Problem Set

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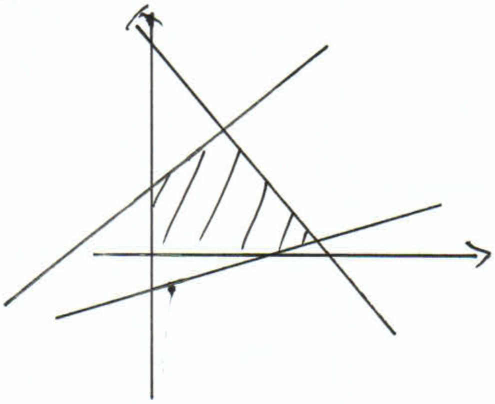
8.1



Then the local maximizer is $\left(\frac{37}{7}, \frac{5}{7}\right)$

The optimal values is $\frac{165}{7}$

8.2.

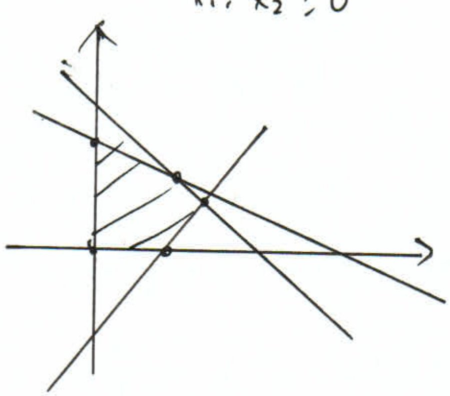
i) $\max\ 3x_1 + x_2$

s.t. $x_1 + 3x_2 \le 15$

$2x_1 + 3x_2 \le 18$

$x_1 - x_2 \le 4$

$x_1, x_2 \ge 0$



Now for the vertices

① $(0, 0)$

$y_1 = 0$

② $(0, 5)$

$y_2 = 5$

③ $(4, 0)$

$y_3 = 12$

④ $(3, 4)$

$y_4 = 9 + 4 = 13$

⑤ $(6, 2)$

$y_5 = 3 \times 6 + 2 = 20$

$\Rightarrow$ The optimal point is $(6, 2)$

optimal value is 20

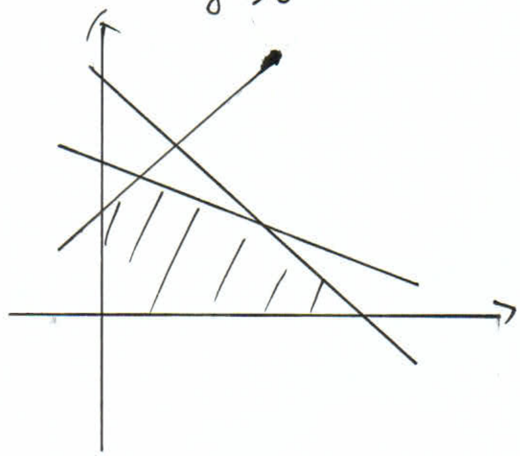
ii) $\max\ 4x + 6y$

s.t. $-x + y \le 11$

$x + y \le 27$

$2x + 5y \le 90$

$x, y \ge 0$



Now consider the vertices

① $(0, 0)$

$V_1 = 0$

② $\cancel{(0, 27)}$ $(27, 0)$

$V_2 = 27 \times 4 = \cancel{162}\ 108$

③ $(0, 11)$

$V_3 = 66$

④ $\left(\frac{25}{7}, \frac{102}{7}\right)$

$V_4 = \frac{100}{7} + \frac{612}{7}$

$= \frac{712}{7}$

⑤ $\cancel{\left(\frac{55}{3}, \frac{26}{3}\right)}$ $(15, 12)$

$\cancel{V_5 = \frac{220}{3} + \frac{156}{3}}$

$= \frac{376}{3}$

$V_5 = 60 + 72 = 132$

$\Rightarrow$ optimal point is $\cancel{\left(\frac{55}{3}, \frac{26}{3}\right)}$ $(15, 12)$

optimal value $\cancel{\frac{376}{3}}$ 132

~~8.7~~

8.5. i) $\max\ 3x_1 + x_2$

s.t. $x_1 + 3x_2 \le 15$

$2x_1 + 3x_2 \le 18$

$x_1 - x_2 \le 4$

$x_1, x_2 \ge 0$

$$\begin{array}{l} \zeta = 3x_1 + x_2 \\ \hline w_1 = 15 - x_1 - 3x_2 \\ w_2 = 18 - 2x_1 - 3x_2 \\ w_3 = 4 - x_1 + x_2 \end{array} \Rightarrow \begin{array}{l} \zeta = 12 - 3w_3 + 4x_2 \\ \hline w_1 = 11 + w_3 - 4x_2 \\ w_2 = 10 + 2w_3 - 5x_2 \\ x_1 = 4 - w_3 + x_2 \end{array} \Rightarrow \begin{array}{l} \zeta = 20 - \frac{7}{5}w_3 - \frac{4}{5}w_2 \\ \hline w_1 = 3 - \frac{3}{5}w_3 + \frac{4}{5}w_2 \\ x_2 = 2 + \frac{2}{5}w_3 - \frac{1}{5}w_2 \\ x_1 = 6 - \frac{3}{5}w_3 - \frac{1}{5}w_2 \end{array}$$

$\Rightarrow$ optimal point $(6, 2)$ optimum value is 20.

ii) $\max \quad 4x + 6y$

$$\text{s.t.} \quad -x + y \le 11$$
$$x + y \le 27$$
$$2x + 5y \le 90$$
$$x, y \ge 0$$

$$\begin{array}{lcl} \zeta & & 4x + 6y \\ \hline w_1 & = & 11 + x - y \\ w_2 & = & 27 - x - y \\ w_3 & = & 90 - 2x - 5y \end{array} \quad \Rightarrow \quad \begin{array}{lcl} \zeta & & 108 - 4w_2 + 2y \\ \hline w_1 & = & 38 - w_2 - 2y \\ x & = & 27 - w_2 - y \\ w_3 & = & 36 + 2w_2 - 3y \end{array}$$

$$\Rightarrow \quad \begin{array}{lcl} \zeta & = & 132 - \frac{8}{3}w_2 - \frac{2}{3}w_3 \\ \hline w_1 & = & 14 - \frac{7}{3}w_2 + \frac{2}{3}w_3 \\ x & = & 15 - \frac{5}{3}w_2 + \frac{1}{3}w_3 \\ \cancel{w}y & = & 12 + \frac{2}{3}w_2 - \frac{1}{3}w_3 \end{array}$$

$\Rightarrow$ optimal point is $(15, 12)$

optimum value is 132.

8.7

i) $\max \quad x_1 + 2x_2$

$$\text{s.t.} \quad -4x_1 - 2x_2 \le -8$$
$$-2x_1 + 3x_2 \le 6$$
$$x_1 \le 3$$
$$x_1, x_2 \ge 0$$

$\xi \quad x_1 + 2x_2 \qquad\qquad \xi \qquad -x_0$

$$w_1 = -8 + 4x_1 + 2x_2 \quad \Rightarrow \quad w_1 = -8 + 4x_1 + 2x_2 + x_0$$
$$w_2 = 6 + 2x_1 - 3x_2 \qquad\qquad w_2 = 6 + 2x_1 - 3x_2 + x_0$$
$$w_3 = 3 - x_1 \qquad\qquad w_3 = 3 - x_1 + x_0$$

$$\Rightarrow \quad \xi = -8 + 4x_1 + 2x_2 + w_1$$
$$x_0 = 8 - 4x_1 - 2x_2 + w_1$$
$$w_2 = 14 - 2x_1 - 5x_2 - w_1$$
$$w_3 = 11 - 5x_1 - 2x_2 - w_1$$

$\Rightarrow \quad \xi = \qquad -x_0$

$\Rightarrow$ Starting point $(2, 0)$.

$$x_1 = 2 - \frac{1}{2}x_2 + \frac{1}{4}w_1 - \frac{1}{4}x_0$$
$$w_2 = \cancel{14 - 2x} \; 10 - 4x_2 - \frac{3}{2}w_1 + \frac{1}{2}x_0$$
$$w_3 = 1 + \frac{1}{2}x_2 + \frac{5}{4}w_1 \cancel{+ \frac{8}{4}x_0} - \frac{5}{4}x_0$$
$$-\frac{9}{4}w_1$$

then we have

ζ

$$x_1 = 2 - \frac{1}{2}x_2 + \frac{1}{4}w_1.$$

$$w_2 = 10 - 4x_2 - \frac{3}{2}w_1$$

$$w_3 = 1 + \frac{1}{2}x_2 \cancel{+} - \frac{9}{4}w_1$$

$\zeta =$

$$x_1 = \frac{3}{4}.$$

$$x_2 = \frac{5}{2} - \frac{1}{4}w_2 - \frac{3}{8}w_1$$

$\overset{W_3}{\cancel{x_3}} =$

ii)

$$\max \quad 5x_1 + 2x_2$$
$$\text{s.t.} \quad 5x_1 + 3x_2 \le 15$$
$$3x_1 + 5x_2 \le 15$$
$$4x_1 - 3x_2 \le -12$$
$$x_1, x_2 \ge 0$$

$$\begin{aligned}
\zeta &= \qquad\qquad - x_0 \\
w_1 &= 15 - 5x_1 - 3x_2 + x_0 \\
w_2 &= 15 - 3x_1 - 5x_2 + x_0 \\
w_3 &= -12 - 4x_1 + 3x_2 + x_0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\zeta &= -12 - 4x_1 + 3x_2 + w_3 \\
w_1 &= 27 + 3x_1 - 6x_2 - w_3 \\
w_2 &= 27 + x_1 - 8x_2 - w_3 \\
x_0 &= 12 + 4x_1 - 3x_2 - w_3
\end{aligned}$$

$$\Rightarrow \quad
\begin{aligned}
\zeta &= -\frac{15}{8} - \frac{29}{8}x_1 - \frac{3}{8}w_2 + \frac{5}{8}w_3 \\
w_1 &= -\frac{33}{4} + \frac{9}{4}x_1 + \frac{3}{4}w_2 - \frac{1}{4}w_3 \\
x_2 &= \frac{27}{8} + \frac{1}{8}x_1 - \frac{1}{8}w_2 - \frac{1}{8}w_3 \\
x_0 &= \frac{15}{8} + \frac{29}{8}x_1 + \frac{3}{8}w_2 - \frac{5}{8}w_3
\end{aligned}
\qquad
\begin{aligned}
\zeta &= \qquad\qquad - x_0 \\
w_1 &= -9 + \frac{4}{5}x_1 + \frac{3}{5}w_2 + \frac{2}{5}x_0 \\
x_2 &= 3 - \frac{3}{5}x_1 - \frac{1}{5}w_2 + \frac{1}{5}x_0 \\
w_3 &= 3 + \frac{29}{5}x_1 + \frac{3}{5}w_2 - \frac{8}{5}x_0
\end{aligned}$$

$$\Rightarrow 5x_1 + 2x_2 = 5x_1 + 6 - \frac{6}{5}x_1 - \frac{2}{5}w_2 \cancel{+ \frac{2}{5}}$$

$$\Rightarrow \quad
\begin{aligned}
\zeta &= 6 + \frac{19}{5}x_1 - \frac{2}{5}w_2 \\
w_1 &= -9 + \frac{4}{5}x_1 + \frac{3}{5}w_2 \\
x_2 &= 3 - \frac{3}{5}x_1 - \frac{1}{5}w_2 \\
w_3 &= 3 + \frac{29}{5}x_1 + \frac{3}{5}w_2
\end{aligned}
\qquad
\begin{aligned}
\zeta &= \frac{169}{25} - \frac{19}{3}x_2 - \frac{21}{5}w_2 \\
w_1 &= -5 - \frac{4}{3}x_2 + \frac{1}{3}w_2 \\
x_1 &= 5 - \frac{5}{3}x_2 - \frac{1}{3}w_2 \\
w_3 &= 32 - \frac{29}{3}x_2 - \frac{4}{3}w_2
\end{aligned}$$

then the optimal point is $(5, 0)$. optimum value is $\frac{169}{25}$.

iii) $\max \quad -3x_1 + x_2$

s.t. $x_2 \le 4$

$-2x_1 + 3x_2 \le 6$

$x_1, x_2 \ge 0$

$\zeta = -3x_1 + x_2$ $\qquad\qquad$ $\zeta = 4 - 3x_1 - w_1$

$w_1 = \cancel{6 - 2}\ 4 - x_2$ $\qquad\qquad$ $x_2 = 4 - w_1$

$w_2 = 6 + 2x_1 - 3x_2$ $\qquad\qquad$ $w_2 = 6 + 2x_1 - 12 + 3w_1$

$= -6 + 2x_1 + 3w_1$

$\Rightarrow$ optimal point is ~~tt~~ $(0, 4)$

optimum value is 4.

8.17.

Pf: Primal: $\max \ c^T x$

$$\text{s.t. } Ax \precsim b$$
$$x \succsim 0$$

Dual: ~~max~~ $\min \ b^T y$

$$\text{s.t. } A^T y \succsim c$$
$$y \succsim 0$$

Then the dual of the dual is

$$\max \ c^T . z .$$
$$\text{s.t. } A . z \precsim b .$$
$$z \succsim 0$$

This is exactly the same as the primal problem.

8.13.

Pf: ① Suppose $x = 0$ is <u>not</u> an optimal point.

Then $\exists \ \bar{x} \succsim 0$ s.t $c^T . x > 0$.

Additionally $A\tilde{x} \precsim 0$. $\Rightarrow$ for $\forall \lambda \in \mathbb{R}^+$. $A\lambda\tilde{x} \precsim 0$.

$\Rightarrow \quad c^T . (\lambda\tilde{x}) = \lambda . (c^T . \tilde{x})$

$\Rightarrow$ the problem is unbounded.

② suppose the problem is bounded.

According to the fundamental theorem of linear optimization and. Strong duality. theorem.

the optimal point of the primal problem.

$$C^T \cdot x^* = 0^T \cdot y = 0$$

we know $c^T \cdot 0 = 0$.

$\Rightarrow$ 0 is an optimal point.

8.18.

Primal. Problem.

min. $3x_1 + 5x_2 + 4x_3$.

s.t. $2x_1 + x_2 + 2x_3 \geq 1$

$x_1 + 3x_2 + 3x_3 \geq 1$

$x_1, x_2, x_3 \geq 0$

Dual Problem.

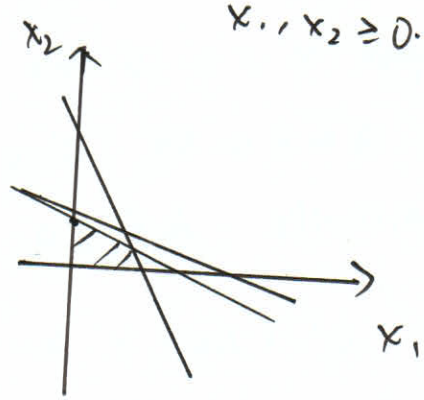
max $x_1 + x_2$

s.t. $2x_1 + x_2 \leq 3$

$x_1 + 3x_2 \leq 5$

$2x_1 + 3x_2 \leq 4$.

$x_1, x_2 \geq 0$.



For the primal problem.

~~$3x_1 + 5x_2 + 4x_3$~~

~~$x_1 = 1$~~

According to the simplex algorithm we know that the optimal point is $(\frac{1}{4}, 0, \frac{1}{4})$. optimum value is $\frac{7}{4}$.

for the vertices.

① $(0, 0)$

$y = 0$

② $(\frac{3}{2}, 0)$

$y = \frac{3}{2}$

③ $(0, \frac{4}{3})$

$y = \frac{4}{3}$

④ $(\frac{5}{4}, \frac{1}{2})$

$y = \frac{7}{4}$.

$\Rightarrow$ optimal point $(\frac{5}{4}, \frac{1}{2})$

optimum value is $\frac{7}{4}$.

1 9.3

1.1 i)

1. Gradient descent method firstly chooses a descent direction that leads to the largest decrease in function value according to the first order approximation. Then it chooses an appropriate step size. This method is fit for computing a well differentiable objective function. The strength of this method is that the algorithm converges fast for certain functions. The weakness for this method is that it takes computational time to compute the step size instead of following iterations.

2. The newton and quasi-newton method compute a newton step, which is a second order approximation, and then generate a step size. This method is fit for a function that is second order differentiable. The strength of this method is that it converges quadratically. The weakness of this method is that it needs to compute the second derivative; additionally it requires the algorithm to start from a starting point that is close enough to the extrema. A potential improvement is using quasi-newton method to approximate the hessian matrix.

3. Conjugate gradient method approximate Hessian matrix in n step to mitigate the computational complexity of computing hessian matrix. This method is fit for solving large quadratic optimization problem. It's especially good at computing sparse matrix.

2 9.10

Proof. Now for arbitrary x_0, we have $x_1 = x_0 - (D^2 f(x_0))^{-1} Df(x_0) = x_0 - Q^{-1}(Qx_0 - b) = x_0 - x_0 + Q^{-1}b = Q^{-1}b$. This is exactly the optimal point. $\square$