Problem_set_2_suchanek_jan

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Problem Set 2 Jan Suchánek

1 Problem 1

Find complete solution (without any software) of these difference equations and discuss stability of equilibrium:

1.1 a)

$$2y_{n+1} + 0.6y_n = 13$$

First, I divide the equation by 2 so that coefficient by y_{n+1} is equal to one.

$$y_{n+1} + 0.3y_n = 6.5$$

I will start by deriving solution for homogeneous equation. Characteristic polynomial for homogeneous equation:

$$\lambda + 0.3 = 0$$

From this is easily derived that $FS = \{(-0.3)^n\}$ (Theorem about fundamental system of difference equations), which brings me to solution for homogeneous equation:

$$y_h(n) = c \cdot (-0.3)^n,\ c \in \mathbb{R}, n \in \mathbb{N}$$

Now I will continue by deriving particular solution of this equation. From Theorem about particular solution can be seen that desired particular solution is equal to real constant.

$$y_p(n) = d,\ d \in \mathbb{R}$$

I will plug particular solution into original equation divided by two to derive value of d:

$$y_{n+1} + 0.3y_n = d + 0.3d = 1.3d = 6.5 \rightarrow d = 5 \rightarrow y_p(n) = 5$$

In difference equations, following holds:

$$y(n) = y_h(n) + y_p(n)$$

Therefore I can easily obtain solution:

$$y(n) = y_h(n) + y_p(n) = c \cdot (-0.3)^n + 5,\ c \in \mathbb{R}, n \in \mathbb{N}$$

Stability of equilibrium:

In this case it is pretty obvious that solution is oscillatory but damped (unless c = 0). Therefore equilibrium ($y^* = 5$) is stable.

1.2 b)

$$2y_{n+2} - 6y_{n+1} + 5y_n = 1$$

First, I divide the equation by 2 so that coefficient by y_{n+2} is equal to one.

$$y_{n+2} - 3y_{n+1} + 2.5y_n = 0.5$$

I will start by deriving solution for homogeneous equation. Characteristic polynomial for homogeneous equation:

$$\lambda^2 - 3\lambda + 2.5 = 0$$

I use discriminant to derive solutions for homogeneous equation:

$$D = (-3)^2 - 4 \cdot 2.5 = -1 \ \rightarrow \ \lambda_{1,2} = \frac{3 \pm \sqrt{-1}}{2}$$

Solution of this equation with positive imaginary part is $1.5 \cdot (1 + \frac{i}{3})$. In order to derive Fundamental system by applying Theorem about fundamental system, I need to solve following:

$$1.5 \cdot (1 + \frac{i}{3}) = \mu(cosv + isinv) \rightarrow 3sinv = cosv$$

$v \in [0, 2\pi),\ \mu \in \mathbb{R}$. But both $sinv$ and $cosv$ need to be positive (Theorem about FS), which restrict v to $[0, \frac{\pi}{2})$. From $3sinv = cosv$ I can compute v:

$$tanv = \frac{1}{3} \rightarrow v \approx 0.32175$$

Now value of μ can be easily extracted by pluging back value of v:

$$\mu = \frac{1.5}{cosv} \approx= 1.58114$$

In order to make my solution as little confusing as possible, I will refer to μ and v by their symbols instead of their approximated values. By applying Theorem about fundamental system I get that:

$$FS = \{\mu^n \cdot cos(\nu n), \mu^n \cdot sin(\nu n)\}$$

Therefore solution for homogeneous equation

$$y_h(n) = a \cdot \mu^n \cdot cos(\nu n) + b \cdot \mu^n \cdot sin(\nu n),\ a, b \in \mathbb{R}$$

Now I will continue by deriving particular solution of this equation. From Theorem about particular solution can be seen that desired particular solution is equal to real constant.

$$y_p(n) = d,\ d \in \mathbb{R}$$

I will plug particular solution into original equation divided by two to derive value of d:

$$y_{n+2} - 3y_{n+1} + 2.5y_n = d - 3d + 2.5d = 0.5d = 0.5 \rightarrow d = 1 \rightarrow y_p(n) = 1$$

In difference equations, following holds:

$$y(n) = y_h(n) + y_p(n)$$

Therefore I can easily obtain solution:

$$y(n) = y_h(n) + y_p(n) = a \cdot \mu^n \cdot cos(\nu n) + b \cdot \mu^n \cdot sin(\nu n) + 1$$

where

$$a, b \in \mathbb{R}, n \in \mathbb{N},\ \nu = arctg\left(\frac{1}{3}\right) \approx 0.32175,\ \mu = \frac{3}{2 \cdot cos\left(arct\left(\frac{1}{3}\right)\right)} \approx 1.58114$$

Stability of equilibrium:

If both a and b are equal to zero, solution is constant.

If one of the coefficients is equal to zero and the other is nonzero: Values for both sine and cosine are restricted to interval $[-1, 1]$ and μ is strictly larger than one. Thus in this case, solution will be oscillatory and explosive. Therefore equilibrium will be unstable.

If both a and b are nonzero: With μ being strictly larger than zero and both sine and cosine being restricted to $[-1, 1]$, solution will be oscillatory and explosive. Therefore equilibrium will be unstable.

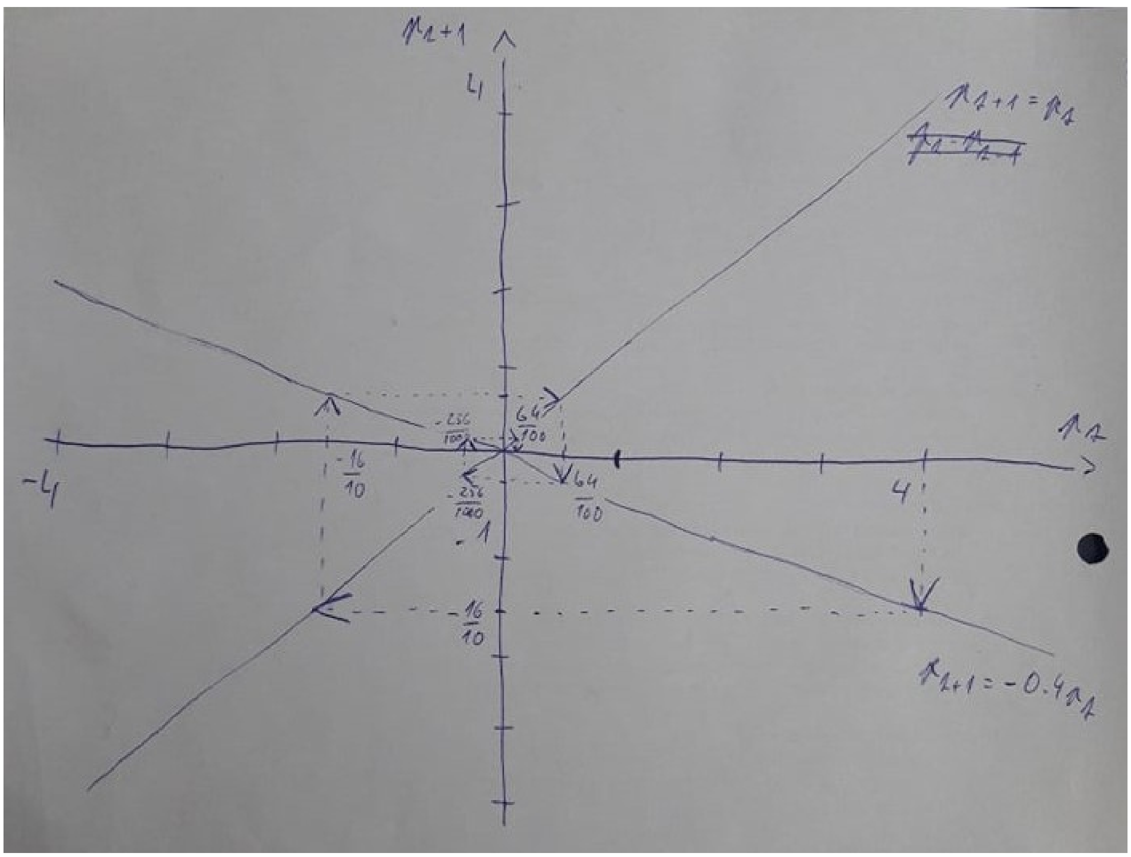
2 Problem 2 - Cobbweb model

My surname is “Suchánek”. Therefore according to the table, a = 4; b = 1; c = 4; d = 0.4. My personalized equation is therefore as follows:

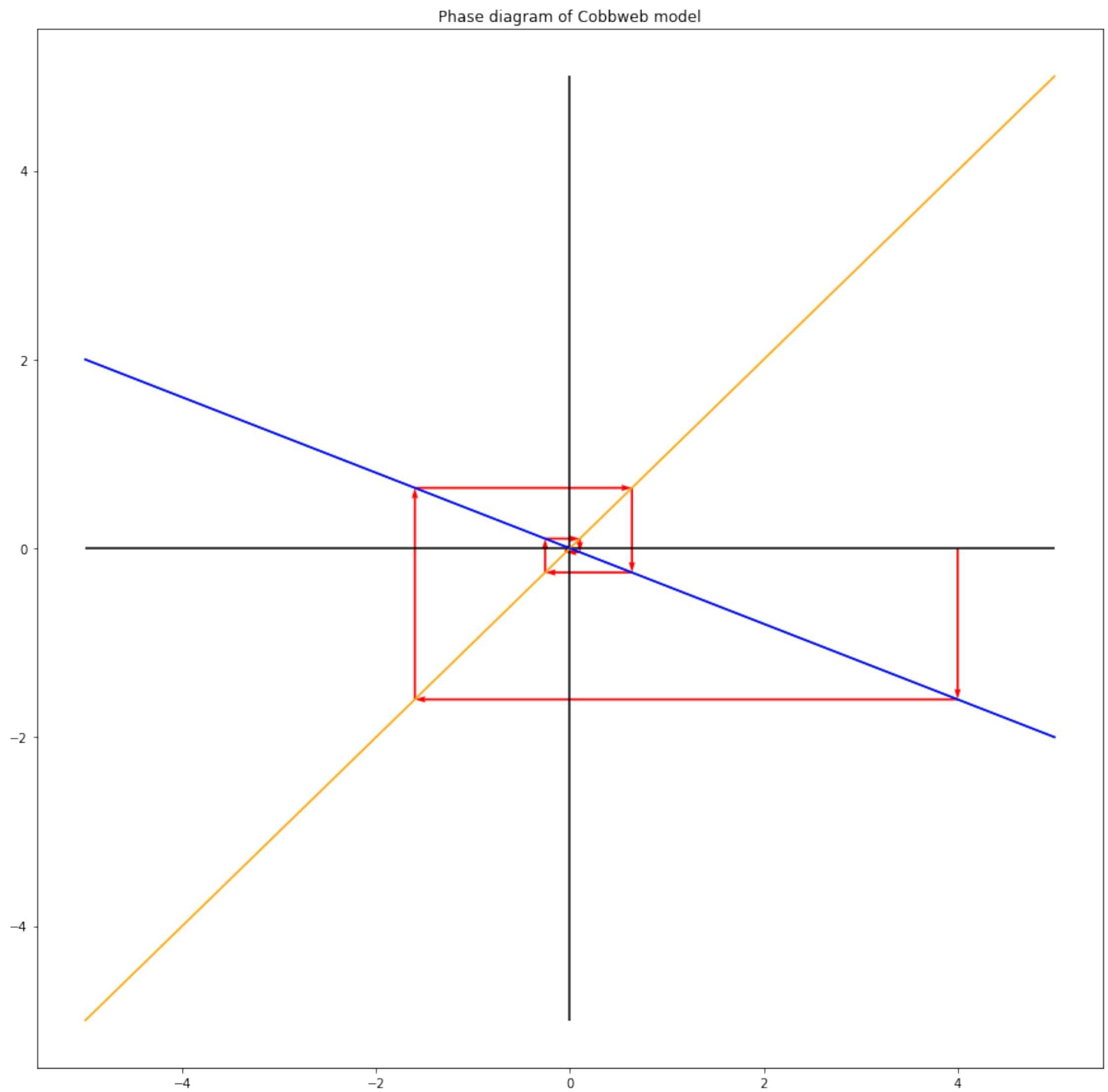
$$p_{t+1} = -0.4p_t$$
$$p_t = (-0.4)^t \cdot p_0, \quad t \in \mathbb{N}, p_0 \in \mathbb{R}$$

2.1 Phase Diagram

Hand-written:



Python phase diagram:



3 Problem 3 - Solow Model

Central equation of the Solow model:

$$\dot{k}(k) = s \cdot a \cdot k^{\alpha} - (n + \delta) \cdot k(t)$$

3.1 a) Calculation of fixed points (equilibria)

I know that in equilibrium $\dot{k}(t)$ must be equal to zero, because in equilibrium, value of $k(t)$ is staying the same. This gives me following equation:

$$0 = s \cdot a \cdot k^{\alpha} - (n + \delta) \cdot k$$

From this equation equilibrium value of k can be easily extracted:

$$s \cdot a \cdot k^{\alpha} = (n + \delta) \cdot k$$

$$k^{\alpha-1} = \frac{n + \delta}{s \cdot a}$$

$$k^{*} = \left(\frac{n + \delta}{s \cdot a}\right)^{\frac{1}{\alpha-1}}$$

If we take into account that $a = 3$, $s = 0.1$, $\delta = 0.25$, $n = 0.1$ and $\alpha = 0.3$, we get

$$k^{*} = \left(\frac{0.1 + 0.25}{0.1 \cdot 3}\right)^{\frac{1}{0.3-1}} \approx 0.8023$$

3.2 b) Establish whether equilibria are stable or unstable

We get only one equilibrium point. α is lower than 1, therefore for small k, value of $\dot{k}$ will be positive (k cannot be lower than zero - it makes sense, as level of capital per capita cannot be negative). Of course, with larger k function $\dot{k}$ begins to decrease. Eventually, it is equal to zero in equilibrium k^{*}. Then it continues to decrease. For $k < k^{*}$ is value of $\dot{k}$ positive, therefore value of $k < k^{*}$ will grow until it reaches k^{*}. For $k > k^{*}$ is value of $\dot{k}$ negative, therefore value of $k > k^{*}$ will fall until it reaches k^{*}. Therefore equilibrium point is stable.

Other way to solve this problem: function $\dot{k}$ is obviously differentiable. Therefore we can take first derivative with respect to k and see monotonicity of $\dot{k}$. In equilibrium point, $\dot{k}$ is equal to zero, around equlibrium point are values of $\dot{k}$ nonzero.

If equilibrium is stable, first derivative will be negative around equilibrium ($\dot{k}$ will be decreasing - values of k will be attracted to equilibrium).

If equilibrium is unstable, first derivative will be positive around equilibrium ($\dot{k}$ will be increasing - values of k will be repelled from equilibrium).

$$\frac{\partial \dot{k}(k)}{\partial k} = s \cdot a \cdot \alpha \cdot k^{\alpha-1} - (n + \delta)$$

Now I can plug in formula for k^{*} from part a) of this exercise:

$$\frac{\partial \dot{k}(k)}{\partial k} = s \cdot a \cdot \alpha \cdot k^{\alpha-1} - (n+\delta) = s \cdot a \cdot \alpha \cdot \left(\frac{n + \delta}{s \cdot a}\right)^{\frac{\alpha-1}{\alpha-1}} - (n+\delta) = \alpha \cdot (n+\delta) - (n+\delta) = (n+\delta) \cdot (1-\alpha) < 0$$

because $n + \delta > 0$ and $\alpha < 1 \rightarrow \alpha - 1 < 0$. Therefore it does not matter what values of parameters are given (as long as they are in line with assumptions of Solow model), equilibrium will be positive.

4 Problem 4

My name is Jan Suchánek. However, taht would give me "strange" system, therefore I will rename myself for the purpose of this problem and change my first name to "Bob". That gives me:

$a = -3$

$b = 3$

$c = -1$

$d = -2$

$e = 2$

$f = 1$

That gives me following system:

$$\dot{x}(t) = -3x(t) + 3y(t) + 2$$
$$\dot{y}(t) = -x(t) - 2y(t) + 1$$

In equilibrium, both values of $x(t)$ and $y(t)$ are neither increasing or decreasing, they are staying the same. If they are neither decreasing or increasing, their first derivatives must be equal to zero. Therefore left hand-sides of previous two equations must be equal to zero. That implies

$$\dot{x}(t) = 0 = -3x(t) + 3y(t) + 2 \ \rightarrow \dot{x}(t) = 0 \text{ when } y = x - \frac{2}{3}$$

$$\dot{y}(t) = 0 = -x(t) - 2y(t) + 1 \ \rightarrow \dot{y}(t) = 0 \text{ when } y = \frac{-x+1}{2}$$

Last two formulas describe equilibrium lines. In their intersection lies the equilibrium. I can find the it by solving these two equations.

$$x - \frac{2}{3} = \frac{-x+1}{2}$$

$$x = \frac{7}{9} \ \rightarrow y = \frac{1}{9}$$

In addition, we can see how solutions behave outside of equilibrium. For example, $x(t)$ is increasing exactly when $\dot{x}(t) > 0$ and vice versa (works analogically for y). That gives us following:

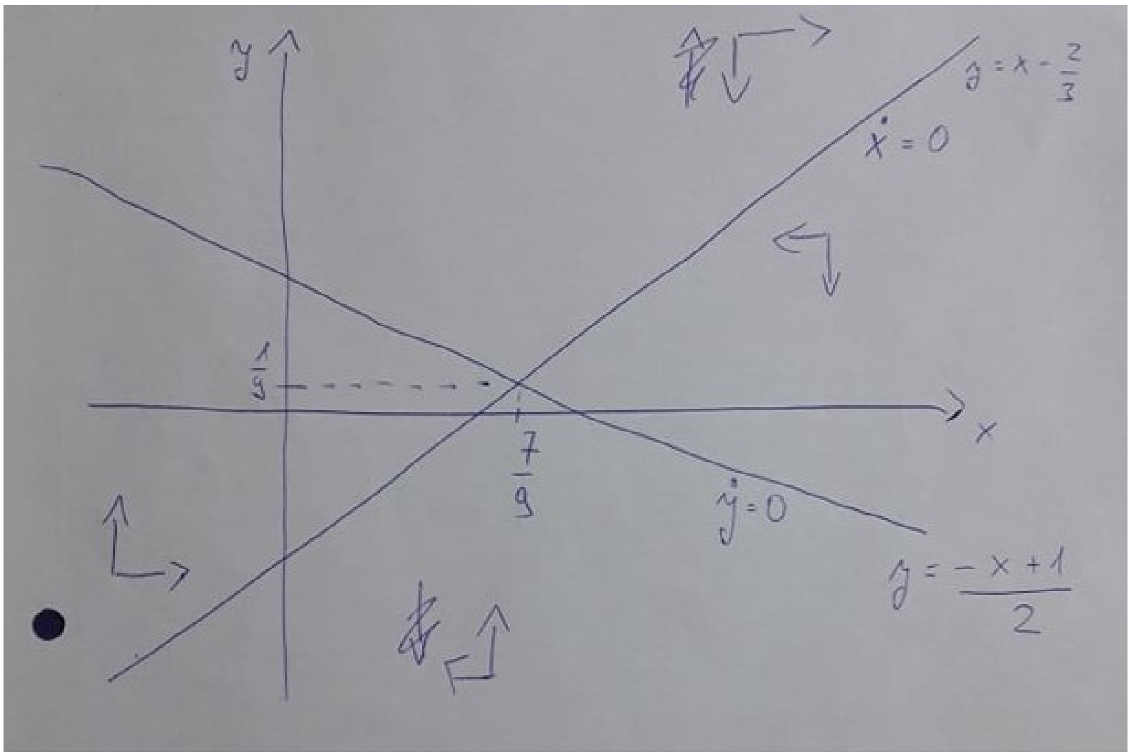
$$0 < \dot{x}(t) = -3x(t) + 3y(t) + 2 \ \rightarrow \ \text{x is increasing when } y > x - \frac{2}{3}$$

$$0 > \dot{x}(t) = -3x(t) + 3y(t) + 2 \ \rightarrow \ \text{x is decreasing when } y < x - \frac{2}{3}$$

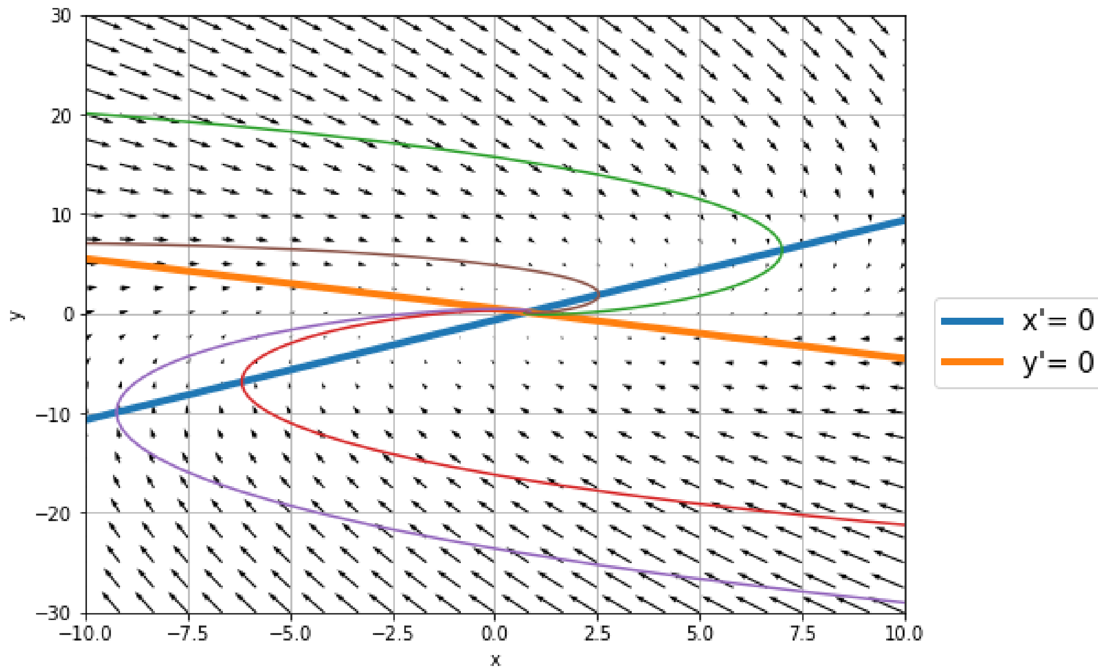
$$0 < \dot{y}(t) = -x(t) - 2y(t) + 1 \;\rightarrow\; \text{y is increasing when } y < \frac{-x+1}{2}$$

$$0 > \dot{y}(t) = -x(t) - 2y(t) + 1 \;\rightarrow\; \text{y is decreasing when } y > \frac{-x+1}{2}$$

Using all this information, I can provide a poor drawing of phase diagram:



Python phase diagram:



Now i can calculate eigen values, which is not very hard. I can rewrite the system like this:

$$\dot{x} + 3x - 3y = 2$$

$$x + \dot{y} + 2y = 1$$

That gives me following matrix, which has determinant equal to zero. λ satisfying this are the eigenvalues:

$\begin{bmatrix} \lambda+3 & -3 \\ -2 & \lambda+2 \end{bmatrix}$

$$0 = (\lambda+3)\cdot(\lambda+2) - (-3)\cdot 2 = \lambda^2 + 5\lambda + 6 + 3 = \lambda^2 + 5\lambda + 9$$

Determinant is equal to -11. therefore the eigenvalues are:

$$\lambda_{1,2} = \frac{-5 \pm i \cdot \sqrt{11}}{2}$$

According to last slide in presentation about systems of difference equations is fixed point proper node and it is stable.