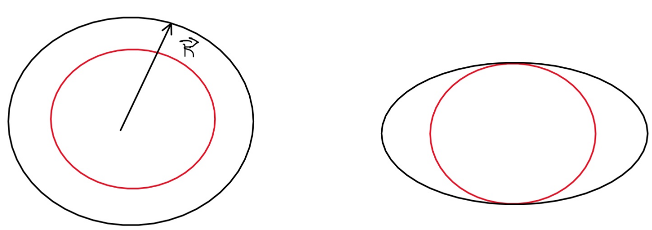

Figure 11.2: ^{3} B phase: Fermi surface has isotropic gap . A phase: Fermi suface hass nonal point.

- ^3He A phase

  The d vector is $d_x = \vec{k}\cdot(\hat{e}_1 + \hat{e}_2)$, namly $\vec{d} = \sqrt{\frac{3}{2}}(\hat{k}_x + \mathrm{i}k_y, 0, 0)$. There has no spin angular momentum on the A phase. However, the orbital angular momentum is 1. There has very rich physic on this phase [1].

11.6 Solution of edge modes

11.7 Cohenrence factor

11.8 Electromagnetic response

11.8.1 Linear response

In this section we discus the eletromagnetic response to superconducotr. We use linear response theory to study paramagnetic curent and diamagnetic current. The gauge potential is coupled into kinetic energy term

$$H = \int d^3x\psi^\dagger(x)\frac{1}{2m}\left(-\mathrm{i}\hbar\nabla + \frac{e}{c}\vec{A}\right)^2\psi(x) + \int d^3x\psi(x)^\dagger(x)\psi^\dagger(x')V(x'-x)\psi(x')\psi(x) \tag{11.75}$$

The hamiltonian relying on vector potential A is just

$$H_1 = \frac{\mathrm{i}e\hbar}{mc}\int d^x\psi^\dagger(x)\left(\vec{A}\cdot\nabla + \nabla\cdot\vec{A}\right)\psi(x) = \mathrm{i}\mu_B\int d^x\psi^\dagger(3)\left(\vec{A}\cdot\nabla + \nabla\cdot\vec{A}\right)\psi(x) \tag{11.76}$$

On the momentum space, we derive the hamiltonian (11.76)

$$\begin{aligned} H_1 &= \frac{1}{2}\mathrm{i}\mu_B\int d^3\psi^\dagger(x)\left(\vec{A}\cdot\nabla + \nabla\cdot\vec{A}\right)\psi(x) = \frac{1}{2}\mathrm{i}\mu_B\int d^3x\sum_{k_1,k,k_2} c^\dagger_{k_1}e^{-\mathrm{i}k_1x}\left(\vec{A}(q)e^{\mathrm{i}qx}\cdot\nabla c_{k_2}e^{\mathrm{i}k_2x} + \nabla\cdot(\vec{A}(q)e^{\mathrm{i}qx}c_{k_2}e^{\mathrm{i}k_2x})\right) \\ &= -\mu_B\sum_{k,q}\vec{A}(q)\cdot(\vec{k}+\frac{q}{2})c^\dagger_{k+q}c_k(\vec{k}) \end{aligned} \tag{11.77}$$

The hamiltonian (11.77) could be viewed as perturbation . If we use Feymann diagram to express the (11.76), then it turns into

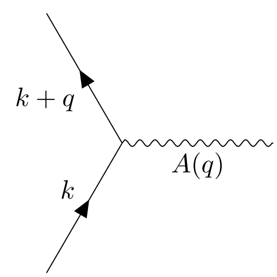

Figure 11.3: Electron souples with photon at vertex . Every vertex contributes to factor μ_B

Furthermore, we consider take Columb gauge $\nabla\cdot\vec{A} = 0$, then the perturbation hamiltonian (11.77) becomes into

$$H_I = -\frac{1}{2}\mu_B\sum_{k,q}(\vec{q}\cdot\vec{A}(q))\left(c^\dagger_{k+q\uparrow}c_{k\uparrow} - c^\dagger_{-k\downarrow}c_{-(k+q)\downarrow}\right) \tag{11.78}$$

In the superconductor region, we use Bogliubov particle formalism to discuss problem . Hence, we substitute the Hamltonian (11.78) with Bogliubov particle operator α_k to discuss problem .

$$
\begin{aligned}
H_I &= -\frac{1}{2}\mu_B \sum_{k,q} (\vec{k}\cdot\vec{A}(q)) \left(c^\dagger_{k+q\uparrow} c_{k\uparrow} - c^\dagger_{-k\downarrow} c_{-(k+q)\downarrow} \right) \\
&= -\frac{1}{2}\mu_B \sum_{k,q} (\vec{k}\cdot\vec{A}(q)) \left[(u_{k+q}\alpha^\dagger_{k+q\uparrow} - v_{k+q}\alpha_{-(k+q)\downarrow})(u_k\alpha_{k\uparrow} - v_k\alpha^\dagger_{-k\downarrow}) - (k+q \to -k, \uparrow \to \downarrow) \right] \\
&= -\frac{1}{2}\mu_B \sum_{k,q,\sigma} \sum_{k,q} (\vec{k}\cdot\vec{A}(q)) \left[(u_{k+q}u_k + v_{k+q}v_k)(\alpha^\dagger_{k+q\uparrow}\alpha_{k\uparrow} - \alpha^\dagger_{k+q\downarrow}\alpha_{k\downarrow}) + (-u_{k+q}v_k + u_k v_{k+q}) \left(\alpha^\dagger_{k+q\uparrow}\alpha^\dagger_{-k\downarrow} - \alpha_{k\uparrow}\alpha_{-(k+q)\downarrow} \right) \right]
\end{aligned}
\tag{11.79}
$$

The current $\vec{j}(r)$ could be derived from (11.77).

$$
\vec{j}(r) = \frac{\delta H}{\delta \vec{A}(r)} = \frac{e\hbar}{2mi} \left(\psi^\dagger \nabla \psi - (\nabla\psi)^\dagger \psi \right) - \frac{e^2}{mc^2} \psi^\dagger \vec{A} \psi = \vec{j}_1(r) + \vec{j}_2(r) \tag{11.80}
$$

The first term called paramagnetic current , which exists on noraml metal but vanishes on superconductor. The second term called diamagnetic curent. We could drive the paramagnetic current from (11.78),namly

$$
\begin{aligned}
j_1(r) &= -\frac{1}{2}\mu_B \sum_{k,q} (\vec{k}2 + \vec{q}) e^{\mathrm{i}\vec{q}\cdot\vec{r}} \left(c^\dagger_{k+q\uparrow} c_{k\uparrow} - c^\dagger_{-k\downarrow} c_{-(k+q)\downarrow} \right) \\
&= -\frac{1}{2}\mu_B \sum_{k,q,\sigma} \sum_{k,q} (\vec{k} + \frac{\vec{q}}{2}) \left[(u_{k+q}u_k + v_{k+q}v_k)(\alpha^\dagger_{k+q\uparrow}\alpha_{k\uparrow} - \alpha^\dagger_{k+q\downarrow}\alpha_{k\downarrow}) + (-u_{k+q}v_k + u_k v_{k+q}) \left(\alpha^\dagger_{k+q\uparrow}\alpha^\dagger_{-k\downarrow} - \alpha_{k\uparrow}\alpha_{-(k+q)\downarrow} \right) \right]
\end{aligned}
\tag{11.81}
$$

We use the first perturbation to calculate the paramagnetic current. If we put the perturbation (11.78) into superconductor hamiltonian (11.21) . We use the first perturbation theory to calculate the superconductor ground state up to first order

$$
\mid \Omega \rangle = \mid \Omega \rangle_0 + \sum_l \mid l \rangle_0 \frac{{}_0\langle l \mid H_1 \mid \Omega \rangle_0}{E_l - E_0} \tag{11.82}
$$

where $\mid \Omega\rangle_0$ is the BCS ground state. The BCS is the vaccum of Bogliubov particles . Hence, the paramagnetic current only contributed by second term. The state $\mid l\rangle_0$ is defined as

$$
\mid l \rangle = \alpha^\dagger_{k+q\uparrow}\alpha^\dagger_{-k\downarrow} \mid \Omega \rangle_0 \tag{11.83}
$$

The BCS ground state $\mid \Omega\rangle$ doesn't contribute to paramagnetic current. It requires to immendiate state to $\mid l\rangle$ to carry current. We substitute current (11.81) into (11.82)

$$
\langle \vec{j}_1(r) \rangle = \sum_l \left[\frac{{}_0\langle \Omega \mid H_1 \mid l \rangle \langle l \mid H_1 \mid \Omega \rangle_0}{E_l - E_0} + \frac{\langle l \mid H_1 \mid \Omega \rangle_0 \langle l \mid j_1(r) \mid \Omega \rangle_0}{E_l - E_0} \right] \tag{11.84}
$$

Combing with (11.81,11.83) , the Eq(11.84) could be simplified into

$$
\langle \vec{j}_1(r) \rangle = \frac{1}{c}\mu_B^2 \sum_l \left[\frac{(-u_{k+q}v_k + u_k v_{k+q})^2}{\xi_{k+q} + \xi_k} (\vec{k} + \frac{\vec{q}}{2})(\vec{k}\cdot\vec{A}(q)) e^{\mathrm{i}\vec{q}\cdot\vec{r}} \right] \tag{11.85}
$$

If we consider the contributon brought by spin freedom , the current will become

$$
\boxed{\langle \vec{j}_1(r) \rangle = 2\frac{1}{c}\mu_B^2 \sum_k \left[\frac{(-u_{k+\frac{q}{2}}v_{k-\frac{q}{2}} + u_{k+\frac{q}{2}} v_{k+\frac{q}{2}})^2}{\xi_{k+\frac{q}{2}} + \xi_{k-\frac{q}{2}}} \vec{k}(\vec{k}\cdot\vec{A}(q)) e^{\mathrm{i}\vec{q}\cdot\vec{r}} \right]} \tag{11.86}
$$

Note:-

$$
\begin{cases}
\langle_l H_1 \mid \Omega\rangle_0 = -\dfrac{1}{2}\mu_B(\vec{k}\cdot\vec{A}(\vec{q}))(-u_{k+q}v_k + u_k v_{k+q}) \\
\langle l \mid \vec{j}_1(r) \mid \Omega\rangle_0 = \dfrac{1}{2}\mu_B(\vec{k}\cdot\vec{A}(\vec{q}))(-u_{k+q}v_k + u_k v_{k+q})e^{i\vec{q}\cdot\vec{r}}
\end{cases} \tag{11.87}
$$

Let's analysis the current direction . The term $(\vec{k}\cdot\vec{A}(q))\vec{k}$ could be viewed as two rank tensor

$$
(\vec{k}\cdot\vec{A}(q))\vec{k} \sim k_x(k_x\hat{i} + k_y\hat{j} + k_z\hat{k}) \tag{11.88}
$$

The current requires to preserve invariant under mirror reflection about xy, xz, yz plane, thereby the current propagate along x direction .

$$
\vec{j}(r) = \frac{2e^2\hbar^2}{m^2c}\left(\frac{1}{4\pi}\int_0^{2\pi}\cos^2\phi d\phi\int_{-1}^{1}\sin^2\theta d\cos\theta\frac{N(0)}{2}\int_{-\infty}^{\infty}\frac{(-u_{k+\frac{q}{2}}v_{k-\frac{q}{2}} + u_{k+\frac{q}{2}}v_{k+\frac{q}{2}})^2}{\xi_{k+\frac{q}{2}} + \xi_{k-\frac{q}{2}}}\vec{A}(q)\right) \tag{11.89}
$$

Note:-

We use (11.23) to calculate the Eq(11.89)

$$
\begin{aligned}
\left(-u_{k+\frac{q}{2}}v_{k-\frac{q}{2}} + u_{k+\frac{q}{2}}v_{k+\frac{q}{2}}\right)^2 &= \frac{1}{4}\left((1+\frac{\varepsilon_{k+\frac{q}{2}}}{\xi_{k+\frac{q}{2}}})(1-\frac{\varepsilon_{k-\frac{q}{2}}}{\xi_{k-\frac{q}{2}}}) + (1-\frac{\varepsilon_{k-\frac{q}{2}}}{\xi_{k-\frac{q}{2}}})(1+\frac{\varepsilon_{k+\frac{q}{2}}}{\xi_{k+\frac{q}{2}}})\right) - \frac{1}{2}\frac{\Delta^2}{\xi_{k+\frac{q}{2}}\xi_{k-\frac{q}{2}}} \\
&= \frac{1}{2}\frac{\xi_{k+\frac{q}{2}}\xi_{k-\frac{q}{2}} - \varepsilon_{k+\frac{q}{2}}\varepsilon_{k-\frac{q}{2}} - \Delta^2}{\xi_{k+\frac{q}{2}}\xi_{k-\frac{q}{2}}}
\end{aligned} \tag{11.90}
$$

We discuss current for the normal metal cases and superconductor cases.

- Normal metal
- Superconductor

11.9 Electromagnetic asorbtion

The elctromagnetic asorbtion perturbative hamiltonian reads as

$$
\Delta H = \sum_{k,q}(k+\frac{q}{2})A(q)c^{\dagger}_{k+q}c_k \tag{11.91}
$$

The vector potential is time reversal odd . [3] Hence, this is case-II response . The initial state is BCS ground state. The final state is connected with Bogliubov particle creation operator $\alpha_k\alpha_{k'}$. We can writen down real part of optical conductivity with Fermin golden rule [4]

$$
\Re\sigma(\omega) = \frac{2\pi}{\hbar}\sum_{k,k'} \mid N(k\sigma|k'\sigma') \mid^2 \left((1-f(E_k))(1-f(E_{k'})) - f(E_k)f(E_{k'})\right)\delta(\hbar\omega - E_k - E_{k'}) \tag{11.92}
$$

At the zero temperatue, the Eq(11.92) could be written into

$$
\Re\sigma(\omega) = -\frac{2\pi}{\hbar}N^2(0)\tilde{N}^2\int_{\Delta}^{\infty}dE\int_{\Delta}^{\infty}dE'\frac{EE'}{\sqrt{E^2-\Delta^2}\sqrt{E'^2-\Delta^2}}(uv'-u'v')^2\delta(\hbar\omega - E - E') \tag{11.93}
$$

[3] $\vec{E} = -\frac{\partial\vec{A}}{\partial t}$.

[4] The two Bouliubov particle energy meets with frequency $\delta(\hbar\omega - E_k - E_{k'})$

Note:-

The coherence factor could be calculated as [a]

$$
\begin{aligned}
(uv'-u'v')^2 = u^2v'^2+u'^2v^2-2uvu'v' = \frac{1}{2}(1+\frac{\varepsilon_k}{\xi_k})(1-\frac{\varepsilon_k}{\xi_k})+\frac{1}{2}(1+\frac{\varepsilon_{k'}}{\xi_{k'}})(1-\frac{\varepsilon_{k'}}{\xi_{k'}})-\frac{1}{2}\frac{\Delta^2}{\xi_k\xi_{k'}} \\
=\frac{1}{2}\left(1+\frac{\varepsilon_k\varepsilon_{k'}}{\xi_k\xi_{k'}}-\frac{\Delta^2}{\xi_k\xi_{k'}}\right) \\
=\frac{1}{2}\left(1+\frac{\varepsilon_k\varepsilon_{k'}}{\xi_k\xi_{k'}}-\frac{\Delta^2}{\xi_k\xi_{k'}}\right)
\end{aligned}
\tag{11.94}
$$

[a] $\varepsilon_k, \varepsilon_{k'}$ lies on Fermi surface.

Hence, we could written down the optical conductance for normal metal

$$
\Re\sigma_n(\omega) = -\frac{2\pi}{\hbar}N^2(0)\tilde{N}^2\hbar\omega \tag{11.95}
$$

We consider the relative radio at region $\omega \gg 2\Delta$

$$
\begin{aligned}
\Re\left(\frac{\sigma_s(\omega)}{\sigma_n(\omega)}\right) &= \frac{1}{\hbar\omega}\int_{\Delta}^{\hbar\omega-\Delta}\frac{E(E-\hbar\omega)-\Delta^2}{\sqrt{E^2-\Delta^2}\sqrt{(\hbar\omega-2\Delta)^2-\Delta^2}}dE \\
&= \frac{1}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}}\frac{y(x-y)-\frac{1}{4}}{\sqrt{x^2-\frac{1}{4}}\sqrt{(x-y)^2-\frac{1}{4}}}dy \\
&= \frac{1}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}}\frac{-(x-y)^2-\frac{1}{4}+x^2-xy}{\sqrt{x^2-\frac{1}{4}}\sqrt{(x-y)^2-\frac{1}{4}}}dy \\
&= \frac{1}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}}\sqrt{\frac{x^2-\frac{1}{4}}{(x-y)^2-\frac{1}{4}}}
\end{aligned}
\tag{11.96}
$$

where $x=\frac{\hbar\omega}{2\Delta}, y=\frac{E}{2\Delta}$.

Note:-

We simplify the Eq(11.96) into elliplitic function

$$
\begin{aligned}
\frac{1}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}}\frac{y(x-y)-\frac{1}{4}}{\sqrt{x^2-\frac{1}{4}}\sqrt{(x-y)^2-\frac{1}{4}}}dy &= \frac{1}{x\sqrt{x^2-\frac{1}{4}}}\int_{\frac{1}{2}}^{x-\frac{1}{2}}\frac{y(x-y)-\frac{1}{4}}{\sqrt{(x-y)^2-\frac{1}{4}}} \\
&= \frac{1}{x\sqrt{x^2-\frac{1}{4}}}\int_{\frac{1}{2}}^{x-\frac{1}{2}}\frac{\frac{1}{2}x^2-\frac{1}{2}(x-y)^2-\frac{1}{4}}{\sqrt{(x-y)^2-\frac{1}{4}}}
\end{aligned}
\tag{11.97}
$$

Let $y = x-\sqrt{\left(x-\frac{1}{2}\right)^2\cos^2\theta+\frac{1}{4}\sin^2\theta}$

$$
\begin{aligned}
&\frac{1}{x\sqrt{x^2-\frac{1}{4}}}\int_0^{\frac{\pi}{2}}\frac{\left((x-\frac{1}{2})^2-\frac{1}{4}\right)\sin\theta\cos\theta}{\sqrt{\left(x-\frac{1}{2}\right)^2\cos^2\theta+\frac{1}{4}\sin^2\theta}}\left[\frac{\frac{1}{2}x^2-\frac{3}{8}}{\sqrt{\left((x-\frac{1}{2})^2-\frac{1}{4}\right)\cos^2\theta}}-\frac{1}{2}\sqrt{\left((x-\frac{1}{2})^2-\frac{1}{4}\right)\cos^2\theta}\right] \\
&= \frac{1}{x\sqrt{x^2-\frac{1}{4}}}\sqrt{\left((x-\frac{1}{2})^2-\frac{1}{4}\right)}(\frac{1}{2}x^2-\frac{3}{8})\int_0^1\frac{dx}{\sqrt{\frac{1}{4}-\left((x-\frac{1}{2})^2-\frac{1}{4}\right)^2x^2}}
\end{aligned}
$$

11.9.1 Green function method

To calculate conductance , we calculate the curent-curent correlation function $\Pi_{\mu\nu}$. The vertex function is $\frac{e}{\hbar}\frac{\partial \varepsilon_k}{\partial k_\mu}$

$$\Pi_{\mu\nu}(q, \mathrm{i}\nu_n) = \frac{e^2}{\hbar^2 V}\sum_k \frac{\partial \varepsilon_{k+\frac{q}{2}}}{\partial k_\mu}\frac{\partial \varepsilon_{k-\frac{q}{2}}}{\partial k_\nu}\frac{1}{\beta}\sum_{\mathrm{i}\omega_n}\mathrm{Tr}\left(G(k+q, \mathrm{i}(\nu_n+\omega_n))G(k, \mathrm{i}\omega_n)\right) \tag{11.97}$$

where $G(k, \mathrm{i}\omega_n)$ is the Nambu-Gorkov green function. We use the Matsubara summation to calculate the trace part

$$\begin{aligned}
&\frac{1}{\beta}\sum_{\mathrm{i}\omega_n}\mathrm{Tr}\left(G(k+q, \mathrm{i}(\nu_n+\omega_n))G(k, \mathrm{i}\omega_n)\right) = \frac{1}{\beta}\sum_{\mathrm{i}\omega_n}\mathrm{Tr}\left(\frac{1}{\mathrm{i}(\omega_n+\nu_n) - \sigma_z\varepsilon_{k+q} - \Delta\sigma_1}\frac{1}{\mathrm{i}\omega_n - \sigma_z\varepsilon_k - \Delta\sigma_1}\right)\\
&= \frac{1}{\beta}\sum_{\mathrm{i}\omega_n}\mathrm{Tr}\left(\frac{(\mathrm{i}(\omega_n+\nu_n) + \sigma_z\varepsilon_{k+q} + \Delta\sigma_1)(\mathrm{i}\omega_n + \varepsilon_k\sigma_3 + \Delta\sigma_1)}{[(\mathrm{i}(\omega_n+\nu_n))^2 - \varepsilon_{k+q}^2 - \Delta^2][(\mathrm{i}\omega_n)^2 - \varepsilon_k^2 - \Delta^2]}\right)\\
&= \frac{1}{\beta}\sum_{\mathrm{i}\omega_n}\left(\frac{(\mathrm{i}(\omega_n+\nu_n))(\mathrm{i}\omega_n) + \varepsilon_k\varepsilon_{k+q} + \Delta^2}{[(\mathrm{i}(\omega_n+\nu_n))^2 - \varepsilon_{k+q}^2 - \Delta^2][(\mathrm{i}\omega_n)^2 - \varepsilon_k^2 - \Delta^2]}\right)\\
&= \frac{\xi_{k+q}(\xi_{k+q} - \mathrm{i}\nu_n) + \varepsilon_k\varepsilon_{k+q} + \Delta^2}{\xi_{k+q}\left[(\xi_{k+q} - \mathrm{i}\nu_n)^2 - \xi_k^2\right]}f(\xi_{k+q}) + \frac{\xi_k(\xi_k + \mathrm{i}\nu_n) + \varepsilon_k\varepsilon_{k+q} + \Delta^2}{\xi_k\left[(\xi_k + \mathrm{i}\nu_n)^2 - \xi_{k+q}^2\right]}f(\xi_k)\\
&- \frac{\xi_{k+q}(\xi_{k+q} + \mathrm{i}\nu_n) + \varepsilon_k\varepsilon_{k+q} + \Delta^2}{\xi_{k+q}\left[(\xi_{k+q} + \mathrm{i}\nu_n)^2 - \xi_k^2\right]}(1 - f(\xi_{k+q})) - \frac{\xi_k(\xi_k - \mathrm{i}\nu_n) + \varepsilon_k\varepsilon_{k+q} + \Delta^2}{\xi_k\left[(\xi_k - \mathrm{i}\nu_n)^2 - \xi_{k+q}^2\right]}(1 - f(\xi_k))
\end{aligned} \tag{11.98}$$

We consider the zero temorature limit, the polarization (11.97) will become

$$\Pi_{\mu\nu}(q, \mathrm{i}\nu_n) = -\frac{e^2}{\hbar^2 V}\sum_k \frac{\partial \varepsilon_{k+\frac{q}{2}}}{\partial k_\mu}\frac{\partial \varepsilon_{k-\frac{q}{2}}}{\partial k_\nu}\left[\frac{\xi_{k+q}(\xi_{k+q} + \mathrm{i}\nu_n) + \varepsilon_k\varepsilon_{k+q} + \Delta^2}{\xi_{k+q}\left[(\xi_{k+q} + \mathrm{i}\nu_n)^2 - \xi_k^2\right]} + \frac{\xi_k(\xi_k - \mathrm{i}\nu_n) + \varepsilon_k\varepsilon_{k+q} + \Delta^2}{\xi_k\left[(\xi_k - \mathrm{i}\nu_n)^2 - \xi_{k+q}^2\right]}\right] \tag{11.99}$$

Note:-

$$\begin{aligned}
&\Im\left[\frac{1}{(\xi_k - \nu - \mathrm{i}\varepsilon - \xi_{k+q})(\xi_k - \nu - \mathrm{i}\varepsilon + \xi_{k+q})}\right] = \Im\left[\frac{1}{2\xi_{k+q}}\left(\frac{1}{(\xi_k - \nu - \mathrm{i}\varepsilon - \xi_{k+q})} - \frac{1}{(\xi_k - \nu - \mathrm{i}\varepsilon + \xi_{k+q})}\right)\right]\\
&= \frac{\pi}{2\xi_{k+q}}\left(\delta(\nu + \xi_{k+q} - \xi_k) - \delta(\nu - \xi_k - \xi_{k+q})\right)
\end{aligned} \tag{11.100}$$

$$\begin{aligned}
&\Im\left[\frac{1}{(\xi_k + \nu + \mathrm{i}\varepsilon - \xi_{k+q})(\xi_k + \nu + \mathrm{i}\varepsilon + \xi_{k+q})}\right] = \Im\left[\frac{1}{2\xi_{k+q}}\left(\frac{1}{(\xi_k + \nu + \mathrm{i}\varepsilon - \xi_{k+q})} - \frac{1}{(\xi_k + \nu + \mathrm{i}\varepsilon + \xi_{k+q})}\right)\right]\\
&= \frac{\pi}{2\xi_{k+q}}\left(\delta(\nu + \xi_{k+q} + \xi_k) - \delta(\nu + \xi_k - \xi_{k+q})\right)
\end{aligned} \tag{11.101}$$

The imaginary part of polarization is

$$\Im\Pi_{\mu\mu}(q, \nu) = \frac{\pi e^2}{2V\hbar^2}\sum_k \frac{\partial \varepsilon_{k+\frac{q}{2}}}{\partial k_\mu}\frac{\partial \varepsilon_{k-\frac{q}{2}}}{\partial k_\nu}\left[\frac{\xi_k\xi_{k+q} - \varepsilon_k\varepsilon_{k+q} - \Delta^2}{\xi_k\xi_{k+q}}\right]\left(\delta(\omega + \xi_k + \xi_{k+q}) + \delta(\omega - \xi_k - \xi_{k+q})\right) \tag{11.102}$$

We consider to calculate optical conductance for isotropic s-wave superconductors

$$\Re\sigma(\omega) = \frac{\pi e^2 v_F^2}{6} N^2(0) \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \frac{E}{\sqrt{E^2-\Delta^2}} \frac{E'}{\sqrt{E'^2-\Delta^2}} \frac{EE'-\Delta^2}{EE'} \delta(\omega - E - E') \tag{11.103}$$

Then we consider to derive optical ratio of normal state metal and superconductor

$$\Re\left(\frac{\sigma_s(\omega)}{\sigma_n(\omega)}\right) = \frac{1}{\hbar\omega} \int_{-\infty}^{\infty} dE \frac{E(\omega-E)-\Delta^2}{\sqrt{E^2-\Delta^2}\sqrt{(\omega-E)^2-\Delta^2}} \tag{11.104}$$

Note:-

We notice the we can let $E = \dfrac{\omega + (\omega - 2\Delta)x}{2}$ to simplify (11.104) into Elliptic integralc

$$\begin{aligned}
&\sqrt{E^2-\Delta^2}\sqrt{(\omega-E)^2-\Delta^2} = [(E+\Delta)(E-\Delta)(\omega-E+\Delta)(\omega-E-\Delta)]^{0.5} \\
&= \frac{\omega-2\Delta}{2}(1+x)\frac{\omega+2\Delta+(\omega-2\Delta)x}{2}\frac{(\omega+2\Delta)-(\omega-2\Delta)x}{2}\cdot\frac{\omega-2\Delta}{2}(1-x) \\
&= \left((\frac{\omega}{2})^2-\Delta^2\right)^2(1-x^2)(1-\alpha^3x^2)
\end{aligned} \tag{11.105}$$

where $\alpha = \dfrac{\omega-2\Delta}{\omega+2\Delta}$

$$E(\omega-E)-\Delta^2 = \frac{\omega+(\omega-2\Delta)x}{2}\frac{\omega-(\omega-2\Delta)x}{2} - \Delta^2 = \left((\frac{\omega}{2})-\Delta^2\right)^2(1-\alpha x^2) \tag{11.106}$$

Hence, the Eq(11.104) could be simplified into

$$\begin{aligned}
\Re\left(\frac{\sigma_s(\omega)}{\sigma_n(\omega)}\right) = \frac{\omega-2\Delta}{\hbar\omega}\int_0^1 \frac{1-\frac{1}{\alpha}+\frac{1}{\alpha}(1-\alpha^2x^2)}{\sqrt{(1-x^2)(1-\alpha^2x^2)}} &= (1-y)(1-\frac{y+1}{1-y})K\left(\frac{1-y}{y+1}\right)+(1+y)E(\frac{1-y}{1+y}) \\
&= -2yK\left(\frac{1-y}{y+1}\right)+(1+y)E(\frac{1-y}{1+y})
\end{aligned} \tag{11.107}$$

We consider two limits

$$\begin{cases}
\lim_{y\to 1}\Re\left(\frac{\sigma_s(\omega)}{\sigma_n(\omega)}\right) = 2E(0)-2K(0) = 0 \\
\lim_{y\to 0}\Re\left(\frac{\sigma_s(\omega)}{\sigma_n(\omega)}\right) = E(1) = 1
\end{cases} \tag{11.108}$$

The behaviour of (11.104) is plotted in Figure(11.4).

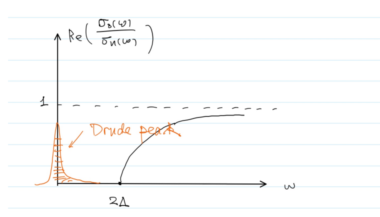

Figure 11.4: The conductance start response from 2Δ. If the frequency is infinite, optical conductance at superconductivity region is equal to normal state.

11.10 Unconventional supercondutor

In this section , we will introduce theory about unconventional superconductor theory. The pairing wavefunctions could be described with space part and spin part

$$\varphi(r_1, r_2) = \phi(r_1, r_2) \otimes \chi(s_1, s_2) \tag{11.109}$$

The system has global $SO(3)$ symmetry. The space wavefunction is spllited into spherical harmonics.

$$\phi(r_1, r_2) = R(\mid r_1 - r_2 \mid) \sum_l \sum_m c_{lm} Y_{lm}(\Omega) \tag{11.110}$$

- If $s = 0$, the pairing wavefunction is isotropic . If the wavefunctions has no node , this wavefunction is called conventional superconductor pairing. If wavefunction has node, this wavefunction is called $s_\pm$ pairing.
- If $l = 1$, the wavefction change sign when it rotates around Fermi surface with angle π. The p wave order parameter is written as

$$\Delta = \sum_{k'} g(\mid k - k' \mid) c^\dagger_{k'\alpha} \left(\mathrm{i}\sigma_2\sigma_\mu\right) c^\dagger_{k'\beta} \tag{11.111}$$

- If $d = 2$, the wavefction change sign when it rotates around Fermi surface with angle $\frac{\pi}{2}$.

The high T_c supercondcutor has been found from 1986. It has been reached consensus that the physics lies on $Cu - O$ plane. We start from real spcace model to discuss superductivity . We write down phenomenological model to describe superconductivity

$$H = -t \sum_{\langle i,j \rangle} c^\dagger_i c_j + h.c + V \sum_{\langle i,j \rangle} \left(c^\dagger_{i\uparrow} c_{j\downarrow} - c^\dagger_{i\downarrow} c_{j\uparrow} \right) \left(c_{i\uparrow} c_{j\downarrow} - c_{i\downarrow} c_{j\uparrow} \right) \tag{11.112}$$

We study this problem into momentum space. The hamiltonian on the momentum space becomes as

$$H = -\sum_k 2t(\cos k_x + \cos k_y) c^\dagger_k c_k + \frac{g}{V} \sum_k 4 \left(\cos k_x \cos k_y + \cos k_x \cos k_y\right) c^\dagger_{-k'\downarrow} c^\dagger_{k'\uparrow} c^\dagger_{k\uparrow} c_{-k\downarrow} \tag{11.113}$$

Note:-

The second term could be transformed as

$$\sum_{\langle i,j \rangle} \left(c^\dagger_{i\uparrow} c_{j\downarrow} - c^\dagger_{i\downarrow} c_{j\uparrow} \right) \left(c_{i\uparrow} c_{j\downarrow} - c_{i\downarrow} c_{j\uparrow} \right) = \sum_{\langle i,j \rangle} \sum_{k_1, k_2, k'_1, k'_2} \left(c^\dagger_{k'_1\uparrow} c_{k'_2\downarrow} - c^\dagger_{k'_1\downarrow} c^\dagger_{k'_2\uparrow} \right) e^{-\mathrm{i}k'_1(\vec{r}_1 - \vec{r}_j)} e^{-\mathrm{i}(\vec{k_1}' + \vec{k}'_2)\cdot\vec{r}_j}$$
$$\left(c^\dagger_{k_1\uparrow} c_{k_2\downarrow} - c^\dagger_{k_1\downarrow} c^\dagger_{k_2\uparrow} \right) e^{-\mathrm{i}k_1(\vec{r}_1 - \vec{r}_j)} e^{-\mathrm{i}(\vec{k_1} + \vec{k}_2)\cdot\vec{r}_j}$$
$$= 4 \sum_{k',k} (\cos k_x \cos k_y + \cos k_x \cos k_y) c^\dagger_{k'\uparrow} c^\dagger_{-k'\uparrow} c_{k\uparrow} c_{-k\uparrow} \tag{11.114}$$

The hamiltonian could be contained s channel and s wave channel if we make mean field theory for (11.113).

$$H_{MF} = -\sum_k 2t(\cos k_x + \cos k_y) c^\dagger_k c_k + + \frac{g}{V} \sum_k \left(\Delta^*_s (\cos k_x + \cos k_y) c_{k\uparrow} c_{-k\downarrow} + \Delta^*_d (\cos k_x - \cos k_y) c_{k\uparrow} c_{-k\downarrow} + h.c \right)$$
$$+ \frac{2g}{V} \left(\mid \Delta_s \mid^2 + \mid \Delta_d \mid^2 \right) \tag{11.115}$$

The s wave pairing and d wave pairing will appear on mean field hamiltonian. If we consider d wave pairing and no s wave pairing, then the hamiltonian (11.115) will turn into

$$H_{MF} = 2 \sum_k \left(c^\dagger_{k\uparrow} \quad c_{-k\downarrow} \right)^{\mathbf{T}} \begin{pmatrix} -t(\cos k_x + \cos k_y) & \Delta^*_d (\cos k_x - \cos k_y) \\ \Delta_d (\cos k_x - \cos k_y) & t(\cos k_x + \cos k_y) \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} + \frac{2g}{V} \sum_k \mid \Delta_d \mid^2 \tag{11.116}$$

Chapter 12

Kosterilitz-Thouless transition

12.1 Algebraic order in the 2D XY model

The XY model is described by classical hamiltonian

$$H = -J \sum_{\langle i,j \rangle} S_i \cdot S_j \tag{12.1}$$

In the low temperature, the anglae difference is very small . We expand hamiltonian (12.1) into

$$\begin{aligned} H &= -J \sum_{\langle i,j \rangle} S_i \cdot S_j \\ &= -J \sum_{\langle i,j \rangle} \left(1 - \frac{1}{2}(\theta_i - \theta_j)^2\right) \\ &= E_0 + \frac{J}{2} \int d^2r (\nabla\phi(r))^2 \end{aligned} \tag{12.2}$$

12.1.1 Average magnetization

We calculate the average magnetization of hamiltonian (12.2)

$$\begin{aligned} \langle S_x \rangle = \langle \cos\theta(0) \rangle &= \frac{\mathrm{Tr}_{\theta_i}\left(e^{-\beta H}\cos\theta(r)\right)}{\mathrm{Tr}_{\theta_i}\left(e^{-\beta H}\right)} \\ &= \Re\left(\frac{1}{Z}\mathrm{Tr}_{\theta_i}\left(e^{-\beta H} e^{\mathrm{i}\theta(0)}\right)\right) \end{aligned} \tag{12.3}$$

We use the path integral to calculate (12.3)

$$\begin{aligned} \mathrm{Tr}_{\theta_i}\left(e^{-\beta H} e^{\mathrm{i}\cos\theta(r)}\right) &= \prod_k \int \mathcal{D}[\theta_k] \exp\left(-\beta\left(E_0 - \frac{Jk^2a^2}{2}\theta_k\theta_{-k} + \mathrm{i}\theta(k)\right)\right) \\ &= Z \prod_k \exp\left(-\frac{2}{Jk^2a^2\beta}\right) \end{aligned} \tag{12.4}$$

Combing with (12.3),12.4), the momentum has UV and IR cutoff , nmaly $k \in [\frac{\pi}{L}, \frac{\pi}{a}]$.

$$\langle S_x \rangle = \exp\left(-\frac{2k_BT}{J}\int \frac{k^2dk}{(2\pi)^2}\frac{1}{k^2}\right) = \exp\left(-\frac{k_BT}{\pi J}\log\frac{L}{a}\right) = \left(\frac{a}{L}\right)^{\frac{k_BT}{\pi J}} \tag{12.5}$$

12.1.2 Correlation length

The correaltion function is defined as

$$G(r) = \langle e^{\mathrm{i}(\theta(r)-\theta(0))} \rangle = \frac{\mathrm{Tr}_{\theta_i}\left(e^{\mathrm{i}(\theta(r)-\theta(0))}e^{-\beta H}\right)}{\mathrm{Tr}_{\theta_i}\left(e^{-\beta H}\right)} \tag{12.6}$$

By the same way , we use path integral to calculate the correlation function

$$\mathrm{Tr}_{\theta_i}\left(e^{\mathrm{i}(\theta(r)-\theta(0))}e^{-\beta H}\right) = \prod_k \int \mathcal{D}[\theta_k] e^{-\beta E_0} \exp\left(-\frac{Jk^2a^2}{2k_BT}\theta_k\theta_{-k} + \theta_k(e^{\mathrm{i}kr}-1)\right) \tag{12.7}$$

In virtue of complex field θ_k, we split it into imaginary part and real part

$$\theta_k = \alpha_k + \mathrm{i}\beta_k \tag{12.8}$$

Hence, the Eq(12.7) turns into

$$\begin{aligned}\mathrm{Tr}_{\theta_i}\left(e^{\mathrm{i}(\theta(r)-\theta(0))}e^{-\beta H}\right) &= \prod_k e^{-\beta E_0}\int \mathcal{D}[\alpha_k]\exp\left(-\frac{Jk^2a^2}{2k_BT}\alpha_k^2 + \alpha_k(e^{\mathrm{i}kr}-1)\right)\int \mathcal{D}[\beta_k]\exp\left(-\frac{Jk^2a^2}{2k_BT} + \mathrm{i}\beta_k(e^{\mathrm{i}kr}-1)\right) \\ &= Z\exp\left(-\sum_k \frac{k_BT(e^{\mathrm{i}kr}-1)}{aJk^2a^2}\right) \end{aligned}\tag{12.9}$$

We consider summation on the bracket

$$\sum_k \frac{k_BT(e^{\mathrm{i}kr}-1)}{aJk^2a^2} \to \frac{k_BT}{4\pi^2 J}dk\int_0^{2\pi}\int_0^{\frac{\pi}{a}} \frac{1-\cos(kr\cos\theta)}{k}dk = \frac{k_BT}{2\pi J}\int_0^{\frac{\pi}{2}}\frac{1-J_0(kr)}{k}dk \tag{12.10}$$

On the ultraviolet region , the Bessel function $J_0(kr)$ tends to zero. The (12.10) could be approximated into [1]

$$\frac{k_BT}{2\pi J}\log\frac{r\pi}{a} \tag{12.11}$$

The behaviour of correlation function of (12.6) admits power law decay

$$G(r) \sim \left(\frac{r}{a}\right)^{-\eta(T)} \qquad \eta(T) = \frac{k_BT}{2\pi J} \tag{12.12}$$

In the low temperature , the correlation function admits long range behaviour. At the high temperature , the correlation length have exponential decay behaviour . The ferromagnetic order would be destroyed into disorder phase. We can make hypothesis that the system undergoes phase transition.

12.1.3 Vortices and entropy

Vortices are topological defects of filed $\theta(r)$ satisfy Laplace equation $\nabla^2\theta(r) = 0$. The nontrial solution of two dimensional Laplace equation is vortex solution.

$$\oint_{\mathcal{C}} \nabla\theta(r)\cdot d\ell = 2\pi n \tag{12.13}$$

[1] Firstly, you should make integral (12.10) into dimensionless integral

where n is the inding number . Can proliferation of vortex destroy ferromagnetic order? Now we will give the argument bases on free enenrgy . Lst's estimate the single voretex energy.

$$\varepsilon_0 = \frac{J}{2}\int d^3r\nabla\theta(r)\cdot dl = \frac{J}{2}\int_0^{2\pi} d\theta \int_a^L \frac{n^2}{r}dr = \pi J n^2 \log\frac{L}{a} \tag{12.14}$$

We can put single vortex into system with $\left(\frac{L}{a}\right)^2$ ways . The entropy could be derived with Boltzmann entropy

$$S = 2k_B \log\frac{L}{a} \tag{12.15}$$

Hence, the free energy to creation of single isolated vortex is

$$\Delta F = \Delta E - T\Delta S = \left(\pi J n^2 - 2k_B T\right)\log\frac{L}{a} \tag{12.16}$$

- $T < \frac{\pi J}{2}$. The creation of single vortex isn't favorable . The system tend to form vortex -anti-vortex bounded state to keep in netral.
- $T > \frac{\pi J}{2}$. The isolated vortex tend to proliferate.

12.2 Columb gas analogy

To proceed renoramlization group analysis, we should write down the partittion function . It's know to us that gradient field has no curl . We decompose the θ into regular part and singular part.

$$\nabla\theta = \vec{u}_{\text{reg}} + \vec{u}_s \tag{12.17}$$

The regular part is free from curl . However , the singular part will contribute votrtex integral . For example, we can take $\theta = \frac{y}{x}$, then this field correpsonds to vortex with winding number one. The singular field satisifes to

$$\oint_{\mathcal{C}} \nabla\theta(r)\cdot d\ell = \int d^2r\hat{z}\cdot\nabla\times(\vec{u}_s) = 2\pi n \tag{12.18}$$

Note:-

We can make ansat that

$$\nabla\times\vec{u}_s = 2\pi\sum_i n_i\delta(r-r_i)\hat{z} \tag{12.19}$$

We can set $\vec{u}_s = \nabla\times\psi\hat{z}$,

$$\nabla\times\vec{u}_s = \nabla\times(\hat{z}\psi) = \nabla\psi\times\hat{z} \tag{12.20}$$

We substitute it into (12.19)

$$\nabla^2\psi = -2\pi\sum_i \delta(r-r_i) \tag{12.21}$$

The solution ψ could be derived as

$$\psi = -\sum n_i \log \mid r - r_i \mid \tag{12.22}$$

In the Columb gas language , the physical meaning of ψ is just scalar potential genrated by charge density $\sum_i n_i \delta(r - r_i)$.

The continuum halmitoian could be written into regular field ϕ and singular field ψ

$$H = -\frac{J}{2}\int d^2r \left(\nabla\phi + \nabla\times(\psi\hat{z})\right)^2 = -\frac{J}{2}\int d^2r \left((\nabla\phi)^2 + 2\nabla\phi\cdot\nabla\times(\psi\hat{z}) + (\nabla\times(\psi\hat{z}))^2\right) \tag{12.23}$$

The first term on the (12.23) is just spin wave part, which could integrate out by gaussian integral. The second term could be written into total partial , which is vanishing on the boundary.

$$\begin{aligned}\int d^2r\nabla\phi\cdot\nabla\times(\psi\hat{z}) &= \int d^2r\nabla\phi\cdot(\nabla\psi\times\hat{z}) = \int d^2r\hat{z}\cdot(\nabla\phi\times\nabla\psi)\\ &= \int d^2r\varepsilon_{ij}\partial_i\phi\partial_j\psi\\ &= \int d^2r\varepsilon_{ij}\partial_i(\phi\partial_j\psi) - \phi\varepsilon_{ij}\partial_i\partial_j\psi\end{aligned} \tag{12.24}$$

Hence, the hamiltonian will simplified into

$$H = -\frac{J}{2}\int d^2r \left(\nabla^2\phi + \nabla^2\psi\right) \tag{12.25}$$

We make partial integral for the singular part

$$\begin{aligned}\int d^r\nabla^2\psi = \int d^2r\nabla\cdot(\psi\nabla\psi) - \phi\nabla^2 a\psi = -2\pi\sum_{i,j} n_i n_j \log \mid n_i - n_j \mid\\ = -H_{\text{core}} - 2\pi\sum_{i<j} n_i n_j \log \mid n_i - n_j \mid\end{aligned} \tag{12.26}$$

The partition function could be obtained as

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}[\phi]e^{-\frac{J}{2}\int d^2r(\nabla\phi(r))^2}\sum_{N=0}^{\infty}\frac{1}{N!^2}\int\int\prod_{i=1}^{2N}\frac{dp_i^2dx_i^2}{h^{2N}}e^{2\pi J\sum\limits_{i<j}n_in_j\log|r_i-r_j|}\\ &= Z_{\text{spin}}\sum_{N=0}^{\infty}\frac{y_0^{2N}}{N!^2}\int\prod_{i=1}^{2N}\frac{dx_i^2}{a^2}\exp\left(2\pi J\sum_{i<j}n_in_j\log\mid r_i - r_j\mid\right)\end{aligned} \tag{12.27}$$

where y_0 is the dimensionless characteristic quantity $y_0 = \dfrac{\sqrt{2\pi m k_B T a^2}}{h}$

12.2.1 RG flow equation

We use the perturbative treatement to this sytem . We consider two chargeds at position s, s'. Our effective hamiltonian is the average of external charge

$$e^{H_{eff}(r-r')} = \langle e^{-2J\pi\log|r-r'|}\rangle \tag{12.28}$$

Let;s use the partition function (12.27) to write down the effective hamiltonian

$$\langle e^{-2K\pi \log|r-r'|}\rangle = \frac{\sum_{N=0}^{\infty} \frac{y_0^{2N}}{N!^2} \int \prod_{i=1}^{2N} \frac{dx_i^2}{a^2} \exp\left(-2J\pi \log \mid r-r' \mid +2\pi J \sum_{i<j} n_i n_j \log \mid r_i - r_j \mid\right)}{\sum_{N=0}^{\infty} \frac{y_0^{2N}}{N!^2} \int \prod_{i=1}^{2N} \frac{dx_i^2}{a^2} \exp\left(2\pi J \sum_{i<j} n_i n_j \log \mid r_i - r_j \mid\right)}$$

$$= \frac{\exp(-2J\pi \log \mid r-r' \mid) \left(1 + y_0^2 \int dsds' e^{-2J\pi \log|s-s'|+2J\pi D(r,r';s,s')}\right)}{\left(1 + y_0^2 \int dsds' e^{-2J\pi \log|s-s'|+2J\pi D(r,r';s,s')}\right)}$$

$$= \exp(-2J\pi \log \mid r-r' \mid) \left(1 + y_0^2 \int dsds' e^{-2J\pi \log s-s'|} \left(e^{3J\pi D(r,r';s,s')} - 1\right)\right) \tag{12.29}$$

where the $D(r,r';s,s')$ is the interaction between external charge and the single dipole.

$$D(r,r';s,s') = \log \mid s-r \mid - \log \mid s-r' \mid + \log \mid s'-r' \mid - \log \mid s'-r \mid \tag{12.30}$$

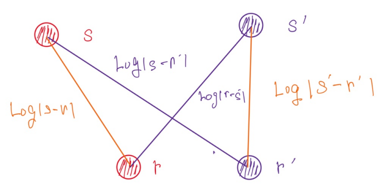

Figure 12.1: The interaction between the external charge and dipole

Note:-

n this short note, we will introduce central coordinate $X = \frac{s+s'}{2}$ and relative coordinate $xs - S6\prime$ to expand the interaction term $D(r,r';s,s')$.

$$\begin{cases} \log \mid s-r \mid = \log \mid X + \frac{x}{2} - r \mid = \log \mid X-r \mid + \frac{x}{2} \cdot \nabla_X \log \mid X-r \mid + \frac{1}{2}\left(\frac{x}{2} \cdot \nabla_X\right)^2 \log \mid X-r \mid + \cdots \\ \log \mid s'-r \mid = \log \mid X - \frac{x}{2} - r \mid = \log \mid X-r \mid - \frac{x}{2} \cdot \nabla_X \log \mid X-r \mid + \frac{1}{2}\left(\frac{x}{2} \cdot \nabla_X\right)^2 \log \mid X-r \mid + \cdots \\ \log \mid s-r' \mid = \log \mid X + \frac{x}{2} - r' \mid = \log \mid X-r' \mid + \frac{x}{2} \cdot \nabla_X \log \mid X-r' \mid + \frac{1}{2}\left(\frac{x}{2} \cdot \nabla_X\right)^2 \log \mid X-r' \mid + \cdots \\ \log \mid s'-r' \mid = \log \mid X - \frac{x}{2} - r' \mid = \log \mid X-r' \mid - \frac{x}{2} \cdot \nabla_X \log \mid X-r' \mid + \frac{1}{2}\left(\frac{x}{2} \cdot \nabla_X\right)^2 \log \mid X-r' \mid + \cdots \end{cases} \tag{12.31}$$

We substitute the (12.31) into (12.30)

$$D(r,r';s,s') = x \cdot \nabla_X \left(\log \mid X-r \mid - \log \mid X'-r' \mid\right) \tag{12.32}$$

We expand the $e^{2J\pi D(r,r';s,s')}$ to second order

$$e^{2J\pi D(r,r';s,s')} = 1 + 2J\pi x \cdot \nabla_X \left(\log \mid X-r \mid - \log \mid X-r' \mid\right) + \frac{1}{2}\left(2J\pi x \cdot \nabla_X \left(\log \mid X-r \mid - \log \mid X-r' \mid\right)\right)^2 + \cdots \tag{12.33}$$

The integral measurement could be expressed by varibales X, x as $\int d^2s d^2s' = \int d^2x d^2X$. We subtitute the (12.33) into (12.29)

$$\int dsds' e^{-2J\pi\log|s-s'|+2J\pi D(r,r';s,s')} = \int d^2xd^2X(x)^{-2J\pi}\left(1+2J\pi x\cdot\nabla_X\log\left|\frac{r-X}{r'-X}\right|+2J^2\pi^2\left(x\cdot\nabla_X\log\left|\frac{r-X}{r'-X}\right|\right)^2+c\dots\right) \tag{12.34}$$

Note:-

Let's analyze the trems on the (12.34)

$$\int d^2xd^2Xe^{-2J\pi\log x}x\cdot\nabla_X\log\left|\frac{r-X}{r'-x'}\right| = \int d^2xe^{-2J\pi\log x}x\cdot\int d^2X\nabla_X\log\left|\frac{r-X}{r'-x'}\right| = 0 \tag{12.35}$$

$$\begin{aligned}&\int d^2xd^2X\left(x\cdot\nabla_X\log\left|\frac{r-X}{r'-X}\right|\right)^2 = \int_0^\infty e^{-2J\pi\log x}x^3dx\int d^2X\\&\int_0^{2\pi}d\theta\left(\cos\theta\nabla_{X_1}\log\left|\frac{r-X}{r'-X}\right|+\sin\theta\nabla_{X_2}\log\left|\frac{r-X}{r'-X}\right|\right)^2\\&=\pi\int d^2xd^2X\left(x\cdot\nabla_X\log\left|\frac{r-X}{r'-X}\right|\right)^2=\pi\int_0^\infty e^{-2J\pi\log x}x^3dx\int d^2X\left(\nabla_X\log\left|\frac{r-X}{r'-X}\right|\right)^2\end{aligned} \tag{12.36}$$

We calculate the last part integral

$$\begin{aligned}\int d^2X\left(\nabla_X\log\left|\frac{r-X}{r'-X}\right|\right)^2 &= \left(\nabla_X\log\left|\frac{r-X}{r'-X}\right|\right)_{X\to\infty} - \int d^2X\log\left|\frac{r-X}{r'-X}\right|\nabla_X^2\log\left|\frac{r-X}{r'-X}\right|\\&= -\int d^2X\log\left|\frac{r-X}{r'-X}\right|(2\pi\delta(X-r)-2\pi\delta(X-r')\mid)\\&= -2\pi\int d^2X(\log\mid r-r\mid-\log\mid r'-r\mid-\log\mid r-r'\mid+\log\mid r'-r'\mid)\\&= 4\pi\log\mid r-r'\mid\end{aligned} \tag{12.37}$$

Hence, the integral (12.36) turns into

$$\begin{aligned}&\exp(-2J\pi\log\mid r-r'\mid)\left(1+y_0^2\int dsds'e^{-2J\pi\log|s-s'|+2J\pi D(r,r';s,s')}\right)=exp(-2J\pi\log\mid r-r'\mid)\\&\left(1+8J^2\pi^4y_0^2\log\mid r-r'\mid\int_1^\infty x^{3-2J\pi}dx+\mathcal{O}(y_0^4)\right)\end{aligned} \tag{12.38}$$

We use the lattice cut-off to revise the divergent integral

We could write down the K_{eff} from the (12.38)

$$K_{\text{eff}} = K - 4\pi^3K^2y_0^2\int_1^\infty dxx^{3-2\pi K} \tag{12.39}$$

To be convenient , we use the K^{-1}

$$K_{\text{eff}}^{-1} = \frac{1}{K}\frac{1}{1-4\pi^2Ky_0^2\int_1^\infty dxx^{3-2\pi K}} \simeq K^{-1}+4\pi^2y_0^2\int_1^\infty dxx^{3-2\pi K} \tag{12.40}$$

- Scale $x : 1 \mapsto b$

$$K_{\text{eff}}^{-1} = \left(K^{-1} + 4\pi^2 y_0^2 \int_1^b dx x^{3-2\pi K}\right) + 4\pi^2 y_0^2 \int_b^\infty dx x^{3-2\pi K} \tag{12.41}$$

- Rescale the $x : x \mapsto x/b$

$$\tilde{K}^{-1} = K^{-1} + 4\pi^3 \tilde{y}_0^2 \int_{1/b}^1 dx x^{3-2\pi K} \qquad \tilde{y}_0 = b^{2-\pi K} y_0 \tag{12.42}$$

We choose the infinitesimal renormalization parameter $b = e^l$

$$\tilde{y}_0 = y_0 \left(1 + (2-\pi K)dl + \mathcal{O}(dl^2)\right) \implies \frac{dy_0}{dl} = (2-\pi K)y_0 \tag{12.43}$$

$$\begin{aligned} \tilde{K}^{-1} &= K^{-1} + 4\pi^3 \tilde{y}_0^2 \int_{1/b}^1 dx x^{3-2\pi K} = K^{-1} + 4\pi^3 \tilde{y}_0^2 \left.\frac{x^{4-\pi K}}{4-\pi K}\right|_{\frac{1}{b}}^1 \\ &= K^{-1} + 4\pi^3 \tilde{y}_0^2 \frac{1-e^{-(4-\pi K)l}}{4-\pi K} \\ &= K^{-1} + 4\pi^3 l \tilde{y}_0^2 \implies \frac{d\tilde{K}^{-1}}{dl} = 4\pi^3 \tilde{y_0}^2 \end{aligned} \tag{12.44}$$

Now, we have derive the *RG* equation. To simplify problem, we focus on the behaviour of fixed point. The RG equation (12.43) tells us that $K = \frac{2}{\pi}$ is fixed point, which gives the critical temperature

$$T = \frac{\pi}{2} J/k_B \tag{12.45}$$

This result is meeted with vortice argument (12.16). We introduce new varibales $t = \frac{\pi}{2} - K^{-1}$ [2] to study the behviour near fixed point.

$$\begin{cases} \dfrac{dt}{dl} = 4\pi^3 y^2 \\ \dfrac{dy}{dl} = \dfrac{4}{\pi} t y \end{cases} \tag{12.46}$$

The Eq(12.46) tells us conserve quantity

$$\frac{d}{dl}\left(t^2 - \pi^4 y^2\right) = 0 \tag{12.47}$$

[2] The variable t is small quantity, $2 - \pi K = \frac{2t}{2+\pi/2} \simeq \frac{4}{\pi} t + \mathcal{O}(t^2)$

Chapter 13

Appendix

13.1 Bogliubov transformation

We consider Bogliubov transformation of Bosons. Let's start from hamiltonian below

$$
\begin{aligned}
H &= \varepsilon_k(b_k^\dagger b_k + b_k b_k^\dagger) + g(b_k b_k + b_k^\dagger b_k^\dagger) = (b_k^\dagger \quad b_k)\begin{pmatrix}\varepsilon_k & g \\ g & \varepsilon_k\end{pmatrix}\begin{pmatrix} b_k \\ b_k^\dagger\end{pmatrix} \\
&= (b_k^\dagger \quad b_k)\begin{pmatrix}\cosh\theta_k & \sinh\theta_k \\ \sinh\theta_k & \cosh\theta_k\end{pmatrix}\begin{pmatrix}\varepsilon_k & g \\ g & \varepsilon_k\end{pmatrix}\begin{pmatrix}\cosh\theta_k & \sinh\theta_k \\ \sinh\theta_k & \cosh\theta_k\end{pmatrix}\begin{pmatrix} b_k \\ b_k^\dagger\end{pmatrix} \\
&= (b_k^\dagger \quad b_k)(\cosh\theta_k + g\sinh\theta_k\sigma_x)(\varepsilon_k + g\sigma_x)(\cosh\theta_k + g\sinh\theta_k\sigma_x)\begin{pmatrix} b_k \\ b_k^\dagger\end{pmatrix} \\
&= (\varepsilon_k\cosh 2\theta_k + g\sinh 2\theta_k) + (g\cosh 2\theta_k + \varepsilon_k\sinh 2\theta_k)\sigma_x
\end{aligned}
\tag{13.1}
$$

We can see that $\tanh 2\theta_k = -\dfrac{g}{\varepsilon_k}$. The hamiltonian can be expressed as

$$
H = \sqrt{\varepsilon_k^2 - g^2}(B_k^\dagger B_k + B_k B_k^\dagger) \tag{13.2}
$$

where

$$
\begin{pmatrix} B_k^\dagger \\ B_k\end{pmatrix} = \begin{pmatrix}\cosh\theta_k & \sinh\theta_k \\ \sinh\theta_k & \cosh\theta_k\end{pmatrix}\begin{pmatrix} B_k \\ B_k^\dagger\end{pmatrix} \tag{13.3}
$$

Bibliography

[1] Anthony J. Leggett. A theoretical description of the new phases of liquid ^{3}He. *Rev. Mod. Phys.*, 47:331–414, Apr 1975.