Sparse Spectral Precision Estimation

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Spectral Representation

Let $\{X_t\}$ be a zero mean stationary time series with autocovariance function $\gamma(h) = E[X_{t+h}X_t]$ with $h \in \mathbb{Z}$. By the spectral representation theorem, there exists a unique process $\{Z(\omega)\}$ with $\omega \in (-\frac{1}{2}, \frac{1}{2}]$ such that X_t can be represented as an infinite sum of complex random exponentials for each frequency ω.

$$X_t = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega t} dZ(\omega),$$

The theory behind this is rich and will be glossed over for this write up but it is important to note that essentially this is a decomposition of variance. Assuming the given time-series has square summable autocovariances we are able to define an isomorphism between ℓ_2 and $\{exp(2\pi i\omega t)\}$. The random variable $dZ(\omega)$ has the following properties that characterize $\{Z(\omega)\}$.

$$dZ(\omega) = \begin{cases} Z(\omega + d\omega) - Z(\omega), & 0 \leq \omega \leq \frac{1}{2} \\ 0, & \omega = \frac{1}{2} \\ dZ^*(-\omega), & \frac{1}{2} \leq \omega \leq 0 \end{cases}$$

- $E[dZ(\omega)] = 0$, $\forall \omega \in (-\frac{1}{2}, \frac{1}{2}]$
- $\text{Cov}(dZ(\omega_1), dZ(\omega_2)) = 0$ when $\omega_1 \neq \omega_2$ where ω_1, $\omega_2 \in (-\frac{1}{2}, \frac{1}{2}]$,
- $E[|dZ(\omega)|^2] = E[dZ(\omega)\overline{dZ(\omega)}] = dF(\omega)$, $\forall \omega \in (-\frac{1}{2}, \frac{1}{2}]$ where $F(\omega)$ is called the integrated spectrum or spectral distribution function.

See that for any two unique frequencies $\omega_1, \omega_2 \in (-\frac{1}{2}, \frac{1}{2}]$ we have that $dZ(\omega_1)$ and $dZ(\omega_2)$ are independent random variables. Since $dZ(\omega)$ is independent on disjoint frequency intervals $[\omega_1, \omega_1 + d\omega]$ and $[\omega_2, \omega_2 + d\omega]$ we can see the relationship between the spectral distribution function $F(\omega)$ and the autocovariance function $\gamma(h)$ of X_t.

$$\begin{aligned} \gamma(h) = E[X_{t+h}X_t^*] &= E\big[\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega_1 t} dZ(\omega_1) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i\omega_2 t}\overline{dZ(\omega_2)}\big] \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi it(\omega_1 - \omega_2)} E[dZ(\omega_1)\overline{dZ(\omega_2)}] \end{aligned}$$

Recall that $E[dZ(\omega_1)\overline{dZ(\omega_2)}] = dF(\omega)$ when $\omega_1 = \omega_2$ and is zero otherwise. The assumed absolute summability of the autocovariance function implies that the spectral density function is differentiable everywhere and we have $dF(\omega) = f(\omega)d\omega$. It is apparent that the sequence $\{\gamma(h)\}$ and the spectral density function $f(\omega)$ are a Fourier pair.

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega_1 t} f(\omega) d\omega \quad \leftrightarrow \quad f(\omega) = \sum_{h \in \mathbb{Z}} \gamma(h) e^{-2\pi iwh}$$

With the connection between the autocovariance and spectral density established at the population level we can now look at the sample level analogue. Given an observed stationary time series it is known that the long term average of the periodogram approximates the spectral density function. In the following derivation the discrete Fourier transform (DFT) is expanded into the component cosine and sine transforms. The fact that these transforms are asymptotically uncorrelated as $n \to \infty$ will be useful later on when estimating the spectral precision matrix.

For an observed time series $\{X_t\}_{t=1}^n$ the folded set of fundamental Fourier frequencies is $\mathcal{W} = \{\frac{j}{n} \mid j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. The expectation of the smoothed periodogram $P(w_n) = d(w_n)d(w_n)^*$ with window size $L = 2M+1$ and DFT $d(w_n) = \sum_{t=1}^n X_t e^{-2\pi i w_n t}$ is as follows. The limit, namely $E[P(w_j)] \to f(\omega_j)$ as $n \to \infty$, shows that the periodogram is an unbiased estimator of the spectral density. Additionally, the smoothed periodogram be a consistent estimator of the spectral density. The set of Fourier frequencies asymptotically becomes more refined with large sample size so that $L \lll N$.

$$
\begin{aligned}
E[P(w_j)] &= \frac{1}{L} E\Bigg[\sum_{|k| \le M} d(w_{j+k}) d(w_{j+k})^*\Bigg], \ \omega_j \in \mathcal{W} \\
&= \frac{1}{L} \sum_{|k| \le M} E\Bigg[\sum_{t=1}^{n}\sum_{s=1}^{n} X_t X_s e^{2\pi i w_j (t-s)}\Bigg] \\
&= \frac{1}{L} \sum_{|k| \le M} \sum_{h=1-n}^{n-1} \sum_{t=1}^{n-|h|} E\big[X_{t+|h|} X_t\big] e^{-2\pi i w_j h} \\
&= \frac{1}{L} \sum_{|k| \le M} \sum_{h=1-n}^{n-1} \gamma(h) e^{-2\pi i w_j h}
\end{aligned}
$$

All of these concepts can be extended to the multivariate case where the the off diagonal entries of the spectral density matrix correspond the the cross spectral density. Let $\boldsymbol{X}_t = (X_{t,1}, \ldots, X_{t,d})^T \in \mathbb{R}^d$ be a zero mean multivariate stationary time series. The spectral density matrix $\boldsymbol{f}(\omega)$ can be shown to be an estimator of the spectral density function of a multivariate time series where the elements of the matrix are such that $\forall i, j \in \mathbb{Z}_d \ \ \gamma_{i,j}(h) \leftrightarrow f_{i,j}(\omega)$.

$$
\boldsymbol{\Gamma}(h) = \begin{pmatrix} \gamma_{1,1}(h) & \ldots & \gamma_{1,d}(h) \\ \vdots & \ddots & \vdots \\ \gamma_{d,1}(h) & \ldots & \gamma_{d,d}(h) \end{pmatrix}, \qquad \boldsymbol{f}(\omega) = \begin{pmatrix} f_{1,1}(\omega) & \ldots & f_{1,d}(\omega) \\ \vdots & \ddots & \vdots \\ f_{d,1}(\omega) & \ldots & f_{d,d}(\omega) \end{pmatrix}, \ \ h \in \mathbb{Z} \text{ and } \omega \in (0, 2\pi]
$$

Note that for the rest of this write up the discussion will focus on the spectral the spectral density matrix and it's inversion $\boldsymbol{\Omega}(\omega) = \boldsymbol{f}(\omega)^{-1}$

Sparse Inverse Spectral Density Matrix Estimation

Introducing fused lasso for estimates of periodogram in order to recover zeros of the spectral precision matrix.