Jan Suchánek

1)

a) $y_{n+1} + 0.8 y_n = 5$

Characteristic polynomial: $\lambda + 0.8 = 0$

$\lambda = -0.8$

$FS = \{(-0.8)^n\}$ ← Fundamental system theorem

$\underline{y_H(n) = c \cdot (-0.8)^n}, \quad c \in \mathbb{R}$

$\underline{y_p(n) = a, \quad a \in \mathbb{R}}$ ← Theorem about particular solution

$$a + 0.8a = 5$$

$$1.8a = 5$$

$$a = \frac{5}{1.8} = \frac{50}{18} = \underline{\frac{25}{9}}$$

$$y(n) = y_H(n) + y_p(n)$$

$$\underline{\underline{y(n) = c \cdot (-0.8)^n + \frac{25}{9}, \quad c \in \mathbb{R}}}$$

Thus equilibrium is stable

It is obvious that solution is oscillatory and asymptotically stable when $c \neq 0$. If $c = 0$, solution is constant => equilibrium is stable.

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1) b)

$$y_{n+2} - 4y_{n+1} + 8y_n = -5$$

ch. p. $\lambda^2 - 4\lambda + 8 = 0$

$$\lambda_{12} = \frac{4 \pm \sqrt{16-32}}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i = 2(1 \pm i)$$

$$= 2 \cdot \frac{2}{\sqrt{2}}\left(\cos\left(\frac{\sqrt{2}}{2}\pi\right) + i \sin\left(\frac{\sqrt{2}}{2}\pi\right)\right)$$

$$= 2\sqrt{2}\left(\cos\left(\frac{\sqrt{2}}{2}\pi\right) + i \sin\left(\frac{\sqrt{2}}{2}\pi\right)\right)$$

Fundamental system theorem

$$FS = \left\{ 2^{\frac{3}{2}n} \cos\left(\frac{\sqrt{2}}{2}\pi n\right), \; 2^{\frac{3}{2}n} \sin\left(\frac{\sqrt{2}}{2}\pi n\right) \right\}$$

$$\underline{y_H(n) = 2^{\frac{3}{2}n}\left(a \cos\left(\frac{\sqrt{2}}{2}\pi n\right) + b \sin\left(\frac{\sqrt{2}}{2}\pi n\right)\right), \; a, b \in \mathbb{R}}$$

$\underline{y_p(n) = c}$, $c \in \mathbb{R}$ Theorem about particular solution

$$c - 4c + 8c = -5$$

$$5c = -5 \Rightarrow \underline{c = -1} \Rightarrow \underline{y_p(n) = -1}$$

$$y(n) = 2^{\frac{3}{2}n}\left(a \cos\left(\frac{\sqrt{2}}{2}\pi n\right) + b \sin\left(\frac{\sqrt{2}}{2}\pi n\right)\right) - 1, \; a, b \in \mathbb{R}$$

Stability of equilibria:

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1) $a = 0 \wedge b = 0$

Solution is constant $\Rightarrow$ stable equilibrium.

2) $(a \neq 0 \wedge b = 0) \vee (a = 0 \wedge b \neq 0)$

$2^{\frac{3}{2}} > 1$ and range of sine and cosine is $\langle -1, 1 \rangle$, thus solution ~~will be~~ is oscillatory and explosive $\Rightarrow$ equilibrium is unstable.

3) $a \neq 0 \wedge b \neq 0$

$2^{\frac{3}{2}} > 1$ and range of sine and cosine is $\langle -1, 1 \rangle$, therefore solution is oscillatory and explosive, thus equilibrium is unstable.

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2) Cobweb model

$q^d_t = a - b p_t$ JS $\Rightarrow$ $a = 4$

$b = 1$

$q^s_t = c + d p_{t-1}$ $c = 4$

$d = 0.4$

$q^d_t = 4 - p_t$ In equilibrium:

$q^s_t = 4 + 0.4 p_{t-1}$

$q^d_t = q^s_t$

$4 - p_t = 4 + 0.4 p_{t-1}$

$0 = p_t + 0.4 p_{t-1}$

Ch. e. $\lambda + 0.4 = 0$

$\lambda = -0.4 \Rightarrow FS = \{(-0.4)^t\}$

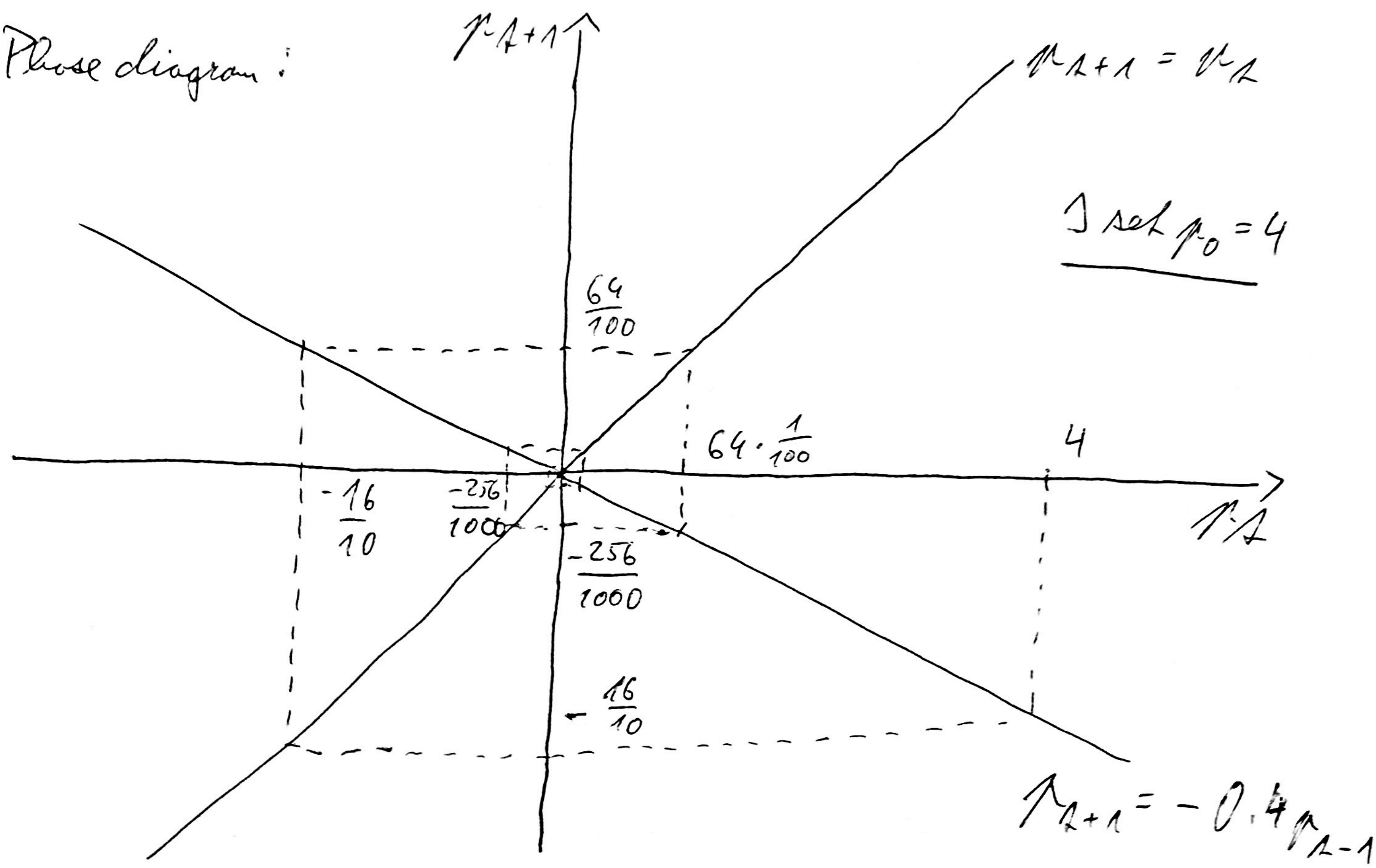
$p(t) = (-0.4)^t \cdot a, a \in \mathbb{R}$

$p(0) = p_0 \Rightarrow \underline{\underline{p(t) = (-0.4)^t \cdot p_0}}, p_0 > 0$

$\underbrace{p_0 > 0}_{\text{reasonable assumption}}$

$\underline{p_t = -0.4 p_{t-1}}$

Phase diagram:



4) My initials "JS" result in "strange" system. Thus I change my initials to "ES". That gives me

$a = -3 \qquad c = -1 \qquad e = -2$

$b = 3 \qquad d = 2 \qquad f = 1$

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Therefore:

$$\dot{x}(t) = -3x(t) + 3y(t) - 2$$

$$\dot{y}(t) = -x(t) + 2y(t) + 1$$

In the equilibrium, $x(t)$ and $y(t)$ are neither increasing or decreasing $\Rightarrow$ their first derivatives are equal to zero:

$$0 = -3x + 3y - 2 \Rightarrow y = \frac{3x+2}{3} = \underline{x + \frac{2}{3}}$$

$$0 = -x + 2y + 1 \Rightarrow \underline{y = \frac{x-1}{2}}$$

Equilibrium lines

On ~~the~~ intersection of equilibrium lines lies the equilibrium:

$$x + \frac{2}{3} = \frac{x-1}{2} \quad / \cdot 6$$

$$6x + 4 = 3x - 3$$

$$3x = -7$$

$$\underline{x = -\frac{7}{3}}$$

$$\underline{y = -\frac{7}{3} + \frac{2}{3} = -\frac{5}{3}}$$

$$\underline{\underline{E\left[-\frac{7}{3}, -\frac{5}{3}\right]}}$$

We can see how solutions behave outside of equilibrium:

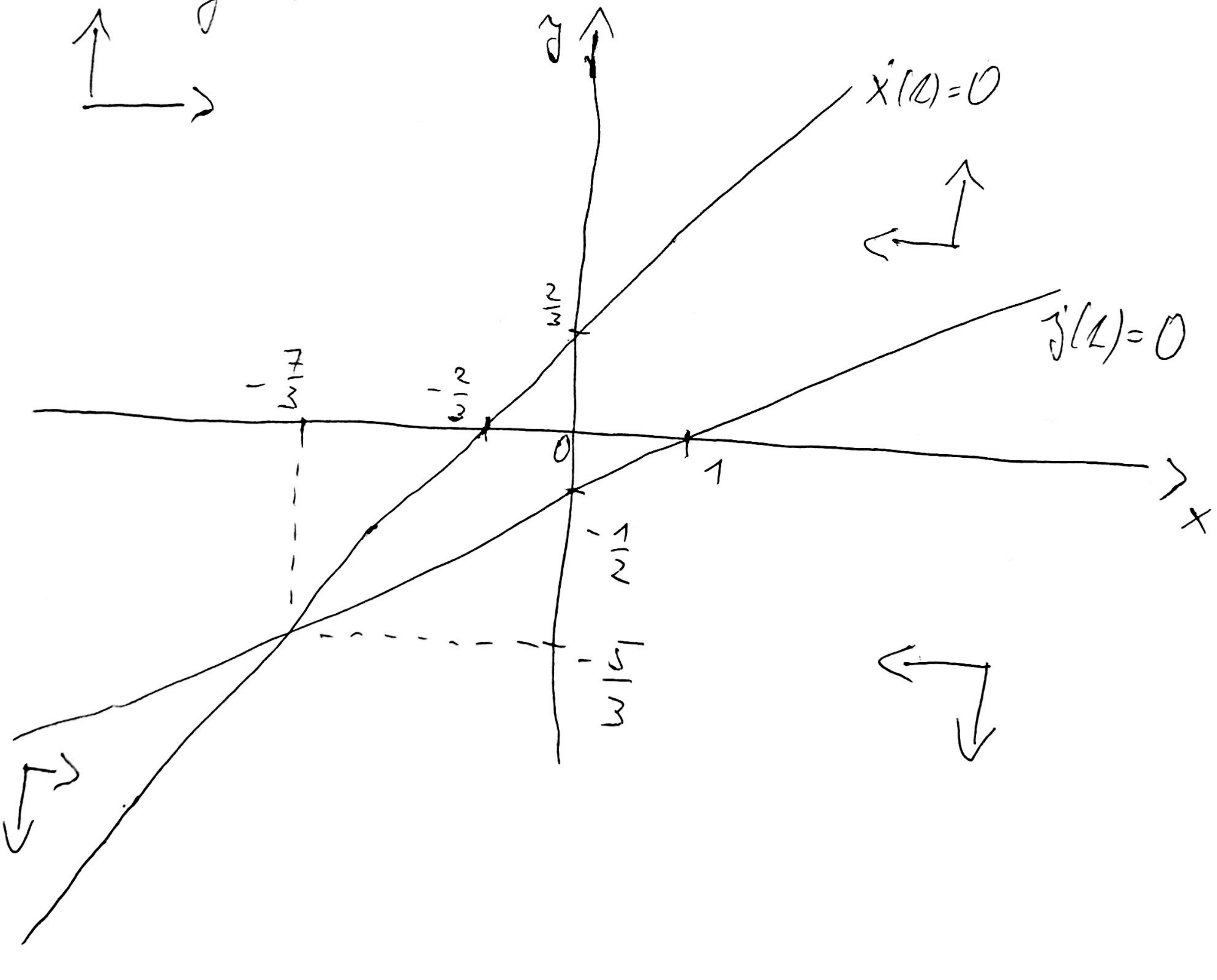
$$0 < \dot{x}(t) \iff 0 < -3x + 3y - 2 \iff y > x + \frac{2}{3}$$

$$0 < \dot{x}(t) \iff 0 > -3x + 3y - 2 \iff y < x + \frac{2}{3}$$

$$0 < \dot{y}(t) \iff 0 < -x + 2y + 1 \iff y > \frac{x - 1}{2}$$

$$0 > \dot{y}(t) \iff 0 > -x + 2y + 1 \iff y < \frac{x - 1}{2}$$

Phase diagram



Eigenvalues:

$$\dot{x} = -3x + 3y - 2 \Rightarrow -2 = \dot{x} + 3x - 3y$$

$$\dot{y} = -x + 2y + 1 \Rightarrow 1 = x + \dot{y} - 2y$$

$$A = \begin{pmatrix} \lambda+3 & -3 \\ 1 & \lambda-2 \end{pmatrix} \Rightarrow \det(A) = \begin{vmatrix} \lambda+3 & -3 \\ 1 & \lambda-2 \end{vmatrix} =$$

$$= \lambda^2 - 2\lambda + 3\lambda - 6 - (-3) = \lambda^2 + \lambda - 6 + 3 = \underline{\lambda^2 + \lambda - 3}$$

$$D = 1 - 4 \cdot (-3) = 13$$

$$\lambda_{12} = \frac{-1 \pm \sqrt{13}}{2} = \begin{cases} \frac{-1+\sqrt{13}}{2} \\ \frac{-1-\sqrt{13}}{2} \end{cases}$$

Saddle ~~stat~~ point, unstable ~~sa~~ saddle.