Problem Set 1

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1 Problem 1

1.1 3.6

Proof. We know that for $i \in I$, since B_i and $Bi + 1$ are pairwise disjoint, we have $P(A \cap B_i) + P(A \cap B_{i+1}) = P((A \cap B_i) \cup (A \cap B_{i+1})) = P(A \cup (B_i \cap B_{i+1}))$. Then by induction we should have the following:

$$\Sigma_{i \in I} P(A \cup B_i) = P(A \cup (\cap_{i \in I} B_i)) = P(A \cup \Omega) = P(A)$$

□

1.2 3.8

Proof. We know that $P(\cup_{k=1}^{n} E_k) = 1 - P((\cup_{k=1}^{n} E_k)^c) = 1 - P(\cap_{k=1}^{n} E_k^c)$. Since E_k are independent events. Then we should have $P(\cap_{k=1}^{n} E_k^c) = \Pi_{k=1}^{n} P(E_k^c) = \Pi_{k=1}^{n}(1 - P(E_k))$. Thus we have the following:

$$P(\cup_{k=1}^{n} E_k) = 1 - \Pi_{k=1}^{n}(1 - P(E_k))$$

□

1.3 3.11

Now we should have $P(s = crime|stested+) = \dfrac{P(s = crime \cap stested+)}{P(stested+)}$. Now we know that if s tested positive, this happens if the sample is the one at the crime scene or it's just a random match. Then we should have $P(stested+) = P(s = crime \cap stested+) + P(random_match) = \dfrac{1}{3000000} + \dfrac{1}{2500000} = \dfrac{5.5}{7500000}$. Then we should have $P(s = crime|stested+) = \dfrac{\frac{1}{2500000}}{\frac{5.5}{7500000}} = \dfrac{30}{55}$.

1.4 3.12

Now we define the following notation: D_i means that the car is behind door number i; O_i denotes the host opens door number i.
Now for the case of three doors, without loss of generality, suppose the contestant choose door number 1 and the host opens door number 3. Thus we can have:

$$P(O_3) = P(O_3|D_1)P(D_1) + P(O_3|D_2)P(D_2) + P(O_3|D_3)P(D_3) = \frac{1}{2} * \frac{1}{3} + 1 * \frac{1}{3} + 0 = \frac{1}{2}$$

Then we should have:

$$P(D_1|O_3) = \frac{P(D_1 \cap O_3)}{P(O_3)} = \frac{P(O_3|D_1)P(D_1)}{P(O_3)} = \frac{1/2 * 1/3}{1/2} = \frac{1}{3}$$
$$P(D_2|O_3) = \frac{P(D_2 \cap O_3)}{P(O_3)} = \frac{P(O_3|D_2)P(D_2)}{P(O_3)} = \frac{1 * 1/3}{1/2} = \frac{2}{3}$$

Then we konw that the contestant has better oppurtunity to win the prize if he or she switches.
Now suppose the number of doors change to 10. Then suppose the contestant chooses door number 1 and the host opend door number 3 to 10. Then we should have:

$$P(O_{3-10}) = 1/9 * 1/10 + 1 * 1/10 = 1/9$$
$$P(D_1|O_{3-10}) = \frac{P(D_1 \cap O_{3-10})}{P(O_{3-10})} = \frac{P(O_{3-10}|D_1)P(D_1)}{P(O_{3-10})} = \frac{1/9 * 1/10}{1/9} = \frac{1}{10}$$
$$P(D_2|O_{3-10}) = \frac{P(D_2 \cap O_{3-10})}{P(O_{3-10})} = \frac{P(O_{3-10}|D_2)P(D_2)}{P(O_{3-10})} = \frac{1 * 1/10}{1/9} = \frac{9}{10}$$

Now the probablity of winning by switching door is now $\frac{9}{10}$.

1.5 3.16

Proof. We should have the following:

$$E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2] = E[X^2] - 2 * E[X] * \mu + \mu^2 = E[X^2] - \mu^2$$

This is by the linearity of expectation. □

1.6 3.33

Proof. $P(|\frac{B}{n} - p| \geq \epsilon) = P(|B - np| \geq n\epsilon)$. Now by Chebyshev's inequality, we know that $P(|B - np| \geq n\epsilon) \leq \frac{\sigma_B^2}{(n\epsilon)^2} = \frac{p(1-p)n}{n^2\epsilon^2} = \frac{p(1-p)}{n\epsilon^2}$ □

2 Problem 2

2.1 a)

Now suppose we roll a dice twice. Let event A be that the total number rolled is 7 points. Let event B be that the we roll 2 for the first time. Let event C be that the we roll 5 for the second time. Now we know that $P(A \cap B) = \frac{1}{36}$; $P(A \cap C) = \frac{1}{36}$; $P(B \cap C) = \frac{1}{6} * \frac{1}{6} = \frac{1}{36}$. But now we have $P(A \cap B \cap C) = P(B \cap C) = \frac{1}{36} \neq \frac{1}{6} * \frac{1}{6} * \frac{1}{6} = \frac{1}{216}$.

2.2 b)

3 Problem 3

Now we have that $\sigma = 1, 2, 3......, 6$. Now we have the probability mass function: $P(d) = log_1 0(1 + \frac{1}{d})$. Then for each $d \in \sigma$, we have a well defined probability. Additionally, we have $\Sigma_{d=1}^{10} log_1 0(1 + \frac{1}{d}) = log_1 0 \frac{2}{1}.\frac{3}{2}.\frac{4}{3}......\frac{10}{9} = log_{10} 10 = 1$. Then we know that Benford's law is a well defined discrete probability.

4 Problem 4

4.1 (a)

We should have $E[X] = \Sigma_{n=1}^{\infty} \frac{1}{2}^n * 2^n = \Sigma_{n=1}^{\infty} 1 \to \infty$.

4.2 (b)

Now we have $E[log(X)] = \Sigma_{n=1}^{\infty} \frac{1}{2}^n * n * log(2) = log(2) * \Sigma_{n=1}^{\infty} \frac{n}{2^n}$. Now consider the partial sum $s_k = \Sigma_{k=1}^{\infty} \frac{k}{2^k}$. Then we know that $s_k - \frac{1}{2} s_k = \Sigma_{k=1}^{n} \frac{1}{2^n}$. Then $\lim_{n \to \infty} \frac{1}{2} s_n = \lim_{n \to \infty} \frac{1/2}{1/2} = 1$. Then we know that $E[log(X)] = 2$.

5 Problem 5

Now denote the random variable X as the returns of the risk-free asset for investing 1 dollar in the risk-free asset. Denote the random variable $A = \begin{cases} 0, From \quad US \\ 1, From \quad Switzerland \end{cases}$; $B = \begin{cases} 0, Invest \quad US \\ 1, Invest \quad Switzerland \end{cases}$. Denote R as the interest rate of the risk-free asset. Then we have the following equations:

$$E(X|A=0,B=0) = r*0.5 + r*0.5 = r$$
$$E[X|A=0,B=1] = 1.25*r*0.5 + r/1.25*0.5 = 1.025r$$

Similarly we should have:

$$E(X|A=1,B=0) = 1.025r$$
$$E[X|A=1,B=1] = r$$

Then we know that both people should invest in oversea asset to maximize their expected returns.

6 Problem 6

6.1 a)

Now suppose we pick a $x_i \in [0,1]$ in Σ with equal probability. Then we define the following random variable $X_i = \begin{cases} 1, x_i \in [0,0.5] \\ 2, x_i \in (0.5,1] \end{cases}$. Repeat this process infinitely. Then define the random variable $X = \Sigma_{i=1}^{\infty} X_i$. Then by central limit theorem and induction, we should have $E[X] = \Sigma_{i=1}^{\infty} E[X_i] = 0$ and $Var[X] \to \infty$.

6.2 b)

Now suppose we pick a $x \in [0,1]$ in Σ with equal probability. Define random variable $X = \begin{cases} 0.05, x \in [0,0.5] \\ 0.1, x \in [0.5,1] \end{cases}$; and $Y = \begin{cases} 0, x \in [0,0.9] \\ 1, x \in [0.9,1] \end{cases}$. Then we know $P(X > Y) = 1 - P(Y \neq 1) = 0.9 > 0.5$. Then we know that $E[X] = 0.05*0.5 + 0.1*0.5 = 0.075$ and $E[Y] = 0.9*0 + 1*0.1 - 0.1 > 0.075 = E[X]$.

6.3 c)

Now we define the following random variables: $X = \begin{cases} 0, x \in [0,0.8] \\ 3, x \in [0.8,1] \end{cases}$; $Y = \begin{cases} 0, x \in [0,0.7] \\ 2, x \in [0.7,1] \end{cases}$; $Z = \begin{cases} 0, x \in [0,0.4] \\ 1, x \in [0.4,1] \end{cases}$; Then we can see that $P(X > Y) > 0; P(X > Z) > 0; P(Y > Z) > 0$. But we can compute the following:

$$E[X] = 0*0.8 + 3*0.2 = 0.6$$
$$E[Y] = 0*0.7 + 2*0.3 = 0.6$$
$$E[Z] = 0*0.4 + 1*0.6 = 0.6$$

Now we have $E[X] = E[Y] = E[Z]$.

7 Problem 7

7.1 (a)

True.
Now we have $F(Y < a) = 0.5 * F(X < a) + 0.5 * F(X > -a) = 2 * 0.5 * F(X < a) = F(X < a)$. This is true for $\forall a \ in \Re$. Then we know that $Y \in N(0, 1)$.

7.2 (b)

True.
Since $Y = X.Z$. Then $|Y| = |X|.|Z| = |X|$. Then we know that $P(|X| = |Y| = 1)$.

7.3 (c)

True.
We can demonstrate this by examining $f(Y|X = a)$. Then we know that In this case, $f(Y = a|X = a) = 0.5 \neq f(Y = a); f(Y = -a|X = a) = 0.5 \neq f(Y = -a)$. Then we know that X and Y are not independent.

7.4 (d)

True.
Now we have $Cov(X, Y) = E[XY] - E[X]E[Y] = E[XY] + 0 = E[XY] = E[X^2 Z]$. Since X and Z are independent, we know that $Cov(X, Y) = E[X^2 Z] = E[X^2]E[Z] = 0$.

7.5 (e)

False.
This can be shown by our examples above. The random variables X and Y have $Cov(X, Y) = 0$, but they are not independent.

8 Problem 8

Now we have $F(m < a) = F(minX_1, X_2, ..., X_n < a) = 1 - F(X_1 > a).F(X_2 > a)...F(X_n > a) = 1 - (1 - a)^n$. Thus the cdf is $F_m(a) = 1 - (1 - a)^n$. Then $f_m(a) = \dfrac{\partial F_m(a)}{\partial a} = n.(1 - a)^{n-1}$. Then we can have $E[m] = \int_0^1 x.f_m(x)dx = \int_0^1 x.n.x.(1 - x)^{n-1}dx = -x(1 - x)^n|_0^1 + \int_0^1 (1 - x)^n dx = -\dfrac{(1 - x)^{n+1}}{n + 1}|_0^1 = \dfrac{1}{n + 1}$

For M, we should have $F(M < a) = F(max\{X_1, X_2, ..., X_n\} < a) = F(X_1 < a).F(X_2 < a)......F(X_n < a) = a^n$. Then we know that $f_M = \dfrac{F_M(a)}{\partial a} = n.a^{n-1}$. Thus we have $E[M] = \int_0^1 x f_M(x)dx = \int_0^1 n.x^{n-1}.xdx = \dfrac{n.x^{n+1}}{n + 1}|_0^1 = \frac{n}{n+1}$.

9 Problem 9

9.1 (a)

Denote the the state of the ith period X_i. Then we compute $P(|\Sigma_{i=1}^{1000} X_i - 500| < 500 * 0.02) = 1 - P(|\Sigma_{i=1}^{1000} X_i - 500| \geq 10) \geq 1 - \frac{n.\sigma^2}{10^2} = 1 -$.

9.2 (b)

We have the following inequality: $P(|\Sigma_{i=1}^{n} X_i/n - 1/2| < 0.01) = 1 - P(|\Sigma_{i=1}^{n} X_i/n - 1/2| \geq 0.01) = 1 - \frac{n * 1/4}{n^2 0.005^2} = 1 - \frac{1}{0.01^2 n} \geq 0.99$. Then $n \geq 1000000$.

10 Problem 10

Proof. Suppose $\theta < 0$. Then we know that $f(x) = e^{\theta x}$ is a convex function. Thus by Jensen's inequality, we know that $E[f(x)] \geq f(E[x]) = e^{\theta E[x]} < 1$. This contradicts with the fact that $E[e^{\theta x} = 1]$. Then we have proved that $\theta > 0$. □