On the problem of the rank of the Mordell-Weil group

Master's Thesis

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Notation

Let E be an elliptic curve over $\mathbb{Q}$ with minimal Weierstrass equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ and conductor N_E.

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$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r,$$

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where $r < \infty$ is the **rank** of the elliptic curve. Furthermore, by Wiles' theorem, there is a modular parametrization over $\mathbb{Q}$

$$\phi_E \colon X_0(N_E) \to E.$$

We denote by $\deg \phi_E$ the smallest degree of these modular parametrizations of E.

Quadratic twist

We define the **quadratic twist** $E^{(D)}$ of E as another elliptic curve which is isomorphic to E over the quadratic extension with discriminant D over $\mathbb{Q}$. The quadratic twist $E^{(D)}$ is given by the equation

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Remark

The conductor $N^{(D)}$ of $E^{(D)}$ is D^2N_E.

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One strategy to address the main question could be to use the following conjecture by Watkins:

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But we have the following result

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$$\log N_E \ll v_2(\deg \phi_E)$$

Faltings' height

We consider the **Néron differential**

$$\omega_E = \frac{\mathrm{d}x}{2y + a_1 x + a_3}$$

Its pullback through the modular parametrization satisfies

$$\phi_E^* \omega_E = c_E f_E(z)\,\mathrm{d}z,$$

where f_E is a generalized newform of weight 2 for $\Gamma_0(N)$, and $c_E \in \mathbb{Q}^*$.

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where f_E is a generalized newform of weight 2 for $\Gamma_0(N)$, and $c_E \in \mathbb{Q}^*$. With this in mind, we set the **Faltings' height** of $E/\mathbb{Q}$ as

$$h(E) = -\frac{1}{2} \log \left(\frac{i}{2} \int_{E(\mathbb{C})} \omega_E \wedge \overline{\omega}_E \right).$$

The Petersson norm

The Petersson norm of f_E relative to $\Gamma_0(N)$ is defined by

$$\|f_E\|_N := \left(\int_{\Gamma_0(N)\backslash\mathfrak{h}} |f(z)|^2 \,\mathrm{d}x \wedge \mathrm{d}y\right)^{1/2}, \qquad z = x + iy \in \mathfrak{h}.$$

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Therefore, we have

$$\deg \phi_E = 4\pi^2 c_E^2 \, \|f_E\|_N^2 \exp(2h(E)).$$

Analytic tool

Theorem (Linnik, 1994)

Let p be the smallest prime satisfying $p \equiv a \mod b$, where a and b are coprime integers. Then,

$$p \ll b^L$$

The arithmetic of twist

Thanks to the work of Esparza-Lozano & Pasten [1] we have the following results

Lemma 1

We have $\delta(E^{(D)}, E) := \exp(2h(E^{(D)}) - 2h(E))$ is a rational number and $|v_2(\delta(E^{(D)}, E))| \leq 3$

Lemma 2

For a fundamental discriminant $D = q$ where q is a prime, we have $\|f_{E^{(q)}}\|^2_{N^{(q)}} \,/\, \|f_E\|^2_N \in \mathbb{Q}^\times$ and

$$v_2(\|f_{E^{(q)}}\|^2_{N^{(q)}} \,/\, \|f_E\|^2_N) + 1 \geq v_2((q-1).$$

Main Theorem

Recall the main result

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Fix an integer $\alpha \geq 0$. By the Dirichlet's theorem, there are primes of the form $2^{\alpha} m + 1$ with $m \geq 0$. We will consider the least of them denoted by q. Then $v_2(q-1) \geq \alpha$. Therefore, for a fundamental discriminant $D = q$ we have

$$v_2(\|f_{E(q)}\|^2_{N(q)} \,/\, \|f_E\|^2_N) \gg \alpha.$$

Thus, $v_2(\deg \phi_{E(q)} / \deg \phi_E) \gg \alpha$.

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Fix an integer $\alpha \geq 0$. By the Dirichlet's theorem, there are primes of the form $2^{\alpha} m + 1$ with $m \geq 0$. We will consider the least of them denoted by q. Then $v_2(q-1) \geq \alpha$. Therefore, for a fundamental discriminant $D = q$ we have

$$v_2(\|f_{E(q)}\|^2_{N(q)} / \|f_E\|^2_N) \gg \alpha.$$

Thus, $v_2(\deg \phi_{E(q)} / \deg \phi_E) \gg \alpha$. In particular,

$$v_2(\deg \phi_{E(q)}) \gg \alpha.$$

On other hand, recall $N^{(D)} = D^2 N_E = q^2 N_E$ so $\log N^{(q)} = 2 \log q + \log N_E$. As N_E is fixed, we denote $\log N_E$ as κ_E.

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$$\log N^{(q)} \leq 2L\alpha \log 2 + \kappa_E.$$

That is, $\log N^{(q)} \ll \alpha$. Finally,

$$\log N^{(q)} \ll \alpha \ll v_2(\deg \phi_{E^{(q)}}).$$

q.e.d.

Conclusion

Watkins' conjecture

$$\operatorname{rank} E(\mathbb{Q}) \leq v_2(\deg \phi_E).$$

$$\not\Downarrow$$

Uniform Bound's conjecture

$$\exists B \geq 0 : \quad \operatorname{rank} E(\mathbb{Q}) \leq B$$

[1] Jose A. Esparza-Lozano and Hector Pasten. "A conjecture of Watkins for quadratics twist". In: **Proceedings of the American Mathematical Society** 152 (2021), pp. 2381–2385.

[2] Andrew Wiles. "Modular Elliptic Curves and Fermat's Last Theorem". In: **Annals of Mathematics** 141.3 (1995), pp. 443–551.

[3] Gary Cornell and Joseph H. Silverman. **Arithmetic Geometry**. Springer New York, 1986.

[4] Mark Watkins. "Computing the modular degree of an elliptic curve.". In: **Experimental Mathematics** 11.4 (2002), pp. 487–502.

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