a) $f_1 \preceq f_2 \preceq f_4 \preceq f_3$

$f_1(n) = n^{0,999\,999} \ln n = n^{1-10^{-6}} \cdot \ln n$

$f_2(n) = 10^7 n$

$f_3(n) = (1+10^{-6})^n$

$f_4(n) = n^2$

① $\lim\limits_{n\to\infty} \frac{n^{1-10^{-6}} \ln n}{10^7 n} = \lim\limits_{n\to\infty} \frac{\cancel{n} \cdot n^{-10^{-6}} \ln n}{10^7 \cancel{n}} = \lim\limits_{n\to\infty} \frac{1}{10^7} \cdot \frac{\ln n}{n^{10^{-6}}}$

$\overset{H}{=} \lim\limits_{n\to\infty} \frac{1}{10^7} \cdot \frac{1}{n \cdot 10^{-6} \cdot n^{10^{-6}-1}} = \lim\limits_{n\to\infty} \frac{1}{10^{\cancel{7}}} \frac{\cancel{10^6} \cdot n^{1-10^{-6}}}{n}$

$= \lim\limits_{n\to\infty} \frac{n^{-10^{-6}}}{10} = \lim\limits_{n\to\infty} \frac{1}{10 \cdot n^{10^{-6}}} = 0 < +\infty$

wolno rosnące funkcja pierwiastkowe

② $\lim\limits_{n\to\infty} \frac{10^7 \cancel{n}}{n^{\cancel{2}}} = 0$

③ $\lim\limits_{n\to\infty} \frac{n^2}{(1+10^{-6})^n} \overset{H}{=} \lim\limits_{n\to\infty} \frac{2 \cdot n}{\ln(1+10^{-6}) \cdot (1+10^{-6})^n}$

$\overset{H}{=} \lim\limits_{n\to\infty} \frac{2}{(\ln(1+10^{-6}))^2 \cdot \underbrace{(1+10^{-6})^n}_{\to\infty}} = 0$

b) $f_1(n) = 2^{100n}$

$f_2(n) = \binom{n}{2}$

$f_3(n) = n\sqrt{n}$

$$f_3(n) \overset{①}{\preceq} f_2(n) \overset{②}{\preceq} f_1(n)$$

① $$\lim_{n\to\infty} \left| \frac{\binom{n}{2}}{2^{100n}} \right| = \lim_{n\to\infty} \frac{n(n-1)}{2\cdot 2^{100n}} = \lim_{n\to\infty} \frac{n^2-n}{2^{100n+1}}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{2n-1}{2^{100n+1}\cdot \ln 2} \overset{H}{=} \lim_{n\to\infty} \frac{2}{\underbrace{2^{100n+1}}_{\to\infty}\cdot(\ln 2)^2} = 0$$

② $$\lim_{n\to\infty} \left| \frac{n^{3/2}}{2^{100n}} \right| \overset{H}{=} \lim_{n\to\infty} \frac{\frac{3}{2}n^{\frac{1}{2}}}{2^{100n}\cdot \ln 2} \overset{H}{=} \lim_{n\to\infty} \frac{\frac{3}{4}n^{-\frac{1}{2}}}{2^{100n}\cdot(\ln 2)^2}$$

$$= \lim_{n\to\infty} \frac{3}{\underbrace{2^{102n}\cdot(\ln 2)^2\cdot\sqrt{n}}_{\to\infty}} = 0$$

c) $f_1(n) = n^{\sqrt{n}}$

$f_2(n) = 2^n$

$f_3(n) = n^{10} \cdot 2^{\frac{n}{2}} = \frac{n^{10} \cdot 2^n}{2^2}$

$f_4(n) = \sum_{i=1}^{n} (i+1)$

$f_4(n)$ - suma ciągu arytmetycznego

$$f_4(n) = \frac{2+n+1}{2} \cdot n = \frac{(3+n)}{2} \cdot n = \frac{n^2+3n}{2}$$

$$f_4(n) \overset{1}{\preceq} f_1(n) \overset{2}{\preceq} f_3(n) \overset{3}{\preceq} f_2(n)$$

1) $$\lim_{n\to\infty} \left| \frac{n^2+3n}{2n^{\sqrt{n}}} \right| = \lim_{n\to\infty} \left| \frac{(n^2+3n) \cdot n^{-\sqrt{n}}}{2} \right| = \lim_{n\to\infty} \left| \frac{n^{2-\sqrt{n}} + 3n^{1-\sqrt{n}}}{2} \right|$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{n^2}{n^{\sqrt{n}}} + \frac{3}{2} \lim_{n\to\infty} \frac{n}{n^{\sqrt{n}}} = 0 + 0 = 0$$

2) $$\lim_{n\to\infty} \left| \frac{n^{\sqrt{n}}}{n^{10} \cdot 2^{\frac{n}{2}}} \right|$$ Równoważnie możemy sprawdzić $$\lim_{n\to\infty} \left| \frac{n^{\sqrt{n}}}{2^n} \right| = \lim_{n\to\infty} \left(\frac{n}{2^{\sqrt{n}}} \right)^{\sqrt{n}}$$

$$\lim_{n\to\infty} \left| \frac{n}{2^{\sqrt{n}}} \right| \overset{H}{=} \lim_{n\to\infty} \left| \frac{1}{2^{\sqrt{n}} \cdot \ln 2 \cdot \frac{1}{2\sqrt{n}}} \right| = \lim_{n\to\infty} \frac{\sqrt{n}}{2^{\sqrt{n}-1} \ln 2} \overset{H}{=}$$

$$\lim_{n\to\infty} \left| \frac{\cancel{\frac{1}{2\sqrt{n}}}}{2^{\sqrt{n}-1} \cdot (\ln 2)^2 \cdot \cancel{\frac{1}{2\sqrt{n}}}} \right| = \lim_{n\to\infty} \frac{1}{(\ln 2)^2 \cdot 2^{\sqrt{n}-1}} = 0$$

3) $$\lim_{n\to\infty} \left| \frac{n^{10} \cdot 2^{\frac{n}{2}}}{2^n} \right| = \lim_{n\to\infty} \left| n^{10} \cdot 2^{\frac{n}{2}-n} \right| = \lim_{n\to\infty} \left| \frac{n^{10}}{2^{\frac{n}{2}}} \right| = 0$$

rośnie wolniej