

Stationarity, Unit root testing
Tools for Modern Macroeconometrics
Lecture 1b
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Stationarity and Unit Root Testing

Why do we need to test for Non-Stationarity?

- The stationarity or otherwise of a series can strongly influence its behaviour and properties - e.g. persistence of shocks will be infinite for nonstationary series
- Spurious regressions. If two variables are trending over time, a regression of one on the other could have a high R^2 even if the two are totally unrelated
- If the variables in the regression model are not stationary, then it can be proved that the standard assumptions for asymptotic analysis will not be valid. In other words, the usual “ t -ratios” will not follow a t -distribution, so we cannot validly undertake hypothesis tests about the regression parameters.

Two types of Non-Stationarity

- Weak form or covariance stationarity
- There are two models which have been frequently used to characterise non-stationarity: the random walk model with drift:

$$y_t = \mu + y_{t-1} + u_t \quad (1)$$

and the deterministic trend process:

$$y_t = \alpha + \beta t + u_t \quad (2)$$

where u_t is iid in both cases.

Stochastic Non-Stationarity

- Note that the model (1) could be generalised to the case where y_t is an explosive process:

$$y_t = \mu + \phi y_{t-1} + u_t$$

where $\phi > 1$.

- Typically, the explosive case is ignored and we use $\phi = 1$ to characterise the non-stationarity because
 - $\phi > 1$ does not describe many data series in economics and finance.
 - $\phi > 1$ has an intuitively unappealing property: shocks to the system are not only persistent through time, they are propagated so that a given shock will have an increasingly large influence.

Stochastic Non-stationarity: The Impact of Shocks

- To see this, consider the general case of an AR(1) with no drift:

$$y_t = \phi y_{t-1} + u_t \quad (3)$$

Let ϕ take any value for now.

- We can write:

$$y_{t-1} = \phi y_{t-2} + u_{t-1}$$

$$y_{t-2} = \phi y_{t-3} + u_{t-2}$$

- Substituting into (3) yields:

$$\begin{aligned} y_t &= \phi(\phi y_{t-2} + u_{t-1}) + u_t \\ &= \phi^2 y_{t-2} + \phi u_{t-1} + u_t \end{aligned}$$

- Substituting again for y_{t-2} :

$$\begin{aligned} y_t &= \phi^2(\phi y_{t-3} + u_{t-2}) + \phi u_{t-1} + u_t \\ &= \phi^3 y_{t-3} + \phi^2 u_{t-2} + \phi u_{t-1} + u_t \end{aligned}$$

- Successive substitutions of this type lead to:

$$y_t = \phi^T y_0 + \phi u_{t-1} + \phi^2 u_{t-2} + \phi^3 u_{t-3} + \dots + \phi^T u_0 + u_t$$

The Impact of Shocks for Stationary and Non-stationary Series

- We have 3 cases:

1. $\phi < 1 \Rightarrow \phi^T \rightarrow 0$ as $T \rightarrow \infty$

So the shocks to the system gradually die away.

2. $\phi = 1 \Rightarrow \phi^T = 1 \forall T$

So shocks persist in the system and never die away. We obtain:

$$y_t = y_0 + \sum_{i=0}^{\infty} u_t \quad \text{as } T \rightarrow \infty$$

So just an infinite sum of past shocks plus some starting value of y_0 .

3. $\phi > 1$. Now given shocks become more influential as time goes on, since if $\phi > 1$, $\phi^3 > \phi^2 > \phi$ etc.

Detrending a Stochastically Non-stationary Series

- Going back to our 2 characterisations of non-stationarity, the r.w. with drift:

$$y_t = \mu + y_{t-1} + u_t \quad (1)$$

and the trend-stationary process

$$y_t = \alpha + \beta t + u_t \quad (2)$$

- **The two will require different treatments to induce stationarity.** The second case is known as deterministic non-stationarity and what is required is detrending.
- The first case is known as stochastic non-stationarity. If we let

$$\Delta y_t = y_t - y_{t-1}$$

and

$$L y_t = y_{t-1}$$

so

$$(1-L) y_t = y_t - L y_t = y_t - y_{t-1}$$

If we take (1) and subtract y_{t-1} from both sides:

$$y_t - y_{t-1} = \mu + u_t$$

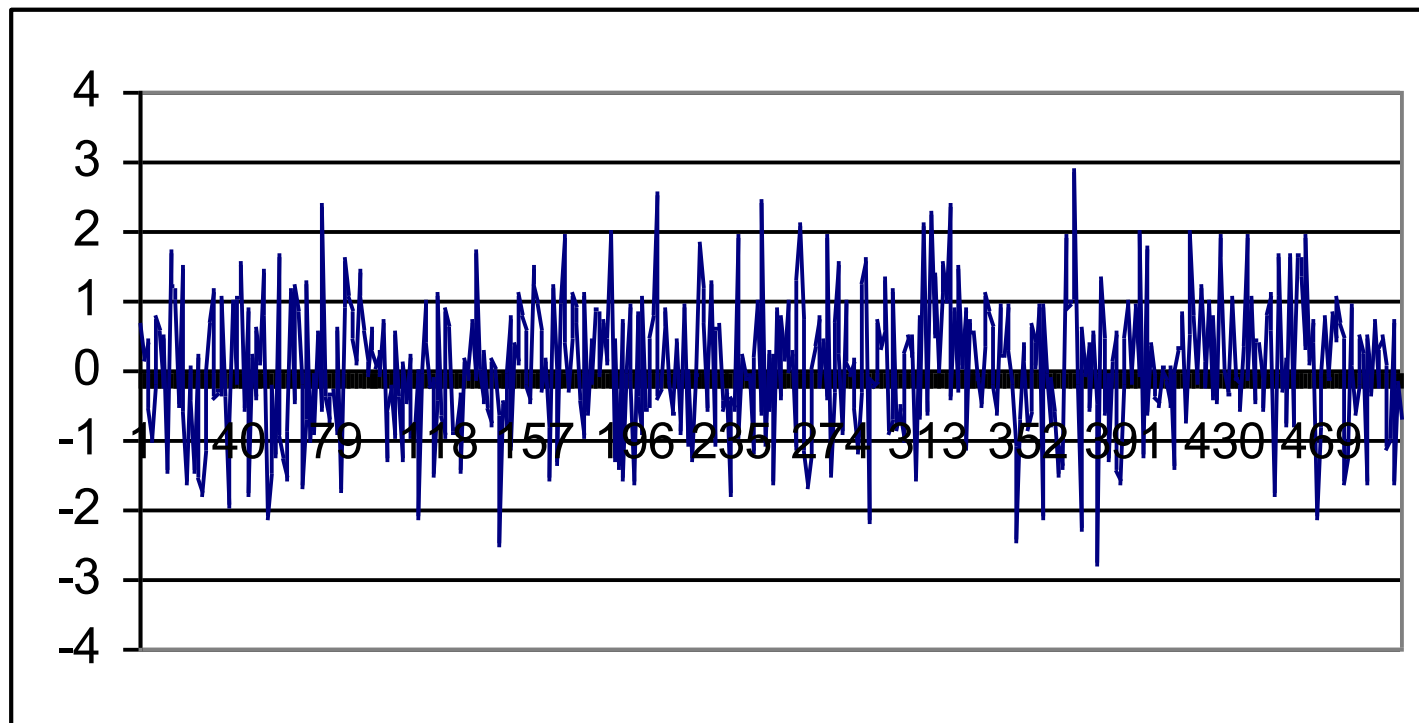
$$\Delta y_t = \mu + u_t$$

We say that we have induced stationarity by “differencing once”.

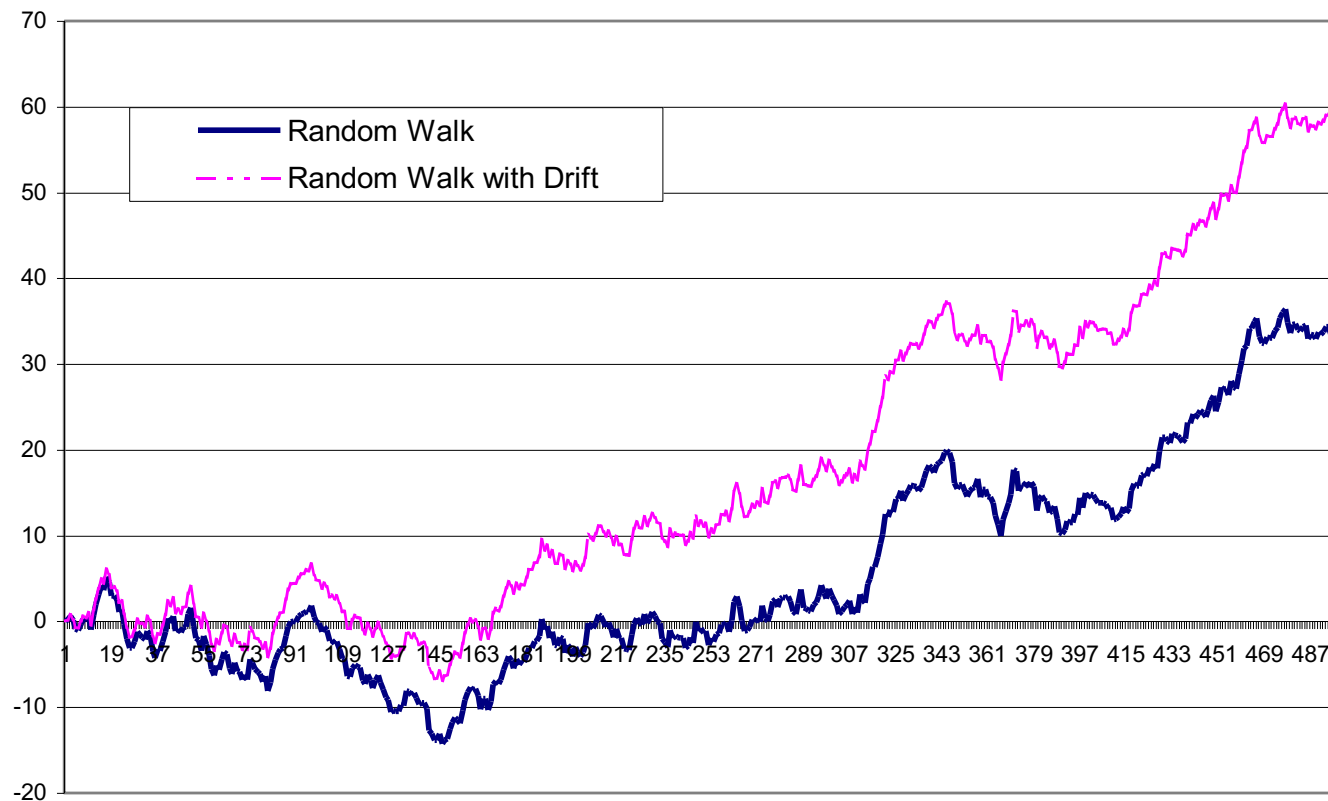
Detrending a Series: Using the Right Method

- Although trend-stationary and difference-stationary series are both “trending” over time, the correct approach needs to be used in each case.
- If we first difference the trend-stationary series, it would “remove” the non-stationarity, but at the expense of introducing an MA(1) structure into the errors.
- Conversely if we try to detrend a series which has stochastic trend, then we will not remove the non-stationarity.
- We will now concentrate on the stochastic non-stationarity model since deterministic non-stationarity does not adequately describe most series in economics or finance.

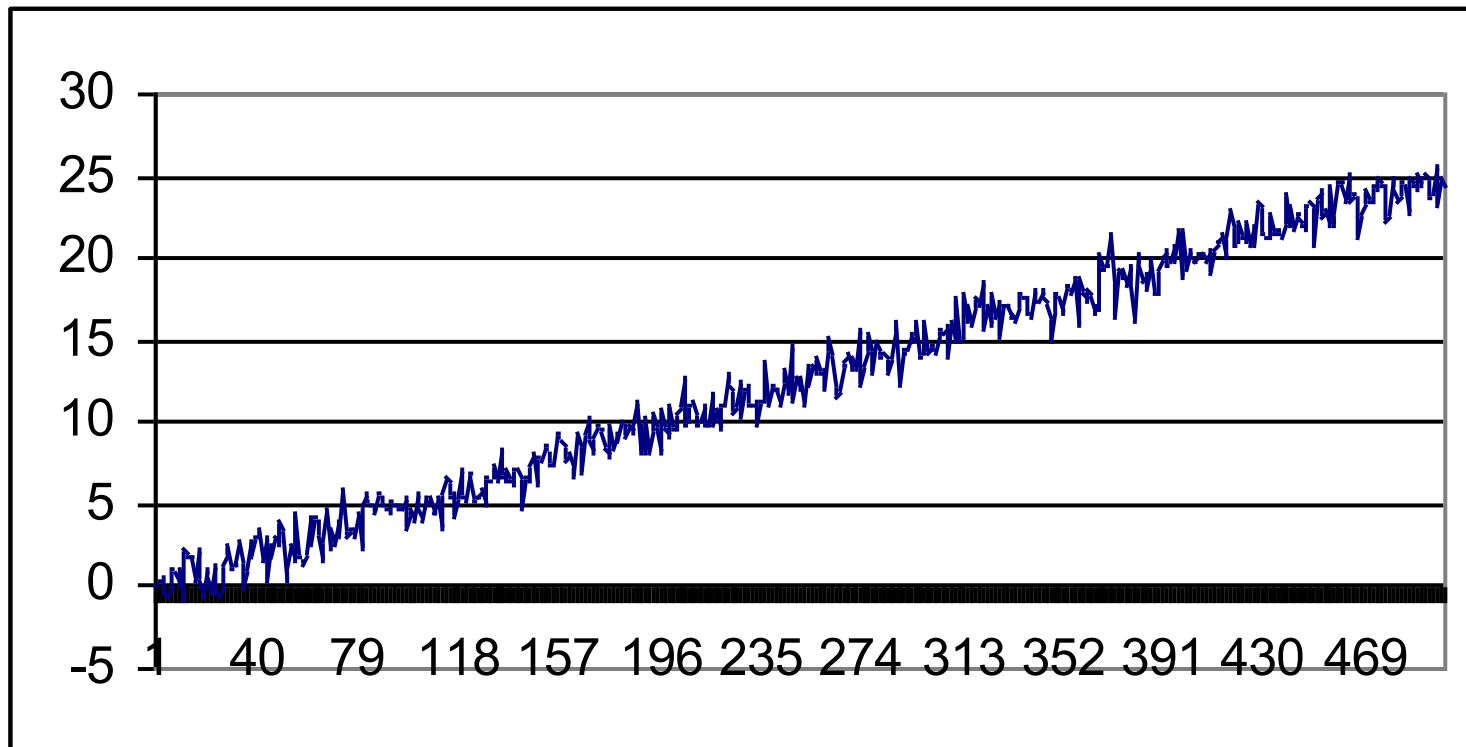
Sample Plots for various Stochastic Processes: A White Noise Process



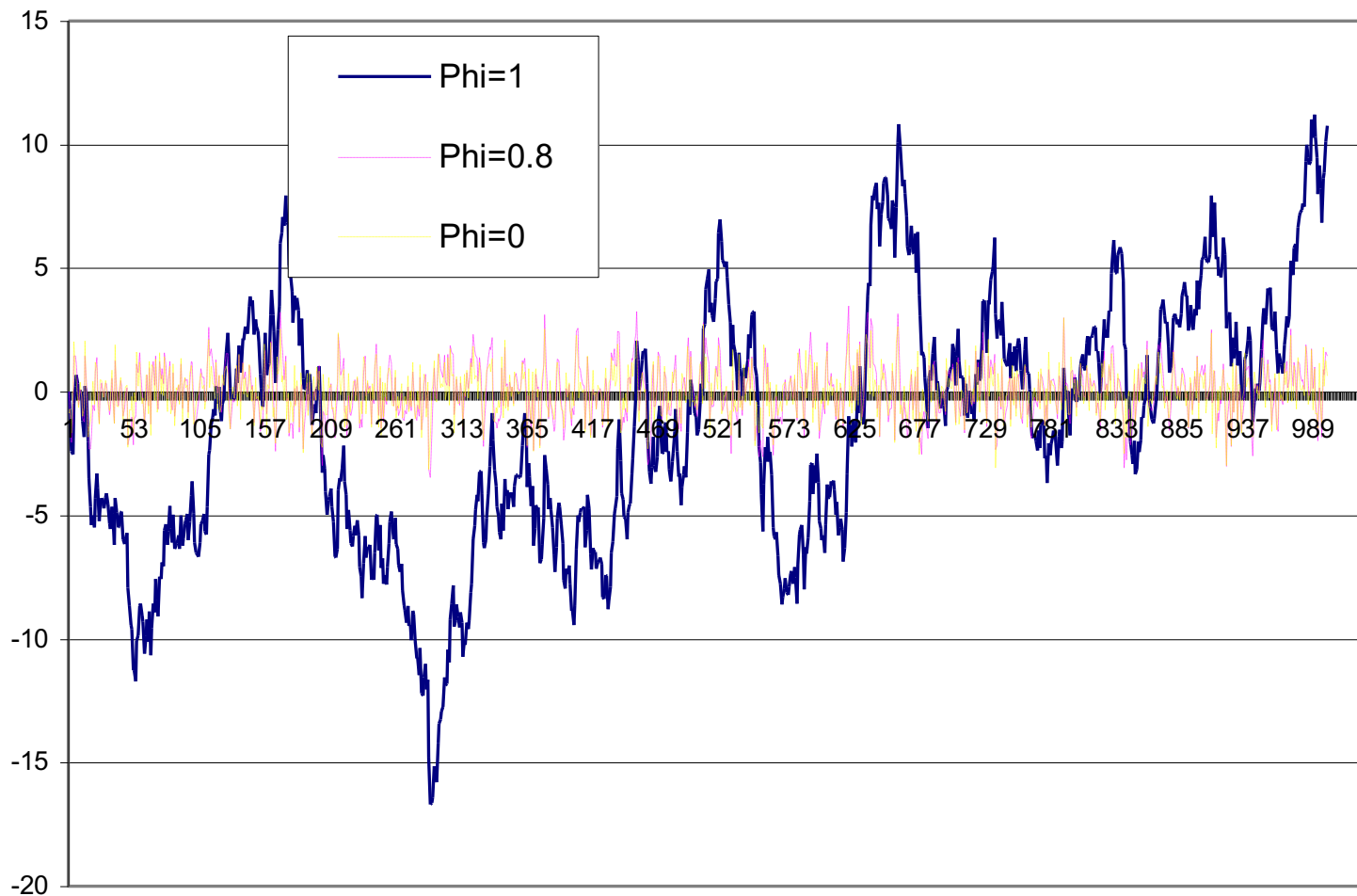
Sample Plots for various Stochastic Processes: A Random Walk and a Random Walk with Drift



Sample Plots for various Stochastic Processes: A Deterministic Trend Process



Autoregressive Processes with differing values of ϕ (0, 0.8, 1)



Definition of Non-Stationarity

- Consider again the simplest stochastic trend model:

$$y_t = y_{t-1} + u_t$$

or

$$\Delta y_t = u_t$$

- We can generalise this concept to consider the case where the series contains more than one “unit root”. That is, we would need to apply the first difference operator, Δ , more than once to induce stationarity.

Definition

If a non-stationary series, y_t must be differenced d times before it becomes stationary, then it is said to be integrated of order d . We write $y_t \sim I(d)$.

So if $y_t \sim I(d)$ then $\Delta^d y_t \sim I(0)$.

An $I(0)$ series is a stationary series

An $I(1)$ series contains one unit root,

$$\text{e.g. } y_t = y_{t-1} + u_t$$

Characteristics of $I(0)$, $I(1)$ and $I(2)$ Series

- An $I(2)$ series contains two unit roots and so would require differencing twice to induce stationarity.
- $I(1)$ and $I(2)$ series can wander a long way from their mean value and cross this mean value rarely.
- $I(0)$ series should cross the mean frequently.
- The majority of economic and financial series contain a single unit root, although some are stationary and consumer prices have been argued to have 2 unit roots.

How do we test for a unit root?

- The early and pioneering work on testing for a unit root in time series was done by Dickey and Fuller (Dickey and Fuller 1979, Fuller 1976). The basic objective of the test is to test the null hypothesis that $\phi=1$ in:

$$y_t = \phi y_{t-1} + u_t$$

against the one-sided alternative $\phi < 1$. So we have

H_0 : series contains a unit root

vs. H_1 : series is stationary.

- We usually use the regression:

$$\Delta y_t = \psi y_{t-1} + u_t$$

so that a test of $\phi=1$ is equivalent to a test of $\psi=0$ (since $\phi-1=\psi$).

Different forms for the DF Test Regressions

- Dickey Fuller tests are also known as τ tests: τ , τ_μ , τ_τ .
- The null (H_0) and alternative (H_1) models in each case are

i) $H_0: y_t = y_{t-1} + u_t$

$$H_1: y_t = \phi y_{t-1} + u_t, \phi < 1$$

This is a test for a random walk against a stationary autoregressive process of order one (AR(1))

ii) $H_0: y_t = y_{t-1} + u_t$

$$H_1: y_t = \phi y_{t-1} + \mu + u_t, \phi < 1$$

This is a test for a random walk against a stationary AR(1) with drift.

iii) $H_0: y_t = y_{t-1} + u_t$

$$H_1: y_t = \phi y_{t-1} + \mu + \lambda t + u_t, \phi < 1$$

This is a test for a random walk against a stationary AR(1) with drift and a time trend.

Computing the DF Test Statistic

- We can write

$$\Delta y_t = u_t$$

where $\Delta y_t = y_t - y_{t-1}$, and the alternatives may be expressed as

$$\Delta y_t = \psi y_{t-1} + \mu + \lambda t + u_t$$

with $\mu = \lambda = 0$ in case i), and $\lambda = 0$ in case ii) and $\psi = \phi - 1$. In each case, the tests are based on the t -ratio on the y_{t-1} term in the estimated regression of Δy_t on y_{t-1} , plus a constant in case ii) and a constant and trend in case iii). The test statistics are defined as

$$\text{test statistic} = \frac{\hat{\psi}}{\hat{SE}(\hat{\psi})}$$

- The test statistic does not follow the usual t -distribution under the null, since the null is one of non-stationarity, but rather follows a non-standard distribution. Critical values are derived from Monte Carlo experiments in, for example, Fuller (1976). Relevant examples of the distribution are shown in table 4.1 below

Critical Values for the DF Test

Significance level	10%	5%	1%
C.V. for constant but no trend	-2.57	-2.86	-3.43
C.V. for constant and trend	-3.12	-3.41	-3.96

Table 4.1: Critical Values for DF and ADF Tests (Fuller, 1976, p373).

The null hypothesis of a unit root is rejected in favour of the stationary alternative in each case if the test statistic is more negative than the critical value.

The Augmented Dickey Fuller (ADF) Test

- The tests above are only valid if u_t is white noise. In particular, u_t will be autocorrelated if there was autocorrelation in the dependent variable of the regression (Δy_t) which we have not modelled. The solution is to “augment” the test using p lags of the dependent variable. The alternative model in case (i) is now written:

$$\Delta y_t = \psi y_{t-1} + \sum_{i=1}^p \alpha_i \Delta y_{t-i} + u_t$$

- The same critical values from the DF tables are used as before. A problem now arises in determining the optimal number of lags of the dependent variable.

There are 2 ways

- use the frequency of the data to decide
- use information criteria

Testing for Higher Orders of Integration

- Consider the simple regression:

$$\Delta y_t = \psi y_{t-1} + u_t$$

We test $H_0: \psi = 0$ vs. $H_1: \psi < 0$.

- If H_0 is rejected we simply conclude that y_t does not contain a unit root.
- But what do we conclude if H_0 is not rejected? The series contains a unit root, but is that it? No! What if $y_t \sim I(2)$? We would still not have rejected. So we now need to test

$$H_0: y_t \sim I(2) \text{ vs. } H_1: y_t \sim I(1)$$

We would continue to test for a further unit root until we rejected H_0 .

- We now regress $\Delta^2 y_t$ on Δy_{t-1} (plus lags of $\Delta^2 y_t$ if necessary).
- Now we test $H_0: \Delta y_t \sim I(1)$ which is equivalent to $H_0: y_t \sim I(2)$.
- So in this case, if we do not reject (unlikely), we conclude that y_t is at least $I(2)$.

The Phillips-Perron Test

- Phillips and Perron have developed a more comprehensive theory of unit root nonstationarity. The tests are similar to ADF tests, but they incorporate an automatic correction to the DF procedure to allow for autocorrelated residuals.
- The tests usually give the same conclusions as the ADF tests, and the calculation of the test statistics is complex.

Criticism of Dickey-Fuller and Phillips-Perron-type tests

- Main criticism is that the power of the tests is low if the process is stationary but with a root close to the non-stationary boundary.

e.g. the tests are poor at deciding if

$$\phi=1 \text{ or } \phi=0.95,$$

especially with small sample sizes.

- If the true data generating process (dgp) is

$$y_t = 0.95y_{t-1} + u_t$$

then the null hypothesis of a unit root should be rejected.