Problem Set 3.

4.2 pf: We should have the matrix representation of D

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Now we have

$$\lambda E - D = \begin{bmatrix} \lambda & 0 & 0 \\ -2 & \lambda & 0 \\ 0 & -1 & \lambda \end{bmatrix}$$

$\Rightarrow \det(\lambda E - D) = \lambda^3. \quad = 0$

$\Rightarrow \lambda = 0$

$$\Rightarrow \lambda E - D = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Then there are one linearly independent eigenvector of D with respect to λ.

$\Rightarrow$ the algebraic multiplicity is 3 and geometric multiplicity is one.

4.4. i) pf: denote $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Since $A = A^H$.

$\Rightarrow$ a and d are real numbers since $a = \bar{a}$ and $d = \bar{d}$

$b = \bar{c}$.

$$\Rightarrow \lambda E - A = \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix}$$

$\Rightarrow \det(\lambda E - A) = \lambda^2 - (a+d)\lambda + ad - bc$

$= \lambda^2 - (a+d)\lambda + ad - \|b\|^2.$

$= 0$

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4ad + 4\|b\|^2}}{2}$$

$\Rightarrow \quad (a+d)^2 - 4ad + 4\|b\|^2$

$\quad = (a-d)^2 + 4\|b\|^2 \geq 0$

$\Rightarrow$ λ must be real.

ii) Now suppose $A = -A^H$

and denote $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow \quad \bar{a} = -a. \qquad \bar{b} = -b.$

$\Rightarrow$ a and b only have imaginary part.

$\bar{b} = -c$ and $\bar{c} = -b \quad \Rightarrow \quad b = -\bar{c}$.

$\lambda E - A = \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix}$

$\det(\lambda E - A) = \lambda^2 - (a+d)\lambda + ad - bc$

Consider $(a+d)^2 - 4ad + 4bc$

$\quad = (a-d)^2 + 4bc$

Since $a-d$ only has imaginary part

$\Rightarrow \quad (a-d)^2 \leq 0$

$\quad 4bc = -4\|c\|^2 \leq 0$

$\Rightarrow \quad (a+d)^2 - 4ad + 4bc \leq 0$

$\Rightarrow$ it can only have imaginary eigenvalue.

4.6.

pf: denote the upper triangular matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{nn} \end{bmatrix}$

$\Rightarrow \lambda E - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ & \lambda - a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & - & \text{—} & \lambda - a_{nn} \end{bmatrix}$

$\Rightarrow \det(\lambda E - A) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})$

$= 0$

$\Rightarrow$ The characteristic polynomial implies.

$\lambda_1 = a_{11}, \ \lambda_2 = a_{22} \ \cdots \ \lambda_n = a_{nn}$

4.8.

i). Now we just have to prove that $\{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$ are linearly independent

Now consider $a\sin(x) + b\cos(x) + c\cdot\sin(2x) + d\cdot\cos(2x)$

$= a\sin(x) + b\cos(x) + 2c\sin(x)\cdot\cos(x) + d\cos^2(x) - d\sin^2(x).$

$= 0.$

when $x=0$ $\quad a\cdot\sin(x) + b\cos(x) + c\sin(2x) + d\cos(2x) = b + d = 0$

when $x = \frac{\pi}{2}$ $\quad a\cdot\sin(x) + b\cos(x) + c\cdot\sin(2x) + d\cos(2x) = a - d = 0.$

when $x = \pi$ $\quad a\cdot\sin(x) + b\cos(x) + c\cdot\sin(2x) + d\cos(2x) = -b + d = 0$

Now we know that $a = b = d = 0$

$\Rightarrow c = 0.$

$\Rightarrow a = b = c = d = 0.$

$\Rightarrow \{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$ are linearly independent.

i). $\sin'(x) = \cos(x)$

$\cos'(x) = -\sin(x)$

$\sin'(2x) = 2\cos(2x)$

$\cos'(2x) = -2\sin(2x)$

then $D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

ii) The two D-invariant subspace are $\{\sin(x), \cos(x)\}$ and

$\{\sin(2x), \cos(2x)\}$.

For $\{\sin(x), \cos(x)\}$. consider $v \in \{\sin(x), \cos(x)\} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}$.

then $D \cdot v = \begin{bmatrix} -b \\ a \\ 0 \\ 0 \end{bmatrix} \in \{\sin(x), \cos(x)\}$.

For $\{\sin(2x), \cos(2x)\}$. consider $w \in \{\sin(2x), \cos(2x)\} = \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix}$

then $D \cdot w = \begin{bmatrix} 0 \\ 0 \\ -2d \\ 2c \end{bmatrix} \in \{\sin(2x), \cos(2x)\}$.

Then we know that they are two D-invarient subspaces.

4.13. Now consider $\lambda E - A = \begin{bmatrix} \lambda - 0.8 & -0.4 \\ -0.2 & \lambda - 0.6 \end{bmatrix}$

$$\det(\lambda E - A) = \lambda^2 - 1.4\lambda + 0.48 - 0.08$$

$$= \lambda^2 - 1.4\lambda + 0.4 \qquad \Rightarrow \lambda_1 = 1, \ \lambda_2 = 0.4.$$

$$= (\lambda - 1)(\lambda - 0.4)$$

$$\lambda_1 E - A = \begin{bmatrix} 0.2 & -0.4 \\ -0.2 & 0.4 \end{bmatrix}$$

then $\begin{bmatrix} 0.2 & -0.4 \\ -0.2 & 0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.2x_1 - 0.4x_2 \\ -0.2x_1 + 0.4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{cases} 0.2x_1 - 0.4x_2 = 0 \\ -0.2x_1 + 0.4x_2 = 0 \end{cases} \Rightarrow x_1 = 2x_2$ then $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector.

$$\lambda_2 E - A = \begin{bmatrix} -0.4 & -0.4 \\ -0.2 & -0.2 \end{bmatrix}$$

$$\begin{bmatrix} -0.4 & -0.4 \\ -0.2 & -0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -0.4x_1 - 0.4x_2 \\ -0.2x_1 - 0.2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{cases} -0.4x_1 - 0.4x_2 = 0 \\ -0.2x_1 - 0.2x_2 = 0 \end{cases} \Rightarrow x_1 = -x_2$ then $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector

with respect to $\lambda_2 = 0.4$.

Then construct $P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

$\det(P) = -2 - 1 = -3.$

Then $P^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

$\Rightarrow P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$

4.15

Pf: Now firstly we try to prove $\{f(\lambda_i)\}_{i=1}^{n}$ are the eigenvalues.

for arbitrary λ_i, denote the corresponding eigenvector as x_i.

then $f(A) \cdot x_i = a_0 I x_i + a_1 A x_i + \cdots + a_n A^n x_i$

$$= (a_0 + a_1 \lambda_i + \cdots + a_n \lambda_i^n) \cdot x_i$$

$$= f(\lambda_i)\, x_i$$

$\Rightarrow$ $f(\lambda_i)$ is an eigenvalue of $f(A)$

Then we prove that there are no other eigenvalues.

Suppose there are other eigenvalues λ'.

Since A is a semi-simple matrix. Then we can form a set of eigenvectors $\{x_1, x_2, \cdots, x_n\}$. that forms a basis of $\mathbb{F}^n$.

Then since there is another eigenvalue λ', then there is another eigenvector x'. And these are distinct eigenvalues, then $x_1, \cdots, x_n, x'$ are linearly independent. But the dimension of $\mathbb{F}^n$ is n. Contradiction

$\Rightarrow$ $\{f(\lambda_i)\}_{i=1}^{n}$ are the eigenvalues of $f(A)$.

4.16.

i) from 4.13, we know

$$P^{-1} A P = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix} \Rightarrow A = P \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix} P^{-1}$$

$$\Rightarrow A^n = P \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0.4^n \end{bmatrix} P^{-1}$$

$$\Rightarrow \lim_{n\to\infty} A^n = P \lim_{n\to\infty} \begin{bmatrix} 1 & 0 \\ 0 & 0.4^n \end{bmatrix} \cdot P^{-1} = P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & [illegible] \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\Rightarrow \lim_{n\to\infty} A^n = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Then $$A^k = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.4^k \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0.4^k \\ 1 & -0.4^k \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} + \frac{0.4^k}{3} & \frac{2}{3} - \frac{2}{3} \cdot 0.4^k \\ \frac{1}{3} - \frac{1}{3} 0.4^k & \frac{1}{3} + \frac{2}{3} 0.4^k \end{bmatrix}$$

$$\Rightarrow A^k - \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{0.4^k}{3} & -\frac{2}{3} 0.4^k \\ -\frac{1}{3} 0.4^k & \frac{2}{3} 0.4^k \end{bmatrix}$$

The 1 norm is maximum column sum.

$$\Rightarrow \left\| A^k - \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \right\|_1 = 2 \cdot \left| \frac{2}{3} 0.4^k \right| = \frac{4}{3} 0.4^k.$$

Then $\forall \varepsilon > 0$, $\exists N = [\log_{0.4} \frac{3}{4} \varepsilon]$. s.t. $\forall k > N$

$$\left\| A^k - \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \right\|_1 < \varepsilon.$$

Then $B = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ is the $\lim_{n\to\infty} A^n$ under 1 norm

ii) Now consider

$$\| A^k - \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \|_\infty = \| \begin{bmatrix} \frac{0.4^k}{3} & -\frac{2}{3} 0.4^k \\ -\frac{0.4^k}{3} & \frac{2}{3} 0.4^k \end{bmatrix} \|_\infty$$

$$= |\frac{1}{3} 0.4^k| + |-\frac{2}{3} 0.4^k|$$

$$= 0.4^k.$$

Then $\forall \varepsilon > 0$, $\exists N = [\log_{0.4} \varepsilon]$ s.t. $\forall k > N$.

$$\| A^k - \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \|_\infty < \varepsilon.$$

iii) By theorem. 4.3.12

$$\lambda_1 = f(1) = 3 + 5 \times 1 + 1^3 = 9.$$

$$\lambda_2 = f(0.4) = 3 + 5 \times 0.4 + 0.4^3 = 3 + 2 + 0.064$$
$$= 5.064.$$

$\Rightarrow \lambda_1 = 9 \quad \lambda_2 = 5.064.$

4.18. pf: Consider the eigenvalues of A^T.

$\Rightarrow |\lambda E - A^T| = |(\lambda E - A^T)^T| = |\lambda E - A|.$

$\Rightarrow$ the eigenvalues of A are the eigenvalues of A^T.

$\Rightarrow$ for λ. $\exists$ eigenvector x s.t. $A^T x = \lambda x$.

$\Rightarrow (A^T x)^T = (\lambda x)^T \Rightarrow x^T A = \lambda x^T.$

$\Rightarrow \exists$ a non zero row vector x^T s.t. $x^T A = \lambda x^T$.

4.20.

pf: Since A and B are orthonormally similar.

then $\exists U$ orthonormal matrix s.t.

$B = U^H A U$.

$\Rightarrow B^H = (U^H A U)^H = U^H A^H (U^H)^H = U^H A U = B$.

$\Rightarrow B$ is hermitian.

4.24.

pf. ① Suppose A is hermitian matrix

then $\langle x, Ax \rangle = x^H A x$.

$\Rightarrow$ Consider $\langle x, Ax \rangle$ as a 1×1 matrix

then $\langle x, Ax \rangle^H = \overline{\langle x, Ax \rangle}$

$\Rightarrow \langle x, Ax \rangle^H = x^H A^H x = x^H A x$

$\Rightarrow \langle x, Ax \rangle = \overline{\langle x, Ax \rangle}$

Then $\langle x, Ax \rangle$ takes on only real values.

$\Rightarrow P(x) = \dfrac{\langle x, Ax \rangle}{\|x\|^2}$ takes only real values if A is hermitian

② suppose A is skew matrix,

then $\langle x, Ax \rangle = x^H A x$

$\langle x, Ax \rangle^H = x^H A^H x = -x^H A x = -\langle x, Ax \rangle$

$\Rightarrow \overline{\langle x, Ax \rangle} = -\langle x, Ax \rangle$

$\Rightarrow \langle x, Ax \rangle$ takes on only imaginary parts

$\Rightarrow \rho(x) = \dfrac{\langle x, Ax \rangle}{\|x\|^2}$ takes on only imaginary values

when A is a skew matrix.

4.25.

i) Consider $(x_1 x_1^H + \cdots + x_n x_n^H) \cdot x_j$

$= x_1 x_1^H x_j + \cdots + x_j x_j^H x_j + \cdots + x_n x_n^H x_j$

Since $\{x_1 \cdots x_n\}$ is orthonormal eigenvectors.

Then $(x_1 x_1^H + \cdots x_n x_n^H) \cdot x_j$

$= x_j x_j^H x_j$

$= x_j$.

This is true for $\forall j \in \{1, \cdots n\}$.

Since $\{x_1 \cdots x_n\}$ is a basis in C^n.

$\Rightarrow \forall x \in C^n$. $(x_1 x_1^H + \cdots + x_n x_n^H) x = x$.

$\Rightarrow I = x_1 x_1^H + \cdots + x_n x_n^H$

ii) Consider $(\lambda_1 x_1 x_1^H + \cdots + \lambda_n x_n x_n^H) x_j$

$= \lambda_j x_j$

This is true for $\forall j \in \{1, \cdots n\}$.

$\Rightarrow \lambda_1 x_1 x_1^H + \cdots + \lambda_n x_n x_n^H$ has eigenvalues $\lambda_1 \cdots \lambda_n$.

and corresponding eigenvectors $x_1 \cdots x_n$

$\Rightarrow$ denote $P = [x_1 \ \cdots \ x_n]$

$$B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

then $\lambda_1 x_1 x_1^H + \cdots + \lambda_n x_n x_n^H$

$= P^H B P$

$= A$

4.27.

Pf: denote $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$

Now consider e_j.

$e_j^H \cdot A \ e_j = a_{jj} > 0$ by definition of positive definite matrix.

This is true for $\forall j \in \{1, 2 \cdots n\}$

Then all diagonal entries are real positive

4.28. $AB = \begin{bmatrix} \sum_{i=1}^{n} a_{1i} b_{i1} & & \\ & \ddots & \\ & & \sum_{i=1}^{n} a_{ni} b_{i1} \end{bmatrix}$

$\Rightarrow tr(AB) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji} b_{ij}$.

$tr(A) = \sum_{i=1}^{n} a_{ii}$ $\quad tr(B) = \sum_{j=1}^{n} b_{jj}$ $\rightarrow tr(A) tr(B) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ii} b_{jj}$

Since A and B are positive semi-definite.

then $\exists X, Y \in M_n(F)$ s.t. $A = X^H X$ $B = Y^H Y$.

$\Rightarrow tr(AB) = tr(X^H X Y^H Y) = tr($

$= tr((XY^H)^H XY^H)$

denote $XY^H = Z$ then

$tr(AB) = tr(Z^H Z)$

$= \sum_{i=1} \sum_{j=1} \bar{z}_{ij} z_{ij}$.

$= \sum_{i=1} \sum_{j=1} \| z_{ij} \|^2$

≥ 0

We know that $\langle A, B \rangle = tr(A^H B)$

$\cancel{\ \cdot tr(B) = tr(X^H X) \cdot tr(Y^H Y)}$

~~By cauchy shawrz inequality~~

$\cancel{tr(X^H Y)^2 \leq tr(X^H X)\, tr(Y^H Y)}$

$\cancel{tr((YX^H)^H (YX^H)) \leq \sqrt{tr(YX^H)}}$

Now we have.

$tr(AB) \leq \sqrt{tr(A^2)\, tr(B^2)} = \sqrt{(\sum_{i=1}^{n} \lambda_i^2)(\sum_{i=1}^{n} \eta_i^2)}$

Since A and B are positive semidefinite $\Rightarrow \lambda_i \geq 0$ $\eta_i \geq 0$ $\forall i$.

$\Rightarrow tr(AB) \leq \sqrt{(\sum_{i=1}^{n} \lambda_i^2)(\sum_{i=1}^{n} \eta_i^2)}$

$\leq \sqrt{(\sum_{i=1}^{n} \lambda_i)^2 (\sum_{i=1}^{n} \eta_i)^2}$

$= tr(A)\, tr(B)$

$\Rightarrow tr(AB) \leq tr(A)\, tr(B)$

To sum up $0 \leq tr(AB) \leq tr(A)\, tr(B)$

4.31.

i). $\|A\|_2 = \sup_{|x|=1} \|Ax\|_2 = \sup_{|x|=1} \sqrt{x^H A^H A x}.$

$A^H A$ is symetric $\Rightarrow$ $A^H A$ is normal

Hence it's orthonormally diagonalizable.

Then let $\{x_1, x_2 \cdots x_n\}$ be a set of orthonormal eigenvectors of $A^H A$

then for $\forall x \in \mathbb{F}^n$ with $\|x\|_2 = 1$ $\quad x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$

$\Rightarrow |x|_2^2 = a_1^2 + a_2^2 + \cdots + a_n^2 = 1$

$\Rightarrow \quad x^H A^H A x = \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2.$

Then since $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$

then $\sup x^H A^H A x = \lambda_1.$

$\Rightarrow \sup_{|x|=1} \|Ax\|_2 = \sqrt{\lambda_1} = \sigma_1.$

$\Rightarrow \|A\|_2 = \sigma_1.$

ii) $\|A^{-1}\|_2 = \sup_{|x|} \sqrt{x^H (A^H A)^{-1} x}.$

For $\forall B \in M_n(F)$ with B being symetric. suppose B has eigenvalues $\lambda_1 - \lambda_n$. then

B^{-1} has eigenvalues $\lambda_1^{-1} - - \lambda_2^{-1}$

Then for $A^H A$, since $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$

$\Rightarrow d_1^{-1} \leq d_2^{-1} \leq \cdots \leq d_n^{-1}$

$\Rightarrow \sigma_1^{-1} \leq \sigma_2^{-1} \leq \cdots \leq \sigma_n^{-1}$

By the result in i) we have $\|A^{-1}\|_2 = \sigma_n^{-1}$

iii) Firstly prove $\|A_2\|_2^2 = \|A^H A\|_2$.

$\|A_2\|^2 = (\sup \sqrt{x^H A^H A x})^2 = \sup x^H A^H A x = \sigma_1^2 = d_1$

$\|A^H A\|_2 = \sup \sqrt{x^H A^H A A^H A x}$.

Now $A^H A A^H A = (A^H A)^2$.

$\Rightarrow A^H A A^H A$ has eigenvalues $d_1^2, d_2^2, \dots d_n^2$

$\Rightarrow d_1^2 \geq d_2^2 \geq \dots \geq d_n^2$

$\Rightarrow \|A^H A\|_2 = \sqrt{d_1^2} = d_1 = \|A_2\|^2$.

iv) Since U and V are orthonormal

$\Rightarrow \forall x\, y \in \mathbb{F}^m \langle x, y\rangle = \langle Ux, Uy\rangle$.

$\forall v, w \in \mathbb{F}^n, \langle v, w\rangle = \langle Vx, Vy\rangle$

$\Rightarrow \|UAV\|_2 = \sup_{|x|} \|UAVx\|_2 = \sup_{|x|=1} \sqrt{\langle UAVx, UAVx\rangle} = \sup_{|x|=1} \sqrt{\langle AVx, AVx\rangle}$

Since V is orthonormal. $V(\mathbb{F}^n) = R(V) = \mathbb{F}^n$.

additionall for $\forall x \in \mathbb{F}^n$ with $\|x\|_2 = 1$ $\quad \langle Vx, Vx\rangle = \langle x, x\rangle = 1$

$\Rightarrow Vx = y$ and $\|y\| = 1$

$$\Rightarrow \|UAV\|_2 = \sup_{\|x\|=1} \sqrt{\langle AVx, AVx\rangle} = \sup_{\|y\|=1} \sqrt{\langle Ay, Ay\rangle} = \|A\|_2.$$

4.32.

i). We know $\|A\|_F = \sqrt{tr(A^H A)}$.

$$\begin{aligned}
\Rightarrow \|UAV\|_F &= \sqrt{tr(V^H A^H U^H U A V)} \\
&= \sqrt{tr(V^H A^H A V)} \\
&= \sqrt{tr(A^H A V V^H)} \\
&= \sqrt{tr(A^H A)} \\
&= \|A\|_F .
\end{aligned}$$

ii) Now apply Singular value decomposition to. A.

Suppose $A = U \Sigma V$ where U and V are orthonormal and

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & 0 \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ 0 & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \ 0 \end{bmatrix}$$

according to i). $\quad \|A\|_F = \|U\Sigma V\|_F = \|\Sigma\|_F$

$$\Rightarrow \|\Sigma\|_F = (\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2)^{\frac{1}{2}}$$

$$\Rightarrow \|A\|_F = \|\Sigma\|_F = (\sigma_1^2 + \cdots + \sigma_r^2)^{\frac{1}{2}}.$$

4.33

Pf: Now for $\forall x, y$. since U and V are orthonormal matrix.

$\exists\, v, w$ s.t. $Uv = y$ $\quad Vw = x$.

and. $|v| = 1$, $|w| = 1$ by definition of orthonormal transformati

$$\Rightarrow \sup_{|x|=1,\, |y|=1} |y^H A x| = \sup_{|v|=1,\, |w|=1} |v^H U^H U \Sigma V^H V w|.$$

$$= \sup_{|v|=1,\, |w|=1} |v^H \Sigma w|.$$

$$= \sup_{|v|=1,\, |w|=1} \left| \sum_{i=1}^{n} \sigma_i \bar{v}_i w_i \right|.$$

Since $|v| = 1$ and $|w| = 1$, then $\bar{v}_i w_i \leq 1$.

$$\Rightarrow \sup_{|x|=1,\, |y|=1} |y^H A x| = \sigma_1 \quad \text{where we set } v_1 = w_1 = 1$$

$$\Rightarrow \sup_{|x|=1,\, |y|=1} |y^H A x| = \sigma_1 = \|A\|_2.$$

4.36.

Consider $A = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$

Then $A^H A = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 5 & 6 \\ 6 & 8 \end{bmatrix}$

$\det(\lambda E - A) = \det\left(\begin{bmatrix} \lambda-2 & -2 \\ -1 & \lambda-2 \end{bmatrix}\right) = (\lambda-2)^2 - 2 = \lambda^2 - 4\lambda + 2.$

$\Rightarrow \lambda_1 = \frac{4+\sqrt{8}}{2} = 2+\sqrt{2}$

$\lambda_2 = \frac{4-\sqrt{8}}{2} = 2-\sqrt{2}$

$\det(\eta E - A^H A) = \det\left(\begin{bmatrix} \eta-5 & -6 \\ -6 & \eta-8 \end{bmatrix}\right) = \eta^2 - 13\eta + 4$

$\eta_1 = \frac{13+\sqrt{153}}{2}$ $\quad \eta_2 = \frac{13-\sqrt{153}}{2}$.

and. $\eta_1 \neq \lambda_1^2 \neq \lambda_2^2$

$\eta_2 \neq \lambda_1^2 \neq \lambda_2^2$.

$\Rightarrow$ singular values are not equal to any of A's eigenvalues.

Additionally $\det(A) = 2 \neq 0$

4.38.

i). $AA^+A = A$.

$A = U_1 \Sigma_1 V_1^H$

$A^+ = V_1 \Sigma_1^{-1} U_1^H$

$\Rightarrow$ $A^+A = V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H$

$= V_1 V_1^H = E.$

$\Rightarrow AA^+A = AE = A.$

ii) $A^{+} A A^{+}$

from i) we know $A^{+}A = E$

$$\Rightarrow A^{+}AA^{+} = EA^{+} = A^{+}.$$

iii) $(AA^{+})^{H} = AA^{+}$

$$AA^{+} = U_1 \Sigma_1 V_1^{H} V_1 \Sigma_1^{-1} U_1^{H} = U_1 \Sigma_1 \Sigma_1^{-1} U_1^{H} = U_1 \Sigma_1 \Sigma_1^{-1} U_1^{H}$$

$$\Rightarrow (AA^{+})^{H} = U_1 (\Sigma_1^{-1})^{H} V_1^{H} V_1 \Sigma_1^{H} U_1^{H}.$$

$$= U_1 \Sigma_1^{-1} V_1^{H} V_1 \Sigma_1 U_1^{H}.$$

$$= U_1 \Sigma_1^{-1} \Sigma_1 U_1^{H}$$

$$= AA^{+}.$$

iv) $(A^{+}A)^{H} = A^{+}A.$

$$A^{+}A = V_1 \Sigma_1^{-1} U_1^{H} U_1 \Sigma V_1^{H}$$

$$(A^{+}A)^{H} = V_1 \Sigma_1^{H} U_1^{H} U_1 (\Sigma_1^{-1})^{H} V_1^{H}$$

$$= V_1 \Sigma_1 U_1^{H} U_1 \Sigma_1^{-1} V_1^{H}$$

$$= V_1 \Sigma_1 \Sigma_1^{-1} V_1^{H}$$

$$= V_1 \Sigma_1^{-1} \Sigma_1 V_1^{H}$$

$$= V_1 \Sigma_1^{-1} U_1^{H} U_1 \Sigma_1 V_1^{H}$$

$$= A^{+}A.$$

v). We know $R(A) = N(A^H)\perp$.

$$AA^\dagger = A \cdot V_1 \Sigma_1^{-1} U_1^H$$

$$= A V_1 \Sigma_1^{-1} (\Sigma_1^{-1})^H V_1^H A^H.$$

$\Rightarrow \forall x \in N(A^H)$.

$$AA^\dagger x = A V_1 \Sigma_1^{-1} (\Sigma_1^{-1})^H V_1^H (A^H x)$$

$$= 0$$

$\forall y \in N(A^H)\perp$

$$A A^\dagger x = U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} (\Sigma_1^{-1})^H V_1^H (A^H y)$$

$$= U_1^H U_1 x$$

$$= x.$$

$\Rightarrow \quad AA^\dagger = Proj_{N(A^H)^\perp} = Proj_{R(A)}$

iv) We know $R(A^H) = N(A)\perp$

then $\forall x \in N(A)$

$$A^\dagger A x = A^\dagger (Ax) = A^\dagger 0 = 0.$$

$\forall y \in N(A)\perp.$

$$A^\dagger A y = V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H y$$

$$= y.$$

$\Rightarrow \quad A^\dagger A = Proj_{N(A)\perp} = Proj_{R(A^H)}$