in Appendix C. Chapter 4 extends Lyapunov stability to time-varying systems and shows how it can be useful in the analysis of perturbed system. This leads into the notion of input-to-state stability. Chapter 5 deals with a special class of systems that dissipates energy. One point we emphasize is the connection between passivity and Lyapunov stability. Chapter 6 looks at input-output stability and shows that it can be established using Lyapunov functions. The tools of Chapters 5 and 6 are used in Chapter 7 to derive stability criteria for the feedback connection of two stable systems.

The next six chapters deal with nonlinear control. Chapter 8 presents some special nonlinear forms that play significant roles in the design of nonlinear controllers. Chapters 9 to 13 deal with nonlinear control problems, including nonlinear observers. The nonlinear control techniques we are going to study can be categorized into five different approaches to deal with nonlinearity. These are:

- Approximate nonlinearity
- Compensate for nonlinearity
- Dominate nonlinearity
- Use intrinsic properties
- Divide and conquer

Linearization is the prime example of approximating nonlinearities. Feedback linearization that cancels nonlinearity is an example of nonlinearity compensation. Robust control techniques, which are built around the classical tool of high-gain feedback, dominate nonlinearities. Passivity-based control is an example of a technique that takes advantage of an intrinsic property of the system. Because the complexity of a nonlinear system grows rapidly with dimension, one of the effective ideas is to decompose the system into lower-order components, which might be easier to analyze and design, then build up back to the original system. Backstepping is an example of this divide and conquer approach.

Four appendices at the end of the book give examples of nonlinear state models, mathematical background, procedures for constructing composite Lyapunov functions, and proofs of some results. The topics in this book overlap with topics in some excellent textbooks, which can be consulted for further reading. The list includes [10, 53, 63, 66, 92, 118, 129, 132, 144]. The main source for the material in this book is [74], which was prepared using many references. The reader is advised to check the Notes and References section of [74] for a detailed account of these references.

1.4 Exercises

1.1 A mathematical model that describes a wide variety of single-input–single-ouput nonlinear systems is the nth-order differential equation

$$y^{(n)} = g\left(t, y, \dot{y}, \ldots, y^{(n-1)}, u\right)$$

where u is the input and y the output. Find a state model.

1.2 The nonlinear dynamic equations for a single-link manipulator with flexible joints [135], damping ignored, is given by

$$I\ddot{q}_1 + MgL\sin q_1 + k(q_1 - q_2) = 0, \qquad J\ddot{q}_2 - k(q_1 - q_2) = u$$

where q_1 and q_2 are angular positions, I and J are moments of inertia, k is a spring constant, M is the total mass, L is a distance, and u is a torque input.

(a) Using q_1, $\dot{q}_1$, q_2, and $\dot{q}_2$ as state variables, find the state equation.

(b) Show that the right-hand side function is globally Lipschitz when u is constant.

(c) Find the equilibrium points when $u = 0$.

1.3 A synchronous generator connected to an infinite bus is represented by [103]

$$M\ddot{\delta} = P - D\dot{\delta} - \eta_1 E_q \sin\delta, \qquad \tau\dot{E}_q = -\eta_2 E_q + \eta_3 \cos\delta + E_F$$

where δ is an angle in radians, E_q is voltage, P is mechanical input power, E_F is field voltage (input), D is damping coefficient, M is inertial coefficient, τ is time constant, and η_1, η_2, and η_3 are positive parameters.

(a) Using δ, $\dot{\delta}$, and E_q as state variables, find the state equation.

(b) Show that the right-hand side function is locally Lipschitz when P and E_F are constant. Is it globally Lipschitz?

(c) Show that when P and E_F are constant and $0 < P < \eta_1 E_F/\eta_2$, there is a unique equilibrium point in the region $0 \le \delta \le \pi/2$.

1.4 The circuit shown in Figure 1.1 contains a nonlinear inductor and is driven by a time-dependent current source. Suppose the nonlinear inductor is a Josephson junction [25] described by $i_L = I_0 \sin k\phi_L$, where ϕ_L is the magnetic flux of the inductor and I_0 and k are constants.

(a) Using ϕ_L and v_C as state variables, find the state equation.

(b) Show that the right-hand side function is locally Lipschitz when i_s is constant. Is it globally Lipschitz?

(c) Find the equilibrium points when $i_s = I_s$ (constant) with $0 < I_s < I_0$.

1.5 Repeat the previous exercise when the nonlinear inductor is described by $i_L = k_1\phi_L + k_2\phi_L^3$, where k_1 and k_2 are positive constants. In Part (b), $I_s > 0$.

1.6 Figure 1.2 shows a vehicle moving on a road with grade angle θ, where v the vehicle's velocity, M is its mass, and F is the tractive force generated by the engine. Assume that the friction is due to Coulomb friction, linear viscous friction, and a drag force proportional to v^2. Viewing F as the control input and θ as a disturbance input, find a state model of the system.

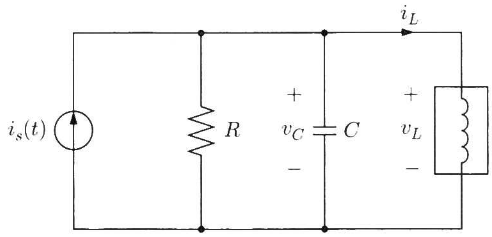

Figure 1.1: Exercises 1.4 and 1.5.

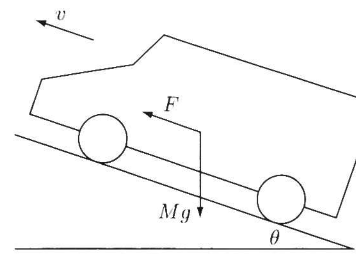

Figure 1.2: Exercise 1.6.

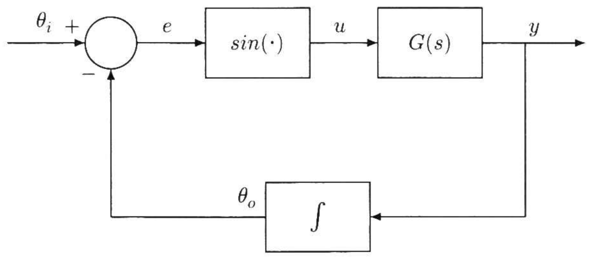

Figure 1.3: Exercise 1.7.

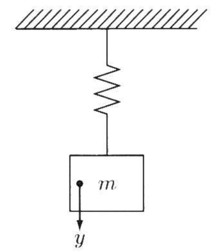

Figure 1.4: Exercise 1.8.

1.7 A phase-locked loop [45] can be represented by the block diagram of Figure 1.3. Let $\{A, B, C\}$ be a minimal realization of the scalar, strictly proper transfer function $G(s)$. Assume that all eigenvalues of A have negative real parts, $G(0) \neq 0$, and $\theta_i =$ constant. Let z be the state of the realization $\{A, B, C\}$.

(a) Show that the closed-loop system can be represented by the state equations

$$\dot{z} = Az + B \sin e, \qquad \dot{e} = -Cz$$

(b) Find the equilibrium points of the system.

1.8 Consider the mass–spring system shown in Figure 1.4. Assuming a linear spring and nonlinear viscous damping described by $c_1\dot{y} + c_2\dot{y}|\dot{y}|$, find a state equation that describes the motion of the system.

The next three exercises give examples of hydraulic systems [27].

1.9 Figure 1.5 shows a hydraulic system where liquid is stored in an open tank. The cross-sectional area of the tank, $A(h)$, is a function of h, the height of the liquid level above the bottom of the tank. The liquid volume v is given by $v = \int_0^h A(\lambda)\, d\lambda$. For a liquid of density ρ, the absolute pressure p is given by $p = \rho g h + p_a$, where p_a is the atmospheric pressure (assumed constant) and g is the acceleration due to

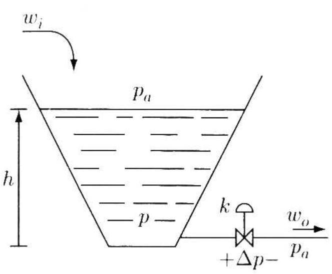

Figure 1.5: Exercise 1.9.

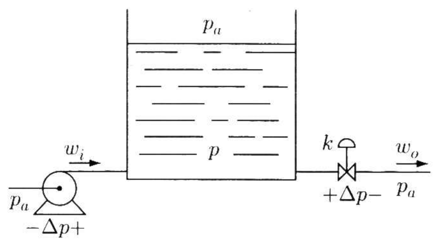

Figure 1.6: Exercise 1.10.

gravity. The tank receives liquid at a flow rate w_i and loses liquid through a valve that obeys the flow-pressure relationship $w_o = k\sqrt{p - p_a}$. The rate of change of v satisfies $\dot{v} = w_i - w_o$. Take w_i to be the control input and h to be the output.

(a) Using h as the state variable, determine the state model.

(b) Using $p - p_a$ as the state variable, determine the state model.

(c) Find a constant input that maintains a constant output at $h = r$.

1.10 The hydraulic system shown in Figure 1.6 consists of a constant speed centrifugal pump feeding a tank from which liquid flows through a pipe and a valve that obeys the relationship $w_o = k\sqrt{p - p_a}$. The pressure-flow characteristic of the pump is given by $p - p_a = \beta\sqrt{1 - w_i/\alpha}$ for some positive constants α and β. The cross-sectional area of the tank is uniform; therefore, $v = Ah$ and $p = p_a + \rho g v/A$, where the variables are defined in the previous exercise.

(a) Using $(p - p_a)$ as the state variable, find the state model.

(b) Find the equilibrium points of the system.

1.11 The valves in the hydraulic system of Figure 1.7 obey the flow relationships $w_1 = k_1\sqrt{p_1 - p_2}$ and $w_2 = k_2\sqrt{p_2 - p_a}$. The pump characteristic is $p_1 - p_a = \beta\sqrt{1 - w_p/\alpha}$. All variables are defined in the previous two exercises.

(a) Using $(p_1 - p_a)$ and $(p_2 - p_a)$ as the state variables, find the state equation.

(b) Find the equilibrium points of the system.

1.12 For each of the following systems, investigate local and global Lipschitz properties. Assume that input variables are continuous functions of time.

(a) The pendulum equation (A.2).

(b) The mass-spring system (A.6).

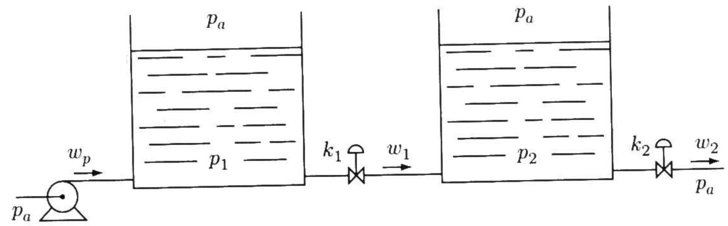

Figure 1.7: The hydraulic system of Exercise 1.11.

(c) The tunnel diode circuit (A.7).

(d) The van der Pol oscillator (A.13).

(e) The boost converter (A.16).

(f) The biochemical reactor (A.19) with ν defined by (A.20).

(g) The DC motor (A.25) when f_e and f_ℓ are linear functions.

(h) The magnetic levitation system (A.30)–(A.32).

(i) The electrostatic actuator (A.33).

(j) The two-link robot manipulator (A.35)–(A.37).

(k) The inverted pendulum on a cart (A.41)–(A.44).

(l) The TORA system (A.49)–(A.52).

1.13 Find a global diffeomorphism $z = T(x)$ that transforms the system

$$\dot{x}_1 = x_2 + g_1(x_1), \quad \dot{x}_2 = x_3 + g_2(x_1, x_2), \quad \dot{x}_3 = g_3(x) + g_4(x)u, \quad y = x_1$$

into

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3, \quad \dot{z}_3 = a(z) + b(z)u, \quad y = z_1$$

Assume g_1 to g_4 are smooth.

1.14 Find a diffeomorphism $z = T(x)$ that transforms the system

$$\dot{x}_1 = \sin x_2, \quad \dot{x}_2 = -x_1^2 + u, \quad y = x_1$$

into

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = a(z) + b(z)u, \quad y = z_1$$

and give the definitions of a and b.

trajectories of a saddle define a separatrix. Let the eigenvalues of the linearization be $\lambda_1 > 0 > \lambda_2$ and the corresponding eigenvectors be v_1 and v_2. The stable and unstable trajectories of the nonlinear saddle will be tangent to the stable eigenvector v_2 and the unstable eigenvector v_1, respectively, as they approach the equilibrium point p. Therefore, the two unstable trajectories can be generated from the initial points $x_0 = p \pm \alpha v_1$, where α is a small positive number. Similarly, the two stable trajectories can be generated from the initial points $x_0 = p \pm \alpha v_2$. The major parts of the unstable trajectories will be generated by solution in forward time, while the major parts of the stable ones will be generated by solution in reverse time.

2.6 Exercises

2.1 For each of the following systems,

(a) Find all equilibrium points and determine their types.

(b) Construct and discuss the phase portrait.

(1) $\dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = x_1 - x_2^3$

(2) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 - x_2 - 0.5(x_1 + x_2)^2$

(3) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = 2x_1 + 3x_1^2 + x_1^3 - x_2$

(4) $\dot{x}_1 = -x_2 + \frac{1}{4}x_2^3, \qquad \dot{x}_2 = 4x_1 - 2x_2$

(5) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + x_2(1 - x_1^2 + 0.1x_1^4)$

(6) $\dot{x}_1 = -x_1 + 2x_2 + x_1x_2 + x_2^2, \qquad \dot{x}_2 = -x_1 - x_1^2 - x_1x_2$

(7) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -(x_1 + x_2)/(x_1 + 2)$ (defined over the set $\{x_1 > -2\}$)

(8) $\dot{x}_1 = -x_2, \qquad \dot{x}_2 = 2x_1 + 3x_2 + 2\,\mathrm{sat}(-3x_2)$

(9) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \frac{1}{16}x_1^3 + x_2 - \frac{1}{3}x_2^3$

2.2 Consider the tunnel-diode circuit of Example 2.2. Keep all parameters the same except u. Construct and discuss the phase portrait when

(a) $u = 0.4$;

(b) $u = 2$.

2.3 Consider the biochemical reactor (A.19) with ν defined by (A.20). Let $\alpha = 23$, $\beta = 0.39$, and $\gamma = 0.57$. Construct and discuss the phase portrait in the region $\{x_1 \geq 0,\ x_2 \geq 0\}$ when

(a) $u = 0.5$;

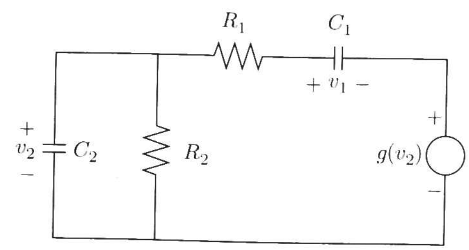

Figure 2.15: Exercise 2.5.

(b) $u = 1$;

(c) $u = 1.5$.

2.4 Construct and discuss the phase portrait of the negative resistance oscillator (A.10) when

(a) $h(v) = -v + v^3 - v^5/5 + v^7/98$ and $\varepsilon = 1$.

(b) $h(v) = -v + v^3 - v^5/5 + v^7/84$ and $\varepsilon = 1$.

2.5 An equivalent circuit of the Wien–Bridge oscillator is shown in Figure 2.15 [26], where $g(v_2)$ is a nonlinear voltage-controlled voltage source.

(a) With $x_1 = (C_1 v_1 + C_2 v_2)/(C_1 + C_2)$ and $x_2 = v_2$ as state variables, show that the state model is given by

$$\dot{x}_1 = -\frac{1}{R_2(C_1 + C_2)} x_2, \quad \dot{x}_2 = \frac{C_1 + C_2}{C_1 C_2 R}(x_1 - x_2) + \frac{1}{C_2 R_1} g(x_2) - \frac{1}{C_2 R_2} x_2$$

(b) Let $C_1 = C_2 = C$, $R_1 = R_2 = R$ and $g(v) = 3.234v - 2.195v^3 + 0.666v^5$. Construct and discuss the phase portrait in the $\tau = t/(CR)$ time scale.

2.6 A reduced-order model of the electrostatic actuator (A.33) can be obtained if the time constant T is sufficiently small. Writing the $\dot{x}_3$-equation as $T\dot{x}_3 = (\cdot)$, setting $T = 0$, and using the resultant algebraic equation to eliminate x_3, we obtain

$$\dot{x}_1 = x_2, \qquad \dot{x}_1 = -x_1 - 2\zeta x_2 + \frac{4u^2}{27(1 - x_1)^2}$$

where $0 \le x_1 \le 1 - \delta$. Let u be a constant input, $u(t) \equiv U < 1$.

(a) Show that there are two equilibrium points and determine their types.

(b) Construct and discuss the phase portrait when $\zeta = \frac{1}{2}$ and $U = \frac{3}{4}$.

2.7 Consider the inverted pendulum equation (A.47) with $a = 1$.

(a) With $u = 0$, find all equilibrium points and determine their types.

(b) The energy $E = 1 + \cos x_1 + \frac{1}{2}x_2^2$ is defined such that $E = 2$ at the upward equilibrium position $(0, 0)$ and $E = 0$ at the downward one $(\pi, 0)$. To swing up the pendulum we need to pump up the energy. Show that the control $u = k(2 - E)x_2 \cos(x_1)$, with $k > 0$, will make $\dot{E} > 0$ whenever $E < 2$ and $x_2 \cos x_1 \neq 0$.

(c) Consider the closed-loop system under the foregoing control. Construct and discuss the phase portrait when $k = 0.1$.

2.8 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -0.5x_1 + 1.5x_2 + 0.5u$$

Construct and discuss the phase portrait for $u = 0$, the feedback control $u = 0.9x_1 - 3.2x_2$, and the constrained feedback control $u = \text{sat}(0.9x_1 - 3.2x_2)$.

2.9 Construct and discuss the phase portrait of the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 - x_2 + \mu \text{ sat}\left(-2x_1 + \tfrac{1}{2}x_2\right)$$

for each of the values $\mu = 0.25$, 1, and 3.

2.10 The elementary processing units in the central nervous system are the neurons. The FitzHugh-Nagumo model [49] is a dimensionless model that attempts to capture the dynamics of a single neuron. It is given by

$$\dot{u} = u - \tfrac{1}{3}u^3 - w + I, \qquad \dot{w} = \varepsilon(b_0 + b_1 u - w)$$

where u, w, and $I \geq 0$ are the membrane voltage, recovery variable, and applied current, respectively. The constants ε, b_0 and b_1 are positive.

(a) Find all equilibrium points and determine their types when $b_1 > 1$.

(b) Repeat part (a) when $b_1 < 1$.

(c) Let $\varepsilon = 0.1$, $b_0 = 2$ and $b_1 = 1.5$. For each of the values $I = 0$ and $I = 2$, construct the phase portrait and discuss the qualitative behavior of the system.

(d) Repeat (c) with $b_1 = 0.5$.

2.11 Consider the magnetic levitation system (A.29) under the feedback control $u = -1 + \text{sat}\left(\frac{1}{2} - x_1 - x_2\right)$.

(a) Find all equilibrium points and determine their types when $c = 1$ and when $c \neq 1$ but $|(1 - c)/c| < 1$.

(b) Construct and discuss the phase portrait when $b = 0.01$ and $c = \frac{2}{3}$, taking into consideration the constraint $0 \le x_1 \le 5$.

2.12 [77] Consider the nonlinear system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = 0.1x_2^3 + u$$

under the feedback control $u = -\gamma^2 x_1 - \gamma x_2$, with $\gamma > 0$. The linearization at the origin has the eigenvalues $(\gamma/2)(-1 \pm j\sqrt{3})$, so the response of the linearization can be made faster by increasing γ. To study the effect of increasing γ on the behavior of the nonlinear system, it is convenient to apply the change of variables $z_1 = \sqrt{\gamma}x_1$, $z_2 = x_2/\sqrt{\gamma}$, and $\tau = \gamma t$.

(a) Show that the transformed state equation in the τ time scale is given by

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = -z_1 - z_2 + 0.1z_2^3$$

(b) Construct and discuss the phase portrait in the z_1–z_2 plane.

(c) Discuss the effect of increasing γ on the phase portrait in the x_1–x_2 plane.