Linear Systems Theory

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Comments and information about typos are very welcome.
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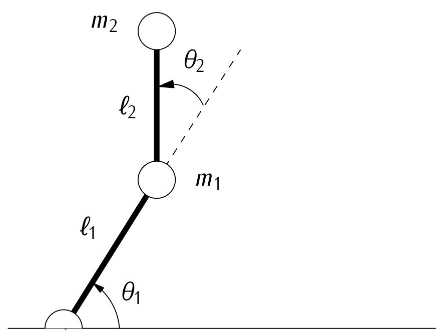
Errata

1) In page 7, the MATLAB® command should read

```
sys_ss=ss(A,B,C,D,...
          'InputName', {'input1', 'input2',...},...
          'OutputName',{'output1','output2',...},...
          'StateName', {'state1', 'state2',...});
```

2) In page 15 in Figure 2.3, the angle θ_2 is incorrectly drawn, it should be drawn as follows:



Moreover, the matrix $M(q)$ and the vector $G(q)$ in Example 2.2 should be as follows:

$$M(q) := \begin{bmatrix} m_2\ell_2^2 + 2m_2\ell_1\ell_2\cos\theta_2 + (m_1+m_2)\ell_1^2 & m_2\ell_2^2 + m_2\ell_1\ell_2\cos\theta_2 \\ m_2\ell_1\ell_2\cos\theta_2 + m_2\ell_2^2 & m_2\ell_2^2 \end{bmatrix}$$
$$G(q) := \begin{bmatrix} m_2 g\ell_2\cos(\theta_1+\theta_2) + (m_1+m_2)g\ell_1\cos\theta_1 \\ m_2 g\ell_2\cos(\theta_1+\theta_2) \end{bmatrix}.$$

3) In page 17, equation (2.8) should read as follows:

$$\dot{x} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} v, \qquad y = \begin{bmatrix} I & 0 \end{bmatrix} x, \qquad x := \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \in \mathbb{R}^{2k}.$$

4) In page 18, the first equation in Section 2.4.4 should read as follows:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = M^{-1}(x_1)\Big(- B(x_1,x_2)x_2 - G(x_1) + u\Big).$$

5) In page 17 in Figure 2.5, the label y at the right should be replaced by $\begin{bmatrix} q \\ \dot{q} \end{bmatrix}$.

6) In page 28, the equation just above definition 3.2 should read

$$\hat{y}(z) = \mathcal{Z}[y(t)] \coloneqq \sum_{t=0}^{\infty} z^{-t} y(t), \quad z \in \mathbb{C}.$$

7) In page 30 in the last equation in Note 4, dt should be replaced by $d\bar{t}$, as in:

$$\mathcal{L}[(x \star y)(t)] = \int_0^{\infty} e^{-s\tau} x(\tau) \Big(\int_0^{\infty} e^{-s\bar{t}} y(\bar{t}) d\bar{t} \Big) d\tau = \hat{x}(s)\hat{y}(s). \qquad \square$$

8) In page 34 in the sidebar example in Section 4.3.2, the matrix N_3 should be

$$N_3 = \left[\begin{smallmatrix} -24 & 3 \\ 1 & \frac{1}{2} \end{smallmatrix} \right].$$

9) In page 35 in the proof of Proposition 4.1, the second equation should read as follows:

$$Z_n = \frac{1}{s} Z_{n-1}, \quad Z_{n-1} = \frac{1}{s} Z_{n-2}, \quad \ldots \quad, \quad Z_2 = \frac{1}{s} Z_1 \quad \Rightarrow \quad Z_k = \frac{1}{s^{k-1}} Z_1.$$

10) In page 38, the first command in MATLAB® Hint 17 should read

```
z=0;p=[-1,-3];k=2;
```

11) In page 48, the last equation should read

$$\begin{aligned} A^{n+1} &= -a_1 A^n - a_2 A^{n-1} - \cdots - a_{n-1} A^2 - a_n A \\ &= a_1 (a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I) - a_2 A^{n-1} - \cdots - a_{n-1} A^2 - a_n A \\ &= (a_1^2 - a_2) A^{n-1} + (a_1 a_2 - a_3) A^{n-2} + \cdots + (a_1 a_{n-1} - a_n) A + a_1 a_n I. \end{aligned}$$

12) In page 50, the forth equation should read

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{\alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n}{(s-\lambda_1)^{m_1} (s-\lambda_2)^{m_2} \cdots (s-\lambda_k)^{m_k}} \right] &= a_{11} e^{\lambda_1 t} + a_{12}\, t\, e^{\lambda_1 t} + \cdots + \frac{a_{1m_1} t^{m_1 - 1} e^{\lambda_1 t}}{(m_1 - 1)!} \\ &\quad + \cdots + a_{k1} e^{\lambda_k t} + a_{k2}\, t\, e^{\lambda_k t} + \cdots + \frac{a_{km_k} t^{m_k - 1} e^{\lambda_k t}}{(m_k - 1)!}. \end{aligned}$$

13) In page 56, the condition 3. in Theorem 7.2 is incorrect. It should read as follows:

Theorem 7.2. *For an $n \times n$ matrix A, the following three conditions are equivalent:*

1. *A is semisimple.*
2. *A has n linearly independent eigenvectors.*
3. *There is a nonzero polynomial* without repeated roots *that annihilates A; i.e., there is a nonzero polynomial $p(s)$ without repeated roots for which $p(A) = 0$.* □

Note that condition 3 provides a simple procedure to check for diagonalizability. Since every polynomial that annihilates A must have each eigenvalue of A as a root (perhaps with different multiplicities), one simply needs to compute all the distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ ($k \leqslant n$) of A and then check if the polynomial $p(s) = (s - \lambda_1) \cdots (s - \lambda_k)$ annihilates A.

14) In page 57, equation (7.1) should read

$$A^t = \underbrace{P^{-1}JP\ P^{-1}JP\ \cdots\ P^{-1}JP}_{t \text{ times}} = P^{-1}J^tP = P^{-1}\begin{bmatrix} J_1^t & 0 & \cdots & 0 \\ 0 & J_2^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_\ell^t \end{bmatrix} P,$$

15) In page 57, the following clarifying sentence should be added after the expression for J_i^t: *"The formula above presumes that* $0! = 1$ *and that any entries in* J_i^t *that would contain factorials of negative numbers should be set to zero."*

16) In page 58, the first equation should read

$$e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$

17) In page 65 in Definition 8.1, the condition for exponential stability should read

$$\|x(t)\| \leqslant ce^{-\lambda(t-t_0)}\|x(t_0)\|, \qquad \forall t \geqslant 0.$$

18) In page 66, the second inequality in the **Attention!** box below Theorem 8.1 should read

$$\|x(t)\| = \|e^{A(t-t_0)}x_0\| \leqslant \|e^{A(t-t_0)}\|\,\|x_0\| \leqslant c\,e^{-\lambda(t-t_0)}\|x_0\|, \qquad \forall t \in \mathbb{R}.$$

19) In page 69 in the proof of Theorem 8.2, the argument regarding how condition 4 implies condition 5, should consider selecting $Q = I$ (instead of $Q = -I$ as stated).

20) In page 70 at the end of the proof of Theorem 8.2, the definition of λ should read

$$\lambda := \frac{\lambda_{\min}[Q]}{\lambda_{\max}[P]}.$$

21) In page 72, the third equation should read

$$v(t+1) = x'(t+1)Px(t+1) = x(t)'A'PAx(t).$$

22) In page 72, in the last equation in Section 8.6 should read

$$v(t+1) = x'(t)(P-Q)x(t) = v(t) - x'(t)Qx(t), \qquad \forall t \geqslant 0.$$

23) In page 72, equation (8.15) should read

$$\|x(t) - x^{\text{eq}}\| \leqslant ce^{-\lambda(t-t_0)}\|x(t_0) - x^{\text{eq}}\|, \qquad \forall t \geqslant t_0.$$

24) In page 75, Theorem 8.6 should read "If the matrix A in the linearized system (8.13) has one or more eigenvalues with strictly positive real part, then ..."

25) In page 76, the first note should read "This equilibrium point is $x^{\text{eq}} = [\pi\ 0]'$, $u^{\text{eq}} = 0$, $y^{\text{eq}} = \pi$, and therefore $\delta x := x - x^{\text{eq}} = x - [\pi\ 0]'$, $\delta u := u - u^{\text{eq}} = u$, $\delta y := y - y^{\text{eq}} = y - \pi$."

26) In page 76 and 77, squares are missing in the expression for the eigenvalues of A. Specifically, in page 76, it should read as follows: *"The eigenvalues of A are given by*

$$\det(\lambda I - A) = \lambda\Big(\lambda + \frac{b}{m\ell^2}\Big) + \frac{g}{\ell} = 0 \quad \Leftrightarrow \quad \lambda = -\frac{b}{2m\ell^2} \pm \sqrt{\Big(\frac{b}{2m\ell^2}\Big)^2 - \frac{g}{\ell}},$$

and therefore the linearized system is exponentially stable, because

$$-\frac{b}{2m\ell^2} \pm \sqrt{\Big(\frac{b}{2m\ell^2}\Big)^2 - \frac{g}{\ell}}$$

has a negative real part"; and in page 77, it should read as follows: *"The eigenvalues of A are given by*

$$\det(\lambda I - A) = \lambda\Big(\lambda + \frac{b}{m\ell^2}\Big) - \frac{g}{\ell} = 0 \quad \Leftrightarrow \quad \lambda = -\frac{b}{2m\ell^2} \pm \sqrt{\Big(\frac{b}{2m\ell^2}\Big)^2 + \frac{g}{\ell}},$$

and therefore the linearized system is unstable, because

$$-\frac{b}{2m\ell^2} + \sqrt{\Big(\frac{b}{2m\ell^2}\Big)^2 + \frac{g}{\ell}} > 0."$$

27) In page 82, the first equation should read

$$u_T(\tau) \coloneqq \begin{cases} 0 & 0 \leqslant \tau < T \\ e_j & \tau \geqslant T \end{cases} \qquad \forall \tau \geqslant 0.$$

28) In page 83, the last equation before Theorem 9.2 should read

$$\sup_{t\geqslant 0} \int_0^t |\bar{g}_{ij}(t-\tau)| d\tau = \sup_{t\geqslant 0} \int_0^t |\bar{g}_{ij}(\rho)| d\rho = \int_0^\infty |\bar{g}_{ij}(\rho)| d\rho.$$

29) In page 84, the last equation should read

$$g_{ij}(t) = \mathcal{L}^{-1}\big[\hat{g}_{ij}(s)\big] = a_{11} e^{\lambda_1 t} + a_{12}\, t\, e^{\lambda_1 t} + \cdots + \frac{a_{1m_1} t^{m_1-1} e^{\lambda_1 t}}{(m_1-1)!} + \cdots + a_{k1} e^{\lambda_k t} + a_{k2}\, t\, e^{\lambda_k t} + \cdots + \frac{a_{km_k} t^{m_k-1} e^{\lambda_k t}}{(m_k-1)!}.$$

30) In page 86, the equation in Exercise 9.1 should read

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x + u.$$

31) In page 89, the two equations for J_{LQR} should read

$$\begin{aligned} J_{\text{LQR}} &\coloneqq \int_0^\infty x'C'QCx + u'Ru\; dt \\ &= H\big(x(\cdot); u(\cdot)\big) + \int_0^\infty x'C'QCx + u'Ru + (Ax+Bu)'Px + x'P(Ax+Bu)\; dt \\ &= H\big(x(\cdot); u(\cdot)\big) + \int_0^\infty x'(A'P + PA + C'QC)x + u'Ru + 2u'B'Px\; dt, \end{aligned}$$

and

$$J_{\text{LQR}} = H\big(x(\cdot); u(\cdot)\big) + \int_0^\infty x'(A'P + PA + C'QC - PBR^{-1}B'P)x + (u' + x'K')R(u + Kx)\; dt.$$

32) In page 90, the closed-loop equation below (10.3) should read

$$\dot{x} = Ax - BKx = (A - BR^{-1}B'P)x.$$

33) In page 96, the sentence just above equation (11.2) should read *"Similarly, determining the controllable subspace amounts to finding for which vectors $x_0 \in \mathbb{R}^n$ the equation ..."*

34) In page 99, the last equation should read

$$\left.\begin{aligned} \dim\ker W' + \dim(\ker W')^{\perp} &= m \\ \dim\ker W' + \dim\operatorname{Im} W' &= m \end{aligned}\right\} \quad \Rightarrow \quad \dim(\ker W')^{\perp} = \dim\operatorname{Im} W' = \operatorname{rank} W = \dim\operatorname{Im} W$$

35) In page 102, the equation after (11.9) should read

$$\begin{aligned} \int_{t_0}^{t_1} \|\bar{u}(\tau)\|^2 d\tau &= \int_{t_0}^{t_1} \|\overbrace{B(\tau)'\Phi(t_1,\tau)'\eta_1}^{u(\tau)} + v(\tau)\|^2 d\tau \\ &= \eta_1' W_R(t_0,t_1)\eta_1 + \int_{t_0}^{t_1} \|v(\tau)\|^2 d\tau + 2\eta_1' \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)v(\tau)d\tau. \end{aligned}$$

36) In page 112, the second to last equation should read

$$B'x, B'A'x, \ldots, B'(A')^{n-1}x,$$

37) In page 114, equation (12.8) should read

$$(A'x)^* Wx + x^* WA'x = \lambda^* x^* Wx + \lambda x^* Wx = 2\Re[\lambda]x^* Wx.$$

38) In page 114, the last sentence of the paragraph that includes equation (12.8) should read *"We conclude that every eigenvector of A' is not in the kernel of B', which implies controllability by the eigenvector test."*

39) In page 115, the last equation should read

$$(A'x)^* Px + x^* PA'x = \lambda^* x^* Px + \lambda x^* Px = 2\Re[\lambda]x^* Px = -\|B'x\|^2.$$

40) In page 120, the third sentence in the proof of Theorem 13.2 should read *"Since the number of nonzero rows of $\bar{C}$ is exactly $\bar{n}$, all these rows must be linearly independent."*

41) In page 121, the last equation should read

$$T(s) = \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} (sI - A_c)^{-1} & \times \\ 0 & (sI - A_u)^{-1} \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} = C_c(sI - A_c)^{-1}B_c + D.$$

42) In page 123, equation (14.1) should read

$$\begin{aligned} \begin{bmatrix} \dot{x}_c / x_c^+ \\ \dot{x}_u / x_u^+ \end{bmatrix} &= \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u, & x_c \in \mathbb{R}^{\bar{n}},\ x_u \in \mathbb{R}^{n-\bar{n}}, \\ y &= \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + Du, & u \in \mathbb{R}^k, y \in \mathbb{R}^m, \end{aligned}$$

where $\bar{n}$ denotes the dimension of the controllable subspace $\mathcal{C}$ of the original system.

43) In page 126, the second equation in the proof of Theorem 14.3 should read

$$(A'x)^* Px + x^* PA'x = \lambda^* x^* Px + \lambda x^* Px = 2\Re[\lambda]x^* Px.$$

44) In page 127, the three-line equation in the middle of the page should read

$$\begin{aligned}
&\bar{A}\bar{P}+\bar{P}\bar{A}' - \bar{B}\bar{B}' \\
&\quad = \begin{bmatrix} A_{\mathrm{c}} & A_{12} \\ 0 & A_{\mathrm{u}} \end{bmatrix} \begin{bmatrix} P_{\mathrm{c}} & 0 \\ 0 & \rho P_{\mathrm{u}} \end{bmatrix} + \begin{bmatrix} P_{\mathrm{c}} & 0 \\ 0 & \rho P_{\mathrm{u}} \end{bmatrix} \begin{bmatrix} A'_{\mathrm{c}} & 0 \\ A'_{12} & A'_{\mathrm{u}} \end{bmatrix} - \begin{bmatrix} B_{\mathrm{c}} \\ 0 \end{bmatrix} \begin{bmatrix} B'_{\mathrm{c}} & 0 \end{bmatrix} \\
&\quad = -\begin{bmatrix} Q_{\mathrm{c}} & -\rho A_{12} P_{\mathrm{u}} \\ -\rho P_{\mathrm{u}} A'_{12} & \rho Q_{\mathrm{u}} \end{bmatrix}
\end{aligned}$$

45) In page 128, Theorem 14.6 should read as follows:

Theorem 14.6 (Eigenvalue assignment). *Assume that the system*

$$\dot{x}/x^{+} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k \tag{AB-CLTI}$$

is controllable. Given any set of n complex numbers $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ in which complex values appear on conjugate pairs, there exists a state feedback matrix $K \in \mathbb{R}^{k\times n}$ such that the closed-loop system $\dot{x}/x^{+} = (A - BK)x$ has eigenvalues equal to the λ_i. □

Note the requirement that the eigenvalues must appear in complex conjugate pairs to get a real-valued matrix K.

46) In page 141, the second equation should read

$$y(t) = C(t)\Phi(t, t_0)x_0 + \sum_{\tau=t_0}^{t-1} C(t)\Phi(t, \tau+1)B(\tau)u(\tau) + D(t)u(t), \qquad \forall t_0 \leqslant t < t_1.$$

47) In page 142, the two equations for the state given in Theorem 15.4 should read

$$x(t_0) = W_O(t_0, t_1)^{-1} \sum_{t=t_0}^{t_1-1} \Phi(t, t_0)' C(t)' \tilde{y}(t),$$

and

$$x(t_1) = W_{Cn}(t_0, t_1)^{-1} \sum_{t=t_0}^{t_1-1} \Phi(t, t_1)' C(t)' \bar{y}(t).$$

48) In pages 142–143, Section 15.8, the word "controllable" should be throughout replaced by "reachable" and vice-versa. Additionally, the controllability gramians W_C and $\bar{W}_C$ should be replaced by the reachability gramians W_R and $\bar{W}_R$.

49) In page 143, equation (15.7) should read

$$\dot{\bar{x}} = A'\bar{x} + C'\bar{u}, \qquad \bar{y} = B'\bar{x} + D'\bar{u}, \qquad \bar{x} \in \mathbb{R}^n,\ \bar{u} \in \mathbb{R}^m,\ \bar{y} \in \mathbb{R}^k.$$

50) In page 145, the last equation of Section 15.9 should read

$$W = \sum_{\tau=0}^{\infty} (A')^{\tau} C' C A^{\tau} = \lim_{t_1 - t_0 \to \infty} W_O(t_0, t_1).$$

51) In page 145, the system in Exercise 15.1 should be

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x, \qquad y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} x.$$

The exercise still makes sense with a u in the output equation, but it is much less interesting.

52) In page 152, equation (16.4a) should read

$$\begin{bmatrix} \dot{x}_o/x_o^+ \\ \dot{x}_u/x_u^+ \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_u \end{bmatrix} \begin{bmatrix} x_o \\ x_u \end{bmatrix} + \begin{bmatrix} B_o \\ B_u \end{bmatrix} u, \qquad x_o \in \mathbb{R}^{n-\bar{n}},\ x_u \in \mathbb{R}^{\bar{n}},$$

$$y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} x_o \\ x_u \end{bmatrix} + Du, \qquad u \in \mathbb{R}^k, y \in \mathbb{R}^m,$$

where $\bar{n}$ denotes the dimension of the unobservable subspace $\mathcal{UO}$ of the original system.

53) In page 152, the definition for detectability should read

Definition 16.1 (Detectable system). The pair (A, C) is *detectable* if it is algebraically equivalent to a system in the standard form for unobservable systems (16.4) with $\bar{n} = 0$ (i.e., A_u nonexistent) or with A_u a stability matrix.

54) In page 154, Theorem 16.9 should read as follows:

Theorem 16.9. *Assume that the pair (A, C) is observable. Given any set of n complex numbers $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ in which complex values appear on conjugate pairs, there exists an output injection matrix $L \in \mathbb{R}^{n\times m}$ such that $A - LC$ has eigenvalues equal to the λ_i.* □

Note the requirement that the eigenvalues must appear in complex conjugate pairs to get a real-valued matrix L.

55) In page 155, equation (16.10) should read

$$\begin{bmatrix} \dot{x}/x^+ \\ \dot{e}/e^+ \end{bmatrix} = \begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}.$$

56) In page 156, the system in Exercise 16.9 should be

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x, \qquad y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} x.$$

The exercise still makes sense with a u in the output equation, but it is much less interesting.

57) In page 160, the last sentence of Section 17.2 should read as follows: *"But since $\bar{C}$ has only $\bar{n} < n$ rows, its rank must be lower than n..."*

58) In page 162, the equation below (17.13) should read

$$Ax = \lambda x \quad \Leftrightarrow \quad \begin{cases} -\sum_{i=1}^n \alpha_i x_i = \lambda x_1 \\ x_i = \lambda x_{i+1}, & i \in \{1, 2, \dots, n-1\}. \end{cases}$$

59) In page 169, the third sentence after equation (18.1) should read as follows: *"We recall that the number of roots of $d(s)$ (i.e., the number of poles) is equal to the dimension of a minimal realization for $\hat{g}(s)$."*

60) In page 171, the last sentence in the proof of Lemma 18.1 should read as follows: *"However, at the roots of $D_r(s)$, all $r \times r$ submatrices are singular and therefore have linearly dependent rows and columns."*

61) In page 184, the second section of Section 19.6 should read as follows: *"Since $L(s)$ and $R(s)$ are unimodular, $\hat{G}(s)$ has an inverse if and only if $SM_{\hat{G}}(s)$ is invertible, which happens only when $m = r$."*

62) In page 185, the last equation should read

$$\hat{c}(s) = \hat{g}(s)^{-1} \frac{\hat{q}(s)}{1 - \hat{q}(s)} = \frac{k}{s} \hat{g}(s)^{-1}.$$

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