Problem Set 5

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1 7.1

Proof. Now for arbitrary $x, y \in conv(S)$. We have the following:

$$x = \lambda_1 x_{i_1} + ... + \lambda_{i_m} x_{i_m}$$
$$y = \eta_1 x_{j_1} + ... + \eta_{i_n} x_{i_n}$$

Then for any convex combination of x and y, we should have:

$$\lambda x + (1 - lambda)y = \lambda\lambda_1 x_{i_1} + ... + \lambda\lambda_{i_m} x_{i_m} + (1-\lambda)\eta_1 x_{j_1} + ... + (1-\lambda)\eta_{i_n} x_{i_n} \in conv(S)$$

This is true for arbitray x, y and $\lambda \in [0, 1]$. Thus we know that $conv(S)$ is convex. □

2 7.2

2.1 a)

Denote the hyperplane as P. Now for arbitrary $x, y \in P$ and $\lambda \in [0, 1]$, we have $z = \lambda x + (1-\lambda)y$. Then $< z, a >=< \lambda x + (1-\lambda)y, a >= \lambda < x, a > +(1-\lambda) < y, a >= b$. Then $z \in P$. Thus we know that P is convex.

3 7.4

3.1 i)

$||x-y||^2 =< x-y, x-y >=< x-p+p-y, x-p+p-y >=< x-p, x-p > + < x-p, p-y > + < p-y, x-p > + < p-y, p-y >= ||x-p||^2 + ||y-p||^2 + 2 < x-p, p-y >$.

3.2 ii)

Since $< x-p, p-y >\geq 0$. Additionally, $p \neq y$, then $||y-p||^2 > 0$. Then we know that $||x-y||^2 > ||x-p||^2$. Then we konw that $||x-y|| > ||x-p||$.

3.3 iii)

$$
\begin{aligned}
||x-z||^2 &= <x-z, x-z> \\
&= <x-\lambda y-(1-\lambda)p, x-\lambda y-(1-\lambda)p> \\
&= <x-p, x-p> + <x-p, \lambda p-\lambda y> + <\lambda p-\lambda y, x-p> + \lambda^2 <y-p, y-p> \\
&= ||x-p||^2 + 2\lambda <x-p, p-y> + \lambda^2 ||y-p||^2
\end{aligned}
$$

4 7.6

Denote the set $\{x \in \Re^n | f(x) < c\}$ as A. Then for arbitrary $x, y \in A$ and $\lambda \in [0,1]$, denote $z = \lambda x + (1-\lambda)y$. Since the function is convex, then we should have $f(z) \leq \lambda f(x) + (1-\lambda)f(y) \leq c$. Then $z \in A$. Thus the set A is a convex set.

5 7.7

For arbitrary $x, y \in C$ and $\eta \in [0,1]$, denote $z = \eta x + (1-\eta)y$. Then

$$f(z) = \Sigma_{i=1}^k \lambda_i f_i(z) \leq \Sigma_{i=1}^k \lambda_i \eta f_i(x) + (1-eta) f_i(y) = \eta f(x) + (1-\eta) f(y)$$

Then we know that $f(.)$ is a convex function.

6 7.13

Proof. Suppose not. Then there exist $x, y \in \Re^n$ such that $f(x) \neq f(y)$. Suppose $f(x) = a, f(y) = b$ and without loss of generality assume $a > b > 0$. Then we let z be such that $x = \lambda z + (1-\lambda)y$. Then we know that $f(x) \leq \lambda f(z) + (1-\lambda)f(y)$. Thus $f(z) \geq \dfrac{f(x)-(1-\lambda)f(y)}{\lambda}$. Now take λ to 0, then we know that $f(z)$ tends to infinity. This contradicts with f being bounded. Thus we know that f has to be constant on $\Re^n$. □

7 7.20

Since f and $-f$ are both convex, then we know that for arbitrary $x, y \in \Re^n$ and $\lambda \in [0,1]$, we should have with $z = \lambda x + (1-\lambda)y$ $f(z) = \lambda f(x) + (1-\lambda)f(y)$.
Firstly we want to show that for arbitrary $a \in \Re$, $f(ax) - f(0) = a(f(x) - f(0))$. When $a = -1$, we should have $f(0) = 0.5f(x) + 0.5f(-x)$ then it must be that $f(-x) - f(0) = -1(f(x) - f(0))$. When $a \in [0,1]$, we know that $f(ax) = af(x) + (1-a)f(0)$. Then $f(ax) - f(0) = a(f(x) - f(0))$. When $a \in (1, \infty)$, we should have $\frac{1}{a}()f(x) - f(0)) = f(ax) - f(0)$. Thus $f(ax) - f(0) = a(f(x) - f(0))$. When $a \in (-\infty, 0)$, we can apply the above conclusion to achieve the same result. Then we know that it must be that $f(ax) - f(0) = a(f(x) - f(0))$.

Then we want to show that $f(ax+by)-f(0)=a(f(x)-f(0))+b(f(y)-f(0))$. With the results above, we just have to show that $f(x+y)-f(0)=(f(x)-f(0))+(f(y)f(0)$. Now consider we should have the following results with the conclustion reached above:

$$f(x+y)+f(x-y)=2*f(x)$$
$$f(x+y)+f(y-x)=2*f(y)$$
$$f(x-y)+f(y-x)=2*f(0)$$

Then we can solve for the above equations:

$$f(x+y)-f(0)=f(x)-f(0)+(f(y)-f(0))$$
$$f(x-y)-f(0)=f(x)-f(0)-(f(y)-f(0))$$
$$f(y-x)-f(0)=f(y)-f(0)-(f(x)-f(0))$$

Now we have that $f(x+y)-f(0)=(f(x)-f(0))+(f(y)f(0)$.
With the above results we know that $f(x)=L(x)+f(0)$.

8 7.21

$\Rightarrow$
Suppose x^* is a local minimizer for the problem

$$\begin{aligned}\min&\phi(f(x))\\ s.t.&G(x)\leq 0\\ &H(x)=0\end{aligned}$$

. Then there is a neighbourhood $B_\delta(x^*)$, such that $\forall x\in B_\delta(x^*)$, $\phi(f(x^*))\leq\phi(f(x))$. Since $\phi(.)$ is a strictly increasing function, we konw that $f(x^*)\leq f(x)$ with the constraint $G(x)\leq 0; H(x)=0$. Then it must be that x^* is a local minimizer for the problem:

$$\begin{aligned}\min&f(x)\\ s.t.&G(x)\leq 0\\ &H(x)=0\end{aligned}$$

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$\Leftarrow$ Suppose x^* is a local minimizer for the problem

$$\begin{aligned}\min&f(x)\\ s.t.&G(x)\leq 0\\ &H(x)=0\end{aligned}$$

. Then there is a neighbourhood $B_\delta(x^*)$, such that $\forall x\in B_\delta(x^*)$, $f(x^*)\leq f(x)$. Since $\phi(.)$ is a strictly increasing function, we konw that $\phi(f(x^*))\leq\phi(f(x))$ with the constraint $G(x)\leq 0; H(x)=0$. Then it must be that x^* is a local minimizer for the problem:

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