ECEn 774
Final Exam, Winter 2016
Due by 11:59PM April 20

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The exam is open book, open notes, open computer. Computer simulations can be in Matlab, Python, or any programming language of choice.

For each problem turn in handwritten derivations and proofs, final computer simulation plots, and computer code.

Problem 1	______/ 20
Problem 2	______/ 20
Problem 3	______/ 20
Problem 4	______/ 20
Problem 5	______/ 20
Total	______/ 100

I certify that the solutions to this exam represents my own work, and that I did not consult with any other individual about the problems on the exam.

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Signature

1. PD controller

$\dot{p} = v$

$\dot{v} = \frac{-u_1 v|v|}{m} - \frac{u_2 v|v|^3}{m} + \frac{d}{m}T$ all parameters known to the controller

let $e = p_{ref} - p$

$\dot{x}_1 = x_2$ $\quad x_1 = p - p_{ref}$ $\quad x_2 = v$

$\dot{x}_2 = \frac{-u_1 x_2|x_2|}{m} - \frac{u_2 x_2|x_2|^3}{m} + \frac{d}{m}T$

Lyapunov

$$V = \frac{1}{2}x^T x \Rightarrow \dot{V} = \frac{1}{2}\dot{x}^T x + \frac{1}{2}x^T\dot{x} = \dot{x}^T x = (\dot{x}_1 \quad \dot{x}_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= x_1 x_2 - \frac{u_1 x_2^2|x_2|}{m} - \frac{u_2 x_2^2|x_2|^3}{m} + \frac{d}{m}x_2 T$$

$\boxed{\text{pick} \quad T = -k_1 x_1 - k_2 x_2}$ ← PD controller

$$\Rightarrow \dot{V} = x_1 x_2 - \frac{u_1 x_2^2|x_2|}{m} - \frac{u_2 x_2^2|x_2|^2}{m} + \frac{d}{m}x_2(-k_1 x_1 - k_2 x_2)$$

$$= -\left(k_1\frac{d}{m} + 1\right)x_1 x_2 - \frac{u_1 x_2^2|x_2|}{m} - \frac{u_2 x_2^2|x_2|^2}{m} - k_2\frac{d}{m}x_2^2 \le 0$$

Lasalle Invariance principle

let $E = \{\dot{V} = 0\} = \{(x_1 = 0, x_2 = 0)\}$

$M = \{(x_1 = 0, x_2 = 0)\}$ Thus $E = M \Rightarrow$ by Lasalle Invariance principle,

$x_1 = p - p_{ref} \to 0$

2. Backstepping

$$\dot{\rho} = v + v_c$$

$$\dot{v} = \frac{-u_1 v|v|}{m} - \frac{u_2 v|v|^3}{m} + \frac{d}{m}T$$

all parameters are known to the controller.

change of variable

$$z = \rho - \rho_{ref}$$

$$\dot{z} = \dot{\rho} = v + v_c$$

Lyapunov

$$V_1 = \frac{1}{2} z^2$$

$$\dot{V}_1 = z\dot{z} = z(v + v_c + \xi - \xi)$$

$$= z(v_c + \xi) + z(v - \xi)$$

$$\boxed{\text{pick } \xi = -v_c - k_1 z} \longrightarrow \dot{\xi} = -k_1 \dot{z} = -k_1(v + v_c)$$

$$\Rightarrow \dot{V}_1 = -k_1 z^2 + z(v - \xi)$$

$$V_2 = V_1 + \frac{1}{2}(v - \xi)^2$$

$$\dot{V}_2 = -k_1 z^2 + (v - \xi)(\dot{v} - \dot{\xi} + z)$$

$$= -k_1 z^2 + (v - \xi)\left(-\frac{u_1 v|v|}{m} - \frac{u_2 v|v|^3}{m} + \frac{d}{m}T - \dot{\xi} + z\right)$$

$$\boxed{\text{pick } T = \frac{m}{d}\left(\frac{u_1 v|v|}{m} + \frac{u_2 v|v|^3}{m} + \dot{\xi} - z - k_2(v - \xi)\right)}$$

$$\Rightarrow \dot{V}_2 = -k_1 z^2 - k_2 (v - \xi)^2 < 0$$

Tuning parameters

k_1, k_2

3. Feedback linearizing Controller

$$\dot{p} = V + V_c$$

$$\dot{V} = -\frac{u_1 V|V|}{m} - \frac{u_2 V|V|^3}{m} + \frac{\alpha}{m} T$$

Change of Variables

$$z_1 = p - p_{ref}$$

$$z_2 = V + V_c$$

$$\dot{z}_1 = V + V_c = z_2$$

$$\dot{z}_2 = -\frac{u_1 V|V|}{m} - \frac{u_2 V|V|^3}{m} + \frac{\alpha}{m} T$$

$$\boxed{\text{Pick } T = \frac{m}{\alpha}\left(\frac{u_1 V|V|}{m} + \frac{u_2 V|V|^3}{m} + \mathcal{V}\right)}$$

$$\Rightarrow \begin{array}{l} \dot{z}_1 = z_2 \\ \dot{z}_2 = \mathcal{V} \end{array} \Rightarrow \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathcal{V}$$

$$\boxed{\text{Pick } \mathcal{V} = -k_1 z_1 - k_2 z_2}$$

$$\Rightarrow \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to 0 \Rightarrow \begin{pmatrix} p - p_{ref} \\ V + V_c \end{pmatrix} \to 0 \Rightarrow \begin{pmatrix} p \\ V \end{pmatrix} \to \begin{pmatrix} p_{ref} \\ -V_c \end{pmatrix}$$

$$\boxed{\text{Tuning parameters } k_1, k_2}$$

4. Sliding Mode

$$\dot{p} = V + V_c$$

$$\dot{V} = \frac{-u_1 V|V|}{m} - \frac{u_2 V|V|^3}{m} + \frac{\alpha}{m} T,$$

$5 \le m \le 20$

$0 < u_1 \le 50$

$0 < u_2 \le 30$

$0 < \alpha \le 1$

$-1 \le V_c \le 1$ known to the controller

$$S = \dot{e} + k_s e$$

$$e = p - p_{ref}$$

$$\dot{e} = \dot{p} = V + V_c$$

change of variables

$$x_1 = e \quad \Rightarrow \quad \dot{x}_1 = \dot{e} = V + V_c$$

$$x_2 = \dot{e} \qquad\quad \dot{x}_2 = \ddot{e} = \dot{V} = -\frac{u_1 V|V|}{m} - \frac{u_2 V|V|^3}{m} + \frac{\alpha}{m} T$$

Sliding Surface

$S = x_2 + a x_1 \Rightarrow$ when $S = 0$, $x_2 = \dot{x}_1 = -a x_1$ which means $x_1 \to 0$ on the sliding surface.

$$\dot{S} = \dot{x}_2 + a\dot{x}_1 = \dot{V} + a(V + V_c)$$

Lyapunov

$$\mathcal{V} = \frac{1}{2} S^2$$

$$\dot{\mathcal{V}} = S\dot{S} = S\left(\dot{V} + a(V + V_c)\right) = S\left(-\frac{u_1 V|V|}{m} - \frac{u_2 V|V|^3}{m} + \frac{\alpha}{m} T + a(V + V_c)\right)$$

$$= \frac{\alpha S}{m}\left(-\frac{u_1 V|V|}{\alpha} - \frac{u_2 V|V|^3}{\alpha} + T + \frac{m}{\alpha} a(V + V_c)\right)$$

Assume $\left| -\frac{u_1 V|V|}{\alpha} - \frac{u_2 V|V|^3}{\alpha} + \frac{m}{\alpha} a(V + V_c)\right| \le \rho(x)$

$$\rho(x) = \left| -\frac{50 V|V|}{0.1} - \frac{30 V|V|^3}{0.1}\right| + \left|\frac{20}{0.1} a(V + 1)\right|$$

$$\dot{\mathcal{V}} \le \frac{\alpha S}{m} T + \frac{\alpha}{m}|S| \rho(x)$$

$\boxed{\text{pick } T = -(\rho(x) + \beta_0)\, sgn(S)} \Rightarrow \boxed{T = -(\rho(x) + \beta_0)\, sat\left(S/\varepsilon\right)}$ due to chattering

$$\Rightarrow \dot{\mathcal{V}} \le -\frac{\alpha}{m} \beta_0 |S|$$

$\boxed{\text{Tuning parameters } \beta_0,\ a = k_e}$

let $W = \sqrt{2\mathcal{V}} = |S|$

$$\dot{W} = \frac{\dot{\mathcal{V}}}{W} \le \frac{-\frac{\alpha}{m}\beta_0 |S|}{|S|} = -\frac{\alpha}{m}\beta_0$$

Thus, the trajectory reaches the manifold $S = 0$ in finite time.

5. Baseline + Adaptive

1. linearize, Baseline controller

$$\dot{p} = V + V_c$$

$$\dot{V} = -\frac{u_1 V|V|}{m} - \frac{u_2 V|V|^3}{m} + \frac{d}{m}T$$

equilibrium

$$0 = V + V_c \Rightarrow V_e = -V_c$$

$$0 = u_1 V_c|-V_c| + u_2 V_c|-V_c|^3 + dT \Rightarrow$$

$$\boxed{x_e = (0, -V_c) \qquad T_e = \frac{-u_1 V_c|-V_c| - u_2 V_c|-V_c|^3}{d}}$$

$$\begin{pmatrix} \dot{p} \\ \dot{V} \end{pmatrix} = \begin{pmatrix} V + V_c \\ -\frac{u_1 V|V|}{m} - \frac{u_2 V|V|^3}{m} \end{pmatrix}\begin{pmatrix} p \\ V \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{d}{m} \end{pmatrix} T$$

Linearizing about x_e, T_e

$$\dot{x}_p = \begin{pmatrix} 0 & 1 \\ 0 & \frac{2uV_c}{m} + \frac{4u_2 V_c^3}{m} \end{pmatrix} x_p + \begin{pmatrix} 0 \\ \frac{d}{m} \end{pmatrix} T$$

$$x_I = \int (p - p_{cmd})\,dt$$

augmented state $x = (x_p^T, x_I)^T$

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{2u_1 V_c}{m} + \frac{4u_2 V_c^3}{m} & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} p \\ V \\ x_I \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \frac{d}{m} \\ 0 \end{pmatrix}}_{B} T + \underbrace{\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}}_{B_r} p_{cmd}$$

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ V \\ x_I \end{pmatrix}$$

Used LQR for the Baseline Controller

(2) Referente model, adaptive Controller

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{2u_1 V_c}{m} + \frac{4u_2 V_c^3}{m} & 0 \\ 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{d}{m} \\ 0 \end{pmatrix} T + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} p_{cmd}$$

$$\dot{x} = Ax + Bu + B_r Z_r$$

closed-loop Baseline system

$$\dot{x} = \underbrace{(A - Bk)}_{A_{ref}} x + B_r Z_r$$

Observer-like reference model

$$\dot{x}_{ref} = A_{ref} x_{ref} + B_{ref} Z_{ref} + L_v (x - x_{ref}), \quad L_v = \frac{v+1}{v} S_v$$

In the end,

$$u_{ad} = -\hat{\theta}^T \phi(x)$$

where $\phi(x) = \begin{pmatrix} u_{b1} \\ 1 \\ p \\ V \\ V + V_c \\ \frac{-u_1 V|V|}{m} \\ \frac{-u_2 V|V|^3}{m} \\ \frac{d}{m} \end{pmatrix}$

Tuning parameters

Q, R in LQR

Γ, ν

The adaptive parameters are adjusted according to

$$\dot{\hat{\theta}} = \text{proj}(\hat{\theta}, \Gamma \phi u e^T) S_v^{-1} B)$$