3.8 Exercises

3.1 Show that the scalar system $\dot{x} = (k - x^2)/x$, with $k > 0$, has two asymptotically stable equilibrium points and find their regions of attraction.

3.2 Consider the scalar system $\dot{x} = -g(x)$, where $g(x)$ is locally Lipschitz and

$$g(0) = 0; \quad xg(x) > 0, \quad \forall\, x \neq 0 \ \text{ and } \ x \in (-a, a)$$

It is shown in Section 3.1 that the origin is asymptotically stable. In this exercise we arrive at the same conclusion using Lyapunov functions.

(a) Show that the asymptotic stability conditions of Theorem 3.3 are satisfied with $V(x) = \frac{1}{2}x^2$ or $V(x) = \int_0^x g(y)\, dy$.

(b) If $xg(x) > 0$ for all $x \neq 0$, show that the global asymptotic stability conditions of Theorem 3.3 are satisfied with $V(x) = \frac{1}{2}x^2$ or $V(x) = \frac{1}{2}x^2 + \int_0^x g(y)\, dy$.

(c) Under what conditions on g can you show global asymptotic stability using $V(x) = \int_0^x g(y)\, dy$? Give an example of g where the origin of $\dot{x} = -g(x)$ is globally asymptotic stability but $V(x) = \int_0^x g(y)\, dy$ does not satisfy the global asymptotic stability conditions of Theorem 3.3.

3.3 For each of the following scalar systems, determine if the origin is unstable, stable but not asymptotically stable, asymptotically stable but not globally asymptotically stable, or globally asymptotically stable.

(1) $f(x) = -x$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$

(2) $f(x) = x$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$

(3) $f(x) = \sin x$

(4) $f(x) = -\sin x$

(5) $f(x) = -x - \sin x$

3.4 Euler equations for a rotating rigid spacecraft are given by [105]

$$J_1\dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 + u_1, \quad J_2\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 + u_2, \quad J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2 + u_3$$

where $\omega = \text{col}(\omega_1, \omega_2, \omega_3)$ is the angular velocity vector along the principal axes, $u = \text{col}(u_1, u_2, u_3)$ is the vector of torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of inertia.

(a) Show that with $u = 0$ the origin $\omega = 0$ is stable. Is it asymptotically stable?

(b) Let $u_i = -k_i\omega_i$, where k_1 to k_3 are positive constants. Show that the origin of the closed-loop system is globally asymptotically stable.

3.5 For each of the following systems, determine whether the origin is stable, asymptotically stable or unstable.

(1) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = -2\sin x_1 - 2x_2 - 2x_3$

(2) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = 2\sin x_1 - 2x_2 - 2x_3$

(3) $\dot{x}_1 = x_2 + x_3, \qquad \dot{x}_2 = -\sin x_1 - x_3, \qquad \dot{x}_3 = -\sin x_1 + x_2$

3.6 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - 2x_2, \qquad \text{where } h(x_1) = x_1\left(2 + \frac{x_1^2}{1 + x_1^2}\right)$$

Verify that $V(x) = \int_0^{x_1} h(\sigma)\ d\sigma + \frac{1}{2}(x_1 + x_2)^2$ is positive definite and radially unbounded; then use it to show that the origin is globally exponentially stable.

3.7 Consider the system

$$\dot{x}_1 = -x_1 + x_3, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = -x_2 - x_2^3 - x_3$$

(a) Is the origin exponentially stable?

(b) Using $V(x) = bx_1^2 + 3x_2^2 + 2x_2x_3 + 2x_3^2 + x_2^4$, where $b > 0$ is to be chosen, show that the origin is globally asymptotically stable.

3.8 (**[132]**) Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\tanh(x_1 + x_2)$$

(a) Show that

$$V(x) = \int_0^{x_1} \tanh(\sigma)\ d\sigma + \int_0^{x_1 + x_2} \tanh(\sigma)\ d\sigma + x_2^2$$

is positive definite for all x and radially unbounded.

(b) Show that the origin is globally asymptotically stable.

3.9 (Krasovskii's Method) Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. Assume that $f(x)$ is continuously differentiable and its Jacobian $[\partial f/\partial x]$ satisfies

$$P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \le -I, \quad \forall\ x \in R^n, \quad \text{where } P = P^T > 0$$

(a) Using the representation $f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)x\ d\sigma$, show that

$$x^T P f(x) + f^T(x) P x \le -x^T x, \quad \forall\ x \in R^n$$

(b) Using $V(x) = f^T(x)Pf(x)$, show that the origin is globally asymptotically stable.

3.10 (**[121]**) A closed-loop system under optimal stabilizing control is given by

$$\dot{x} = f(x) - kG(x)R^{-1}(x)G^T(x)\left(\frac{\partial V}{\partial x}\right)^T$$

where $V(x)$ is a continuously differentiable, positive definite function that satisfies the Hamilton–Jacobi–Bellman equation

$$\frac{\partial V}{\partial x}f(x) + q(x) - \tfrac{1}{4}\frac{\partial V}{\partial x}G(x)R^{-1}(x)G^T(x)\left(\frac{\partial V}{\partial x}\right)^T = 0$$

$q(x)$ is a positive semidefinite function, $R(x)$ is a positive definite matrix for all x, and k is a positive constant. Show that the origin is asymptotically stable when

(1) $q(x)$ is positive definite and $k \geq 1/4$.

(2) $q(x)$ is positive semidefinite, $k > 1/4$, and the only solution of $\dot{x} = f(x)$ that stays identically in the set $\{q(x) = 0\}$ is the trivial solution $x(t) \equiv 0$.

3.11 The system

$$\dot{x}_1 = -x_1 + x_3, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = -g_1(x_1) - g_2(x_2) - x_3$$

where g_1 and g_2 are locally Lipschitz and satisfy $g_i(0) = 0$ and $yg_i(y) > 0$ for all $y \neq 0$, has a unique equilibrium point at the origin.

(a) Verify that $V(x) = \int_0^{x_1} g_1(y)\, dy + \int_0^{x_2} g_2(y)\, dy + \frac{1}{2}x_3^2$ is positive definite for all x and use it to show asymptotic stability of the origin.

(b) Under what additional conditions on g_1 and g_2 can you show that the origin is globally asymptotically stable.

(c) Under what additional conditions on g_1 and g_2 can you show that the origin is exponentially stable?

3.12 An unforced mass-spring system (Section A.2) with nonlinear viscous friction and nonlinear spring is modeled by

$$m\ddot{y} + b(1 + c|\dot{y}|)\dot{y} + g(y) = 0$$

where $g(y) = k(1 - a^2y^2)y$, with $|ay| < 1$, for a softening spring, $g(y) = k(1 + a^2y^2)y$ for a hardening spring, and all constants are positive. Take the state variables as $x_1 = y$, $x_2 = \dot{y}$. Using an energy-type Lyapunov function, study the stability of the origin for each spring type.

3.13 Show that the origin of the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \mu x_2 - x_2^3$$

is globally asymptotically stable for $\mu \le 0$ and unstable for $\mu > 0$. Is it exponentially stable when $\mu = 0$?

3.14 Using $V(x) = \frac{1}{2}x^T x$, show that the origin of

$$\dot{x}_1 = -x_1 + x_2 \cos x_1, \qquad \dot{x}_2 = x_1 \cos x_1 - x_2(\cos x_1)^2 + x_3, \qquad \dot{x}_3 = -x_2$$

is globally asymptotically stable. Is it exponentially stable?

3.15 Consider the pendulum equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2 + u$$

under the state feedback control $u = -k_1 \text{ sat}(x_1) - k_2 \text{ sat}(x_2)$, where b, k_1 and k_2 are positive constants and $k_1 > 1$.

(a) Show that the origin is the unique equilibrium point of the closed-loop system.

(b) Using $V(x) = \int_0^{x_1} [\sin\sigma + k_1 \text{ sat}(\sigma)]\, d\sigma + \frac{1}{2}x_2^2$, show that the origin of the closed-loop system is globally asymptotically stable.

3.16 For each of the following systems,

(a) Find all equilibrium points and study their stability using linearization.

(b) Using quadratic Lyapunov functions, estimate the regions of the attraction of each asymptotically stable equilibrium point. Try to make the estimates as large as you can.

(c) Draw the phase portrait of the system to find the exact regions of attraction and compare them with your estimates.

(1) $\dot{x}_1 = -(x_1 + x_1^3) + 2x_2, \qquad \dot{x}_2 = 2x_1 - (x_2 + x_2^3)$

(2) $\dot{x}_1 = x_1 - x_1^3 + x_2, \qquad \dot{x}_2 = x_1 - 3x_2$

(3) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2 + 2x_2^3 - \frac{1}{2}x_2^5$

(4) $\dot{x}_1 = -x_2, \qquad \dot{x}_2 = 2x_1 + 3x_2 + 2 \text{ sat}(-3x_2)$

3.17 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + x_1^5 - x_2$$

(a) Using the Lyapunov function candidate $V(x) = \frac{1}{2}x_1^2 - \frac{1}{6}x_1^6 + \frac{1}{2}x_2^2$, show that the origin is asymptotically stable and estimate the region of attraction.

(b) Draw the phase portrait of the system to find the exact region of attraction and compare it with your estimate.

3.18 Consider the system

$$\dot{x}_1 = x_1^3 - x_2, \qquad \dot{x}_2 = x_1 - x_2$$

Show that the origin is asymptotically stable? Is it exponentially stable? Is it globally asymptotically stable? If not, estimate the region of attraction.

3.19 For each of the following systems, show that there is an equilibrium point at the origin and investigate its stability. If the origin is asymptotically stable, determine if it is globally asymptotically stable; if it is not so, estimate the region of attraction.

(1) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 - x_2$

(2) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3$

(3) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - 2x_2 - (x_1^2 + x_2^2)\,\tanh(x_1 + x_2)$

(4) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \frac{1}{3}x_1^3 - 2x_2$

(5) $\dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = -x_1^3 - x_2$

3.20 Show that the origin of the system

$$\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -0.1x_1^3 - x_2 - 10\sin x_3, \qquad \dot{x}_3 = x_2 - x_3$$

is asymptotically stable and estimate the region of attraction.

3.21 ([148]) Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2 - (x_1 + 2x_2)(1 - x_2^2)$$

Show that the origin is asymptotically stable and find an estimate of the region of attraction that includes the point $x = (-1, 1)$. **Hint:** Use the variable gradient method to find a quadratic Lyapunov function $V(x) = x^T P x$ such that $\dot{V}(x)$ is negative definite in the set $\{|x_2| \leq 1\}$.