Homework Problem Set #3

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January 2020

Let R be a ring with identity $1 \neq 0$, let F be a field, and let x be an indeterminate over F.

9.1.7 Prove that a polynomial ring in more than one variable over R is not a Principal Ideal Domain.

9.2.1 Let $f(x) \in F[x]$ be a polynomial of degree $n \geq 1$ and let bars denote passage to the quotient $F[x]/f(x)$. Prove that for each $\overline{g(x)}$ there is a unique polynomial $g_0(x)$ of degree $\leq n-1$ such that $\overline{g(x)} = \overline{g_0(x)}$ (equivalently, the elements $\overline{1}, \overline{x}, ..., \overline{x^{n-1}}$ are a *basis* of the vector space $F[x]/((f(x))$ in particular, the dimension of this space is n). [Use the Division Algorithm.].

9.2.2 Let F be a finite field of order q and let $f(x)$ be a polynomial in $F[x]$ of degree $n \geq 1$. Prove that $F[x]/(f(x))$ has q^n elements. [Use the preceding exercise.]

9.2.3 Let $f(x)$ be a polynomial in $F[x]$. Prove that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible. [Use Proposition 7, Section 8.2.]

9.3.3 Let F be a field. Prove that the set R of polynomials in $F[x]$ whose coefficient of x is equal to 0 is a subring of $F[x]$ and that R is not a U.F.D.

[Show that $x^6 = (x^2)^3 = (x^3)^2$ gives two distinct factorizations of x^6 into irreducibles.]