Stationary linear models, ARMA, ARIMA

Tools for Modern Macroeconometrics

Lecture 1

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Univariate Time Series Models

- Where we attempt to predict returns using only information contained in their past values.

Some Notation and Concepts

- <u>A Strictly Stationary Process</u>

A strictly stationary process is one where

$$P\{y_{t_1} \le b_1, \ldots, y_{t_n} \le b_n\} = P\{y_{t_1+m} \le b_1, \ldots, y_{t_n+m} \le b_n\}$$

i.e. the probability measure for the sequence $\{y_t\}$ is the same as that for $\{y_{t+m}\}$ $\forall$ m.

- <u>A Weakly Stationary Process</u>

If a series satisfies the next three equations, it is said to be weakly or covariance stationary

1. $E(y_t) = \mu$, $t = 1,2,\ldots,\infty$
2. $E(y_t - \mu)(y_t - \mu) = \sigma^2 < \infty$
3. $E(y_{t_1} - \mu)(y_{t_2} - \mu) = \gamma_{t_2 - t_1}$ $\forall$ t_1 , t_2

Univariate Time Series Models (cont'd)

- So if the process is covariance stationary, all the variances are the same and all the covariances depend on the difference between t_1 and t_2. The moments

$$E(y_t - E(y_t))(y_{t+s} - E(y_{t+s})) = \gamma_s \, , \; s = 0,1,2, \ldots$$

  are known as the covariance function.
- The covariances, γ_s, are known as autocovariances.

- However, the value of the autocovariances depend on the units of measurement of y_t.
- It is thus more convenient to use the autocorrelations which are the autocovariances normalised by dividing by the variance:

$$\tau_s = \frac{\gamma_s}{\gamma_0} \, , \; s = 0,1,2, \ldots$$

- If we plot τ_s against s=0,1,2,... then we obtain the autocorrelation function or correlogram.

A White Noise Process

- A white noise process is one with (virtually) no discernible structure. A definition of a white noise process is:

$$E(y_t) = \mu$$
$$Var(y_t) = \sigma^2$$
$$\gamma_{t-r} = \begin{cases} \sigma^2 & if \quad t = r \\ 0 & \quad otherwise \end{cases}$$

- Thus the autocorrelation function will be zero apart from a single peak of 1 at $s = 0$. $\tau_s \sim$ approximately N(0,1/T) where T = sample size

- We can use this to do significance tests for the autocorrelation coefficients by constructing a confidence interval.

- For example, a 95% confidence interval would be given by $\pm .196 \times \frac{1}{\sqrt{T}}$. If the sample autocorrelation coefficient, $\hat{\tau}_s$, falls outside this region for any value of s, then we reject the null hypothesis that the true value of the coefficient at lag s is zero.

Joint Hypothesis Tests

- We can also test the joint hypothesis that all m of the τ_k correlation coefficients are simultaneously equal to zero using the Q-statistic developed by Box and Pierce:

$$Q = T\sum_{k=1}^{m} \tau_k^2$$

where T = sample size, m = maximum lag length

- The Q-statistic is asymptotically distributed as a χ_m^2.

- However, the Box Pierce test has poor small sample properties, so a variant has been developed, called the Ljung-Box statistic:

$$Q^* = T(T+2)\sum_{k=1}^{m} \frac{\tau_k^2}{T-k} \sim \chi_m^2$$

- This statistic is very useful as a portmanteau (general) test of linear dependence in time series.

An ACF Example

- Question:

  Suppose that a researcher had estimated the first 5 autocorrelation coefficients using a series of length 100 observations, and found them to be (from 1 to 5): 0.207, -0.013, 0.086, 0.005, -0.022.

  Test each of the individual coefficient for significance, and use both the Box-Pierce and Ljung-Box tests to establish whether they are jointly significant.

- Solution:

  A coefficient would be significant if it lies outside (-0.196,+0.196) at the 5% level, so only the first autocorrelation coefficient is significant.

  Q=5.09 and Q^*=5.26

  Compared with a tabulated $\chi^2(5)$=11.1 at the 5% level, so the 5 coefficients are jointly insignificant.

Moving Average Processes

- Let u_t (t=1,2,3,...) be a sequence of independently and identically distributed (iid) random variables with E(u_t)=0 and Var(u_t)=σ_ε^2, then

  $$y_t = \mu + u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \ldots + \theta_q u_{t-q}$$

  is a q^{th} order moving average model MA(q).

- Its properties are

  $E(y_t)=\mu$; $Var(y_t) = \gamma_0 = (1+\theta_1^2 + \theta_2^2+\ldots+\theta_q^2)\sigma^2$

  Covariances

  $$\gamma_s = \begin{cases} (\theta_s + \theta_{s+1}\theta_1 + \theta_{s+2}\theta_2 + \ldots + \theta_q\theta_{q-s})\sigma^2 & for \quad s = 1,2,\ldots,q \\ 0 \quad for \quad s > q \end{cases}$$

Example of an MA Problem

1. Consider the following MA(2) process:

$$X_t = u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2}$$

where ε_t is a zero mean white noise process with variance σ^2.

(i) Calculate the mean and variance of X_t

(ii) Derive the autocorrelation function for this process (i.e. express the autocorrelations, τ_1, τ_2, ... as functions of the parameters θ_1 and θ_2).

(iii) If θ_1 = -0.5 and θ_2 = 0.25, sketch the acf of X_t.

Solution

(i) If $E(u_t)=0$, then $E(u_{t-i})=0\ \forall\ i$.

So

$$E(X_t) = E(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2}) = E(u_t) + \theta_1 E(u_{t-1}) + \theta_2 E(u_{t-2}) = 0$$

$$\begin{aligned}
\text{Var}(X_t) &= E[X_t - E(X_t)][X_t - E(X_t)] \\
\text{but } E(X_t) &= 0\text{, so} \\
\text{Var}(X_t) &= E[(X_t)(X_t)] \\
&= E[(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2})(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2})] \\
&= E[u_t^2 + \theta_1^2 u_{t-1}^2 + \theta_2^2 u_{t-2}^2 + \text{cross-products}]
\end{aligned}$$

But E[cross-products]=0 since $\text{Cov}(u_t, u_{t-s})=0$ for $s \neq 0$.

Solution (cont'd)

So $\mathrm{Var}(X_t) = \gamma_0 = \mathrm{E}\ [\ u_t^2 + \theta_1^2 u_{t-1}^2 + \theta_2^2 u_{t-2}^2\ ]$

$= \sigma^2 + \theta_1^2 \sigma^2 + \theta_2^2 \sigma^2$

$= (1 + \theta_1^2 + \theta_2^2)\sigma^2$

(ii) The acf of X_t.

$\gamma_1 = \mathrm{E}[X_t - \mathrm{E}(X_t)][X_{t-1} - \mathrm{E}(X_{t-1})]$

$= \mathrm{E}[X_t][X_{t-1}]$

$= \mathrm{E}[(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2})(u_{t-1} + \theta_1 u_{t-2} + \theta_2 u_{t-3})]$

$= \mathrm{E}[(\theta_1 u_{t-1}^2 + \theta_1 \theta_2 u_{t-2}^2)]$

$= \theta_1 \sigma^2 + \theta_1 \theta_2 \sigma^2$

$= (\theta_1 + \theta_1 \theta_2)\sigma^2$

$$
\begin{aligned}
\gamma_2 &= E[X_t - E(X_t)][X_{t-2} - E(X_{t-2})] \\
&= E[X_t][X_{t-2}] \\
&= E[(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2})(u_{t-2} + \theta_1 u_{t-3} + \theta_2 u_{t-4})] \\
&= E[(\theta_2 u_{t-2}^2)] \\
&= \theta_2 \sigma^2
\end{aligned}
$$

$$
\begin{aligned}
\gamma_3 &= E[X_t - E(X_t)][X_{t-3} - E(X_{t-3})] \\
&= E[X_t][X_{t-3}] \\
&= E[(u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2})(u_{t-3} + \theta_1 u_{t-4} + \theta_2 u_{t-5})] \\
&= 0
\end{aligned}
$$

So $\gamma_s = 0$ for $s > 2$.

We have the autocovariances, now calculate the autocorrelations:

$$\tau_0 = \frac{\gamma_0}{\gamma_0} = 1$$

$$\tau_1 = \frac{\gamma_1}{\gamma_0} = \frac{(\theta_1 + \theta_1\theta_2)\sigma^2}{(1+\theta_1^2+\theta_2^2)\sigma^2} = \frac{(\theta_1 + \theta_1\theta_2)}{(1+\theta_1^2+\theta_2^2)}$$

$$\tau_2 = \frac{\gamma_2}{\gamma_0} = \frac{(\theta_2)\sigma^2}{(1+\theta_1^2+\theta_2^2)\sigma^2} = \frac{\theta_2}{(1+\theta_1^2+\theta_2^2)}$$

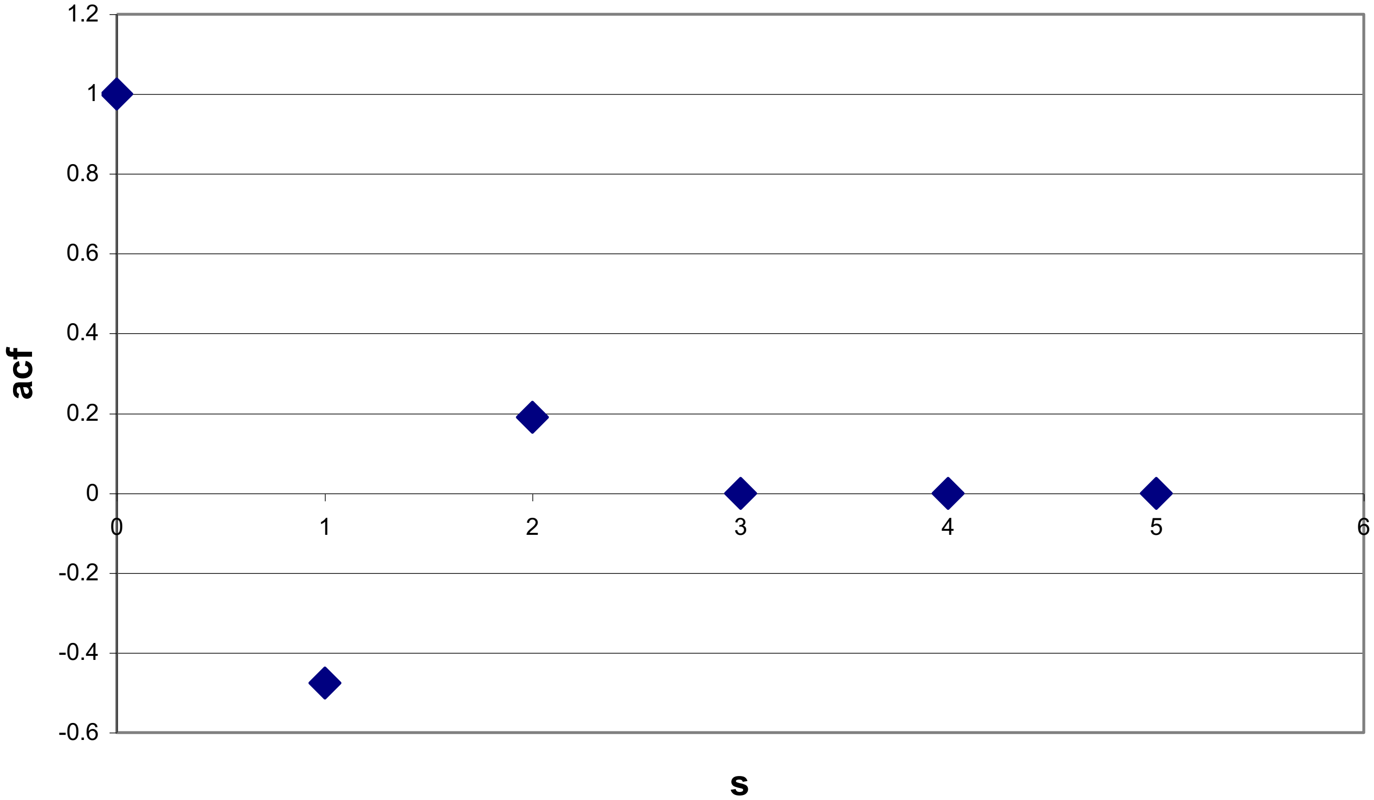
$$\tau_3 = \frac{\gamma_3}{\gamma_0} = 0$$

$$\tau_s = \frac{\gamma_s}{\gamma_0} = 0 \ \forall\ s > 2$$

(iii) For $\theta_1 = -0.5$ and $\theta_2 = 0.25$, substituting these into the formulae above gives $\tau_1 = -0.476$, $\tau_2 = 0.190$.

ACF Plot

Thus the acf plot will appear as follows:



Autoregressive Processes

- An autoregressive model of order *p*, an AR(*p*) can be expressed as

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + u_t$$

- Or using the lag operator notation:

$Ly_t = y_{t\text{-}1}$ $\qquad\qquad\qquad L^i y_t = y_{t\text{-}i}$

$$y_t = \mu + \sum_{i=1}^{p} \phi_i y_{t-i} + u_t$$

- or $y_t = \mu + \sum_{i=1}^{p} \phi_i L^i y_t + u_t$

or $\phi(L) y_t = \mu + u_t$ $\qquad$ where $\qquad \phi(L) = 1 - (\phi_1 L + \phi_2 L^2 + \ldots \phi_p L^p)$.

The Stationary Condition for an AR Model

- The condition for stationarity of a general AR(p) model is that the roots of $1-\phi_1 z-\phi_2 z^2 - \ldots - \phi_p z^p = 0$ all lie outside the unit circle.
- A stationary AR(p) model is required for it to have an MA(∞) representation.
- Example 1: Is $y_t = y_{t-1} + u_t$ stationary?

  The characteristic root is 1, so it is a unit root process (so non-stationary)
- Example 2: Is $y_t = 3y_{t-1} - 0.25y_{t-2} + 0.75y_{t-3} + u_t$ stationary?

  The characteristic roots are 1, 2/3, and 2. Since only one of these lies outside the unit circle, the process is non-stationary.

Wold's Decomposition Theorem

- States that any stationary series can be decomposed into the sum of two unrelated processes, a purely deterministic part and a purely stochastic part, which will be an MA(∞).

- For the AR(p) model, $\phi(L)y_t = u_t$, ignoring the intercept, the Wold decomposition is

$$y_t = \psi(L)u_t$$

where,

$$\psi(L) = (1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p)^{-1}$$

The Moments of an Autoregressive Process

- The moments of an autoregressive process are as follows. The mean is given by

$$E(y_t) = \frac{\phi_0}{1 - \phi_1 - \phi_2 - \ldots - \phi_p}$$

- The autocovariances and autocorrelation functions can be obtained by solving what are known as the Yule-Walker equations:

$$\begin{aligned}
\tau_1 &= \phi_1 + \tau_1\phi_2 + \ldots + \tau_{p-1}\phi_p \\
\tau_2 &= \tau_1\phi_1 + \phi_2 + \ldots + \tau_{p-2}\phi_p \\
\vdots & \qquad\qquad \vdots \qquad\qquad \vdots \\
\tau_p &= \tau_{p-1}\phi_1 + \tau_{p-2}\phi_2 + \ldots + \phi_p
\end{aligned}$$

- If the AR model is stationary, the autocorrelation function will decay exponentially to zero.

Sample AR Problem

- Consider the following simple AR(1) model

$$y_t = \mu + \phi_1 y_{t-1} + u_t$$

(i) Calculate the (unconditional) mean of y_t.

For the remainder of the question, set μ=0 for simplicity.

(ii) Calculate the (unconditional) variance of y_t.

(iii) Derive the autocorrelation function for y_t.

Solution

(i) Unconditional mean:

$$E(y_t) = E(\mu+\phi_1 y_{t-1})$$
$$=\mu + \phi_1 E(y_{t-1})$$

But also

$$\text{So } E(y_t) = \mu + \phi_1 (\mu + \phi_1 E(y_{t-2}))$$
$$= \mu + \phi_1 \mu + \phi_1^2 E(y_{t-2}))$$

$$E(y_t) = \mu + \phi_1 \mu + \phi_1^2 E(y_{t-2}))$$
$$= \mu + \phi_1 \mu + \phi_1^2 (\mu + \phi_1 E(y_{t-3}))$$
$$= \mu + \phi_1 \mu + \phi_1^2 \mu + \phi_1^3 E(y_{t-3})$$

Solution (cont'd)

An infinite number of such substitutions would give

$$E(y_t) = \mu(1+\phi_1+\phi_1^2+...) + \phi_1^{\infty}y_0$$

So long as the model is stationary, i.e. , then $\phi_1^{\infty} = 0$.

So $E(y_t) = \mu(1+\phi_1+\phi_1^2+...) = \dfrac{\mu}{1-\phi_1}$

(ii) Calculating the variance of y_t: $y_t = \phi_1 y_{t-1} + u_t$

From Wold's decomposition theorem:

$$y_t(1-\phi_1 L) = u_t$$

$$y_t = (1-\phi_1 L)^{-1} u_t$$

$$y_t = (1+\phi_1 L + \phi_1^2 L^2 + ...)u_t$$

Solution (cont'd)

So long as $\left|\phi_1\right| < 1$, this will converge.

$$y_t = u_t + \phi_1 u_{t-1} + \phi_1^{\,2} u_{t-2} + \ldots$$

Var(y_t) = E[y_t-E(y_t)][y_t-E(y_t)]

but E(y_t) = 0, since we are setting μ = 0.

Var(y_t) = E[(y_t)(y_t)]

$= \text{E}[\ \left(u_t + \phi_1 u_{t-1} + \phi_1^{\,2} u_{t-2} + ..\right)\left(u_t + \phi_1 u_{t-1} + \phi_1^{\,2} u_{t-2} + ..\right)\ ]$

$= \text{E}[(u_t^{\,2} + \phi_1^{\,2} u_{t-1}^{\,2} + \phi_1^{\,4} u_{t-2}^{\,2} + \ldots + cross - products)]$

$= \text{E}[(u_t^{\,2} + \phi_1^{\,2} u_{t-1}^{\,2} + \phi_1^{\,4} u_{t-2}^{\,2} + \ldots)]$

$= \ \sigma_u^2 + \phi_1^2 \sigma_u^2 + \phi_1^4 \sigma_u^2 + \ldots$

$= \ \sigma_u^2 (1 + \phi_1^2 + \phi_1^4 + \ldots)$

$= \ \dfrac{\sigma_u^2}{(1 - \phi_1^2)}$

Solution (cont'd)

(iii) Turning now to calculating the acf, first calculate the autocovariances:

$$\gamma_1 = \text{Cov}(y_t, y_{t-1}) = \text{E}[y_t\text{-E}(y_t)][y_{t-1}\text{-E}(y_{t-1})]$$

Since a_0 has been set to zero, $\text{E}(y_t) = 0$ and $\text{E}(y_{t-1}) = 0$, so

$$\gamma_1 = \text{E}[y_t y_{t-1}]$$

$$\gamma_1 = \text{E}[(u_t + \phi_1 u_{t-1} + \phi_1{}^2 u_{t-2} + \ldots)(u_{t-1} + \phi_1 u_{t-2} + \phi_1{}^2 u_{t-3} + \ldots)]$$

$$= \text{E}[\ \phi_1\ u_{t-1}{}^2 + \phi_1{}^3 u_{t-2}{}^2 + \ldots + cross - products]$$

$$= \phi_1\sigma^2 + \phi_1^3\sigma^2 + \phi_1^5\sigma^2 + \ldots$$

$$= \frac{\phi_1\sigma^2}{(1-\phi_1^2)}$$

Solution (cont'd)

For the second autocorrelation coefficient,

$\gamma_2 = \text{Cov}(y_t, y_{t\text{-}2}) = \text{E}[y_t\text{-E}(y_t)][y_{t\text{-}2}\text{-E}(y_{t\text{-}2})]$

Using the same rules as applied above for the lag 1 covariance

$$\begin{aligned}
\gamma_2 &= \text{E}[y_t y_{t\text{-}2}] \\
&= \text{E}[(u_t + \phi_1 u_{t-1} + \phi_1{}^2 u_{t-2} + \ldots)(u_{t-2} + \phi_1 u_{t-3} + \phi_1{}^2 u_{t-4} + \ldots)] \\
&= \text{E}[\phi_1{}^2 u_{t-2}{}^2 + \phi_1{}^4 u_{t-3}{}^2 + \ldots + cross - products] \\
&= \phi_1^2 \sigma^2 + \phi_1^4 \sigma^2 + \ldots \\
&= \phi_1^2 \sigma^2 (1 + \phi_1^2 + \phi_1^4 + \ldots) \\
&= \frac{\phi_1^2 \sigma^2}{(1 - \phi_1^2)}
\end{aligned}$$

Solution (cont'd)

- If these steps were repeated for γ_3, the following expression would be obtained

$$\gamma_3 = \frac{\phi_1^3 \sigma^2}{(1-\phi_1^2)}$$

and for any lag *s*, the autocovariance would be given by

$$\gamma_s = \frac{\phi_1^s \sigma^2}{(1-\phi_1^2)}$$

The acf can now be obtained by dividing the covariances by the variance:

Solution (cont'd)

$$\tau_0 = \frac{\gamma_0}{\gamma_0} = 1$$

$$\tau_1 = \frac{\gamma_1}{\gamma_0} = \frac{\left(\dfrac{\phi_1 \sigma^2}{(1-\phi_1^2)}\right)}{\left(\dfrac{\sigma^2}{(1-\phi_1^2)}\right)} = \phi_1$$

$$\tau_2 = \frac{\gamma_2}{\gamma_0} = \frac{\left(\dfrac{\phi_1^2 \sigma^2}{(1-\phi_1^2)}\right)}{\left(\dfrac{\sigma^2}{(1-\phi_1^2)}\right)} = \phi_1^2$$

$$\tau_3 = \phi_1^3$$

...

$$\tau_s = \phi_1^s$$

The Partial Autocorrelation Function (denoted τ_{kk})

- Measures the correlation between an observation k periods ago and the current observation, after controlling for observations at intermediate lags (i.e. all lags < k).

- So τ_{kk} measures the correlation between y_t and y_{t-k} after removing the effects of y_{t-k+1} , y_{t-k+2} , …, y_{t-1} .

- At lag 1, the acf = pacf always

- At lag 2, $\tau_{22} = (\tau_2 - \tau_1^2) / (1 - \tau_1^2)$

- For lags 3+, the formulae are more complex.

The Partial Autocorrelation Function (denoted τ_{kk}) (cont'd)

- The pacf is useful for telling the difference between an AR process and an ARMA process.

- In the case of an AR(p), there are direct connections between y_t and y_{t-s} only for $s \leq p$.

- So for an AR(p), the theoretical pacf will be zero after lag p.

- In the case of an MA(q), this can be written as an AR(∞), so there are direct connections between y_t and all its previous values.

- For an MA(q), the theoretical pacf will be geometrically declining.

ARMA Processes

- By combining the AR(p) and MA(q) models, we can obtain an ARMA(p,q) model:

$$\phi(L)y_t = \mu + \theta(L)u_t$$

where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p$

and $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q$

or $y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \ldots + \theta_q u_{t-q} + u_t$

with $E(u_t) = 0;\ E(u_t^2) = \sigma^2;\ E(u_t u_s) = 0,\ t \neq s$

The Invertibility Condition

- Similar to the stationarity condition, we typically require the MA(q) part of the model to have roots of $\theta(z)$=0 greater than one in absolute value.

- The mean of an ARMA series is given by

$$E(y_t) = \frac{\mu}{1 - \phi_1 - \phi_2 - \ldots - \phi_p}$$

- The autocorrelation function for an ARMA process will display combinations of behaviour derived from the AR and MA parts, but for lags beyond q, the acf will simply be identical to the individual AR(p) model.

Summary of the Behaviour of the acf for AR and MA Processes

An autoregressive process has

- a geometrically decaying acf
- number of spikes of pacf = AR order

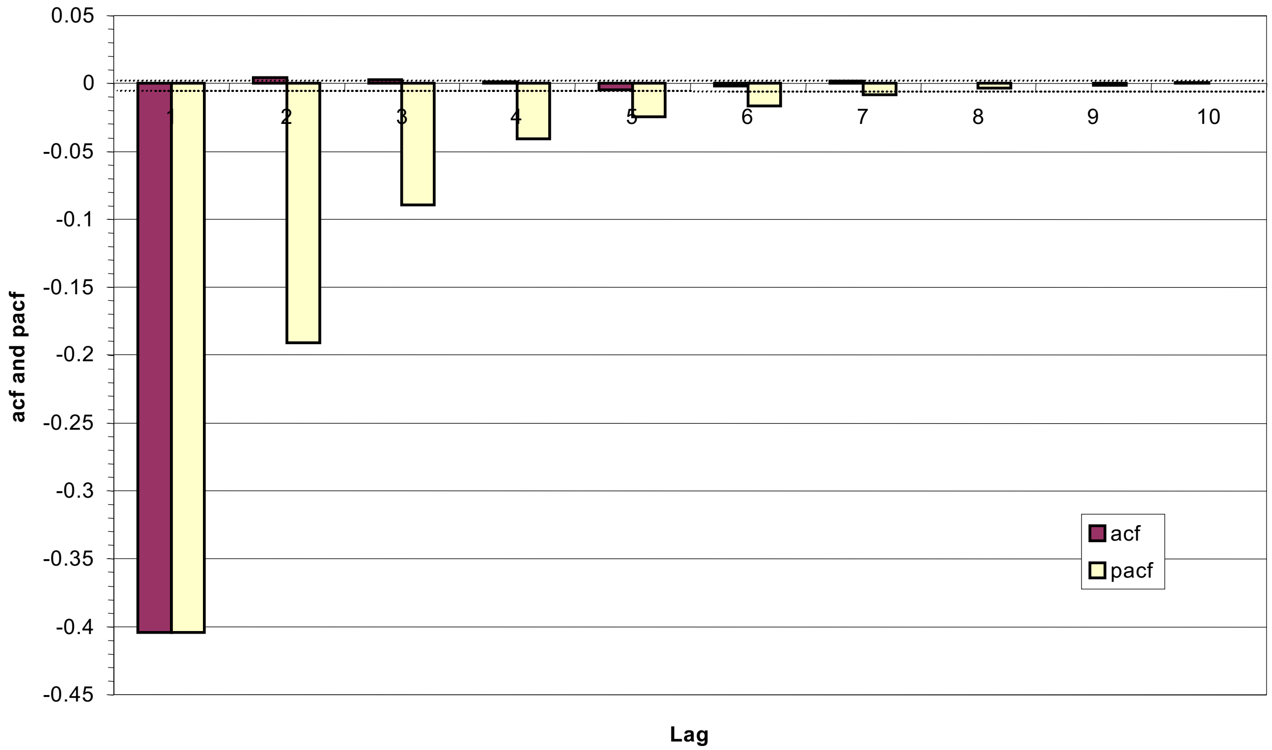
A moving average process has

- Number of spikes of acf = MA order
- a geometrically decaying pacf

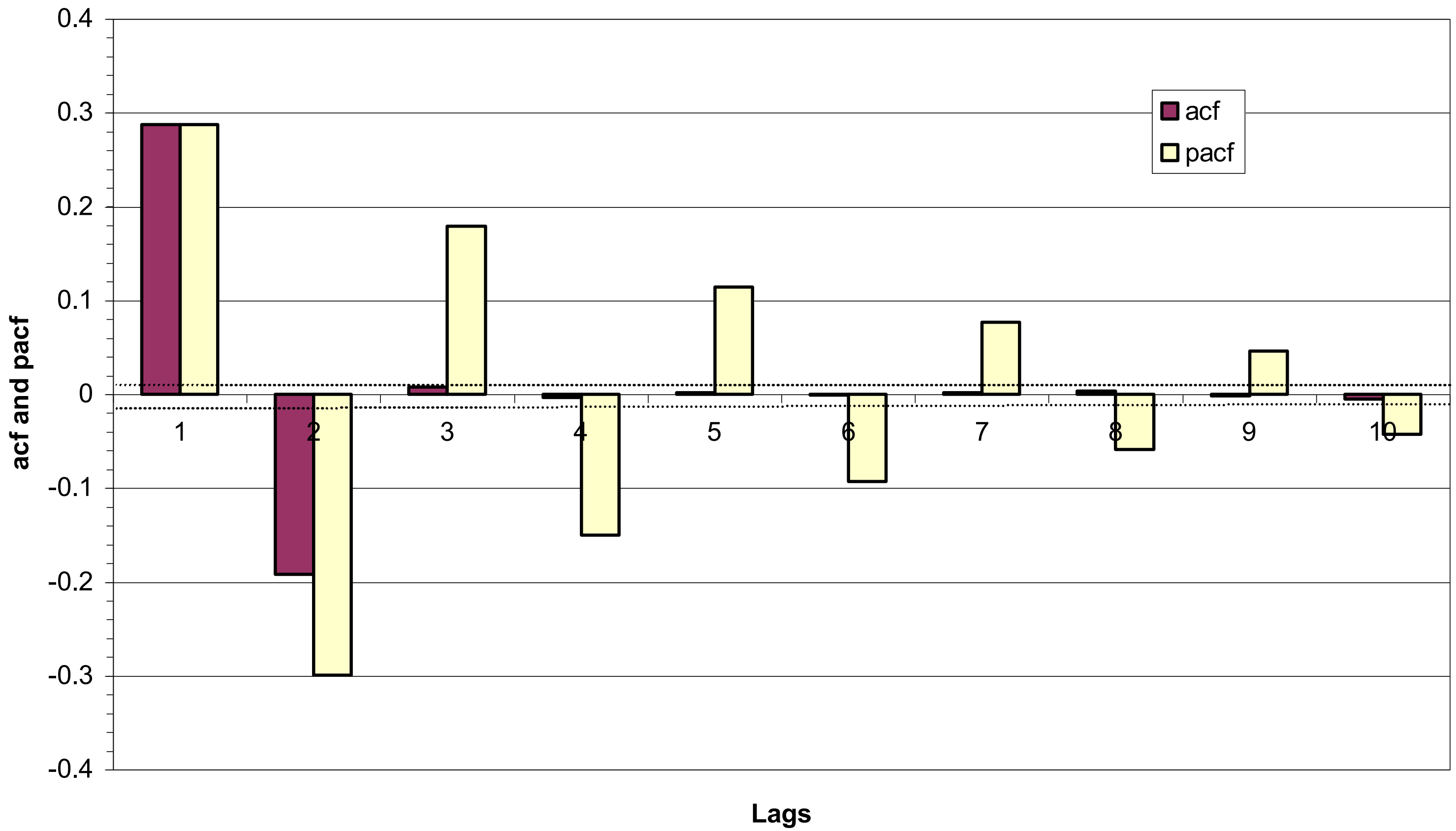
Some sample acf and pacf plots for standard processes

The acf and pacf are not produced analytically from the relevant formulae for a model of that type, but rather are estimated using 100,000 simulated observations with disturbances drawn from a normal distribution.

ACF and PACF for an MA(1) Model: $y_t = -0.5u_{t-1} + u_t$

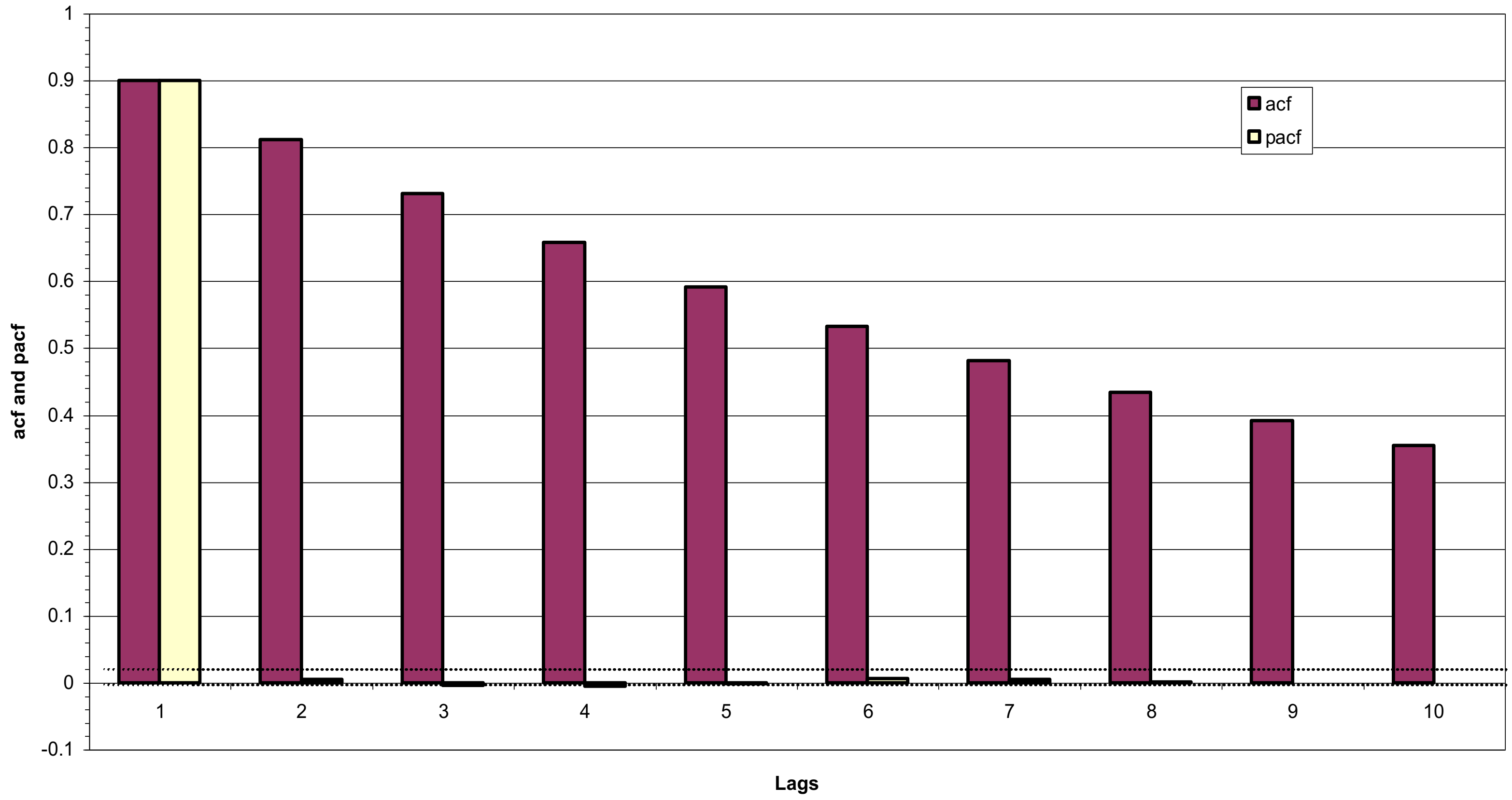


ACF and PACF for an MA(2) Model:
$y_t = 0.5u_{t-1} - 0.25u_{t-2} + u_t$

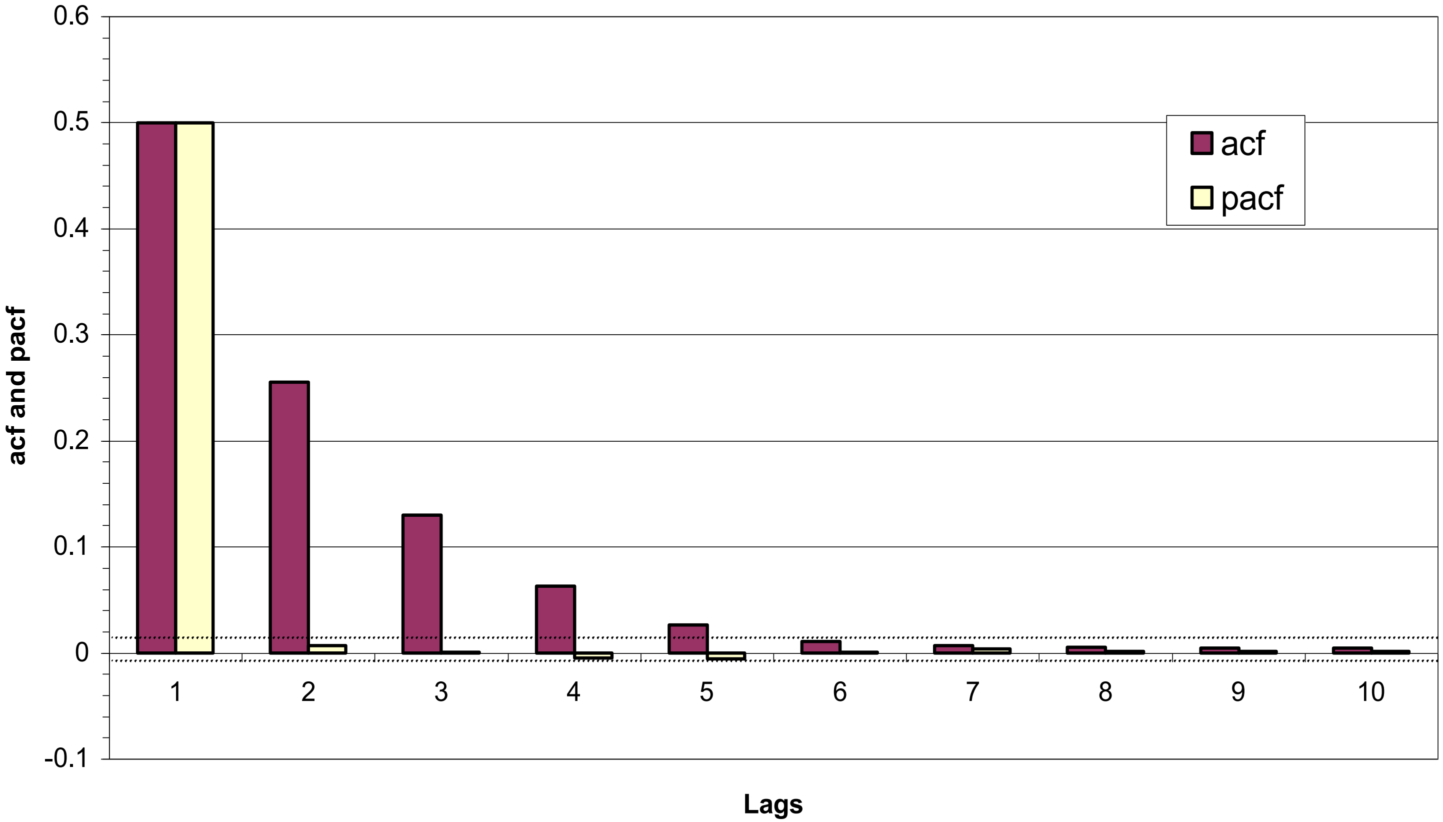


ACF and PACF for a slowly decaying AR(1) Model:

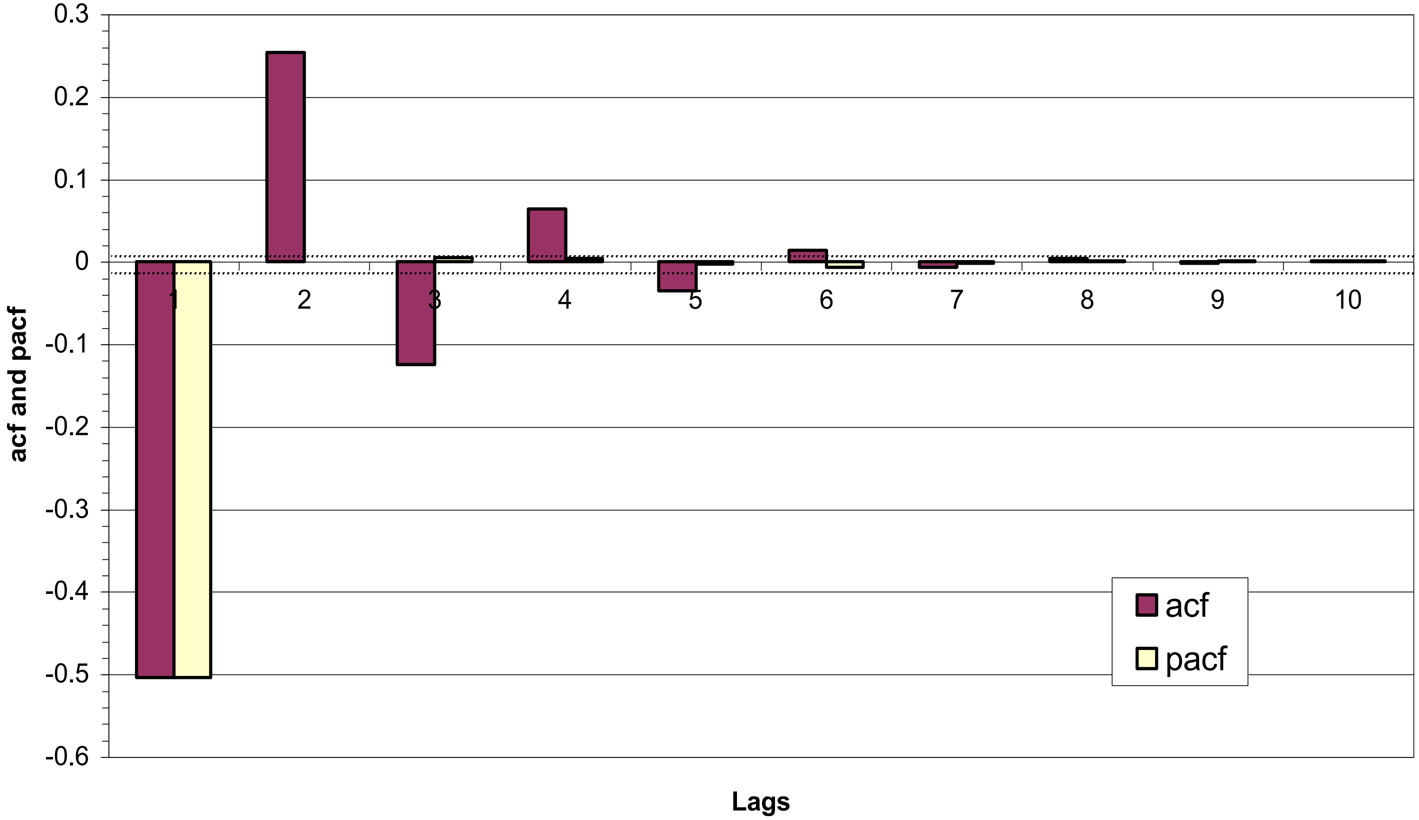
$y_t = 0.9y_{t-1} + u_t$



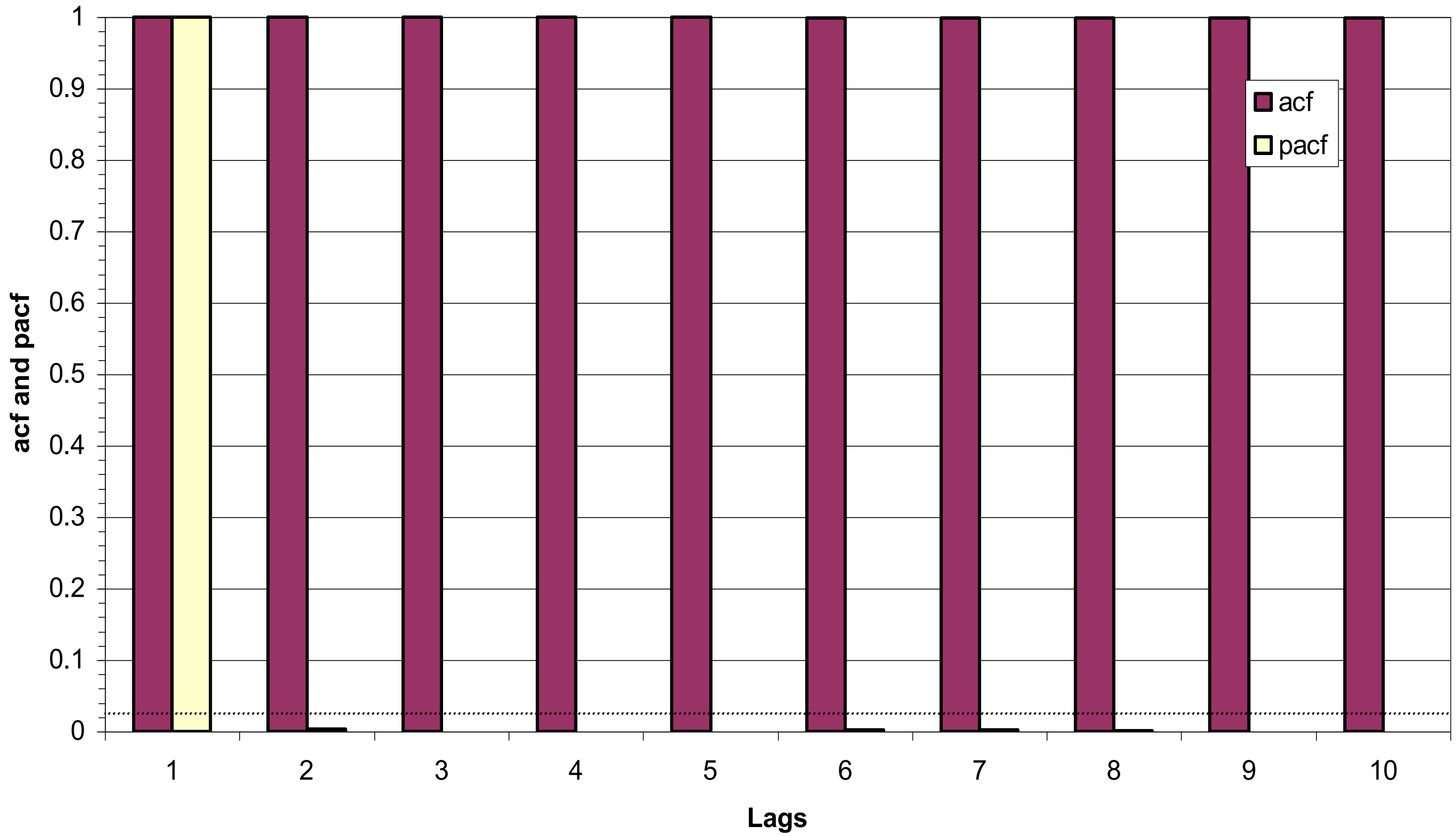
ACF and PACF for a more rapidly decaying AR(1) Model: $y_t = 0.5y_{t-1} + u_t$



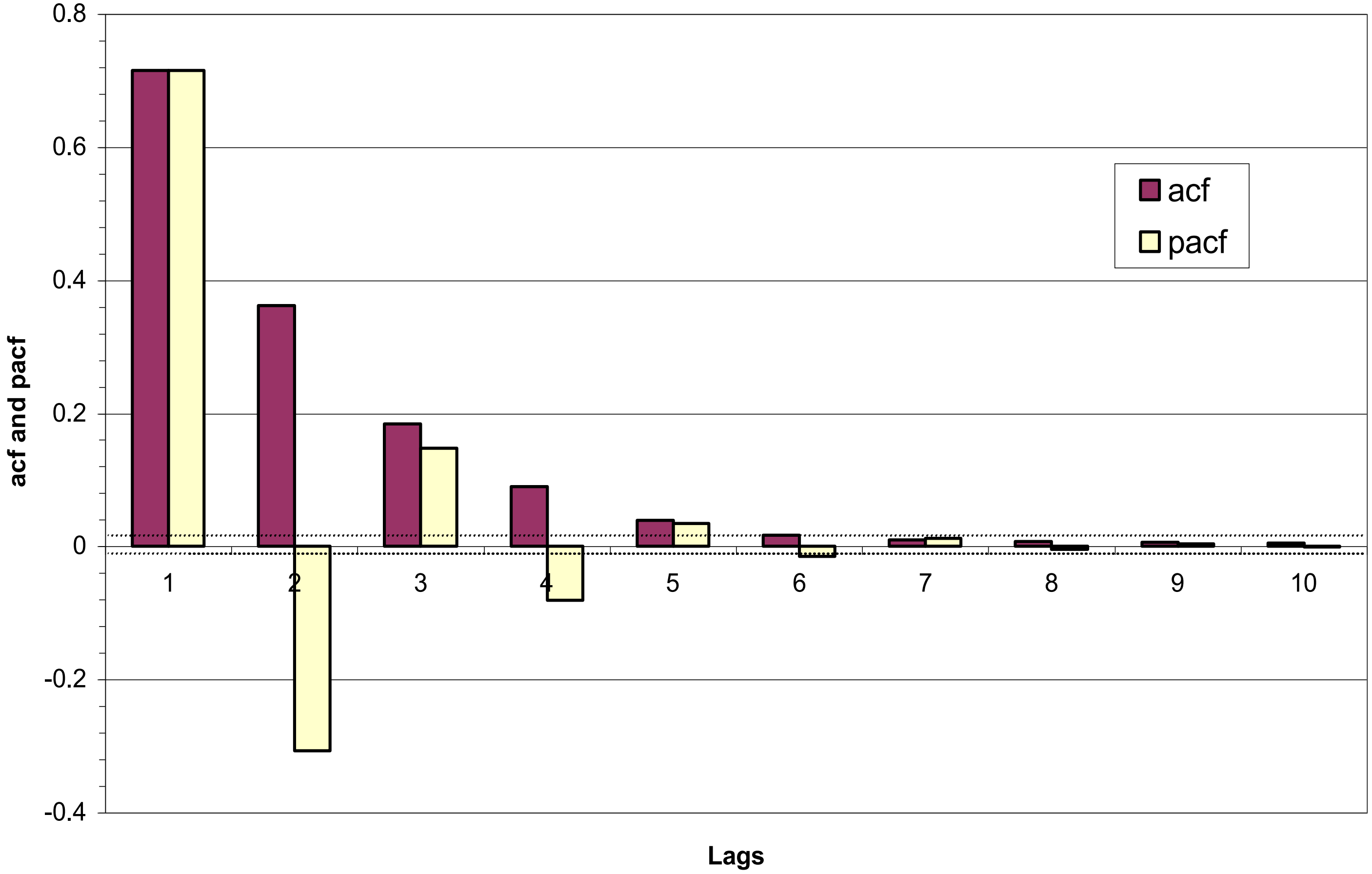
ACF and PACF for a more rapidly decaying AR(1) Model with Negative Coefficient: $y_t = -0.5y_{t-1} + u_t$



ACF and PACF for a Non-stationary Model (i.e. a unit coefficient): $y_t = y_{t-1} + u_t$



ACF and PACF for an ARMA(1,1): $y_t = 0.5y_{t-1} + 0.5u_{t-1} + u_t$



Building ARMA Models
- The Box Jenkins Approach

- Box and Jenkins (1970) were the first to approach the task of estimating an ARMA model in a systematic manner. There are 3 steps to their approach:

  1. Identification
  2. Estimation
  3. Model diagnostic checking

<u>Step 1:</u>

- Involves determining the order of the model.
- Use of graphical procedures
- A better procedure is now available

Building ARMA Models
- The Box Jenkins Approach (cont'd)

Step 2:

- Estimation of the parameters
- Can be done using least squares or maximum likelihood depending on the
  model.

Step 3:

- Model checking

Box and Jenkins suggest 2 methods:

- deliberate overfitting
- residual diagnostics

Some More Recent Developments in ARMA Modelling

- Identification would typically not be done using acf's.
- We want to form a parsimonious model.

- Reasons:
  - variance of estimators is inversely proportional to the number of degrees of freedom.
  - models which are profligate might be inclined to fit to data specific features

- This gives motivation for using information criteria, which embody 2 factors
  - a term which is a function of the RSS
  - some penalty for adding extra parameters
- The object is to choose the number of parameters which minimises the information criterion.

Information Criteria for Model Selection

- The information criteria vary according to how stiff the penalty term is.
- The three most popular criteria are Akaike's (1974) information criterion (AIC), Schwarz's (1978) Bayesian information criterion (SBIC), and the Hannan-Quinn criterion (HQIC).

$$AIC = \ln(\hat{\sigma}^2) + 2k / T$$

$$SBIC = \ln(\hat{\sigma}^2) + \frac{k}{T}\ln T$$

$$HQIC = \ln(\hat{\sigma}^2) + \frac{2k}{T}\ln(\ln(T))$$

where $k = p + q + 1$, T = sample size. So we min. IC s.t. $p \leq \overline{p}, q \leq \overline{q}$

SBIC embodies a stiffer penalty term than *AIC*.

- Which IC should be preferred if they suggest different model orders?
  - *SBIC* is strongly consistent but (inefficient).
  - *AIC* is not consistent, and will typically pick "bigger" models.

ARIMA Models

- As distinct from ARMA models. The I stands for integrated.

- An integrated autoregressive process is one with a characteristic root on the unit circle.

- Typically researchers difference the variable as necessary and then build an ARMA model on those differenced variables.

- An ARMA(p,q) model in the variable differenced d times is equivalent to an ARIMA(p,d,q) model on the original data.

Forecasting in Econometrics

- Forecasting = prediction.
- An important test of the adequacy of a model.

e.g.

- Forecasting tomorrow's return on a particular share
- Forecasting the price of a house given its characteristics
- Forecasting the riskiness of a portfolio over the next year
- Forecasting the volatility of bond returns

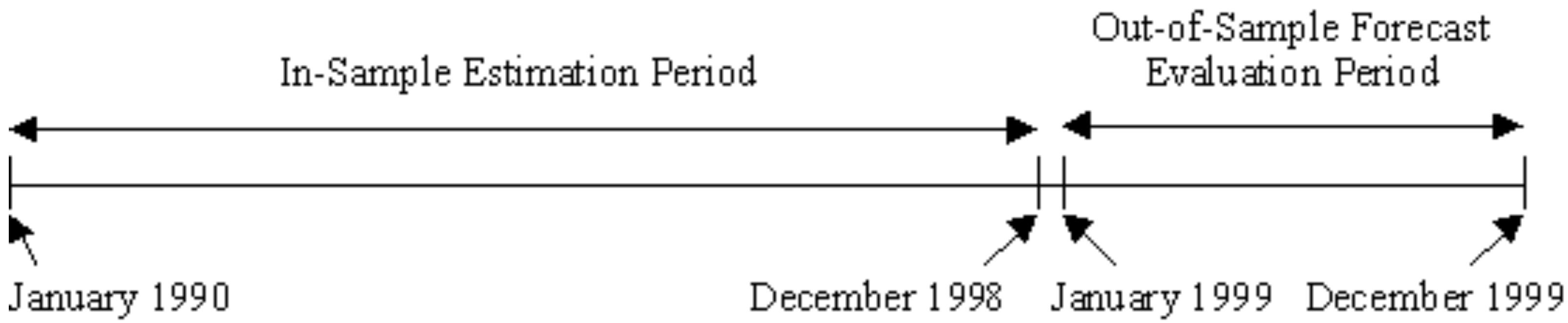
- We can distinguish two approaches:
  - Econometric (structural) forecasting
  - Time series forecasting

- The distinction between the two types is somewhat blurred (e.g, VARs).

In-Sample Versus Out-of-Sample

- Expect the "forecast" of the model to be good in-sample.

- Say we have some data - e.g. monthly FTSE returns for 120 months: 1990M1 – 1999M12. We could use all of it to build the model, or keep some observations back:



- A good test of the model since we have not used the information from 1999M1 onwards when we estimated the model parameters.

How to produce forecasts

- Multi-step ahead versus single-step ahead forecasts
- Recursive versus rolling windows

- To understand how to construct forecasts, we need the idea of conditional expectations:

$$E(y_{t+1} \mid \Omega_t)$$

- We cannot forecast a white noise process: $E(u_{t+\mathrm{s}} \mid \Omega_t) = 0 \ \forall \ s > 0$.

- The two simplest forecasting "methods"

  1. Assume no change : $f(y_{t+s}) = y_t$
  2. Forecasts are the long term average $f(y_{t+s}) = \bar{y}$

Models for Forecasting

- Structural models

e.g. $y = X\beta + u$

$$y_t = \beta_1 + \beta_2 x_{2t} + \ldots + \beta_k x_{kt} + u_t$$

To forecast y, we require the conditional expectation of its future value:

$$E\left(y_t \middle| \Omega_{t-1}\right) = E\left(\beta_1 + \beta_2 x_{2t} + \ldots + \beta_k x_{kt} + u_t\right)$$
$$= \beta_1 + \beta_2 E\left(x_{2t}\right) + \ldots + \beta_k E\left(x_{kt}\right)$$

But what are $\mathrm{E}(x_{2t})$ etc.? We could use $\bar{x}_2$, so

$$E\left(y_t\right) = \beta_1 + \beta_2 \bar{x}_2 + \ldots + \beta_k \bar{x}_k$$
$$= \bar{y}\,!!$$

Models for Forecasting (cont'd)

- Time Series Models

  The current value of a series, y_t, is modelled as a function only of its previous values and the current value of an error term (and possibly previous values of the error term).

- Models include:
  - simple unweighted averages
  - exponentially weighted averages
  - ARIMA models
  - Non-linear models – e.g. threshold models, GARCH, bilinear models, etc.

Forecasting with ARMA Models

The forecasting model typically used is of the form:

$$f_{t,s} = \mu + \sum_{i=1}^{p} \phi_i f_{t,s-i} + \sum_{j=1}^{q} \theta_j u_{t+s-j}$$

where $f_{t,s} = y_{t+s}$, $s \leq 0$; $u_{t+s} = 0$, $s > 0$

$= u_{t+s}$, $s \leq 0$

Forecasting with MA Models

- An MA(q) only has memory of q.

  e.g. say we have estimated an MA(3) model:

$$y_t = \mu + \theta_1 u_{t-1} + \theta_2 u_{t-2} + \theta_3 u_{t-3} + u_t$$

$$y_{t+1} = \mu + \theta_1 u_t + \theta_2 u_{t-1} + \theta_3 u_{t-2} + u_{t+1}$$

$$y_{t+2} = \mu + \theta_1 u_{t+1} + \theta_2 u_t + \theta_3 u_{t-1} + u_{t+2}$$

$$y_{t+3} = \mu + \theta_1 u_{t+2} + \theta_2 u_{t+1} + \theta_3 u_t + u_{t+3}$$

- We are at time t and we want to forecast 1,2,..., s steps ahead.

- We know y_t , y_{t-1}, ..., and u_t , u_{t-1}

Forecasting with MA Models (cont'd)

$$f_{t,1} = \mathrm{E}(y_{t+1\,|\,t}) \quad = \quad \mathrm{E}(\mu + \theta_1 u_t + \theta_2 u_{t-1} + \theta_3 u_{t-2} + u_{t+1})$$
$$= \quad \mu + \theta_1 u_t + \theta_2 u_{t-1} + \theta_3 u_{t-2}$$

$$f_{t,2} = \mathrm{E}(y_{t+2\,|\,t}) \quad = \quad \mathrm{E}(\mu + \theta_1 u_{t+1} + \theta_2 u_t + \theta_3 u_{t-1} + u_{t+2})$$
$$= \quad \mu + \theta_2 u_t + \theta_3 u_{t-1}$$

$$f_{t,3} = \mathrm{E}(y_{t+3\,|\,t}) \quad = \quad \mathrm{E}(\mu + \theta_1 u_{t+2} + \theta_2 u_{t+1} + \theta_3 u_t + u_{t+3})$$
$$= \quad \mu + \theta_3 u_t$$

$$f_{t,4} = \mathrm{E}(y_{t+4\,|\,t}) \quad = \quad \mu$$

$$f_{t,s} = \mathrm{E}(y_{t+s\,|\,t}) \quad = \quad \mu \qquad \forall\, s \geq 4$$

Forecasting with AR Models

- Say we have estimated an AR(2)

$$y_t = \mu + \phi_1 y_{t\text{-}1} + \phi_2 y_{t\text{-}2} + u_t$$
$$y_{t+1} = \mu + \phi_1 y_t + \phi_2 y_{t\text{-}1} + u_{t+1}$$
$$y_{t+2} = \mu + \phi_1 y_{t+1} + \phi_2 y_t + u_{t+2}$$
$$y_{t+3} = \mu + \phi_1 y_{t+2} + \phi_2 y_{t+1} + u_{t+3}$$

$$\begin{aligned} f_{t,1} = \mathrm{E}(y_{t+1\,|\,t}) &= \mathrm{E}(\mu + \phi_1 y_t + \phi_2 y_{t\text{-}1} + u_{t+1}) \\ &= \mu + \phi_1 \mathrm{E}(y_t) + \phi_2 \mathrm{E}(y_{t\text{-}1}) \\ &= \mu + \phi_1 y_t + \phi_2 y_{t\text{-}1} \end{aligned}$$

$$\begin{aligned} f_{t,2} = \mathrm{E}(y_{t+2\,|\,t}) &= \mathrm{E}(\mu + \phi_1 y_{t+1} + \phi_2 y_t + u_{t+2}) \\ &= \mu + \phi_1 \mathrm{E}(y_{t+1}) + \phi_2 \mathrm{E}(y_t) \\ &= \mu + \phi_1 f_{t,1} + \phi_2 y_t \end{aligned}$$

Forecasting with AR Models

$$f_{t,3} = \mathrm{E}(y_{t+3\,|\,t}) = \mathrm{E}(\mu + \phi_1 y_{t+2} + \phi_2 y_{t+1} + u_{t+3})$$
$$= \mu + \phi_1 \mathrm{E}(y_{t+2}) + \phi_2 \mathrm{E}(y_{t+1})$$
$$= \mu + \phi_1 f_{t,2} + \phi_2 f_{t,1}$$

- We can see immediately that

$f_{t,4} = \mu + \phi_1 f_{t,3} + \phi_2 f_{t,2}$ etc., so

$$f_{t,s} = \mu + \phi_1 f_{t,s-1} + \phi_2 f_{t,s-2}$$

- Can easily generate ARMA(p,q) forecasts in the same way.

How can we test whether a forecast is accurate or not?

•For example, say we predict that tomorrow's return on the FTSE will be 0.2, but the outcome is actually -0.4. Is this accurate? Define $f_{t,s}$ as the forecast made at time t for s steps ahead (i.e. the forecast made for time t+s), and y_{t+s} as the realised value of y at time t+s.

- Some of the most popular criteria for assessing the accuracy of time series forecasting techniques are:

$$MSE = \frac{1}{N}\sum_{t=1}^{N} (y_{t+s} - f_{t,s})^2$$

MAE is given by $$MAE = \frac{1}{N}\sum_{t=1}^{N} \left| y_{t+s} - f_{t,s} \right|$$

Mean absolute percentage error: $$MAPE = 100 \times \frac{1}{N}\sum_{t=1}^{N} \left| \frac{y_{t+s} - f_{t,s}}{y_{t+s}} \right|$$

How can we test whether a forecast is accurate or not? (cont'd)

- It has, however, also recently been shown (Gerlow *et al.*, 1993) that the accuracy of forecasts according to traditional statistical criteria are not related to trading profitability.

- A measure more closely correlated with profitability:

$$\% \text{ correct sign predictions} = \frac{1}{N}\sum_{t=1}^{N} z_{t+s}$$

where $z_{t+s} = 1$ if $(x_{t+s} \,.\, f_{t,s}) > 0$

$z_{t+s} = 0$ otherwise

Forecast Evaluation Example

- Given the following forecast and actual values, calculate the MSE, MAE and percentage of correct sign predictions:

Steps Ahead	Forecast	Actual
1	0.20	-0.40
2	0.15	0.20
3	0.10	0.10
4	0.06	-0.10
5	0.04	-0.05

- MSE = 0.079, MAE = 0.180, % of correct sign predictions = 40