Homework 1

ECON 7023: Econometrics II
Maghfira Ramadhani
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Part 1

Problem 3.5

Let $\{y_i : i = 1, 2, 3, \ldots\}$ be an independent, identically distributed sequence with $\mathrm{E}(y_i^2) < \infty$. Let $\mu = \mathrm{E}(y_i)$ and $\sigma^2 = \mathrm{Var}(y_i)$.

a. Let $\bar{y}_N$ denote the sample average based on a sample size of N. Find $\mathrm{Var}[\sqrt{N}(\bar{y}_N - \mu)]$.
Answer:
By definition of sample mean, we have $\mathrm{Var}(\bar{y}_N) = \mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^N y_i\right)$. Recall property of variance that $\mathrm{Var}(ax) = a^2\mathrm{Var}(x)$ for random variable x and constant a. Since N is constant, thus we have $\mathrm{Var}(\bar{y}_N) = \left(\frac{1}{N}\right)^2 \mathrm{Var}\left(\sum_{i=1}^N y_i\right)$. It is given that $y_i \overset{iid}{\sim} (\mu, \sigma^2)$ for $i \in \{1, 2, .., n\}$. Therefore $\mathrm{Var}(y_i) = \sigma^2$ and is constant, and since y_i's are independent we have $\mathrm{Var}\left(\sum_{i=1}^N y_i\right) = \sum_{i=1}^N \mathrm{Var}(y_i)$. Finally, combining these results, we have

$$\mathrm{Var}(\bar{y}_N) = \left(\frac{1}{N}\right)^2 \sum_{i=1}^N \mathrm{Var}(y_i) = \left(\frac{1}{N}\right)^2 N\sigma^2 = \sigma^2/N.$$

Now, we can calculate $\mathrm{Var}[\sqrt{N}(\bar{y}_N - \mu)]$. Since N and μ are constant, we have

$$\mathrm{Var}[\sqrt{N}(\bar{y}_N - \mu)] = \mathrm{Var}[\sqrt{N}(\bar{y}_N)] = (\sqrt{N})^2\mathrm{Var}(\bar{y}_N) = N(\sigma^2/N) = \sigma^2.$$

□

b. What is the asymptotic variance of $\sqrt{N}(\bar{y}_N - \mu)$?
Answer:
By Central Limit Theorem, since $y_i \overset{iid}{\sim} (\mu, \sigma^2)$, then we have $\sqrt{N}(\bar{y}_N - \mu) \overset{a}{\sim} \mathrm{Normal}(0, \sigma^2)$. So, the asymptotic variance of $\sqrt{N}(\bar{y}_N - \mu)$ is $\mathrm{Avar}[\sqrt{N}(\bar{y}_N - \mu)] = \sigma^2$.

□

c. What is the asymptotic variance of $\bar{y}_N$? Compare this with $\mathrm{Var}(\bar{y}_N)$.
Answer:
From the result in (c) we have $\mathrm{Avar}[\sqrt{N}(\bar{y}_N - \mu)] = \sigma^2$. Since N and μ are constant we have $\mathrm{Avar}[\sqrt{N}(\bar{y}_N - \mu)] = \mathrm{Avar}[\sqrt{N}(\bar{y}_N)]$. Recall property of variance that $\mathrm{Var}(ax) = a^2\mathrm{Var}(x)$ for random variable x and constant a. Applying this property to the asymptotic variance we have $\mathrm{Avar}(\bar{y}_N)) = \frac{1}{N}\mathrm{Avar}[\sqrt{N}(\bar{y}_N)] = \frac{1}{N}\mathrm{Avar}[\sqrt{N}(\bar{y}_N - \mu)] = \sigma^2/N$. We get the same result as in (a).

□

d. What is the asymptotic standard deviation of $\bar{y}_N$?
Answer:
By definition, the asymptotic standard deviation is the square root of the asymptotic variance that we have from previous result, formally, $\sqrt{\mathrm{Avar}(\bar{y}_N)} = \sqrt{\sigma^2/N} = \sigma/\sqrt{N}$.

□

e. How would you obtain the asymptotic standard error of $\bar{y}_N$?
Answer:
Applying **Definition 3.10** from the textbook, the asymptotic standard error of $\bar{y}_N$ is $\mathrm{se}(\bar{y}_N) = \hat{\sigma}/\sqrt{N}$, with $\hat{\sigma}$ a consistent estimator for σ, formally, $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$. Because we deal with y_i as a single random variable, the consistent estimator for σ^2 will be $\hat{\sigma}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{y_N})^2$. Then take the $|\sqrt{\hat{\sigma}^2}|$ as the $\hat{\sigma}$.

Proof. We need to show that $\hat{\sigma}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{y_N})^2$ is a consistent and unbiased estimator for σ^2. We know that $y_i \overset{iid}{\sim} (\mu, \sigma^2)$. From the Weak Law of Large Number, we have $\mathrm{E}(\bar{y}_N) = \mu$. From previous result, we already have the following properties $\mathrm{Var}(\bar{y}_N) = \sigma^2/N$. Recall that $\mathrm{Var}(\bar{y}_N) = \mathrm{E}(\bar{y}_N^2) - \mathrm{E}(\bar{y}_N)^2$, thus we have $\mathrm{E}(\bar{y}_N^2) = \mathrm{Var}(\bar{y}_N) + \mathrm{E}(\bar{y}_N)^2 = \sigma^2/N + \mu^2$. We also know that $\mu = \mathrm{E}(y_i)$ and $\sigma^2 = \mathrm{Var}(y_i)$, thus we have $\mathrm{E}(y_i^2) = \mathrm{Var}(y_i) + \mathrm{E}(y_i)^2 = \sigma^2 + \mu^2$. Now take expectation of our estimator

$$
\begin{aligned}
\mathrm{E}(\hat{\sigma}^2) &= \frac{1}{N-1}\mathrm{E}\left(\sum_{i=1}^{N}(y_i - \bar{y_N})^2\right) \\
&= \frac{1}{N-1}\mathrm{E}\left(\sum_{i=1}^{N}(y_i^2 - 2y_i\bar{y_N} + \bar{y_N}^2)\right) \\
&= \frac{1}{N-1}\mathrm{E}\left(\sum_{i=1}^{N}y_i^2 - 2\bar{y_N}\sum_{i=1}^{N}y_i + \sum_{i=1}^{N}\bar{y_N}^2\right) \\
&= \frac{1}{N-1}\mathrm{E}\left(\sum_{i=1}^{N}y_i^2 - 2N\bar{y_N}^2 + N\bar{y_N}^2\right) \\
&= \frac{1}{N-1}\mathrm{E}\left(\sum_{i=1}^{N}y_i^2 - N\bar{y_N}^2\right) \\
&= \frac{1}{N-1}\left[\mathrm{E}\left(\sum_{i=1}^{N}y_i^2\right) - N\mathrm{E}(\bar{y_N}^2)\right] \\
&= \frac{1}{N-1}[N(\sigma^2 + \mu^2) - N(\sigma^2/N + \mu^2)] \\
&= \frac{1}{N-1}(N-1)\sigma^2 \\
&= \sigma^2.
\end{aligned}
$$

We show that the estimator is unbiased. Now we need to show consistency. By Weak Law of Large Number, we have $\bar{y}_N \xrightarrow{p} \mu$ Again, by Weak Law of Large Number we have $\frac{1}{N}\sum_{i=1}^{N}y_i^2 \xrightarrow{p} \mathrm{E}(y_i^2)$. Recall our estimator

$$
\hat{\sigma}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{y_N})^2 = \frac{1}{N-1}\left(\sum_{i=1}^{N}y_i^2 - N\bar{y_N}^2\right) = \frac{N}{N-1}\left(\frac{1}{N}\sum_{i=1}^{N}y_i^2 - \bar{y_N}^2\right). \tag{1}
$$

Take the *plim* of equation (1) we have

$$
\begin{aligned}
plim\hat{\sigma}^2 &= plim\left(\frac{N}{N-1}\left(\frac{1}{N}\sum_{i=1}^{N}y_i^2 - \bar{y_N}^2\right)\right) \\
&= \lim_{N\to\infty}\left(\frac{N}{N-1}\right)\left(plim\left(\frac{1}{N}\sum_{i=1}^{N}y_i^2\right) - plim(\bar{y_N})^2\right) \quad \text{[by Slutsky's theorem]} \\
&= 1\cdot(\mathrm{E}(y_i^2) - \mathrm{E}(y_i)^2) \\
&= \mathrm{Var}(y_i) = \sigma^2.
\end{aligned}
$$

We showed that the estimator is consistent. $\square$

Problem 4.1

Consider a standard log(*wage*) equation for men under the assumption that all explanatory variables are exogenous:

$$\begin{aligned}&\log(wage) = \beta_0 + \beta_1 married + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma} + u, \\ &\mathrm{E}(u|married, educ, \mathbf{z}) = 0,\end{aligned} \tag{4.49}$$

where $\mathbf{z}$ contains factors other than marital status and education that can affect wages. When β_1 is small, $100 \cdot \beta_1$ is approximately the ceteris paribus difference in wages between married and unmarried men. When β_1 is large, it might be preferable to use the exact percentage difference in $\mathrm{E}(wage|married, educ, \mathbf{z})$. Call this θ_1.

a. Show that if u is independent of all explanatory variables in equation (4.49), then $\theta_1 = 100 \cdot [\exp(\beta_1) - 1]$. (Hint: Find $\mathrm{E}(wage|married, educ, \mathbf{z})$ for $married = 1$ and $married = 0$, and find the percentage difference). A natural, consistent, estimator of θ_1 is $\hat{\theta}_1 = 100 \cdot [\exp(\hat{\beta}_1) - 1]$, where $\hat{\beta}_1$ is the OLS estimator from equation (4.49).
Answer:
Rewrite equation (4.49),

$$\begin{aligned}&\log(wage) = \beta_0 + \beta_1 married + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma} + u \\ &\Leftrightarrow wage = \exp\left(\beta_0 + \beta_1 married + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma} + u\right) \\ &\Leftrightarrow wage = \exp\left(u\right)\exp\left[\beta_0 + \beta_1 married + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma}\right).\end{aligned} \tag{2}$$

Now take the expectation of (2) with respect to all explanatory variables, we have

$$\mathrm{E}(wage|married, educ, \mathbf{z}) = \mathrm{E}(\exp\left(u\right)|married, educ, \mathbf{z})\exp\left(\beta_0 + \beta_1 married + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma}\right). \tag{3}$$

If u is independent of all explanatory variables, then $exp(u)$ will also be independent with all those variables, i.e, $\mathrm{E}(\exp\left(u\right)|married, educ, \mathbf{z}) = E(\exp(u)) = c$, where c is a constant. Now from (3) we have

$$\mathrm{E}(wage|married, educ, \mathbf{z}) = c\exp\left(\beta_0 + \beta_1 married + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma}\right). \tag{4}$$

Then, we can compute the exact percentage different $\theta_1 = 100 \cdot \frac{\mathrm{E}(wage|married=1) - \mathrm{E}(wage|married=0)}{\mathrm{E}(wage|married=0)}$ from (4), we have

$$\begin{aligned}\theta_1 &= 100 \cdot \left[\frac{c\exp\left(\beta_0 + \beta_1(1) + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma}\right) - c\exp\left(\beta_0 + \beta_1(0) + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma}\right)}{c\exp\left(\beta_0 + \beta_1(0) + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma}\right)}\right] \\ &= 100 \cdot \left[\frac{\exp\left(\beta_0 + \beta_1 + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma}\right)}{\exp\left(\beta_0 + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma}\right)} - 1\right] \\ &= 100 \cdot [\exp\left(\beta_1\right) - 1].\end{aligned}$$

□

b. Use the delta method (see Section 3.5.2) to show that asymptotic standard error of $\hat{\theta}_1$ is $100 \cdot [\exp(\hat{\beta}_1)] \cdot \mathrm{se}(\hat{\beta}_1)$.
Answer:
From previous result we have $\theta_1 : \beta_1 \to \mathbb{R}$ defined as $\theta_1 = 100 \cdot [\exp\left(\beta_1\right) - 1]$. We can use the OLS estimator $\hat{\beta}_1$ from (4.49) to estimate $\hat{\theta}_1 = 100 \cdot [\exp(\hat{\beta}_1) - 1]$. We are interested in finding $\mathrm{se}(\hat{\theta}_1)$. By Delta Method we have

$$\mathrm{Avar}[\hat{\theta}_1(\hat{\beta}_1)] = \left(\frac{\partial\hat{\theta}_1}{\partial\hat{\beta}_1}\right)^2 \mathrm{Avar}(\hat{\beta}_1), \tag{5}$$

with the Jacobian in this case is only 1×1, i.e, a scalar. From (5) we take the square root to find $\text{se}(\hat{\theta}_1)$, we have

$$\begin{aligned}\text{se}[\hat{\theta}_1(\hat{\beta}_1)] &= \left(\frac{\partial \hat{\theta}_1}{\partial \hat{\beta}_1}\right) \text{se}(\hat{\beta}_1) \\ &= 100 \cdot \exp(\hat{\beta}_1) \cdot \text{se}(\hat{\beta}_1).\end{aligned}$$

□

c. Repeat parts a and b by finding the exact percentage change in $\text{E}(wage|married, educ, \mathbf{z})$ for any given change in $educ$, $\Delta educ$. Call this θ_2. Explain how to estimate θ_2 and obtain its asymptotic standard error.
Answer:
The step involved is similar with the previous result. Recall (4), we have

$$\text{E}(wage|married, educ, \mathbf{z}) = c \exp\left(\beta_0 + \beta_1 married + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma}\right).$$

Then, we can compute the exact percentage different for an additional years of education $\Delta educ$, that is $\theta_2 = 100 \cdot \frac{\text{E}(wage|educ=\epsilon+\Delta educ)-\text{E}(wage|educ=\epsilon)}{\text{E}(wage|educ=\epsilon)}$ from (4), we have

$$\begin{aligned}\theta_2 &= 100 \cdot \left[\frac{c \exp\left(\beta_0 + \beta_1 married + \beta_2(\epsilon + \Delta educ) + \mathbf{z}\boldsymbol{\gamma}\right) - c \exp\left(\beta_0 + \beta_1 married + \beta_2(\epsilon) + \mathbf{z}\boldsymbol{\gamma}\right)}{c \exp\left(\beta_0 + \beta_1 married + \beta_2(\epsilon) + \mathbf{z}\boldsymbol{\gamma}\right)}\right] \\ &= 100 \cdot \left[\frac{c \exp\left(\beta_0 + \beta_1 married + \beta_2(\epsilon + \Delta educ) + \mathbf{z}\boldsymbol{\gamma}\right)}{c \exp\left(\beta_0 + \beta_1 married + \beta_2(\epsilon) + \mathbf{z}\boldsymbol{\gamma}\right)} - 1\right] \\ &= 100 \cdot \left[\exp\left(\beta_2 \Delta educ\right) - 1\right].\end{aligned}$$

We can use the OLS estimator $\hat{\beta}_2$ from (4.49) to estimate $\hat{\theta}_2 = 100 \cdot [\exp(\beta_2)\Delta educ - 1]$ for a given change in $educ$, $\Delta educ$.

□

To estimate the asymptotic standard error we can use the Delta Method. We have that $\theta_2 : \beta_2 \to \mathbb{R}$ defined as $\theta_2 = 100 \cdot [\exp(\beta_2 \Delta educ) - 1]$. Here we treat $\Delta educ$ as a constant because it is given. By Delta Method we have

$$\text{Avar}[\hat{\theta}_2(\hat{\beta}_2)] = \left(\frac{\partial \hat{\theta}_2}{\partial \hat{\beta}_2}\right)^2 \text{Avar}(\hat{\beta}_2), \tag{6}$$

with the Jacobian in this case is only 1×1, i.e, a scalar. From (6) we take the square root to find $\text{se}(\hat{\theta}_2)$, we have

$$\begin{aligned}\text{se}[\hat{\theta}_2(\hat{\beta}_2)] &= \left(\frac{\partial \hat{\theta}_2}{\partial \hat{\beta}_2}\right) \text{se}(\hat{\beta}_2) \\ &= 100 \cdot |\Delta educ| \cdot \exp(\hat{\beta}_2 \Delta educ) \cdot \text{se}(\hat{\beta}_2).\end{aligned}$$

□

d. Use the data in NLS80.RAW to estimate equation (4.49), where $\mathbf{z}$ contains the remaining variables in equation (4.29) (except ability, of course). Find $\hat{\theta}_1$ and its standard error; find $\hat{\theta}_2$ and its standard error when $\Delta educ = 4$.
Answer:
From the data we get these regression result in Table 1.

Table 1: Regression result for (4.49)

	$\log(wage)$
years of work experience	0.014***
	(0.003)
years with current employer	0.012***
	(0.003)
=1 if married	0.199***
	(0.040)
=1 if live in south	-0.091***
	(0.027)
=1 if live in SMSA	0.184***
	(0.027)
=1 if black	-0.188***
	(0.037)
years of education	0.065***
	(0.006)
Constant	5.395***
	(0.113)
Observations	935

Standard errors in parentheses

Data: NLS80.DTA

Wooldrige (2011)

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Recall the previous result, we can calculate

$$\begin{aligned}
\hat{\theta}_1 &= 100 \cdot [\exp(\hat{\beta}_1) - 1] = 100 \cdot [\exp(0.199) - 1] = 22.018\% \\
\text{se}[\hat{\theta}_1] &= 100 \cdot \exp(\hat{\beta}_1) \cdot \text{se}(\hat{\beta}_1) = 100 \cdot \exp(0.199) \cdot (0.040) = 4.881\% \\
\hat{\theta}_2 &= 100 \cdot [\exp(\hat{\beta}_2 \Delta educ) - 1] = 100 \cdot [\exp(0.065 \cdot 4) - 1] = 29.693\% \\
\text{se}[\hat{\theta}_2] &= 100 \cdot |\Delta educ| \cdot \exp(\hat{\beta}_2 \Delta educ) \cdot \text{se}(\hat{\beta}_2) = 100 \cdot 4 \cdot \exp(0.065 \cdot 4) \cdot (0.006) = 3.113\%.
\end{aligned}$$

□

Problem 4.2

a. Show that, under random sampling and the zero conditional mean assumption $\mathrm{E}(u|\mathbf{x}) = 0, \mathrm{E}(\hat{\boldsymbol{\beta}}|\mathbf{X}) = \boldsymbol{\beta}$ if $\mathbf{X}'\mathbf{X}$ is nonsingular.
Answer:

Recall our OLS estimator

$$\hat{\boldsymbol{\beta}} = \left(N^{-1}\sum_{i=1}^{N} \mathbf{x}_i'\mathbf{x}_i\right)^{-1} \left(N^{-1}\sum_{i=1}^{N} \mathbf{x}_i' y_i\right) = \boldsymbol{\beta} + \left(N^{-1}\sum_{i=1}^{N} \mathbf{x}_i'\mathbf{x}_i\right)^{-1} \left(N^{-1}\sum_{i=1}^{N} \mathbf{x}_i' u_i\right),$$

or in matrix notation

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{y}) = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{u}), \tag{7}$$

with $\mathbf{X}$ a $N \times K$ data matrix of regressors with ith row $\mathbf{x}_i$, $\mathbf{y}$ a $N \times 1$ data vector with ith element y_i, $\mathbf{u}$ a $N \times 1$ matrix of errors. Now take the expectation of (7) with respect to $\mathbf{X}$, we have

$$\mathrm{E}(\hat{\boldsymbol{\beta}}|\mathbf{X}) = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathrm{E}(\mathbf{u}|\mathbf{X}) = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{0} = \boldsymbol{\beta}. \tag{8}$$

In order for equation (8) to hold, we need the zero conditional mean assumption, that is given, and we also need $(\mathbf{X}'\mathbf{X})$ to be nonsingular so that the inverse, $(\mathbf{X}'\mathbf{X})^{-1}$, exists.

□

b. In addition to the assumption from part a, assume that $\mathrm{Var}(u|\mathbf{x}) = \sigma^2$. Show that $\mathrm{Var}(\hat{\boldsymbol{\beta}}|\mathbf{X}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$.
Answer:
Recall previous result from (a) in equation (7) we have

$$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{u}).$$

In this $\boldsymbol{\beta}$ a vector of constants because it is a population property. Thus, if we take the conditional variance, we have

$$\begin{aligned}
\mathrm{Var}(\hat{\boldsymbol{\beta}}|\mathbf{X}) &= \mathrm{Var}((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}) \\
&= [(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathrm{Var}(u|\mathbf{X})[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']' \\
&= [(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathrm{Var}(u|\mathbf{X})[\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}]
\end{aligned} \tag{9}$$

In equation (9) we need to get rid of $\mathrm{Var}(\mathbf{u}|\mathbf{X})$, so recall that

$$\begin{aligned}
\mathrm{Var}(\mathbf{u}|\mathbf{X}) &= \mathrm{E}[\mathbf{u}'\mathbf{u}|\mathbf{X}] - \mathrm{E}[\mathbf{u}|\mathbf{X}]^2 \\
&= \mathrm{E}[\mathbf{u}'\mathbf{u}|\mathbf{X}],
\end{aligned}$$

which is a variance covariance matrix, since it is given in (a) that $\mathrm{E}[\mathbf{u}|\mathbf{X}] = 0$. We are also given $\mathrm{Var}(u_i|\mathbf{x}_i) = \mathrm{E}(u_i^2|\mathbf{x}_i) = \sigma^2$. Note that since $\mathbf{x}_i$ is iid across i and that the matrix is symmetric, we have

$$\mathrm{E}(u_i^2|\mathbf{x}_i) = \mathrm{E}(u_i^2|\mathbf{x}_i, \mathbf{x}_j) = \mathrm{E}(u_i^2|\mathbf{X}) = \sigma^2,$$

where u_i^2 is a diagonal of variance covariance matrix. To transform the matrix quadratic form in (9) to scalar multiplication, we need the covariance part of the matrix to be zero, i.e., $\mathrm{E}(u_i u_j|\mathbf{X}) = \mathrm{E}(u_i u_j|\mathbf{x}_i, \mathbf{x}_j) = 0$. Note that since u_i and $\mathbf{x}_i$ are iid across i, by Law of Iterated Expectation we have

$$\begin{aligned}
\mathrm{E}(u_i u_j|\mathbf{x}_i, \mathbf{x}_j) &= \mathrm{E}(u_i u_j|\mathbf{x}_i, \mathbf{x}_j, u_j) && \text{[since } u_i, u_j \text{ are independent]}\\
&= u_j \mathrm{E}(u_i|\mathbf{x}_i, \mathbf{x}_j, u_j)\\
&= u_j \mathrm{E}(u_i|\mathbf{x}_i, \mathbf{x}_j)\\
&= u_j \cdot 0 && \text{[from given assumption in point (a)]}\\
&= 0.
\end{aligned}$$

Now we have that $\mathrm{Var}(\mathbf{u}|\mathbf{X}) = \mathrm{E}[\mathbf{u}'\mathbf{u}|\mathbf{X}]$ is a diagonal matrix with the diagonal elements σ^2. Thus from (9) the matrix inside the sandwich form can be transformed into scalar multiplication. Finally, we have

$$\begin{aligned}
\mathrm{Var}(\hat{\boldsymbol{\beta}}|\mathbf{X}) &= [(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathrm{Var}(u|\mathbf{X})[\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}]\\
&= \sigma^2[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}]\\
&= \sigma^2[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{I}]\\
&= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}.
\end{aligned}$$

□

Problem 4.4

Show that estimator $\hat{\mathbf{B}} \equiv N^{-1}\sum_{i=1}^N \hat{u}_i^2 \mathbf{x}_i'\mathbf{x}_i$ is consistent for $\mathbf{B} = \mathrm{E}(u^2\mathbf{x}'\mathbf{x})$ by showing that

$$N^{-1}\sum_{i=1}^N \hat{u}_i^2 \mathbf{x}_i'\mathbf{x}_i = N^{-1}\sum_{i=1}^N u_i^2 \mathbf{x}_i'\mathbf{x}_i + o_p(1).$$

(Hint: Write $\hat{u}_i^2 = u_i^2 - 2\mathbf{x}_i u_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + [\mathbf{x}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})]^2$, and use the facts that sample averages are $O_p(1)$ when expectations exist and that $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = o_p(1)$. Assume that all necessary expectations exist and are finite.)
Answer:
Rewrite $\hat{\mathbf{B}}$ as follow

$$\begin{aligned}
\hat{\mathbf{B}} &= N^{-1}\sum_{i=1}^N \hat{u}_i^2 \mathbf{x}_i'\mathbf{x}_i\\
&= N^{-1}\sum_{i=1}^N \left[u_i^2 - 2\mathbf{x}_i u_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + [\mathbf{x}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})]^2\right]\mathbf{x}_i'\mathbf{x}_i\\
&= N^{-1}\sum_{i=1}^N u_i^2 \mathbf{x}_i'\mathbf{x}_i - 2N^{-1}\sum_{i=1}^N \mathbf{x}_i u_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\mathbf{x}_i'\mathbf{x}_i + N^{-1}\sum_{i=1}^N [\mathbf{x}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})]^2\mathbf{x}_i'\mathbf{x}_i.
\end{aligned}$$

Denote the first, second, and last term respectively as P, Q, R. Rewrite the second term as follow

$$\begin{aligned}
Q &= -2N^{-1}\sum_{i=1}^N \mathbf{x}_i u_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\mathbf{x}_i'\mathbf{x}_i\\
&= -2\left[\sum_{j=1}^N N^{-1}\left(\sum_{i=1}^N x_{ij}u_i(\hat{\beta}_j - \beta_j)\mathbf{x}_i'\mathbf{x}_i\right)\right]\\
&= -2\left[\sum_{j=1}^N (\hat{\beta}_j - \beta_j)N^{-1}\left(\sum_{i=1}^N x_{ij}u_i\mathbf{x}_i'\mathbf{x}_i\right)\right].
\end{aligned}$$

Note that $\hat{\beta}_j - \beta_j = o_p(1)$, and $N^{-1}\left(\sum_{i=1}^N x_{ij}u_i\mathbf{x}_i'\mathbf{x}_i\right) = O_p(1)$. Therefore, we have the second term $Q = o_p(1)\cdot O_p(1) = o_p(1)$. Then, rewrite the third term as follow

$$\begin{aligned}
R &= N^{-1}\sum_{i=1}^N [\mathbf{x}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})]^2\mathbf{x}_i'\mathbf{x}_i \\
&= \sum_{k=1}^N\sum_{j=1}^N\left[(\hat{\beta}_k - \beta_k)(\hat{\beta}_j - \beta_j)N^{-1}\left(\sum_{i=1}^N [x_{ij}x_{ik}]\mathbf{x}_i'\mathbf{x}_i\right)\right].
\end{aligned}$$

Similarly, note that $\hat{\beta}_j - \beta_j = o_p(1)$, and $N^{-1}\left(\sum_{i=1}^N [x_{ij}x_{ik}]\mathbf{x}_i'\mathbf{x}_i\right) = O_p(1)$. Therefore, we have the third term $R = o_p(1)\cdot o_p(1)\cdot O_p(1) = o_p(1)$. Finally, we have

$$\begin{aligned}
\hat{\mathbf{B}} &= P + Q + R \\
&= P + o_p(1) + o_p(1) \\
&= N^{-1}\sum_{i=1}^N u_i^2\mathbf{x}_i'\mathbf{x}_i + o_p(1).
\end{aligned}$$

□

Problem 4.9

Consider a linear model where the dependent variable is in logarithmic form, and the lag of $\log(y)$ is also an explanatory variable:

$$\log(y) = \beta_0 + \mathbf{x}\boldsymbol{\beta} + \alpha_1\log(y_{-1}) + u, \quad \mathrm{E}(u|\mathbf{x}, y_{-1}) = 0,$$

where the inclusion of y_{-1} might be to control for correlation between policy variables in $\mathbf{x}$ and a previous value of y; see Example 4.4.

a. For estimating $\boldsymbol{\beta}$, why do we obtain the same estimator if the *growth* in y, $\log(y) - \log(y_{-1})$, is used instead as the dependent variable?
Answer:
Rearrange the equation by defining $\Delta\log(y) = \log(y) - \log(y_{-1})$, we have

$$\Delta\log(y) = \beta_0 + \mathbf{x}\boldsymbol{\beta} + (\alpha_1 - 1)\log(y_{-1}) + u.$$

So if we regress using the dependent variable $\Delta\log(y)$ we will have the same estimator for $\boldsymbol{\beta}$. The only change in the estimator will happen with the regression on $\log(y_{-1})$.

□

b. Suppose that there are no covariates $\mathbf{x}$ in the equation. Show that, if the distributions of y and y_{-1} are identical, then $|\alpha_1| < 1$. This is the *regression-to-the-mean* phenomenon in a dynamic setting. (Hint: Show that $\alpha_1 = \mathrm{Corr}[\log(y), \log(y_{-1})]$.)
Answer:
Based on the prompt we will have the following linear model,

$$\log(y) = \beta_0 + \alpha_1\log(y_{-1}) + u.$$

It is also given that the distribution of y and y_{-1} are identical, formally, $y, y_{-1} \overset{iid}{\sim} (\mu, \sigma^2)$ for some finite mean μ, and variance σ^2. Recall the coefficient for simple regression, for this case we have

$$\alpha_1 = \frac{\mathrm{Cov}[\log(y), \log(y_{-1})]}{\mathrm{Var}[\log(y_{-1})]} \tag{10}$$

Recall the correlation relation with covariance and variance, we have

$$\begin{aligned} \mathrm{Corr}[\log(y), \log(y_{-1})] &= \frac{\mathrm{Cov}[\log(y), \log(y_{-1})]}{\sqrt{\mathrm{Var}[\log(y_{-1})]\mathrm{Var}[\log(y)]}} \\ \Leftrightarrow \mathrm{Cov}[\log(y), \log(y_{-1})] &= \mathrm{Corr}[\log(y), \log(y_{-1})]\sqrt{\mathrm{Var}[\log(y_{-1})]\mathrm{Var}[\log(y)]} \end{aligned} \tag{11}$$

Combining equations (10) and (11), and our assumption, $\mathrm{Var}[\log(y_{-1})] = \mathrm{Var}[\log(y)] = \sigma^2$, we have

$$\begin{aligned} \alpha_1 &= \frac{\mathrm{Corr}[\log(y), \log(y_{-1})]\sqrt{\mathrm{Var}[\log(y_{-1})]\mathrm{Var}[\log(y)]}}{\mathrm{Var}[\log(y_{-1})]} \\ &= \frac{\mathrm{Corr}[\log(y), \log(y_{-1})]\sqrt{\sigma^2 \cdot \sigma^2}}{\sigma^2} \\ &= \mathrm{Corr}[\log(y), \log(y_{-1})] \end{aligned}$$

Since we know $-1 < \alpha_1 = \mathrm{Corr}[\log(y), \log(y_{-1})] < 1$, thus we can conclude that $|\alpha_1| < 1$.

□

Part 2

Show that for a regression model, if a regressor x_j is measured with error, then it will be endogenous.
Answer:
Consider a regression with j variables:

$$y = \beta_0 + \beta_1 x_1 + \ldots + \beta_j x_j^* + v, \tag{12}$$

where $y, x_1, \ldots, x_{j-1}$ are observable, but x_j^* is not, instead we observe x_j that is measured with error. We assume that v has zero mean, i.e $\mathrm{E}(v) = 0$, and is uncorrelated with $x_1, \ldots, x_{j-1}, x_j^*$. We also assume v is uncorrelated with x_j, i.e $\mathrm{Cov}(x_j, v) = 0$. Now define the measurement error of the population, u_j, as follows

$$x_j = x_j^* + u_j \Leftrightarrow x_j^* = x_j - u_j. \tag{13}$$

We assume that u_j has zero mean, i.e $\mathrm{E}(u_j) = 0$, and is uncorrelated with $x_1, \ldots, x_{j-1}, x_j^*$. Now substituting equation (13) into (12) we have

$$\begin{aligned} y &= \beta_0 + \beta_1 x_1 + \ldots + \beta_j (x_j - u_j) + v \\ \Leftrightarrow y &= \beta_0 + \beta_1 x_1 + \ldots + \beta_j x_j + (v - \beta_j u_j) \\ \Leftrightarrow y &= \beta_0 + \beta_1 x_1 + \ldots + \beta_j x_j + e \end{aligned} \tag{14}$$

Let's denote $e = v - \beta_j u_j$ in equation (14), and we assume u_j and v are independent. With this measurement error, equation is the actual regression that we will be estimating rather that the original one. Previously, we have assume that $\mathrm{E}(v) = 0$, and $\mathrm{E}(u_j) = 0$, and are uncorrelated with each x_i including x_j, thus, we also have $\mathrm{E}(e) = 0$. We want to assume that x_j is exogenous, i.e $\mathrm{Cov}(x_j, e) = \mathrm{E}(x_j e) = 0$. But we will have a problem of endogeneity in this case, that is

$$\begin{aligned} \mathrm{Cov}(x_j, e) &= \mathrm{E}(x_j e) \\ &= \mathrm{E}((x_j^* + u_j)(v - \beta_j u_j)) \\ &= \mathrm{E}(x_j^* v) + \mathrm{E}(u_j v) - \beta_j \mathrm{E}(x_j^* u_j) - \beta_j \mathrm{E}(u_j^2) \\ &= 0 + 0 + 0 - \beta_j \mathrm{E}(u_j^2) && [\text{because } x_j^* \perp\!\!\!\perp v, x_j^* \perp\!\!\!\perp u_j, u_j \perp\!\!\!\perp v] \\ &= -\beta_j \mathrm{Var}(u_j) \\ &\neq 0. && [\text{if } \beta_j \neq 0, \mathrm{Var}(u_j) \neq 0] \end{aligned}$$

Not that in the presence of measurement error then $\mathrm{Var}(u_j) \neq 0$. Thus in our case of explanatory variable measured with error, we have endogeneity.

□