Final Examination

Linear System Theory ECE 773/ME 733

April 21, 2017

Note: Depending how how you decide to solve each problem the following matlab commands (among others) may be helpful: lqr, kalman, initial, step, lsim, damp, lqe, vpa, and sigma.

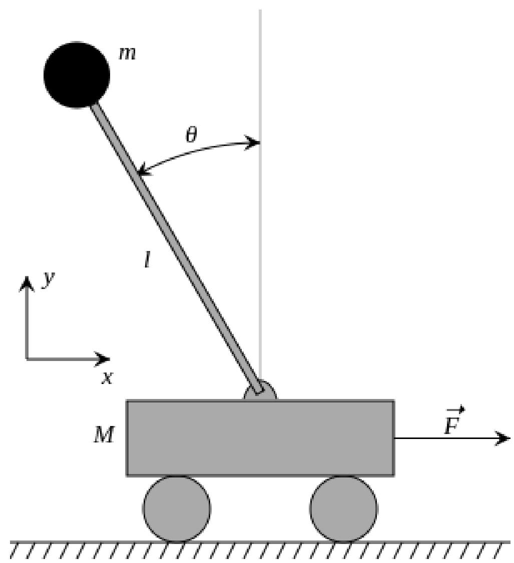

Figure 1: Inverted pendulum on a cart. Photo source: `https://en.wikipedia.org/wiki/Inverted_pendulum`.

Question 1 (20 pt)

Consider the linearized system of an inverted pendulum on a cart, where θ is the pendulum angle, and p is the position of the base. The pendulum has two possible inputs, the force applied to the base F, and a torque applied at the pendulum pivot τ.

A linearized model of the system around the equilibrium point $\theta = \dot{\theta} = \dot{p} = 0$ and with specific choices for the masses (M and m), pendulum length l, and friction constants is as follows:

$$\begin{bmatrix} \dot{p} \\ \dot{\theta} \\ \ddot{p} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -0.10 & 0.10 \\ 0 & 11 & 0.10 & -0.11 \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.10 & -0.10 \\ -0.10 & 1.10 \end{bmatrix} \begin{bmatrix} F \\ \tau \end{bmatrix}$$

Note: This model (A and B matrices) is available on learning suite so you do not have to type it into Matlab.

a) Suppose that only the torque input $\tau(t)$ can be used to control the system. Show that the system isn't completely controllable from this input. Is the system stabilizable? Determine the eigenvalue(s) corresponding with the uncontrollable mode(s). Which states are most dominate in the uncontrollable mode(s)? Does this physically make sense? Why?

b) Suppose we can measure either cart position or pendulum angle, but not both. Which single measurement (if either) will ensure that the system remains completely observable? If your analysis finds any unobservable modes in either of the two cases, identify them and determine if they are detectable.

c) Suppose instead that we have the two simultaneous output measurements $y^T = [\dot{p}, \theta]$. Is the system completely observable in this case? If not identify the number (frequency) and directions of the transmission zeros of the transfer function matrix.

d) Determine two initial conditions x_0 such that $x(0) = x_0$, for which the unforced responses will decay to zero as time increases, i.e. $x(t) \to 0$ as $t \to \infty$. Describe these motions physically.

Question 2 (20 pt)

Consider again the inverted pendulum of Question 1. But, suppose we only use the force input $F(t)$ to control the system.

a) Find a feedback control law that makes the closed-loop system converge exponentially to the equilibrium state x=0 from any initial condition. The resulting closed-loop system should have poles at -1, -2, and $-1 \pm$ j.

b) Plot an example solution and the corresponding force input F(t) when the initial condition is $x_0^T = [1, -0.2, 2, -0.1]$.

c) Suppose that instead of implementing $F(t) = -Kx(t)$ we implement $F(t) = -Ke(t)$ where $e(t) = x(t) - x_d(t)$ is the tracking error and the desired state of the system is given by $x_d(t)^T = [p_d(t), 0, 0, 0]$ with $p_d(t)$ describing the desired position. Thus we wish the controller to allow us to move the cart to any specified position, while keeping the pendulum upright.

   Using the gain matrix from part a), if $p_d(t)$ is sinusoidal, estimate the range of frequencies which this feedback system can track with no more than 10% error (i.e. the amplitude of the error $p(t) - p_d(t)$ should be no greater than 1/10 the amplitude of $p_d(t)$).

d) Can you hold a (nonzero) constant desired position with zero error? If not, what percentage steady-state error will you see for a constant p_d?

   Hint: For the last two parts it may be helpful to find the SISO transfer function relating $p_d(t)$ to the error $p(t) - p_d(t)$.

Question 3 (10 pt)

Assume there is a transmission zero, z_i, for $G(s)$ with $z_i = \lambda_k$ being an eigenvalue of A for some k. For this problem you may let $D = 0$ and assume $G(s)$ is a square matrix. Show the following:

a) Let v_k be the eigenvector corresponding to λ_k . If the initial state conditions $x(t_0) = x_0 = \alpha v_k$, where $\alpha \in \mathbb{R}$, then show that (1) the kth mode is unobservable and (2) the direction of the zero does not matter.

b) If ω_k is the left eigenvector corresponding to λ_k (i.e. $\omega_k A = \lambda_k \omega_k$). Show that the kth mode is uncontrollable.

Question 4 (20 pt)

For this question you will use the model f16.mat provided on learning suite which contains a perturbation model for the longitudinal dynamics of an F16 aircraft. The states of the model are (in this order) Δv (airspeed change), $\Delta\alpha$ (angle of attack change), $\Delta\theta$ (pitch angle change), and Δq (pitch rate change). The longitudinal inputs are δ_t (throttle change) and δ_e (elevator deflection). The equilibrium states of the underlying nonlinear dynamics for the longitudinal model are: airspeed 500 ft/sec, angle of attack 2.3 deg, pitch angle 17.2 deg, and pitch rate 0 deg/sec. The throttle input is physically limited to ± 5, while the elevator input is limited to ± 25 deg.

In this problem we will be designing an LQG controller. As a first step, scale the physical variables to be equivalent using Bryson's method. To this end, suppose that $R = rR_1$ where $r > 0$ is a scalar and R_1 implements Bryson's scaling method. Similarly, suppose that $Q = qQ_1$ where q is a positive scalar and and Q_1 normalizes the state deviations to their equilibrium values. Since the equilibrium for pitch rate is zero, take the normalizing factor of 0.5 deg/sec. Please note that the angles in the provided data (including the elevator deflection angle) are in radians, not degrees.

a) Set $q = 1$ and let r vary. Plot the dampling ratio vs. oscillation frequency for the closed-loop short period mode obtained using the LQR optimal controller when $r = 1000, 100, 10, 7$. Include the open-loop data point as a reference. What happens to the short period dynamics (mode with higher frequency and higher damping) for smaller values of r?

b) Choose an R matrix that maximizes use of the elevator and throttle deflection without exceeding their bounds given an initial perturbation of $x_0^T = [20, .01, -.01, .02]$. Make plots of the open-loop (uncontrolled) and controlled responses to verify your design. Comment on the differences. Note: you may need to use different time scales to see how the two systems behave and different times.

c) Suppose we now want to implement a Kalman filter observer. And we wish to control the airspeed and the flight path angle $\Delta\gamma = \Delta\theta - \Delta\alpha$ and that (only) these variables are measured.

Use a Kalman filter observer design to implement a compensator (please use zpk when displaying your compensator for readability). Use the LQR feedback gains computed in part b. Assume the sensors have noise characteristics quantified by $W_n = diag\{1, 10^{-5}\}$ and process noise characteristics $W_d = B_{long} D B_{long}^T$ where $D = 10^{-4} I$.

d) Is your controller able to track constant desired inputs $y_D^T = [(\Delta v)_D \ (\Delta\gamma)_D]$? If not, modify your design to one that will guarantee perfect tracking for any constant input.