

# Homework Problem Set #8

Harvey Mudd College

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Let  $R$  be a ring with identity  $1 \neq 0$ , let  $F$  be a field, and let  $x$  be an indeterminate over  $F$ .

## 14.1.1

(a) Show that if the field  $K$  is generated over  $F$  by the elements  $\alpha_1, \dots, \alpha_n$  then an automorphism  $\sigma$  of  $K$  fixing  $F$  is uniquely determined by  $\sigma(\alpha_1), \dots, \sigma(\alpha_n)$ . In particular show that an automorphism fixes  $K$  if and only if it fixes a set of generators for  $K$ .

(b) Let  $G \leq \text{Gal}(K/F)$  be a subgroup of the Galois group of the extension  $K/F$  and suppose  $\sigma_1, \dots, \sigma_k$  are generators for  $G$ . Show that the subfield  $E/F$  is fixed by  $G$  if and only if it is fixed by the generators  $\sigma_1, \dots, \sigma_k$ .

**14.1.4** Prove that  $\mathbf{Q}(\sqrt{2})$  and  $\mathbf{Q}(\sqrt{3})$  are not isomorphic.

**14.2.1** Determine the minimal polynomial over  $\mathbf{Q}$  for the element  $\sqrt{2} + \sqrt{5}$ .

**14.2.11** Suppose  $f(x) \in \mathbf{Z}[x]$  is an irreducible quartic whose splitting field has Galois group  $\mathbf{S}_4$  over  $\mathbf{Q}$  (there are many such quartics, cf. Section 6). Let  $\theta$  be a root of  $f(x)$  and set  $K = \mathbf{Q}(\theta)$ . Prove that  $K$  is an extension of  $\mathbf{Q}$  of degree 4 which has no proper subfields. Are there any Galois extensions of  $\mathbf{Q}$  of degree 4 with no proper subfields?