Homework Problem Set #3

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Let R be a ring with identity $1 \neq 0$, let F be a field and let x be an indeterminate over F.

9.1.2 Let $p(x, y, z) = 2x^2y - 3xy^3z + 4y^2z^5$ and $q(x, y, z) = 7x^2 + 5x^2y^3z^4 - 3x^2z^3$ be polynomials in $\mathbb{Z}[x, y, z]$.

(a) Write each of p and q as a polynomial in x with coefficients in $\mathbb{Z}[x, y]$.
(b) Find the degree of each of p and q.
(c) Find the degree of p and q in each of the three variables x, y and z.
(d) Compute pq and find the degree of pq in each of the three variables x, y and z.
(e) Write pq as a polynomial in the variable z with coefficients in $\mathbb{Z}[x, y]$.

2. Repeat the preceding exercise under the assumption that the coefficients of p and q are in $\mathbb{Z}/3\mathbb{Z}$.

9.1.6 Prove that (x, y) is not a principal ideal in $\mathbb{Q}[x, y]$.

9.1.7 Prove that a polynomial ring in more than one variable over R is not a Principal Ideal Domain.

9.2.1 Let $f(x) \in F[x]$ be a polynomial of degree $n \geq 1$ and let bars denote passage to the quotient $F[x]/f(x)$. Prove that for each $\overline{g(x)}$ there is a unique polynomial $g_0(x)$ of degree $\leq n-1$ such that $\overline{g(x)} = \overline{g_0(x)}$ (equivalently, the elements $\overline{1}, \overline{x}, ..., \overline{x^{n-1}}$ are a *basis* of the vector space $F[x]/((f(x))$ in particular, the dimension of this space is n). [Use the Division Algorithm.].

9.2.2 Let F be a finite field of order q and let $f(x)$ be a polynomial in $F[x]$ of degree $n \geq 1$. Prove that $F[x]/(f(x))$ has q^n elements. [Use the preceding exercise.]

9.2.3 Let $f(x)$ be a polynomial in $F[x]$. Prove that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible. [Use Proposition 7, Section 8.2.]

9.2.4 Let F be a finite field. Prove that $F[x]$ contains infinitely many primes. (Note that over an infinite field the polynomials of degree 1 are an infinite set of primes in the ring of polynomials).

9.2.6 Describe (briefly) the ring structure of the following rings:

$$\textbf{(a)}\ \mathbb{Z}[x]/(2) \quad \textbf{(b)}\ \mathbb{Z}[x]/(x) \quad \textbf{(c)}\ \mathbb{Z}[x]/(x^2)$$

$$\textbf{(d)}\ \mathbb{Z}[x,y]/(x^2,y^2,2)$$

Show that $\alpha^2 = 0$ or 1 for every α in the last ring and determine those elements with $\alpha^2 = 0$. Determine the characteristics of each of these rings (cf. Exercise 26, Section 7.3).

9.3.3 Let F be a field. Prove that the set R of polynomials in $F[x]$ whose coefficient of x is equal to 0 is a subring of $F[x]$ and that R is not a U.F.D.

[Show that $x^6 = (x^2)^3 = (x^3)^2$ gives two distinct factorizations of x^6 into irreducibles.]