Homework Problem Set #8

Harvey Mudd College

January 2020

Let R be a ring with identity $1 \neq 0$, let F be a field, and let x be an indeterminate over F.

14.1.1

(a) Show that if the field K is generated over F by the elements $\alpha_1, ..., \alpha_n$ then an automorphism σ of K fixing F is uniquely determined by $\sigma(\alpha_1), ..., \sigma(\alpha_n)$. In particular show that an automorphism fixes K if and only if it fixes a set of generators for K.

(b) Let $G \leq Gal(K/F)$ be a subgroup of the Galois group of the extension K/F and suppose $\sigma_1, ..., \sigma_k$ are generators for G. Show that the subfield E/F is fixed by G if and only if it is fixed by the generators $\sigma_1, ..., \sigma_k$.

14.1.4 Prove that $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{3})$ are not isomorphic.

14.2.1 Determine the minimal polynomial over $\mathbf{Q}$ for the element $\sqrt{2} + \sqrt{5}$.

14.2.11 Suppose $f(x) \in \mathbf{Z}[x]$ is an irreducible quartic whose splitting field has Galois group $\mathbf{S_4}$ over $\mathbf{Q}$ (there are many such quartics, cf. Section 6). Let θ be a root of $f(x)$ and set $K = \mathbf{Q}(\theta)$. Prove that K is an extension of $\mathbf{Q}$ of degree 4 which has no proper subfields. Are there any Galois extensions of $\mathbf{Q}$ of degree 4 with no proper subfields?