In [1]:
```
import os
import numpy as np
from np import linalg as la
from np import random as rand
import pandas as pd
import matplotlib.pyplot as plt
plt.rcParams.update({'font.size': 12})
```
executed in 653ms, finished 19:28:01 2020-09-19

Problem 1:

Consider the normal means problem with ε = 1. In class, we have proved that

$$r(\hat{\mu}^{JS}, \mu) = n - (n-2)^2 \mathbb{E}\|Y\|^{-2}$$

where $\hat{\mu}^{JS}$ is the James–Stein estimator. Argue that the risk of the James–Stein estimator depends only on $\|\mu\|$. Make a plot of the risk function with kμk for n = 10 and n = 80. Compare this plot with a plot for the upper bound derived in the notes.

$$r(\hat{\mu}^{JS}, \mu) \leq 2 + \frac{(n-2)\|\mu\|^2}{(n-2)+\|\mu\|^2}$$

In [344]:
```
"""
Testing if hierarchical estimation of expectation
of E[Y]**2 ~ non-central chi-squred is correct.
"""

n_samples = 10000 # CLT
n = 30 # arbitrary
mu = np.ones(n) # arbitrary

def nonC_chi_0(n,mu):
    xi = la.norm(mu)**2
    return rand.noncentral_chisquare(n, xi)

def nonC_chi_1(n,mu):
    xi = la.norm(mu)**2
    poi = rand.poisson(xi/2)
    return rand.chisquare(n + 2*poi)

samples0 = [nonC_chi_0(n,mu) for i in range(n_samples)]
samples1 = [nonC_chi_1(n,mu) for i in range(n_samples)]

plt.figure(figsize = (9,5))
plt.hist(samples0, bins = 50)
plt.hist(samples1, bins = 50, alpha = .7)
plt.title('Checking Poisson Representation of Non-Central Chi-Squared')
plt.legend(['np.rand.noncentral_chisquare', 'hierarchical(poisson)'])
plt.show()
```
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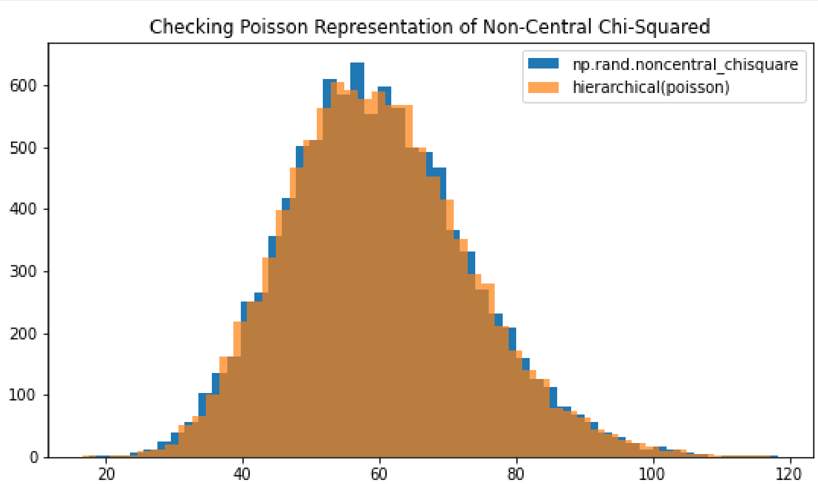

In [403]:
```
def upper_bound(n, mu):
    norm2 = la.norm(mu)**2
    return 2 + (n-2)*norm2/(n-2+norm2)

def nonC_chi(n, mu):
    xi = la.norm(mu)**2
    poi = rand.poisson(xi/2)
    return rand.chisquare(n + 2*poi)

def risk_func(n, mu):
    norm2 = la.norm(mu)**2
    Exp_nonC_chi = np.mean([n-2+2*rand.poisson(norm2/2) for i in range(10)])
    return n - (n-2)**2/Exp_nonC_chi

def q1_plot(n):
    mu_norm = np.linspace(0, 50, 100)
    risk = [risk_func(n, mu_i) for mu_i in mu_norm]
    upper = [upper_bound(n, mu_i) for mu_i in mu_norm]
    plt.figure(figsize = (7,5))
    plt.plot(mu_norm, risk, linewidth = 5)
    plt.plot(mu_norm, upper, linewidth = 5, alpha = .8)
    plt.title(f'James Stein Risk | n = {n}')
    plt.legend(['JS-risk', 'Upper Bound'], loc = 4)
    plt.show()
    print(f'risk(mu_JS, 0) = {risk[0]} <= {upper[0]} = Upper bound')
```
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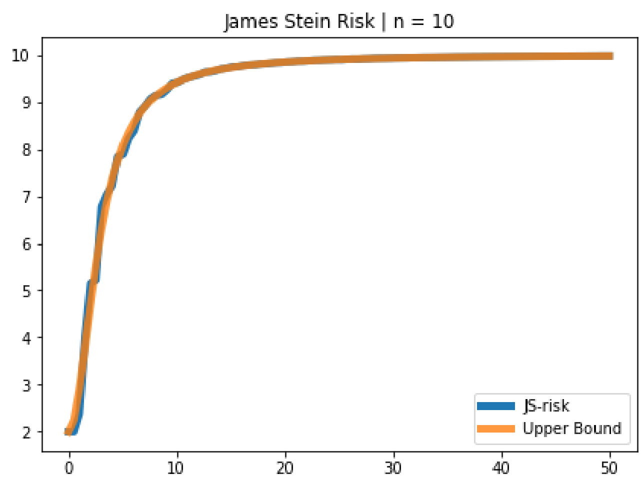
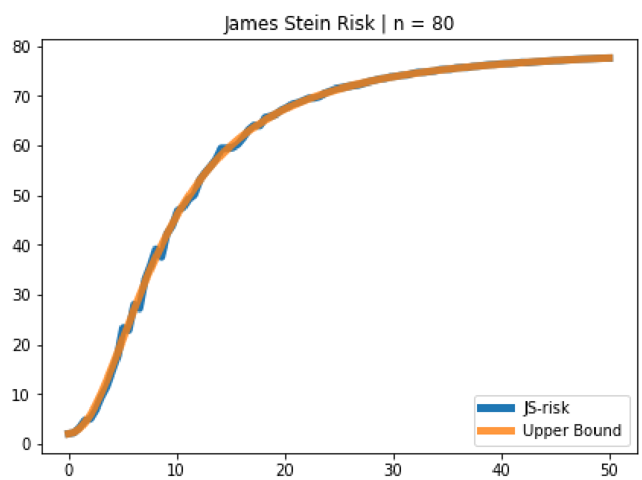
In [404]:
```
q1_plot(10)
q1_plot(80)
```
executed in 202ms, finished 22:07:19 2020-09-19




```
risk(mu_JS, 0) = 2.0 <= 2.0 = Upper bound
```




```
risk(mu_JS, 0) = 2.0 <= 2.0 = Upper bound
```

Note: I am not sure why the risk isn't STRICTLY bounded by the upper bound that we have derived for all $||\mu||$.

In the normal means problem we have the set of observations $Y \sim \mathcal{N}(\mu, \sigma^2 \mathcal{I}_n)$. If we drew one sample from this distribution and asked what the best estimator for μ is, naively we would guess the true paramaters using the observations themselves.

Consider the risk of this estimator $\hat{\mu} = Y$ where $\sigma = 1$.

$$\begin{aligned}\mathbb{E}||\hat{\mu} - Y||^2 &= n + \mathbb{E}||\hat{\mu} - \mu||^2 + 2\mathbb{E}(\hat{\mu} - \mu)^T(\mu - Y) \\ &= n + \mathbb{E}||\hat{\mu} - \mu||^2 - 2\sum_{i=1}^{n} \mathbb{E}[\hat{\mu}_i(Y_i - \mu_i)], \quad \text{because} \quad \mu\mathbb{E}[(Y - \mu)] = 0\end{aligned}$$

and thus by James Stein multivariate lemma we have an unbiased estimate of the risk. In terms of the posed question, the risk is entirely data dependent i.e. does not include any population paramaters.

$$\hat{r}(\hat{\mu}, \mu) = \mathbb{E}||\hat{\mu} - \mu||^2 = \mathbb{E}||\hat{\mu} - Y||^2 + 2\sum_{i=1}^{n} \nabla_i \hat{\mu}_i - n$$

Problem 2

Continuing from the previous problem. Show that the exact value of the risk and the upper bound agree when $||\mu|| = 0$.

I have already shown computationally the value of the risk and the upper bound are equal when $||\mu|| = 0$.

Analytically, notice that whenever $||Y|| = 0$ this implies that we are in fact working with an inverse χ^2_n, not a $\chi^2_n(\xi)$ distribution with respect to the risk function. Note that the expectation of an inverse Chi-Squared distribution only exists when $n > 2$ which is the exact condition for the James Stein estimator.

$$||Y|| = 0 \rightarrow \mathbb{E}||Y||^{-2} = (n-2)^{-1}$$

Thus the risk function is equal to the upper bound.

$$r(\hat{\mu}^{JS}, \mu) = n - (n-2)^2\mathbb{E}||Y||^{-2} = n - \frac{(n-2)^2}{n-2} = 2$$

Problem 3

The file baseball.rtf posted in ecampus contains baseball batting averages data from the 1970 season for 19 major league players. The dataset shows each of 19 players' batting average after 45 at bats, as well as the season batting averages. The idea is to estimate the season averages from the first 45 at bats. A snapshot of some of the relevant columns of the data for 6 players is shown below.

It is not unreasonable to work with the model hi ~ Binomial(45, pi). If we make the variance stabilizing transformation $y_i = \sqrt{45}sin^{-1}(2z_i - 1)$ and $\mu_i = \sqrt{45}sin^{-1}(2p_i - 1)$, then y_i is approximately distributed as $\mathcal{N}(\mu_i, 1)$. Report and compare the MSE for the MLE estimator and the James Stein estimator.

In [351]:
```
path = os.getcwd() + '\hw1_data'
txt = open(path+'\\baseb.txt', 'r')
raw_data = txt.read().replace('\n', '').split(' ')
txt.close()

data = [raw_data[i:i+10] for i in range(0, len(raw_data)-10, 10)]
df = pd.DataFrame(data[1:], columns = data[0])
df[df.columns[2:]] = df[df.columns[2:]].astype('float64')
df.head()
```

executed in 16ms, finished 21:05:44 2020-09-19

Out[351]:

	FirstName	LastName	At-Bats	Hits	BattingAverage	RemainingAt-Bats	RemainingAverage	SeasonAt-Bats	SeasonHits	SeasonAverage
0	Roberto	Clemente	45.0	18.0	0.400	367.0	0.3460	412.0	145.0	0.352
1	Frank	Robinson	45.0	17.0	0.378	426.0	0.2981	471.0	144.0	0.306
2	Frank	Howard	45.0	16.0	0.356	521.0	0.2764	566.0	160.0	0.283
3	Jay	Johnstone	45.0	15.0	0.333	275.0	0.2218	320.0	76.0	0.238
4	Ken	Berry	45.0	14.0	0.311	418.0	0.2727	463.0	128.0	0.276

In [352]:
```
def transform(x):
    return np.sqrt(45)*np.arcsin(2*x-1)

def invert_transform(x):
    return .5*(np.sin(x/np.sqrt(25)) + 1)

df['y'] = transform(df['BattingAverage'])
df['mu'] = transform(df['SeasonAverage'])
df['mu_hat'] = df['y'].mean()+(1-(len(df)-3)/la.norm(df['y'])**2)*(df['y']-df['y'].mean())
```
executed in 6ms, finished 21:05:44 2020-09-19

In [353]:
```
MSE_mle = la.norm(df['y']-df['mu'])**2
MSE_js = la.norm(df['y']-df['mu_hat'])**2

print(f"MSE-MLE = {MSE_mle}\nMSE-JS = {MSE_js}")
```
executed in 4ms, finished 21:05:44 2020-09-19

```
MSE-MLE = 13.711814484625835
MSE-JS = 0.09060119077120204
```

Problem 4

In [354]:
```
from numpy import loadtxt

Y = loadtxt(path+'\\hw1_Y.txt')
X = loadtxt(path+'\\hw1_X.txt', delimiter=",")
true_beta = loadtxt(path+'\\hw1_truebeta.txt')
```
executed in 7ms, finished 21:05:49 2020-09-19

In [355]:
```
from sklearn.linear_model import Ridge, LinearRegression

lmbda_grid = np.arange(0, 2+0.01, 0.01)
estimates = []
MSPE = []
for lmbda in lmbda_grid:
    model = Ridge(alpha=lmbda, fit_intercept = False).fit(X, Y)
    beta_hat = model.coef_
    error = la.norm(X@beta_hat - X@true_beta)**2
    estimates.append(beta_hat)
    MSPE.append(error)
```
executed in 65ms, finished 21:05:50 2020-09-19

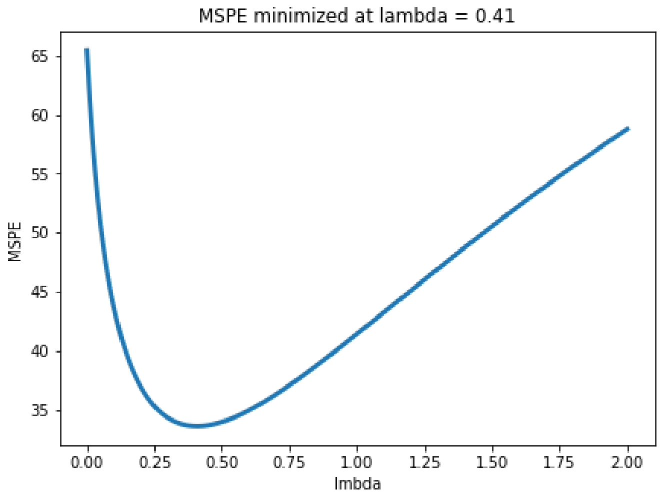
In [356]:
```
OLS_coeffs = LinearRegression(fit_intercept = False).fit(X,Y).coef_

# linear regression coeffs and model fitted with lmbda = 0 are the same, obviously
la.norm(OLS_coeffs - estimates[0])**2
```
executed in 5ms, finished 21:05:50 2020-09-19

Out[356]:
```
2.913013080046369e-22
```

In [357]:
```
optimal_lmbda = lmbda_grid[MSPE.index(min(MSPE))]
```
executed in 3ms, finished 21:05:50 2020-09-19

In [358]:
```
plt.figure(figsize = (7,5))
plt.plot(lmbda_grid, MSPE, linewidth = 3)
plt.xlabel('lmbda')
plt.ylabel('MSPE')
plt.title(f"MSPE minimized at lambda = {round(optimal_lmbda,4)}")
plt.show()
```

executed in 96ms, finished 21:05:51 2020-09-19



Notice that the MSPE is minimized when $\lambda \approx 0.4$. Thus by using shrinkage, we have found a model that has a lower risk than traditional OLS.

In [386]:
```
coeffs = estimates[40] # note that these are estimates when lmbda = .4

plt.figure(figsize = (7,5))
plt.scatter(X@true_beta, X@coeffs, s = 30)
plt.scatter(X@true_beta, X@OLS_coeffs, s = 30)
plt.plot(np.linspace(-8,8), np.linspace(-8,8), color = 'k', linewidth = 2)
plt.xlabel('X@beta_0')
plt.ylabel('estimates')
plt.title(f"Comparing OLS to Ridge Regression")
plt.legend(['x=y','Ridge', 'OLS'])
plt.show()
```

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