

# Homework Problem Set #1

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Let  $R$  be a ring with identity  $1 \neq 0$ .

**7.3.1** Prove that the rings  $2\mathbb{Z}$  and  $3\mathbb{Z}$  are not isomorphic

**7.3.10** Decide which of the following are ideals of the ring  $\mathbb{Z}[x]$ :

- (a) the set of all polynomials whose constant term is a multiple of 3.
- (b) the set of all polynomials whose coefficient of  $x^2$  is a multiple of 3.
- (c) the set of all polynomials whose constant term, coefficient of  $x$  and coefficient of  $x^2$  are zero.
- (d)  $\mathbb{Z}[x]$  (i.e., the polynomials in which only even powers of  $x$  appear).
- (e) the set of polynomials whose coefficients sum to zero.
- (f) the set of polynomials  $p(x)$  such that  $p'(0) = 0$ , where  $p'(x)$  is the usual first derivative of  $p(x)$  with respect to  $x$ .

**7.3.19** Prove that if  $I_1 \subseteq I_2 \subseteq \dots$  are ideals of  $R$  then  $\cup_{n=1}^{\infty} I_n$  is an ideal of  $R$ .

**7.3.26** The *characteristic* of a ring  $R$  is the smallest positive integer  $n$  such that

$$\underbrace{1 + 1 + \dots + 1 = 0}_{n \text{ times}}$$

in  $R$ ; if no such integer exists the characteristic of  $R$  is said to be 0. For example,  $\mathbb{Z}/n\mathbb{Z}$  is a ring of characteristic  $n$  for each positive integer  $n$  and  $\mathbb{Z}$  is a ring of characteristic 0.

(a) Prove that the map  $\mathbb{Z} \rightarrow \mathbb{R}$  defined by

$$k \mapsto \begin{cases} 1 + 1 + \dots + 1 \text{ (} k \text{ times)} & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ -1 - 1 - \dots - 1 \text{ (} -k \text{ times)} & \text{if } k < 0 \end{cases}$$

is a ring homomorphism whose kernel is  $n\mathbb{Z}$ , where  $n$  is the characteristic of  $R$  (this explains the use of the terminology "characteristic 0" instead of the archaic phrase "characteristic  $\infty$ " for rings in which no sum of 1's is zero).

(b) Determine the characteristics of the rings  $\mathbb{Q}$ ,  $\mathbb{Z}[x]$ ,  $\mathbb{Z}/n\mathbb{Z}[x]$ .

(c) Prove that if  $p$  is a prime and if  $R$  is a commutative ring of characteristic  $p$ , then

$$(a + b)^p = a^p + b^p \text{ for all } a, b \in R.$$



**7.3.28** Prove that an integral domain has characteristic  $p$ , where  $p$  is either a prime or 0 (cf. Exercise 26).

**7.4.10** Assume  $R$  is commutative. Prove that if  $P$  is a prime ideal of  $R$ , and  $P$  contains no zero divisors then  $R$  is an integral domain.

### Problem A

- (i) Let  $R$  be an integral domain. As you conjectured in class, prove that the units in  $R[x]$  are precisely the constant polynomials  $p(x) = u$  where  $u$  is a unit in  $R$ .
- (ii) On the other hand, show that  $p(x) = 1 + 2x$  is a unit in  $R[x]$ , where  $R = \mathbb{Z}/4\mathbb{Z}$ .