Problem Set 2.

3.1.

i) Pf: Now we have

$$\|(x+y)\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x, x\rangle + \langle x, -y\rangle + \langle -y, x\rangle + \langle -y, -y\rangle$$

Since this is the real inner product space, then we have

$$\|x+y\|^2 = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle$$

$$\|x-y\|^2 = \langle x, x\rangle + \langle y, y\rangle - 2\langle x, y\rangle$$

then $\|x+y\|^2 - \|x-y\|^2 = 4\langle x, y\rangle$

$$\Rightarrow \langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

ii) Following the specification in i), we should have

$$\|x+y\|^2 + \|x-y\|^2 = 2\langle x, x\rangle + 2\langle y, y\rangle$$
$$= 2\|x\|^2 + 2\|y\|^2.$$

$$\Rightarrow \|x\|^2 + \|y\|^2 = \frac{1}{2}\left(\|x+y\|^2 + \|x-y\|^2\right)$$

3.2.

Pf: $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle y, y\rangle + \langle x, y\rangle + \langle y, x\rangle$

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x, x\rangle + \langle y, y\rangle - \langle x, y\rangle - \langle y, x\rangle$$

$$i\|x-iy\|^2 = i\langle x-iy, x-iy\rangle = i(\langle x, x\rangle - i^2\langle y, y\rangle - i\langle x, y\rangle + i\langle y, x\rangle)$$

$$i\|x+iy\|^2 = i\langle x+iy, x+iy\rangle = i(\langle x, x\rangle - i^2\langle y, y\rangle + i\langle x, y\rangle - i\langle y, x\rangle)$$

Then

$\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2)$

$= 2\langle x,y\rangle + 2\langle y,x\rangle - i\,(2i\langle x,y\rangle - 2i\langle y,x\rangle)$

$= 4\langle x,y\rangle$

$\Rightarrow \langle x,y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2)$

3.3.

i). We have

$$\frac{\langle f(x),g(x)\rangle}{\|f(x)\|\cdot\|g(x)\|} = \frac{\int_0^1 x\cdot x^5\cdot dx}{\sqrt{\int_0^1 x^2\cdot dx}\cdot\sqrt{\int_0^1 (x^5)^2 dx}} = \frac{\frac{1}{7}x^7\Big|_0^1}{\sqrt{\frac{1}{3}x^3\Big|_0^1}\cdot\sqrt{\frac{1}{4}x^4\Big|_0^1}} = \frac{\frac{1}{7}}{\sqrt{\frac{1}{3}}\cdot\sqrt{\frac{1}{4}}} = \frac{\sqrt{33}}{7}.$$

ii) We have

$$\frac{\langle f(x),g(x)\rangle}{\|f(x)\|\cdot\|g(x)\|} = \frac{\int_0^1 x^2\cdot x^4\cdot dx}{\sqrt{\int_0^1 x^4 dx}\cdot\sqrt{\int_0^1 x^8 dx}} = \frac{\frac{1}{7}x^7\Big|_0^1}{\sqrt{\frac{1}{5}x^5\Big|_0^1}\sqrt{\frac{1}{9}x^9\Big|_0^1}} = \frac{3\sqrt{5}}{7}$$

3.8.

i). $\langle \cos(t), \sin(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(t)\cdot\cos(t)\cdot dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(2t)\cdot dt.$

$= \frac{1}{4\pi}\int_{-2\pi}^{2\pi}\sin k\cdot dk = 0.$

$\langle\cos(t),\cos(2t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\cdot\cos(2t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(t+2t)+\cos(2t-t)}{2}\,dt$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(3t)\cdot dt + \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos t\cdot dt$

$= \frac{1}{6\pi}\sin 3t\Big|_{-\pi}^{\pi} + \frac{1}{2\pi}\sin t\Big|_{-\pi}^{\pi} = 0.$

$$\langle \cos(t) . \sin(2t) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t) . \sin(2t) . dt .$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(t+2t) - \cos(t)}{2} dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos 3t \, dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos t \, dt$$

$$= 0$$

$$\langle \sin(t), \cos(2t) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(t) . \cos(2t) \, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} . \frac{\overset{\sin}{\cancel{\cos}}(2t+t) - \overset{\sin}{\cancel{\cos}}(2t-t)}{2}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin 3t . dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin t . dt$$

$$= -\frac{1}{6\pi} \cos 3t \Big|_{-\pi}^{\pi} + \frac{1}{2\pi} \cos t \Big|_{-\pi}^{\pi} = 0$$

$$\langle \sin(t), \sin(2t) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(t) . \sin(2t) . dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(2t+t) - \cos(2t-t)}{2} dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos 3t \, dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos t . dt$$

$$= 0$$

$$\langle \cos(2t), \sin(2t) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2t) . \sin(2t) . dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(4t) . dt$$

$$= 0$$

~~Then we know that~~

Additionally, we have

$$\| \cos(t) \|^2 = \langle \cos(t), \cos(t) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(t) . dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(2t)+1}{2} dt .$$

$$= \frac{1}{\pi} \cdot \frac{t}{2}\Big|_{-\pi}^{\pi} = 1 .$$

$$\|\sin(t)\|^2 = \langle \sin(t), \sin(t) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(t)\, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{(1-\cos(2t))}{2}\, dt$$

$$= \frac{1}{\pi} \cdot \frac{t}{2}\Big|_{-\pi}^{\pi} = 1$$

$$\|\cos(2t)\|^2 = \langle \cos(2t), \cos(2t) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(2t)\, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(4t)+1}{2}\, dt$$

$$= \frac{1}{\pi} \frac{t}{2}\Big|_{-\pi}^{\pi} = 1.$$

$$\|\sin(2t)\|^2 = \langle \sin(2t), \sin(2t) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(2t)\, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos(4t)}{2}\, dt$$

$$= \frac{1}{\pi} \cdot \frac{t}{2}\Big|_{-\pi}^{\pi} = 1$$

$$\|\cos(2t)\|^2 = \langle \cos(2t), \cos(2t) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(2t)\, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1+\cos(4t)}{2}\, dt = 1$$

$\Rightarrow$ We know that $\{\sin(t), \cos(t), \sin(2t), \cos(2t)\}$ is an orthonormal set.

ii). $\|t\| = \sqrt{\langle t, t \rangle} = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} t^2\, dt} = \sqrt{\frac{1}{\pi} \cdot \frac{1}{3} t^3 \Big|_{-\pi}^{\pi}} = \sqrt{\frac{1}{3}\pi^2 + \frac{1}{3}\pi^2}$

$= \frac{\sqrt{6}}{3} \cdot \pi.$

iii).

Now we have

$$\langle \cos(3t), \sin(t) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(3t).\sin(t).\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cdot \frac{\sin(3t+t) - \sin(3t-t)}{2}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(4t)\,dt) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(2t).dt$$

$$= -\frac{1}{8\pi}\cos(4t)\Big|_{-\pi}^{\pi} + \frac{1}{4\pi}\cos(2t)\Big|_{-\pi}^{\pi}$$

$$= 0$$

$$\langle \cos(3t), \cos(t) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(3t).\cos(t).\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(3t+t) + \cos(3t-t)}{2}\,dt$$

$$= \frac{1}{8\pi}\sin(4t)\Big|_{-\pi}^{\pi} + \frac{1}{4\pi}\sin(2t)\Big|_{-\pi}^{\pi}$$

$$= 0$$

Similarly. we should.

$$\langle \cos(3t), \cos(2t) \rangle = 0$$

$$\langle \cos(3t), \sin(2t) \rangle = 0$$

Then we should have

$$\text{Proj}_x(\cos(3t)) = \langle \cos(3t), \cos(t) \rangle.\cos(t). + \langle \cos(3t), \sin(t) \rangle \sin(t) +$$
$$\langle \cos(3t), \cos(2t) \rangle \cos(2t) + \langle \cos(3t), \sin(2t) \rangle \sin(2t)$$
$$= 0$$

iv).

$$\langle t, \cos(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} t.\cos(t).\,dt$$

$$= \frac{1}{\pi}\, t.\sin(t)\Big|_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(t).\,dt$$

$$= 0$$

$$\langle t, \sin(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} t.\sin(t).\,dt$$

$$= -\frac{1}{\pi}\, t.\cos(t)\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t).dt.$$

$$= 1 - 1. = 0.$$

Similarily, we should have

$$\langle t, \cos(2t)\rangle = 0$$

$$\langle t, \sin 2t)\rangle = 0$$

$$\Rightarrow \quad \mathrm{Proj}_x(t) = 0.$$

3.9.

Pf. Now suppose for arbitrary. $x = [x, y] \in \mathbb{R}^2$.

$\| x \|^2 = \langle x, x \rangle = x^2 + y^2$.

Then $R_\theta(x) = \begin{bmatrix} \cos\theta \cdot x - \sin\theta \cdot y \\ \sin\theta \cdot x + \cos\theta \cdot y \end{bmatrix}$

$\langle R_\theta(x), R_\theta(x) \rangle = x^2 \cdot \cos^2\theta + y^2 \cdot \sin^2\theta - 2xy \cdot \cos\theta \cdot \sin\theta$.

$+ x^2 \cdot \sin^2\theta + y^2 \cdot \cos^2\theta + 2xy \cdot \sin\theta \cdot \cos\theta$.

$= x^2(\sin^2\theta + \cos^2\theta) + y^2(\sin^2\theta + \cos^2\theta)$

$= x^2 + y^2$

3.10.

i) pf:

($\Rightarrow$) suppose $Q \in M_n(F)$ is an orthonormal matrix

Denote. $B = Q^H Q$.

then the i,jth entry of B is $\langle Qe_i, Qe_j \rangle$

$\Rightarrow \langle Qe_i, Qe_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$ by the definition of orthonormal transformation

$\Rightarrow$ $B = Q^H Q = I$. Then by the property of inverse

$\Rightarrow$ $Q^H Q = Q \cdot Q^H = I$

($\Leftarrow$) suppose $Q Q^H = Q^H Q = I$.

Then denote $Q = [q_1 \; q_2 \; \cdots \; q_n]$ where q_i are the columns of Q.

and $Q^H = \begin{bmatrix} q_1^H \\ q_2^H \\ \vdots \\ q_n^H \end{bmatrix}$

Then we know that

$$Q^H \cdot Q = \begin{bmatrix} q_{e_1}^H q_{e_1} & \cdots & q_{e_1}^H q_{e_n} \\ \vdots & \ddots & \vdots \\ q_{e_n}^H q_{e_1} & \cdots & q_{e_n}^H q_{e_n} \end{bmatrix}$$

~~Since Q is a orthonormal matrix, then $\langle q_{e_i}, q_{e_j} \rangle = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$~~

$$\Rightarrow Q^H \cdot Q = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = I.$$

Then we know $\langle q_{e_i}, q_{e_j} \rangle = q_{e_i}^H \cdot q_{e_j} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$\Rightarrow \{q_{e_1}, q_{e_2} \cdots q_{e_n}\}$ is an orthonormal set

Then $q_{e_1}, \cdots q_{e_n}$ is a basis.

Take arbitrary $x, y \in F^n$. We have $x = \sum_{i=1}^n x_i q_{e_i}$, $y = \sum_{i=1}^n y_i q_{e_i}$

$\Rightarrow \langle x, y \rangle = \sum_{i=1}^n \overline{x_i} y_i$

$$\langle Qx, Qy \rangle = \langle [x_1, x_2, \cdots x_n], [y_1, y_2 \cdots y_n] \rangle$$

$$= \sum_{i=1}^n \overline{x_i} y_i$$

$$= \langle x, y \rangle$$

$\Rightarrow Q$ is an orthonormal matrix by definition

ii) pf: $\|Qx\| = \sqrt{\langle Qx, Qx \rangle} = \sqrt{\langle x, x \rangle} = \|x\|$

iii). Take arbitrary $x, y \in \mathbb{F}^n$

Since Q is orthonormal matrix.

$$\langle Q^{-1}x, Q^{-1}y \rangle = \langle Q \cdot Q^{-1}x, QQ^{-1}y \rangle = \langle x, y \rangle$$

$\Rightarrow$ Q^{-1} is orthonormal matrix by definition.

iv) From ii we know that. $Q \cdot Q^H = Q^H \cdot Q = I$.

then suppose $Q = [q_1 \; q_2 \; \cdots \; q_n]$

$$\cancel{Q \cdot Q^H = [q_1 \; q_2 \; \cdots \; q_n] \begin{bmatrix} q_1^H \\ \vdots \\ q_n^H \end{bmatrix}}$$

$$\cancel{= \sum_{i=1}^{n} q_i q_i^H = \sum_{i=1}^{n} \langle q_i, q_i \rangle}$$

$$Q^H \cdot Q = \begin{bmatrix} q_1^H \\ \vdots \\ q_n^H \end{bmatrix} [q_1 \; \cdots \; q_n]$$

$$= \begin{bmatrix} q_1^H \cdot q_1 & \cdots & q_1^H q_n \\ \vdots & \ddots & \vdots \\ q_n^H q_1 & \cdots & q_n^H q_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

$$\Rightarrow \langle q_i, q_j \rangle = \begin{cases} 1, & i = i \\ 0, & i \neq j \end{cases}$$

$\Rightarrow$ $q_1, \cdots, q_n$ are orthonormal.

v) pf: We know $\det(Q^H) = \det(Q)$

$\Rightarrow \det(Q\,Q^H) = \det(Q) \cdot \det(Q^H)$

$= \det(I)$

$= 1$

$\Rightarrow [\det(Q)]^2 = 1$

$\Rightarrow \|\det(Q)\| = 1.$

vi) take ~~ar~~arbitrary $x, y \in \mathbb{F}^n$.

$\langle Q_1 Q_2 x, Q_1 Q_2 y \rangle$

$= \langle Q_1 (Q_2 x), Q_1 (Q_2 y) \rangle$

$= \langle Q_2 x, Q_2 y \rangle$

$= \langle x, y \rangle$

$\Rightarrow Q_1 Q_2$ is orthonormal matrix.

~~3.16~~.

3.11. When apply Gram-Schmidt orthonormalization to linearly dependent vectors, we would end up getting zero vector.

Now suppose $x_1, x_2 \cdots x_{k-1}$ are linearly independent and $x_1, x_2 \cdots x_k$ are linearly dependent.

Then apply Gram-Schmidt Orthonormalization to $x_1, \cdots x_{k-1}$ then we get $q_1, \cdots q_{k-1}$. Then $q_1 \cdots q_{k-1}$ are orthonormal.

Additionally $x_1, \cdots x_{k-1}$ can be expressed as linear combination of $q_1 \cdots q_{k-1}$.

Since $x_1 \cdots x_k$ are linearly dependent, then we know x_k can be expressed as linear combination of $x_1 \cdots x_{k-1}$.

Then x_k can be expressed as linear combination of $q_1 \cdots q_{k-1}$

denote this as $x_k = \sum_{i=1}^{k-1} a_i q_i$

$$\Rightarrow \quad p_k = \sum_{i=1}^{k-1} \langle x_k, q_i \rangle q_i$$

$$= \sum_{i=1}^{k-1} a_i q_i = x_k.$$

$$\Rightarrow \quad q_k = \frac{x_k - p_k}{\| x_k - p_k \|} = 0.$$

3.16.

i). Let $D = \begin{bmatrix} -1 & - & - & 0 \\ | & -1 & & | \\ | & & \ddots & | \\ 0 & - & - & -1 \end{bmatrix}$

Then $QD = [-q_1, -q_2 \cdots -q_m]$.

$$\langle -q_i, -q_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

$\Rightarrow$ QD is orthonormal as well.

Additionally $D^{-1}R = -R$ which is upper triangular as well.

And $QD \cdot D^{-1}R = QR = A$.

Then we know that the QR decomposition is not unique.

ii)

Pf: Now suppose there are two different QR decomposition s.t. $A = Q_1R_1 = Q_2R_2$.
and R_1 and R_2 have only positive diagonal elements

Then we know $Q_1R_1 = Q_2R_2 \Rightarrow R_1(R_2)^{-1} = Q_1^{-1}Q_2$.

(Q_1, Q_2, R_1 and R_2 must be invertible since A is invertible)

Since R_1 and R_2 are upper trigular matrix, then we know R_2^{-1} is upper triangular matrix as well. $\Rightarrow$ $R_1(R_2)^{-1}$ is upper triangular as well.

Since Q_1 and Q_2 are orthonormal, $\Rightarrow$ Q_1^{-1} is orthonormal as well

$\Rightarrow$ $Q_1^{-1}Q_2$ is orthonormal.

$\Rightarrow$ $R_1(R_2)^{-1}$ is orthonormal

Claim: $R_1(R_2)^{-1}$ is diagonal matrix. and $R_1(R_2)^{-1} = I$.

Since $R_1(R_2)^{-1}$ is upper triangular and orthonormal, then the first column of $R_1(R_2)^{-1}$ is

e_1. This is because R_1 and R_2 have positive diagonal elements.

Suppose the first $j-1$ columns are $e_1 \cdots e_{j-1}$.

then denote the j th column as r_j.

then $\forall i \in \{1, 2 \cdots j-1\}$. $\langle r_j, e_i \rangle = 0$.

$\Rightarrow$ the first $j-1$ entries of r_j are zero

Since $\|r_j\| = 1$ and $R_1 . R_2$ have positive diagonal element.

then $r_j = e_j$.

$\Rightarrow$ By Induction, $R_1 (R_2)^{-1} = I$.

$\Rightarrow R_1 = R_2$

Since $Q_1 R_1 = Q_2 R_2 \Rightarrow Q_1 = Q_2$

$\Rightarrow$ If A is invertible, then A has unique QR decomposition with R having positive diagonal elements.

3.17

Pf: We have $A = \hat{Q}\hat{R}$, then $A^H = \hat{R}^H \cdot \hat{Q}^H$.

$\Rightarrow A^H A x = \hat{R}^H \cdot \hat{Q}^H \cdot \hat{Q} \cdot \hat{R} \cdot x$

$A^H b = \hat{R}^H \cdot \hat{Q}^H \cdot b$.

$\hat{Q}$ has orthonormal columns. denote it as $\hat{Q} = [q_1 \cdots q_n]$

$$\Rightarrow \hat{Q}^H \cdot \hat{Q} = \begin{bmatrix} q_1^H \\ \vdots \\ q_n^H \end{bmatrix} [q_1 \cdots q_n] = \begin{bmatrix} q_1^H q_1 & \cdots & q_1^H q_n \\ \vdots & \ddots & \\ q_n^H q_1 & \cdots & q_n^H q_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

$\Rightarrow A^H A x = \hat{R}^H \hat{R} x$

Since we know $n = \text{Rank}(A) \leq \min\{\text{Rank}(\hat{Q}), \text{Rank}(\hat{R})\}$.

Then. $\text{Rank}(\hat{R}) \geq n \Rightarrow \hat{R}$ has full rank.

$\Rightarrow \hat{R}$ is invertible.

$\Rightarrow \hat{R}^H$ is invertible

Then $A^H A x = A^H b \Rightarrow \hat{R}^H \hat{R} x = \hat{R}^H \hat{Q}^H b$

$\Rightarrow \hat{R} x = \hat{Q}^H b$.

3.23.

Pf: $(V, \|\cdot\|)$ is a normed linear space $\Rightarrow$ triangle inequality holds

then. $\|x\| = \|x - y + y\| \leq \|x-y\| + \|y\|$.

$\Rightarrow \|x\| - \|y\| \leq \|x-y\|$.

$\|y\| = \|y - x + x\| \leq \|y-x\| + \|x\| = \|x-y\| + \|y\|$

$\Rightarrow \|y\| - \|x\| \leq \|x-y\|$

To sum up. we should have $|\,\|x\| - \|y\|\,| \leq \|x-y\|$

3.24. $\Rightarrow$) $\|f\|_{L^1} = \int_a^b |f(t)|\,dt$

Since $|f(t)| \geq 0$, then $\int_a^b |f(t)|\,dt \geq 0$.

Claim: if $\int_a^b |f(t)|\,dt = 0$, $f(t) = 0 \quad \forall t \in [a,b] \subset \mathbb{R}$

suppose not, then $\exists t \in [a,b]$ s.t. $f(t) > 0$. denote $f(t) = \varepsilon$.

Since $f(t)$ is continuous, $\exists \delta > 0$ s.t. $\forall x \in B_\delta(t)$ $|f(t) - f(x)| < \frac{\varepsilon}{2}$.

$\Rightarrow \forall x \in B_\delta(t)$, $|f(x)| \geq \frac{\varepsilon}{2}$.

Then $\int_{t-\delta}^{t+\delta} |f(t)|\, dt \geq \int_{t-\delta}^{t+\delta} \frac{\varepsilon}{2}\, dt = \frac{\delta\varepsilon}{2}$

Then $\int_a^b |f(t)|\, dt \geq \int_{t-\delta}^{t+\delta} |f(t)|\, dt = \frac{\delta\varepsilon}{2} > 0$.

Contradiction with $\int_a^b |f(t)|\, dt = 0$

$\Rightarrow f(t) = 0 \quad \forall t \in [a,b]$

Now consider $\|af(t)\|_{L_1}$

$$\|af(t)\|_{L_1} = \int_a^b |af(t)|\, dt$$
$$= \int_a^b |a| \cdot |f(t)|\, dt$$
$$= |a| \int_a^b |f(t)|\, dt$$
$$= |a|\, \|f(t)\|_{L_1}$$

Next consider $\|f(t) + g(t)\|_{L_1}$

$$\|f(t) + g(t)\|_{L_1} = \int_a^b |f(t) + g(t)|\, dt$$

for specific $t \in [a,b]$, $f(t), g(t) \in \mathbb{R}$.

Since triangle inequality holds in $\mathbb{R}$, we have

$|f(t) + g(t)| \leq |f(t)| + |g(t)|$

$$\Rightarrow \int_a^b |f(t) + g(t)|\, dt \leq \int_a^b |f(t)| + |g(t)|\, dt = \int_a^b |f(t)|\, dt + \int_a^b |g(t)|\, dt$$
$$= \|f(t)\|_{L_1} + \|g(t)\|_{L_1}$$

To sum up, $\|f\|_{L_1}$ is a norm on $C([a,b]; \mathbb{F})$

ii) $\|f\|_{L^2} = \left(\int_a^b |f(t)|^2\, dt \right)^{\frac{1}{2}}$

Since $|f(t)|^2 \geq 0$, then $\int_a^b |f(t)|^2\, dt \geq 0 \Rightarrow \left(\int_a^b |f(t)|^2 dt \right)^{\frac{1}{2}} \geq 0$

Since $f(t)$ is continuous function and $g(t) = t^2$ is a continuous function.

$\Rightarrow |f(t)|^2$ is continuous on $[a,b]$.

Then similar to i), we can show that $\exists\, t \in [a,b]$ with δ_t s.t. $\int_{t-\delta_t}^{t+\delta_t} |f(x)|^2\, dx > 0$ if $f(t) \neq 0$ for some t.

Then. $f(t) = 0 \iff \| f(t) \|_{L^2} = 0.$

Now consider $| a f(t) |$.

Then $\| a f(t) \|_{L^2} = (\int_a^b | a f(t) |^2 dt)^{\frac{1}{2}}$

$$= (a^2 \int_a^b | f(t) |^2 dt)^{\frac{1}{2}}$$

$$= |a| (\int_a^b | f(t) |^2 dt)^{\frac{1}{2}}$$

$$= |a| \, \| f(t) \|_{L^2} .$$

Next consider $\| f(t) + g(t) \|_{L^2}$.

$$(\| f(t) + g(t) \|_{L^2})^2 = \int_a^b | f(t) + g(t) |^2 dt .$$

$$= \int_a^b f^2(t) . dt + 2 \int_a^b f(t) . g(t) . dt . + \int_a^b g^2(t) . dt .$$

According to cauchy-shawtz inequality,

$$\langle f(t), g(t) \rangle = \int_a^b f(t) . g(t) . dt \leq \| f(t) \| . \| g(t) \| . = (\int_a^b f(t)^2 dt .)^{\frac{1}{2}} (\int_a^b g(t)^2 dt)^{\frac{1}{2}} .$$

$$\Rightarrow (\| f(t) + g(t) \|_{L^2})^2 = \int_a^b f^2(t) . dt + 2 \int_a^b f(t) . g(t) . dt + \int_a^b g^2(t) . dt .$$

$$\leq \int_a^b f^2(t) . dt + 2 (\int_a^b f^2(t) dt)^{\frac{1}{2}} (\int_a^b g^2(t) . dt)^{\frac{1}{2}}$$

$$+ \int_a^b g^2(t) . dt$$

$$= (\| f(t) \|_{L^2} + \| g(t) \|_{L^2})^2$$

$$\Rightarrow \| f(t) + g(t) \|_{L^2} \leq \| f(t) \|_{L^2} + \| g(t) \|_{L^2} .$$

To sum up, $\| f \|_{L^2}$ is a norm on $C([a,b], \mathbb{F})$

iii) $\| f \|_{L^\infty} = \sup_{x \in [a,b]} | f(x) |$

Since $| f(x) | \geq 0$. then $\| f \|_{L^\infty} \geq 0$.

If $\| f \|_{L^\infty} = 0$, then $| f(x) | \leq 0 \ \forall \ x \in [a,b] \Rightarrow f(x) = 0 \ \ \forall x \in [a,b]$

If $f(x)=0$, then $\sup_{x\in[a,b]} |f(x)| = 0$.

$\Rightarrow \quad \|f\|_{L^\infty} = 0 \iff f(x)=0$.

Now consider $\|a.f\|_{L^\infty} = \sup_{x\in[a,b]} |a.f(x)|$.

$$= a.\sup_{x\in[a,b]} |f(x)|$$

$$= a.\|f\|_{L^\infty}.$$

Next consider $\|f+g\|_{L^\infty}$.

$$\|f+g\|_{L^\infty} = \sup_{x\in[a,b]} |f(x)+g(x)|.$$

Since f and g are continous and $[a,b]$ is compact.

Then. $\exists\ t\in[a,b]$ s.t. $|f(t)+g(t)| = \sup_{x\in[a,b]} |f(x)+g(x)|$

Thus we know $\sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)| \geq |f(t)| + |g(t)|$

$$\geq |f(t)+g(t)| = \sup_{x\in[a,b]} |f(x)+g(x)|$$

Then $\|f+g\|_{L^\infty} \leq \|f\|_{L^\infty} + \|g\|_{L^\infty}$.

3.26.

i). pf:

$\|X\|_1 = \sum_{i=1}^{n} |x_i|$

$\|X\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$.

Then $\|X\|_1^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |X_i|.|X_j|. \geq \sum_{i=1}^{n} |X_i|^2 = \|X\|_2^2$.

$\Rightarrow \quad \|X\|_2 \leq \|X\|_1$.

Now consider $y = [1, 1, \cdots 1]^T$.

According to cauchy-shawtz inequality, using L^2 norm:

denote $z = [|X_1|, \cdots |X_n|]$

Then $\langle z, y\rangle = \sum_{i=1}^{n} |x_i| \leq \|z\|_2 \cdot \|y\|_2 = \sqrt{n} \cdot \|x\|_2$.

$\Rightarrow \quad \|x\|_1 \leq \sqrt{n}\,\|x\|_2$.

To sum up, $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\,\|x\|_1$

ii).

$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$.

$\|x\|_\infty = \sup |x_i|$.

Then $\|x\|_2 \geq \sqrt{\left(\sup_{x\in(a,b)} |x_i|\right)^2} = \|x\|_\infty$.

$\|x\|_2^2 = \sum_{i=1}^{n} x_i^2$.

$\|x\|_\infty^2 = \left(\sup |x_i|\right)^2$

since $x_i^2 \leq (\sup |x_i|)^2$.

then $\sum_{i=1}^{n} x_i \leq n\,(\sup |x_i|)^2$

$\Rightarrow \|x\|_2 \leq \sqrt{n} \cdot \|x\|_\infty$

To sum up, $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \cdot \|x\|_\infty$.

3.28

i). $\|A\|_1 = \sup_{|x|=1} \|Ax\|_1$.

$\|A\|_2 = \sup_{|x|=1} \|Ax\|_2$.

For $\forall x$ s.t. $|x| = 1$ we have $\|Ax\|_2 \leq \|Ax\|_1 \leq \sqrt{n}\,\|Ax\|_2$.

Then $\sup \|Ax\|_2 \leq \sup \|Ax\|_1 \leq \sqrt{n} \sup \|Ax\|_2$.

Then $\frac{1}{\sqrt{n}} \sup \|Ax\|_2 \leq \sup \|Ax\|_1 \leq \sqrt{n} \sup \|Ax\|_2$.

$\Rightarrow \frac{1}{\sqrt{n}} \|A\|_2 \leq \|A\|_1 \leq \sqrt{n} \|A\|_2$.

ii). $\|A\|_2 = \sup_{\|x\|=1} \|Ax\|_2$

$\|A\|_\infty = \sup_{|x|=1} \|Ax\|_\infty$

From 3.27, we know $\forall x$ s.t. $|x|$ we have $\|Ax\|_\infty \leq \|Ax\|_2 \leq \sqrt{n} \|Ax\|_\infty$.

Then $\sup \|Ax\|_\infty \leq \sup \|Ax\|_1 \leq \sqrt{n} \sup \|Ax\|_\infty$.

Thus $\frac{1}{\sqrt{n}} \sup \|Ax\|_\infty \leq \sup \|Ax\|_2 \leq \sqrt{n} \sup \|Ax\|_\infty$.

$\Rightarrow \frac{1}{\sqrt{n}} \|A\|_\infty \leq \|A\|_2 \leq \sqrt{n} \|A\|_\infty$.

3.29.

1). Pf: $\|Q\|_2 = \sup_{|x|=1} \|Qx\|_2$

$= \sup_{|x|=1} \sqrt{\langle Qx, Qx \rangle}$

Since Q is orthonormal, then $\langle Qx, Qx \rangle = \langle x, x \rangle$

$\Rightarrow \|Q\|_2 = \sup_{|x|=1} \sqrt{\langle x, x \rangle} = 1$.

2) Pf: $\|R_x\| = \sup_{|A|=1} \|R_x A\| = \sup_{|A|=1} \|Ax\|$

We know for $\forall A$ s.t. $|A|=1$, $\frac{|Ax|}{|x|} \leq \sup_{|x|=1} |Ax| = 1$

$\Rightarrow |Ax| \leq x \quad \forall A \in M_n(F) \Rightarrow \sup_{|A|=1} \|Ax\| \leq |x|$.

Then we know that $\|Rx\| \leq \|x\|_2$.

Then for any $x \in \mathbb{F}^n$ with $x \neq 0$, we can extend x to a basis of $\mathbb{F}^n$ make x as the first column of the orthonormal set.

Then $q_1 = \frac{x}{\|x\|}$

Take $A = [q_1 \ q_2 \ \cdots \ q_n]$ Then $A^H = \begin{bmatrix} q_1^H \\ \vdots \\ q_n^H \end{bmatrix}$

A^H is also orthonormal

$$A^H \cdot x = \begin{bmatrix} \frac{\langle x, x \rangle}{\|x\|_2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \|x\|_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow \|A^H \cdot x\|_2 = \|x\|_2$

Then the equality holds for $\|Rx\| \leq \|x\|_2$

Hence $\|Rx\| = \|x\|_2$

3.30.

pf: $\|A\|_S = \|SAS^{-1}\| \geq 0$ Since $\|\cdot\|$ is a norm on M_n.

If $A = 0$, then $\|A\|_S = \|0\| = 0$

If $\|A\|_S = 0$, then $\|SAS^{-1}\| = 0$. Since $\|\cdot\|$ is a norm on M_n

$\Rightarrow SAS^{-1} = 0 \Rightarrow S^{-1}SAS^{-1}S = A = 0$.

Now consider scalar multiplication $\|aA\|_S = \ = \|aSAS^{-1}\|$.

$= |a| \cdot \|SAS^{-1}\|$

$= |a| \cdot \|A\|_S$.

Consider $A, B \in M_n$

$$\|A+B\|_S = \|S(A+B)S^{-1}\|$$

$$= \|SAS^{-1} + SBS^{-1}\|$$

Since $\|\cdot\|$ is a norm on M_n. then

$$\|A+B\|_S = \|SAS^{-1} + SBS^{-1}\| \leq \|SAS^{-1}\| + \|SBS^{-1}\|. = \|A\|_S + \|B\|_S.$$

$\Rightarrow$ $\|A\|_S$ is a matrix norm on M_n

3.37

Pf: We should have $L(x^2) = 2$, $L(x) = 1$, $L(1) = 0$.

Then suppose $q_L = ax^2 + bx + c$

We should have $L(x^2) = \langle x^2, q_L\rangle = \int_0^1 x^2(ax^2+bx+c)dx$

$$= \tfrac{1}{5}ax^5\big|_0^1 + \tfrac{1}{4}bx^4\big|_0^1 + \tfrac{1}{3}cx^2\big|_0^1 = 2$$

$$L(x) = \langle x, q_L\rangle = \int_0^1 x\cdot(ax^2+bx+c)\,dx$$

$$= \tfrac{1}{4}a + \tfrac{1}{3}b + \tfrac{1}{2}c = 1$$

$$L(1) = \langle 1, q_L\rangle = \int_0^1 ax^2+bx+c\,dx$$

$$= \tfrac{1}{3}a + \tfrac{1}{2}b + c = 0.$$

$\Rightarrow$ Then solve the system of equations we should get

$$\begin{cases} a = 180 \\ b = -168 \\ c = 24 \end{cases}$$

$\Rightarrow$ $q_L(x) = 180x^2 - 168x + 24$

3.38.

We should have $D(x^2) = 2x. = 0x^2 + 2x + 0$

$D(x) = 1 = 0x^2 + 0x + 1$

$D(1) = 0 = 0x^2 + 0x + 0$

$$\Rightarrow D = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

If we use the L^2 norm. As shown in Example 3.7.9. we know

$D^* = -D.$

$$\Rightarrow D^* = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

3.39.

i). $(S+T)^* = S^* + T^*$ $\qquad (\alpha T)^* = \bar{\alpha}.T^*.$

pf: for $v \in V$ and $w \in W$.

Then we know $\langle (S+T).v, w\rangle = \langle Sv + Tv, w\rangle = \langle Sv, w\rangle + \langle Tv, w\rangle$

$= \langle v, S^* w\rangle + \langle v, T^* w\rangle$

$= \langle v, S^* w + T^* w\rangle$

$= \langle v, (S^* + T^*) w\rangle$

$\Rightarrow (S+T)^* = S^* + T^*.$

$\langle \alpha T v, w\rangle = \bar{\alpha}\langle Tv, w\rangle = \bar{\alpha}\langle v, T^* w\rangle = \langle v, \bar{\alpha} T^* w\rangle$

$\Rightarrow (\alpha T)^* = \bar{\alpha}.T^*.$

ii) for any $v \in V$ and $w \in W$.

$$\langle S^* w, v \rangle = \overline{\langle v, S^* w \rangle} = \overline{\langle Sv, w \rangle} = \langle w, Sv \rangle$$

$$\Rightarrow (S^*)^* = S$$

iii) we should have

$$\langle v, T^* S^* w \rangle = \langle Tv, S^* w \rangle$$

$$= \langle STv, w \rangle$$

$$\Rightarrow (ST)^* = T^* S^*.$$

iv) Consider $T^* (T^{-1})^*$.

for any v_1 and $v_2 \in V$

$$\langle v_1, T^* (T^{-1})^* v_2 \rangle = \langle Tv_1, (T^{-1})^* v_2 \rangle$$

$$= \langle T^{-1} T v_1, v_2 \rangle$$

$$= \langle v_1, v_2 \rangle$$

Since v_1 and v_2 are arbitrary

then $T^* (T^{-1})^* = I$

$$\Rightarrow (T^{-1})^* = (T^*)^{-1}$$

340.

i). pf: take $A, B, C \in M_n(F)$

We have $\langle AB, C \rangle = \langle B, A^* C \rangle$

Additionally, we have $\langle AB, C \rangle = B^H A^H C = \langle B, A^H C \rangle$

$\Rightarrow$ Since B and C are arbitrary

$A^* = A^H.$

ii) From i), we know that $A_1^* = A^H$

$\Rightarrow \langle A_2 A_1^*, A_3 \rangle = \langle A_2 A_1^H, A_3 \rangle$

$= tr(A_1 A_2^H \cdot A_3)$

$= tr(A_2^H A_3 A_1)$

$= \langle A_2, A_3 A_1 \rangle$

iii) Take $X, Y \in M_n(F)$

$\langle T_A(X), Y \rangle = \langle X, T_A^*(Y) \rangle$

$\Rightarrow \langle T_A(X), Y \rangle = tr((AX - XA)^H \cdot Y)$

$= tr(X^H A^H Y - A^H X^H Y)$

$= tr(X^H A^H Y) - tr(A^H X^H Y)$

$= tr(X^H A^H Y) - tr(X^H Y A^H)$

$= tr(X^H (A^H Y - Y A^H))$

$= tr(X^H T_{A^*}(Y))$

$= \langle X, T_{A^*}(Y) \rangle$

$\Rightarrow (T_A)^* = T_{A^*}$

3.44.

pf: Now suppose $Ax = b$ has a solution $x \in \mathbb{F}^n$. and $\exists y \in N(A^H)$ with $\langle y, b \rangle \neq 0$.

We know that $N(A^H) = R(A)^\perp$. $\Rightarrow y \in R(A)^\perp$.

$\Rightarrow \langle y, Ax \rangle = 0$

$\Rightarrow \langle y, b \rangle = 0$ contradiction.

Now suppose $Ax = b$ doesn't have a solution. Wts: $\exists y \in N(A^H)$ with $\langle y, b \rangle \neq 0$.

We know that $F^n = R(A) \oplus R(A)^\perp$.

Then b can be expressed as $b = v + w$ where $v \in R(A)$ and $w \in R(A)^\perp$.

Claim: $w \neq 0$.

Suppose not. Then $b = v \in R(A)$. $\Rightarrow \exists x$ s.t. $Ax = b$. contradiction.

$\Rightarrow w \neq 0$.

Now take $w \in N(A^H)$. then $\langle w, b \rangle = \langle w, w \rangle \neq 0$.

The proof is complete.

3.45

pf: For $\forall A \in \text{Skew}_n(\mathbb{R})$.

$A = A^T$

take arbitrary $B \in \text{Sym}_n(\mathbb{R})$

$$\begin{aligned}\langle A, B \rangle = tr(A^T B) &= tr(B^T A) \\ &= tr(BA) \\ &= tr(B(-A^T)) \\ &= -tr(A^T B)\end{aligned}$$

$\Rightarrow tr(A^T B) = 0$

$\Rightarrow A \in \text{Sym}_n(\mathbb{R})^\perp$

for $\forall A \in \text{Sym}_n(\mathbb{R})^\perp$

for $\forall i, j \in \{1, 2, \dots, n\}$. take $B = \begin{bmatrix} 0 & \cdots & \cdots & \vdots \\ \vdots & \ddots & 1 & \vdots \\ \vdots & 1 & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$ where all elements are 0 except for (i, j) and (j, i) position. Additionally $B \in \text{Sym}_n(\mathbb{R})$

$\Rightarrow tr(A^T B) = a_{ij} + a_{ji} = 0$

$\Rightarrow a_{ij} = -a_{ji}$ for $\forall i, j$.

$\Rightarrow A = -A^T$.

$\Rightarrow A \in \text{Skew}_n(\mathbb{R})$

$\Rightarrow \text{Sym}_n(\mathbb{R})^\perp = \text{Skew}_n(\mathbb{R})$

3.46.

i). pf: $Ax \in R(A)$ is trivial.

Since $x \in N(A^H A)$, then $A^H A x = 0 \Rightarrow A^H (Ax) = 0$

$\Rightarrow Ax \in N(A^H)$ by definition

ii) pf: We know that $A^H = A^*$

$\Rightarrow N(A^H) = R(A)^\perp$

$(\Rightarrow)$ for $\forall x \in N(A^H A)$

$A^H A x = 0$

$\Rightarrow Ax \in R(A)^\perp$

But $Ax \in R(A)$. then $Ax = 0$.

$\Rightarrow x \in N(A)$

$(\Leftarrow)$ suppose $x \in N(A)$

then $Ax = 0 \Rightarrow A^H A x = 0$

$\Rightarrow x \in N(A^H A)$

iii) pf: By rank nullity theorem, we have

$R(A) = n - N(A)$

$R(A^H A) = n - N(A^H A)$

Since $N(A) = N(A^H A)$

$\Rightarrow R(A) = R(A^H A)$

$\Rightarrow \mathrm{Rank}(A) = R(A) = R(A^H A) = \mathrm{Rank}(A^H)$

iv). pf: Since A has linearly independent columns

$\Rightarrow$ $Rank(A) = n$.

$\Rightarrow$ $Dim(N(A)) = n - Rank(A) = n-n = 0$

Since $N(A^H A) = N(A)$

$\Rightarrow$ $Rank(A^H A) = Dim(R(A^H A))$

$= n - Dim(N(A^H A))$

$= n$

$\Rightarrow$ $A^H A$ is full rank

$\Rightarrow$ $A^H A$ is invertible

3.47.

i) Pf: $P^2 = A(A^H A)^{-1} A^H A (A^H A)^{-1} A^H$

$= A(A^H A)^{-1} A^H$

$= P$

ii) $P^H = [A(A^H A)^{-1} A^H]^H$

$= (A^H)^H [(A^H A)^{-1}]^H A^H$

$= A \cdot [(A^H A)^H]^{-1} A^H$

$= A (A^H A)^{-1} A^H$

$= P$

iii) Since A has rank n, then A^H has rank n.

Firstly $Rank(P) \leq \min\{Rank(A), Rank(A^H), Rank((A^H A)^{-1})\} = n$

Now multiply P by A^H $\Rightarrow$ $A^H P = A^H A (A^H A)^{-1} A^H = A^H$.

$\Rightarrow$ $Rank(A^H) \leq \min\{Rank(A^H), Rank(P)\}$.

$\Rightarrow$ $Rank(P) \geq n$ $\Rightarrow$ $Rank(P) = n$.

3.48.

i). pf: Consider $A, B \in M_n(\mathbb{R})$

$$P(A+B) = \frac{A+B+(A+B)^T}{2} = \frac{A+A^T}{2} + \frac{B+B^T}{2} = P(A) + P(B)$$

$\Rightarrow$ P is linear

ii) $$P^2(A) = P(P(A))$$

$$= P\left(\frac{A+A^T}{2}\right)$$

$$= \frac{\frac{A+A^T}{2} + \frac{A^T+A}{2}}{2} = \frac{A+A^T}{2} = P(A)$$

$\Rightarrow P^2 = P$

iii) pf: $$\langle A, P(B)\rangle = tr\left(A^T \cdot \frac{B+B^T}{2}\right)$$

$$= tr\left(\frac{A^TB}{2}\right) + tr\left(\frac{A^TB^T}{2}\right)$$

$$= tr\left(\frac{A^TB}{2}\right) + tr\left(\frac{BA}{2}\right)$$

$$= tr\left(\frac{A^TB}{2}\right) + tr\left(\frac{AB}{2}\right)$$

$$= tr\left(\frac{A+A^T}{2} B\right)$$

$$= \langle P(A), B\rangle$$

$\Rightarrow P^* = P$.

iv) for $\forall A \in N(P)$

$$P(A) = \frac{A+A^T}{2} = 0$$

$\Rightarrow A = -A^T$

$\Rightarrow A \in Skew_n(\mathbb{R})$

$\Rightarrow N(P) \subset Skew_n(\mathbb{R})$

for $\forall A \in Skew_n(\mathbb{R})$

then $A = -A^T$.

then $P(A) = \frac{A+A^T}{2} = 0 \Rightarrow A \in N(P)$

$\Rightarrow Skew_n(\mathbb{R}) \subseteq N(P)$

$\Rightarrow N(P) = Skew_n(\mathbb{R})$

(v) for $\forall B \in R(P)$, $\exists A \in M_n(F)$ s.t. $P(A) = \frac{A+A^T}{2} = B$.

then $B^T = \frac{A^T+A}{2} = B$

$\Rightarrow B = Sym_n(\mathbb{R})$

$\Rightarrow R(P) \subseteq Sym_n(\mathbb{R})$

for $\forall B \in Sym_n(\mathbb{R})$, then since $B \in M_n(F)$

$$P(B) = \frac{B+B^T}{2} = \frac{B+B}{2} = B.$$

$\Rightarrow B \in R(P)$

$\Rightarrow Sym_n(\mathbb{R}) \subseteq R(P)$

$\Rightarrow Sym_n(\mathbb{R}) = R(P)$

Vi) pf: $\|A-P(A)\|_F = \sqrt{tr\left(\left(\frac{A-A^T}{2}\right)^T \cdot \frac{A-A^T}{2}\right)}$.

$$= \sqrt{\frac{1}{4}\left[tr(A^TA) - tr((A^T)^2) - tr(A^2) + tr(AA^T)\right]}$$

$$= \sqrt{\frac{1}{4}\left[2tr(A^TA) - 2tr(A^2)\right]}$$

$$= \sqrt{\frac{tr(A^TA) - tr(A^2)}{2}}$$

3.50. Now take $A = \begin{bmatrix} x_1^2 & y_1^2 \\ \vdots & \vdots \\ x_i^2 & y_i^2 \\ \vdots & \vdots \\ x_n^2 & y_n^2 \end{bmatrix}$ $x = \begin{bmatrix} r \\ s \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$

then $\begin{bmatrix} \hat{r} \\ \hat{s} \end{bmatrix} = (A^TA)^{-1}A^Tb$.