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Continuous ZND (Zhang Neural Dynamics) Model for Generalized Sinkhorn Scaling of Time-Varying Matrix

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***Abstract*—In this paper, we first propose a Zhang neural dynamics (ZND) model for the generalized Sinkhorn scaling of time-varying matrix. Specifically, by using the dimensional reduction technique, a continuous-time ZND model of time-varying matrix scaling is proposed and analyzed. In addition, the corresponding theoretical proofs are given, which prove the theoretical validity of the proposed ZND model. Moreover, two numerical experiments containing a square case and a rectangle case are also conducted. Numerical experiments and results substantiate the effectiveness and accuracy of the proposed ZND model.**

***Index Terms*—time-varying matrix scaling, continuous-time model, Sinkhorn theorem, Zhang neural dynamics (ZND)**

I. Introduction

Since the pioneering work of Sinkhorn [1], the scaling of matrices is a problem that has attracted attention in fields of pure and applied mathematics. The Sinkhorn's theorem (i.e., Sinkhorn theorem) states that square matrix with positive entries can be written in a certain standard form. Specifically speaking, for square matrix A with nonnegative entries that has total support, there exist diagonal matrices D_1 and D_2 with positive diagonal elements (or termed entries) such that D_1AD_2 is doubly stochastic (i.e., the elements of D_1AD_2 are all positive, and all rows as well as all columns of D_1AD_2 sum to 1) [1].

There is a simple iterative method, called Sinkhorn-Knopp algorithm [1], to approach the doubly stochastic matrix by rescaling all rows and all columns of A to sum to 1 alternately. Sinkhorn theorem has been proved with a wide variety of methods, and each may present a variety of possible generalizations [2]–[8]. The generalized matrix scaling also has many different real-world applications, such as statistical justification [2], [3], axiomatic justification [4], transportation planning [5], [6], social accounting matrices analysis [7], and condition numbers decreasing [8]. Note that the previous researches on matrix scaling are almost all time-invariant, due to the complexity of the time-varying problem-solving being too heavy in addition to difficulties. Generally speaking, the time-varying matrix scaling problem is traditionally and usually considered as a time-invariant matrix problem under the assumption of short-time invariance to solve statically.

In real-world applications, many problems actually appear in a time-varying form. In recent two decades, a new method called Zhang neural dynamics (ZND) [9]–[13] has been proposed and widely used to solve time-varying problems. The earliest history of ZND is a form of recurrent neural network which dates back to 12 March 2001 [9]. After years of refining, summarizing and extending, many time-varying (or termed, time-variant) problems have been solved by ZND successfully [14]–[16], substantiating its effectiveness in such fields. With the help of ZND, we thus obtain the solution models of the original time-varying problems formulated in the continuous-time form (i.e., the continuous time-derivative form).

In this paper, we firstly propose a continuous-time Sinkhorn time-varying matrix scaling model on the basis of ZND. Furthermore, we give some theoretical proofs of the continuous-time ZND model, which prove the theoretical validity of the proposed ZND model. Moreover, two numerical experiments of time-varying matrix scaling are carried out, which further support our ZND model and show its effectiveness.

The remainder of this paper is organized into four sections. Firstly, the problem formulation and an equivalent equation system are presented in Section II. In Section III, we adopt the ZND formula and the dimensional reduction technique to propose a continuous-time ZND model for time-varying matrix scaling. In Section IV, we conduct two numerical experiments to show the effectiveness of the continuous-time ZND model. Section V concludes this paper with final remarks. Besides, the main contributions of this paper are listed below.

- The generalized Sinkhorn scaling of time-varying matrix is first proposed and solved, by using the ZND method and the special dimensional reduction technique.

- The theoretical analysis and numerical experiments are provided, which show the effectiveness of the continuous-time ZND model.

II. Preliminary and Problem Formulation

Before we introduce the time-varying matrix scaling problem, we discuss the time-invariant form at the beginning, which is a generalization of Sinkhorn theorem. The following theorem shows the equivalence scaling of matrix in time-invariant form.

Lemma 1. *(Generalized Sinkhorn scaling [17]) Let $A \in \mathbb{R}^{m\times n}$ be a nonnegative matrix, while vectors $\mathbf{r} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ with nonnegative entries. There exist positive diagonal matrices D_1 and D_2, such that D_1AD_2 has the row sums' vector $\mathbf{r}$ and the column sums' vector $\mathbf{c}$, if and only if there exists a matrix B with the row sums' vector $\mathbf{r}$ and the column sums' vector $\mathbf{c}$ with the same pattern as A.*

Based on Lemma 1, we formulate the time-varying matrix scaling problem as follows. Assuming $A(t) \in \mathbb{R}^{m\times n}$ being a time-varying matrix with nonnegative entries, for time-varying vectors $\mathbf{r}(t) \in \mathbb{R}^m$ and $\mathbf{c}(t) \in \mathbb{R}^n$ with nonnegative entries, we try to find the time-varying positive diagonal matrices $D_1(t) \in \mathbb{R}^{m\times m}$ and $D_2(t) \in \mathbb{R}^{n\times n}$ such that

$$\begin{cases} B(t) = D_1(t)A(t)D_2(t), \\ B(t)\mathbf{e}_n = \mathbf{r}(t), \\ \mathbf{e}_m^{\mathrm{T}}B(t) = \mathbf{c}^{\mathrm{T}}(t), \end{cases} \tag{1}$$

where $\mathbf{e}_n$ is a vector in $\mathbb{R}^n$ with all entries equaling one, i.e., $\mathbf{e}_n = (1, \cdots, 1)^{\mathrm{T}}$, and $\mathbf{e}_m$ is a vector in $\mathbb{R}^m$ with all entries equaling one, i.e., $\mathbf{e}_m = (1, \cdots, 1)^{\mathrm{T}}$. Meanwhile, $t \in [0, t_{\mathrm{f}}] \subset [0, +\infty)$, where t_{f} denotes the final time (also termed the task duration).

It is worth mentioning that, if there is no priori condition to guarantee that $D_1(t)$ and $D_2(t)$ exist, we can conduct approximate scaling [17]. In particular, according to the Sinkhorn theorem [1], with the condition that, $\forall t \in [0, t_{\mathrm{f}}] \subset [0, +\infty)$, if $A(t) \in \mathbb{R}^{n\times n}$ is a nonzero time-varying square matrix with nonnegative entries, $\mathbf{r}(t) = \mathbf{e}_n$, and $\mathbf{c}(t) = \mathbf{e}_n$, there exist time-varying diagonal matrices $D_1(t)$ and $D_2(t)$ with positive diagonal elements, such that $D_1(t)A(t)D_2(t)\mathbf{e}_n = \mathbf{r}(t)$ and $\mathbf{e}_n^{\mathrm{T}}D_1(t)A(t)D_2(t) = \mathbf{c}^{\mathrm{T}}(t)$.

III. Continuous-Time ZND Model for Time-Varying Sinkhorn Scaling

In this section, a continuous-time ZND model is designed for time-varying Sinkhorn scaling.

Since $D_1(t)$ and $D_2(t)$ are real positive diagonal matrices, there exist real diagnoal matrices (i.e., square-root matrices) $R_1(t)$ and $R_2(t)$, such that $D_1(t) = R_1(t)R_1(t)$ and $D_2(t) = R_2(t)R_2(t)$. In order to simplify and handle (1), we convert (1) to the following equivalent form:

$$B(t) = R_1(t)R_1(t)A(t)R_2(t)R_2(t). \tag{2}$$

Based on equation system (2), we define two error functions (i.e., Zhang functions) as below:

$$\begin{cases} Z_1(t) = B(t)\mathbf{e}_n - \mathbf{r}(t) \\ \qquad\;\; = R_1(t)R_1(t)A(t)R_2(t)R_2(t)\mathbf{e}_n - \mathbf{r}(t), \\ Z_2(t) = \mathbf{e}_m^{\mathrm{T}}B(t) - \mathbf{c}^{\mathrm{T}}(t) \\ \qquad\;\; = \mathbf{e}_m^{\mathrm{T}}R_1(t)R_1(t)A(t)R_2(t)R_2(t) - \mathbf{c}^{\mathrm{T}}(t). \end{cases} \tag{3}$$

By adopting linear ZND design formula $\dot{Z}_j(t) = -\lambda_j Z_j(t)$ [10], with $\dot{Z}_j(t)$ denoting the time derivative of Z_j, $j = 1, 2$, the following time-derivative equation system is obtained:

$$\begin{cases} 2R_1(t)\dot{R}_1(t)A(t)R_2(t)R_2(t)\mathbf{e}_n \\ + 2R_1(t)R_1(t)A(t)\dot{R}_2(t)R_2(t)\mathbf{e}_n \\ + R_1(t)R_1(t)\dot{A}(t)R_2(t)R_2(t)\mathbf{e}_n - \dot{\mathbf{r}}(t) \\ = -\lambda_1(R_1(t)R_1(t)A(t)R_2(t)R_2(t)\mathbf{e}_n - \mathbf{r}(t)), \\ 2\mathbf{e}_m^{\mathrm{T}}R_1(t)\dot{R}_1(t)A(t)R_2(t)R_2(t) \\ + 2\mathbf{e}_m^{\mathrm{T}}R_1(t)R_1(t)A(t)\dot{R}_2(t)R_2(t) \\ + \mathbf{e}_m^{\mathrm{T}}R_1(t)R_1(t)\dot{A}(t)R_2(t)R_2(t) - \dot{\mathbf{c}}^{\mathrm{T}}(t) \\ = -\lambda_2(\mathbf{e}_m^{\mathrm{T}}R_1(t)R_1(t)A(t)R_2(t)R_2(t) - \mathbf{c}^{\mathrm{T}}(t)). \end{cases} \tag{4}$$

Note that the design parameters λ_1 and λ_2 are positive real numbers. Furthermore, by applying Kronecker-Zehfuss product and vectorization (in short, vec) techniques [18], [19], the above equation system is rewritten as

$$\begin{cases} M_1(t)\mathrm{vec}(\dot{R}_1(t)) + M_2(t)\mathrm{vec}(\dot{R}_2(t)) = \mathrm{vec}(N_1(t)), \\ M_3(t)\mathrm{vec}(\dot{R}_1(t)) + M_4(t)\mathrm{vec}(\dot{R}_2(t)) = \mathrm{vec}(N_2(t)), \end{cases} \tag{5}$$

where

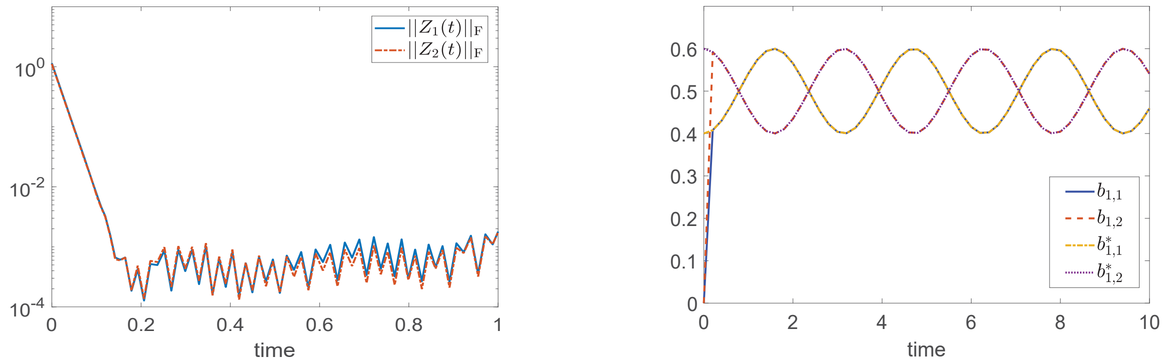
$$\begin{cases} M_1(t) = 2(A(t)R_2(t)R_2(t)\mathbf{e}_n)^{\mathrm{T}} \otimes R_1(t), \\ M_2(t) = 2(R_2(t)\mathbf{e}_n)^{\mathrm{T}} \otimes (R_1(t)R_1(t)A(t)), \\ M_3(t) = 2(A(t)R_2(t)R_2(t))^{\mathrm{T}} \otimes (\mathbf{e}_m^{\mathrm{T}}R_1(t)), \\ M_4(t) = 2R_2^{\mathrm{T}}(t) \otimes (\mathbf{e}_m^{\mathrm{T}}R_1(t)R_1(t)A(t)), \\ N_1(t) = -\lambda_1(R_1(t)R_1(t)A(t)R_2(t)R_2(t)\mathbf{e}_n - \mathbf{r}(t)) \\ \qquad\quad - R_1(t)R_1(t)\dot{A}(t)R_2(t)R_2(t)\mathbf{e}_n + \dot{\mathbf{r}}(t), \\ N_2(t) = -\lambda_2(\mathbf{e}_m^{\mathrm{T}}R_1(t)R_1(t)A(t)R_2(t)R_2(t) - \mathbf{c}^{\mathrm{T}}(t)) \\ \qquad\quad - \mathbf{e}_m^{\mathrm{T}}R_1(t)R_1(t)\dot{A}(t)R_2(t)R_2(t) + \dot{\mathbf{c}}^{\mathrm{T}}(t). \end{cases}$$

In the above, $\otimes$ denotes the Kronecker product, and vec() generates a column vector by stacking all column vectors of a matrix together in MATLAB notation [18]–[20]. To rewrite the equation system (5) into a more compact form, we introduce the vec-permutation matrix. The following theorem is obtained (with its proof given in the Appendix for completeness) [21], [22].

Theorem 1. *For an $n\times n$ time-varying square diagonal matrix $D(t)$, an $n^2\times n$ permutation matrix $\hat{I}_{n^2}$ (or denoted as $\hat{I}_{n^2\times n}$) can be constructed, such that, for any time instant $t \in [0, t_{\mathrm{f}}) \subset [0, +\infty)$, the following formula holds:*

$$\mathrm{vec}(D(t)) = \hat{I}_{n^2}\mathrm{diag}(D(t)),$$

where $\mathrm{diag}(D(t))$ denotes the vector composed of all elements on the main diagonal of $D(t)$.



(a) Residual errors

(b) Element trajectories of $B(t)$

Fig. 1. Trajectories of residual errors and elements of continuous-time ZND model solving Example 1, with two elements' trajectories omitted in right sub-figure due to symmetry, where $b_{i,j}$ corresponds to ZND numerical solution, and $b^*_{i,j}$ corresponds to analytical solution.

By Theorem 1, we have $\mathrm{vec}(\dot{R}_1(t)) = \hat{I}_{m^2}\,\mathrm{diag}(\dot{R}_1(t))$ and $\mathrm{vec}(\dot{R}_2(t)) = \hat{I}_{n^2}\,\mathrm{diag}(\dot{R}_2(t))$. Therefore, the equation system can be rewritten as

$$\begin{cases} M_1(t)\hat{I}_{m^2}\,\mathrm{diag}(\dot{R}_1(t)) + M_2(t)\hat{I}_{n^2}\,\mathrm{diag}(\dot{R}_2(t)) = \mathrm{vec}(N_1(t)), \\ M_3(t)\hat{I}_{m^2}\,\mathrm{diag}(\dot{R}_1(t)) + M_4(t)\hat{I}_{n^2}\,\mathrm{diag}(\dot{R}_2(t)) = \mathrm{vec}(N_2(t)). \end{cases}$$

Then, after arranging, the matrix form of the above equation system is obtained:

$$\begin{pmatrix} M_1(t)\hat{I}_{m^2} & M_2(t)\hat{I}_{n^2} \\ M_3(t)\hat{I}_{m^2} & M_4(t)\hat{I}_{n^2} \end{pmatrix} \begin{pmatrix} \mathrm{diag}(\dot{R}_1(t)) \\ \mathrm{diag}(\dot{R}_2(t)) \end{pmatrix} = \begin{pmatrix} \mathrm{vec}(N_1(t)) \\ \mathrm{vec}(N_2(t)) \end{pmatrix}.$$

For further simplification, let $\lambda = \lambda_1 = \lambda_2$ and $\dot{\mathbf{x}}(t) = (\mathrm{diag}(\dot{R}_1(t)); \mathrm{diag}(\dot{R}_2(t)))$ [20], where $\dot{\mathbf{x}}(t)$ denotes the time derivative of $\mathbf{x}(t)$. Finally, the continuous-time ZND model for generalized Sinkhorn scaling of time-varying matrix is obtained as

$$\dot{\mathbf{x}}(t) = M^{+}(t)\mathbf{n}(t), \tag{6}$$

where

$$\begin{cases} M(t) = \begin{pmatrix} M_1(t)\hat{I}_{m^2} & M_2(t)\hat{I}_{n^2} \\ M_3(t)\hat{I}_{m^2} & M_4(t)\hat{I}_{n^2} \end{pmatrix}, \\ \mathbf{n}(t) = \begin{pmatrix} \mathrm{vec}(N_1(t)) \\ \mathrm{vec}(N_2(t)) \end{pmatrix} = \begin{pmatrix} N_1(t) \\ N_2^{\mathrm{T}}(t) \end{pmatrix}. \end{cases}$$

Besides, $M^{+}(t)$ denotes the pseudo-inverse of $M(t)$ [10].

Theorem 2. *For a smoothly time-varying matrix $A(t) \in \mathbb{R}^{m\times n}$, as well as time-varying vectors $\mathbf{r}(t) \in \mathbb{R}^m$ and $\mathbf{c}(t) \in \mathbb{R}^n$, if there exists a time-varying matrix $B(t) \in \mathbb{R}^{m\times n}$ with the same pattern as $A(t)$ with the row sums' vector $\mathbf{r}(t)$ and the column sums' vector $\mathbf{c}(t)$ [17], then the continuous-time ZND solution model (6), starting from proper initial value $\mathbf{x}(0) \in \mathbb{R}^{m+n}$, converges to the theoretical time-varying solution of equation system (1).*

Proof. In order to obtain the ZND solution (i.e., $D_1(t)$, $D_2(t)$, and $B(t)$) of equation system (1), two vector-valued error functions (i.e., Zhang functions) are defined as $Z_1(t)$ and $Z_2(t)$ in equation system (3), respectively, being column-vector valued and row-vector valued. By adopting the linear ZND design formula to zero out $Z_1(t)$ and $Z_2(t)$, equation system (4) is obtained. From $\dot{Z}_j(t) = -\lambda_j Z_j(t)$, $j = 1, 2$, we know $Z_j(t) = Z_j(0)\exp(-\lambda_j t) \to 0$ [10], achieving (1) exponentially and achieving the generalized Sinkhorn scaling of the time-varying matrix as well. More specifically and in detail, for equation system (4), their element forms are presented in group as follows:

$$\begin{cases} \dot{z}_{1i}(t) = -\lambda_1 z_{1i}(t), \\ \dot{z}_{2j}(t) = -\lambda_2 z_{2j}(t), \end{cases} \tag{7}$$

where $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. To investigate the stability of the two subsystems grouped in (7), two Lyapunov function candidates are defined as

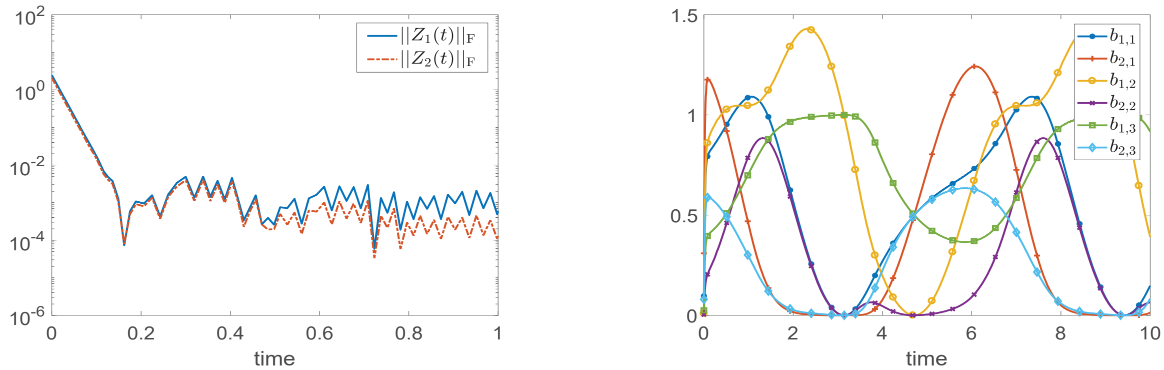
$$\begin{cases} v_{1i}(t) = (z_{1i}(t))^2/2 \geq 0, \\ v_{2j}(t) = (z_{2j}(t))^2/2 \geq 0, \end{cases}$$

with thier time-derivatives as follows:

$$\begin{cases} \dot{v}_{1i}(t) = z_{1i}(t)\dot{z}_{1i}(t) = -\lambda z_{1i}^2(t) \leq 0, \\ \dot{v}_{2j}(t) = z_{2j}(t)\dot{z}_{2j}(t) = -\lambda z_{2j}^2(t) \leq 0. \end{cases}$$

Therefore, we have $\dot{v}_{1i}(t) < 0$ when $z_{1i} \neq 0$, and $\dot{v}_{1i}(t) = 0$ if and only if $z_{1i} = 0$. Besides, $\dot{v}_{2j}(t) < 0$ when $z_{2j} \neq 0$, and $\dot{v}_{2j}(t) = 0$ if and only if $z_{2j} = 0$. According to Lyapunov stability theory [10], equilibrium points $z_{1i} = 0$ and $z_{2j} = 0$ are asymptotically stable. Therefore, z_{1i} and z_{2j} converge to zero; i.e., both $Z_1(t) \in \mathbb{R}^m$ and $Z_2(t) \in \mathbb{R}^n$ are convergent to the zero vectors (i.e., the zero column-vector and the zero row-vector, respectively). Evidently, the solution of (4) converges to the theoretical solution of (1) when $t \to +\infty$ (or saying, $t \gg 0$). Moreover, from the derivation process of (6), we know that it is actually another form of equation system (4), being more robust by using pseudoinverse compared with inverse (especially considering singularity). The proof is thus completed. $\square$

Remark 1. *For the ZND model of generalized time-varying matrix Sinkhorn scaling, where $A(t) \in \mathbb{R}^{m\times n}$, $\mathbf{r}(t) \in \mathbb{R}^m$, and $\mathbf{c}(t) \in \mathbb{R}^n$, let $s = \max(n, m)$. Using big O notation, the computational complexity of each step in obtaining the continuous-time ZND model is $O(s^3)$. The complexities of the computations in each step are considered as follows.*



(a) Residual errors

(b) Element trajectories of $B(t)$

Fig. 2. Trajectories of residual errors and elements of continuous-time ZND model solving Example 2.

- *The computational complexity of obtaining $M_i(t)$ is $O(s^3)$, $i = 1, 2, 3, 4$.*
- *The computational complexity of obtaining $N_i(t)$ is $O(mn)$, $i = 1, 2$, due to the fact that $R_1(t)$ and $R_2(t)$ are the diagonal time-varying matrices. Thus, the computational complexity of constructing $N(t)$ is $O(mn)$.*
- *The computational complexities of multiplying $M_i(t)$, $i = 1, 2, 3, 4$, with the permutation matrix $\hat{I}_{n^2}$ or $\hat{I}_{m^2}$ are correspondingly $O(m^3)$, $O(mn^2)$, $O(m^2n)$, and $O(n^3)$. Thus, the computational complexity of constructing $M(t)$ is $O(s^3)$.*
- *The computational complexity of obtaining $M^+(t)$ by the pseudoinverse operation is $O(s^3)$.*
- *The computational complexity of the multiplication operation between $M^+(t)$ and $N(t)$ is $O(s^2)$.*

Therefore, the total computational complexity of each step of the continuous-time model is $O(s^3)$.

IV. Numerical Verification of Continuous-Time ZND Model

To substantiate the efficacy of the continuous-time ZND model for time-varying matrix scaling, two numerical experiments are carried out in this section.

Example 1

We first consider a simple square case; i.e., $A(t) \in \mathbb{R}^{n\times n}$ is a square real time-varying matrix (also termed matrix stream) with positive elements, $\mathbf{r}(t) = \mathbf{e}_n \in \mathbb{R}^n$, and $\mathbf{c}(t) = \mathbf{e}_n \in \mathbb{R}^n$, with our objective being to find the real positive diagonal matrices $D_1(t) \in \mathbb{R}^{n\times n}$ and $D_2(t) \in \mathbb{R}^{n\times n}$, such that, $\forall t$, $D_1(t)A(t)D_2(t)\mathbf{e}_n = \mathbf{r}(t)$ and $\mathbf{e}_n^{\mathrm{T}} D_1(t)A(t)D_2(t) = \mathbf{c}^{\mathrm{T}}(t)$. This is indeed the Sinkhorn's normal form for the square real matrix stream with positive elements. According to Sinkhorn theorem [1], $\forall t$, there exists $D_1(t) \in \mathbb{R}^{n\times n}$ and $D_2(t) \in \mathbb{R}^{n\times n}$ satisfying the above conditions.

Specifically, let $A(t)$ be the following square real matrix stream with positive elements:

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \in \mathbb{R}^{2\times 2},$$

where

$$a_{11}(t) = \left(\frac{1}{5}\sin^2(t) + \frac{2}{5}\right)(t^2+1)(\sin^2(t)+1),$$

$$a_{12}(t) = \exp(\cos(t))(t^2+1)\left(\frac{1}{5}\cos^2(t) + \frac{2}{5}\right),$$

$$a_{21}(t) = \exp(t)(\sin^2(t)+1)\left(\frac{1}{5}\cos^2(t) + \frac{2}{5}\right),$$

$$a_{22}(t) = \exp(\cos(t))\exp(t)\left(\frac{1}{5}\sin^2(t) + \frac{2}{5}\right).$$

The analytical solution matrices of Sinkhorn normal form for the above case are shown as follows:

$$D_1(t) = \begin{pmatrix} \frac{1}{t^2+1} & 0 \\ 0 & \exp(-t) \end{pmatrix},$$

$$D_2(t) = \begin{pmatrix} \frac{1}{1+\sin^2(t)} & 0 \\ 0 & \exp(-\cos(t)) \end{pmatrix},$$

$$B(t) = \begin{pmatrix} \frac{2}{5} + \frac{1}{5}\sin^2(t) & \frac{2}{5} + \frac{1}{5}\cos^2(t) \\ \frac{2}{5} + \frac{1}{5}\cos^2(t) & \frac{2}{5} + \frac{1}{5}\sin^2(t) \end{pmatrix}.$$

To solve numerically for the Sinkhorn normal form of this time-varying matrix $A(t)$, the continuous-time ZND model is implemented via MATLAB routine "ode45". Note that the initial states are randomly generated by "rand" in MATLAB [20]. As a result, Fig. 1(a) shows the residual errors $\|Z_1(t)\|_{\mathrm{F}}$ and $\|Z_2(t)\|_{\mathrm{F}}$, which show the efficiency and accuracy of the continuous-time ZND model. Thereinto, symbol $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm. Besides, Fig. 1(b) shows the trajectories of the elements of $B(t)$, in which $b_{i,j}$ denotes the (i,j)th element of the numerical solution obtained by using the continuous-time ZND model, and $b^*_{i,j}$ denotes the corresponding element of the analytical solution. It is worth pointing out that we omit the numerical solution trajectories of $b_{2,1}$ and $b_{2,2}$, because $b_{2,1}$ is identical to $b_{1,2}$, and $b_{2,2}$ is identical to $b_{1,1}$. The same situation also appears in the analytical solution.

Example 2

Here is a more general (i.e., a rectangle matrix) case, which is considered as a general scaling for time-varying matrix.

$$A(t)=\begin{pmatrix}\exp(\sin(t)) & \exp(\cos(t)) & \exp(\sin(t))\\ \exp(\cos(t)) & \exp(\sin(t)) & \exp(\cos(t))\end{pmatrix}\in\mathbb{R}^{2\times 3},$$

$$\mathbf{r}(t)=\begin{pmatrix}\sin(t)+2\\ \cos(t)+1\end{pmatrix}\in\mathbb{R}^{2},$$

$$\mathbf{c}(t)=\begin{pmatrix}\cos(t)+1\\ \sin(t)+1\\ 1\end{pmatrix}\in\mathbb{R}^{3}.$$

By adopting the continuous-time ZND model, the numerical experiments are shown in Fig. 2, which provide the evidence to support our ZND model. Because the form of analytical solution in this case cannot be directly expressed, we do not present the analytical solution of this case here.

V. Conclusion

In this paper, we have proposed the Zhang neural dynamics (ZND) model for generalized Sinkhorn scaling of time-varying matrix for the first time. Specifically, by using the dimensional reduction technique and the ZND method, such a continuous-time ZND solution model has been derived and then investigated. We have also provided some theoretical proofs relating to the continuous-time ZND model, which have proved the theoretical validity of the proposed ZND model. Moreover, two numerical experiments of time-varying matrix scaling have been carried out, which have further showed the effectiveness of the ZND model. As future research directions, we will discretize the continuous-time ZND model or apply it to the practical problems handling, such as optimal transport.

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Appendix

The proof of Theorem 1 is given as follows for completeness [21], [22].

Proof. For concision of the proof, with argument t omitted, let D represents the time-varying diagonal matrix $D(t)$ at time instant t. Then the vectorization form of the diagonal square matrix D is

$$\mathrm{vec}(D)=(d_{11}\ 0\ \cdots\ 0\ d_{22}\ 0\ \cdots\ 0\ d_{ii}\ 0\ \cdots\ 0\ d_{nn})^{\mathrm{T}}.$$

Besides, let $\varphi_i=(i-1)n+i$ be the ith subscript of the nonzero element in $\mathrm{vec}(D)$, we have the following equation:

$$\begin{aligned}
\mathrm{vec}(D)=&I_{n^2}\mathrm{vec}(D)\\
=&\begin{array}{cc} & \begin{array}{ccccc}1 & \cdots & \varphi_i & \cdots & \varphi_n\end{array}\\ \begin{array}{c}1\\ \vdots\\ \varphi_i\\ \vdots\\ \varphi_n\end{array} & \begin{pmatrix}1 & \cdots & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & \cdots & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & \cdots & 1\end{pmatrix}\end{array}\\
&\cdot(d_{11}\ 0\ \cdots\ 0\ d_{22}\ 0\ \cdots\ 0\ d_{ii}\ 0\ \cdots\ 0\ d_{nn})^{\mathrm{T}}.
\end{aligned}$$

Since the nondiagonal elements in $\mathrm{vec}(D)$ are zero, the corresponding columns of I_{n^2} have no effect on the computation result. Therefore, we eliminate the nondiagonal elements in $\mathrm{vec}(D)$ by deleting matrix columns and corresponding vector elements, which have no effect on the computation result. Thus, we have

$$\begin{aligned}
\hat{I}_{n^2}\mathrm{vec}(\hat{D})=&\begin{array}{cc} & \begin{array}{ccccc}\varphi_1 & \cdots & \varphi_i & \cdots & \varphi_n\end{array}\\ \begin{array}{c}\varphi_1\\ \vdots\\ \varphi_i\\ \vdots\\ \varphi_n\end{array} & \begin{pmatrix}1 & \cdots & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \cdots & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \cdots & 0 & \cdots & 1\end{pmatrix}\end{array}\\
&\cdot(d_{11}\quad d_{22}\ \cdots\ d_{ii}\ \cdots\ d_{nn})^{\mathrm{T}}\\
=&\hat{I}\mathrm{diag}(D)\\
=&(d_{11}\ 0\ \cdots\ 0\ d_{22}\ 0\ \cdots\ 0\ d_{ii}\ 0\ \cdots\ 0\ d_{nn})^{\mathrm{T}}\\
=&\mathrm{vec}(D),
\end{aligned}$$

i.e., $\mathrm{vec}(D)=\hat{I}_{n^2}\mathrm{diag}(D)$. The proof is thus completed. □

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