MEEN 537 HW3 | Jae Lee

~~4-2~~ ~~4-10~~ ~~4-13~~ ~~4-15~~ ~~4-16~~ ~~4-17~~ ~~extra~~

4-2. $S(a)p = a \times p \qquad p = [p_x\ p_y\ p_z]^T \qquad a = [a_x\ a_y\ a_z]^T$

$$S(a)p = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} a_y p_z - a_z p_y \\ a_z p_x - a_x p_z \\ a_x p_y - a_y p_x \end{bmatrix}$$

$$a \times p = (a_y p_z - a_z p_y)x + (a_z p_x - a_x p_z)y + (a_x p_y - a_y p_x)z = \begin{bmatrix} a_y p_z - a_z p_y \\ a_z p_x - a_x p_z \\ a_x p_y - a_y p_x \end{bmatrix}$$

4-10

$$e^S = I + S + \tfrac{1}{2}S^2 + \tfrac{1}{3!}S^3 + \cdots \qquad RR^T = R^TR = I,\ \det(R) = 1, \quad S \in so(3)$$

$S^T = -S$, $tr(S) = 0$

$$S = \begin{bmatrix} 0 & -S_3 & S_2 \\ S_3 & 0 & -S_1 \\ -S_2 & S_1 & 0 \end{bmatrix} \qquad e^S e^{S^T} = e^{S+S^T} = e^0 = I + 0 + \tfrac{1}{2}0^2 + \cdots = I$$

$$\det(e^S) = e^{Tr(S)} = e^0 = 1$$

Thus, $e^S \in SO(3)$

4-13. $\dfrac{dR}{dt} = \dfrac{\partial R}{\partial \psi}\dfrac{\partial \psi}{\partial t} + \dfrac{\partial R}{\partial \theta}\dfrac{\partial \theta}{\partial t} + \dfrac{\partial R}{\partial \phi}\dfrac{\partial \phi}{\partial t}$, $\quad w = \{c\psi s\theta\dot\phi - s\psi\dot\theta\}i + \{s\psi s\theta\dot\phi + c\psi\dot\theta\}j + \{\dot\psi + c\theta\dot\phi\}k$

$$R = R_{z,\psi}R_{y,\theta}R_{z,\phi} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}\begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta c\psi & -s\psi & c\psi s\theta \\ s\psi c\theta & c\psi & s\psi s\theta \\ -s\theta & 0 & c\theta \end{bmatrix}\begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\psi c\theta c\phi - s\psi s\phi & -c\psi c\theta s\phi - s\psi c\phi & c\psi s\theta \\ s\psi c\theta c\phi + c\psi s\phi & -s\psi c\theta s\phi + c\psi s\phi & s\psi s\theta \\ -s\theta c\phi & s\theta s\phi & c\theta \end{bmatrix}$$

(4.18)

$$\frac{dR}{d\theta}R_{k,\theta} = S(k)R \Rightarrow \frac{dR}{dt} = \frac{dR}{d\theta}\frac{d\theta}{dt} = S(k)R\dot\theta$$

$$\dot R = \dot R_{z,\psi}R_{y,\theta}R_{z,\phi} + R_{z,\psi}\dot R_{y,\theta}R_{z,\phi} + R_{z,\psi}R_{y,\theta}\dot R_{z,\phi} = S(wt)R_{z,\psi}R_{y,\theta}R_{z,\phi} + R_{z,\psi}S(wt)R_{y,\theta}R_{z,\phi} + R_{z,\psi}R_{y,\theta}S(wt)R_{z,\phi}$$

$$= S(\dot\psi k)R_{z,\psi}R_{y,\theta}R_{z,\phi} + R_{z,\psi}S(\dot\theta j)R_{y,\theta}R_{z,\phi} + R_{z,\psi}R_{y,\theta}S(\dot\phi k)R_{z,\phi}$$

$$= S(\dot\psi k)R_{z,\psi}R_{y,\theta}R_{z,\phi} + S(R_{z,\psi}\dot\theta j)R_{z,\psi}R_{y,\theta}R_{z,\phi} + S(R_{z,\psi}R_{y,\theta}\dot\phi k)R_{z,\psi}R_{y,\theta}R_{z,\phi}$$

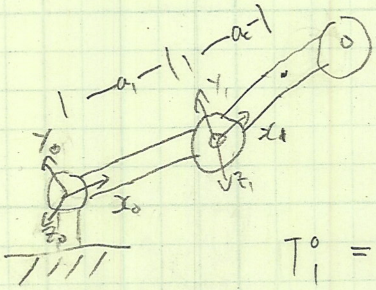
$$= [S(\dot\psi k) + S(R_{z,\psi}\dot\theta j) + S(R_{z,\psi}R_{y,\theta}\dot\phi k)]R = S(w)R$$

So $w = \dot\psi k + R_{z,\psi}\dot\theta j + R_{z,\psi}R_{y,\theta}\dot\phi k$

4-15. $H_1^0 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad V_1(t) = [3\ \ 1\ \ 0]^T$

$$V_1^0 = R_1^0 V_1' = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

4-16



DH	θ_i	d_i	a	α
1	θ_1	0	a_1	0
2	θ_2	0	a_2	0

$$T_1^0 = A_1 = \begin{array}{c} \begin{array}{cccc} & & z_1^0 & O_1^0 \end{array} \\ \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1 c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1 s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

A_2 = same as A_1 but θ_2, a_2

$$T_2^0 = A_1 A_2 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1 c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1 s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cccc} & & & O_2^0 \end{array} \\ \begin{bmatrix} c\theta_1 c\theta_2 - s\theta_1 s\theta_2 & -c\theta_1 s\theta_2 - s\theta_1 c\theta_2 & 0 & a_2 c\theta_1 c\theta_2 - a_2 s\theta_1 s\theta_2 + a_1 c\theta_1 \\ s\theta_1 c\theta_2 + c\theta_1 s\theta_2 & -s\theta_1 s\theta_2 + c\theta_1 c\theta_2 & 0 & a_2 s\theta_1 c\theta_2 + a_2 c\theta_1 s\theta_2 + a_1 s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

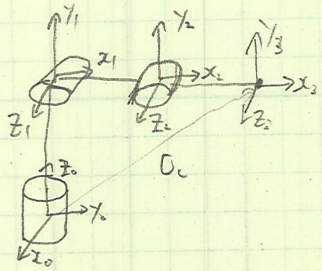
$$O_2^0 = \begin{bmatrix} a_2 c\theta_1 c\theta_2 - a_2 s\theta_1 s\theta_2 + a_1 c\theta_1 \\ a_2 s\theta_1 c\theta_2 + a_2 c\theta_1 s\theta_2 + a_1 s\theta_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

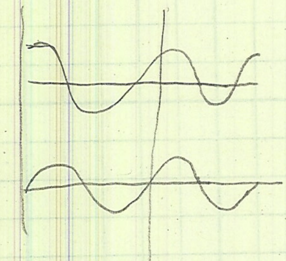
$$J(q) = \begin{bmatrix} z_0^0 \times (O_2^0 - O_0^0) & z_1^0 \times (O_2^0 - O_1^0) \\ z_0^0 & z_1^0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 c_{12} \\ a_2 s_{12} \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

4-17.



Link	θ	d	a	α
1	$\theta_1+\frac{\pi}{2}$	d_1	0	$\frac{\pi}{2}$
2	θ_2	0	a_2	0
3	θ_3	0	a_3	0



$$C(\theta_1+\tfrac{\pi}{2}) = -S\theta_1$$
$$S(\theta_1+\tfrac{\pi}{2}) = C\theta_1$$

$$JV_i = \begin{cases} z_{i-1} \times (O_n - O_{i-1}) & \text{revolute} \\ z_{i-1} & \text{prismatic} \end{cases}$$

$$A_1 = \begin{bmatrix} C(\theta_1+\frac{\pi}{2}) & 0 & S(\theta_1+\frac{\pi}{2}) & 0 \\ S(\theta_1+\frac{\pi}{2}) & 0 & -C(\theta_1+\frac{\pi}{2}) & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_2 C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_2 S\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & a_3 C\theta_3 \\ S\theta_3 & C\theta_3 & 0 & a_3 S\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} -S\theta_1 & 0 & C\theta_1 & 0 \\ C\theta_1 & 0 & S\theta_1 & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_2 C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_2 S\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & a_3 C\theta_3 \\ S\theta_3 & C\theta_3 & 0 & a_3 S\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -S_1C_2 & S_1S_2 & C_1 & -a_2 S_1 C_2 \\ C_1C_2 & -C_1S_2 & S_1 & a_2 C_1 C_2 \\ S_2 & C_2 & 0 & a_2 S_2 + a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_3 & -S_3 & 0 & a_3 C_3 \\ S_3 & C_3 & 0 & a_3 S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J_{11} = \begin{bmatrix} z_0 \times (O_3 - O_0) & z_1 \times (O_3 - O_1) & z_2 \times (O_3 - O_2) \end{bmatrix}$$

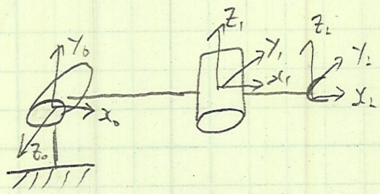
$$= \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 C_1 C_2 + a_3 C_1 C_{23} \\ a_2 S_1 C_2 + a_3 S_1 C_{23} \\ a_2 S_2 + a_3 S_{23} \end{bmatrix} \quad \begin{bmatrix} S_1 \\ -C_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} a_2 C_1 C_2 + a_3 C_1 C_{23} \\ a_2 S_1 C_2 + a_3 S_1 C_{23} \\ a_2 S_2 + a_3 S_{23} \end{bmatrix} \quad \begin{bmatrix} S_1 \\ -C_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} a_3 C_1 C_{23} \\ a_3 S_1 C_{23} \\ a_3 S_{23} \end{bmatrix} \right]$$

$$= \begin{bmatrix} -a_2 S_1 C_2 - a_3 S_1 C_{23} & -a_2 S_2 C_1 - a_3 S_{23} C_1 & -a_3 C_1 S_{23} \\ a_2 C_1 C_2 + a_3 C_1 C_{23} & -a_2 S_1 S_2 - a_3 S_1 S_{23} & -a_3 S_1 S_{23} \\ 0 & a_2 C_2 + a_3 C_{23} & a_3 C_{23} \end{bmatrix}$$

$$\det J_{11} = -a_2 a_3 S_3 (a_2 C_2 + a_3 C_{23})$$

$$\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$$

2. (a)



end effector frame

$$\begin{bmatrix} Z_0^2 \times (O_2^2 - O_0^2) & Z_1^2 \times (O_2^2 - O_1^2) \\ Z_0^2 & Z_1^2 \end{bmatrix}$$

I will need

$T_1^2 \to Z_1^2, O_1^2$

$T_0^2 \to Z_0^2, O_0^2$

from the class, $T_1^0 = A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & a_1 c_1 \\ s_1 & 0 & c_1 & a_1 s_1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $T_2^1 = A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$T_2^0 = \begin{bmatrix} c_1 c_2 & -s_2 c_1 & -s_1 & a_2 c_1 c_2 + a_1 c_1 \\ s_1 c_2 & -s_1 s_2 & c_1 & a_2 s_1 c_2 + a_1 s_1 \\ -s_2 & -c_2 & 0 & -a_2 s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_1 A_2$$

$$J^0(q) = \begin{bmatrix} Z_0^0 \times (O_2^0 - O_0^0) & Z_1^0 \times (O_2^0 - O_1^0) \\ Z_0^0 & Z_1^0 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 c_1 c_2 + a_1 c_1 \\ a_2 s_1 c_2 + a_1 s_1 \\ -a_2 s_2 \end{bmatrix} & \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} a_2 c_1 c_2 \\ a_2 s_1 c_2 \\ -a_2 s_2 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -a_2 s_1 c_2 - a_1 s_1 & -a_2 s_2 c_1 \\ a_2 c_1 c_2 + a_1 c_1 & -a_2 s_1 s_2 \\ 0 & -a_2 s_1 s_1 c_2 - a_2 c_1 c_1 c_2 \\ 0 & -s_1 \\ 0 & c_1 \\ 1 & 0 \end{bmatrix} = J^0(q)$$

Transformation

$$J^2(q) = \begin{bmatrix} R_0^2 & 0 \\ 0 & R_0^2 \end{bmatrix} J^0(q)$$ where $R_0^2 = \begin{bmatrix} c_1 c_2 & s_1 c_2 & -s_2 \\ -s_2 c_1 & -s_1 s_2 & -c_2 \\ -s_1 & c_1 & 0 \end{bmatrix}$

direct calculation

$$J^2(q) = \begin{bmatrix} Z^2_0 \times (O^2_2 - O^2_0) & Z^2_1 \times (O^2_2 - O^2_1) \\ Z^2_0 & Z^2_1 \end{bmatrix}$$

$$T^2_0 = (T^0_2)^{-1} = \begin{bmatrix} (R^0_2)^T & -(R^0_2)^T t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^2_0 & -R^2_0 t \\ 0 & 1 \end{bmatrix}$$

where

$$t = \begin{bmatrix} a_2 C_1 C_2 + a_1 C_1 \\ a_2 S_1 C_2 + a_1 S_1 \\ -a_2 S_2 \end{bmatrix}$$

$$T^2_1 = \begin{bmatrix} C_2 & S_2 & 0 & -a_2 \\ -S_2 & C_2 & 0 & 2a_2 S_2 C_2 \\ 0 & 0 & 1 & 0 \\ & 0 & & 1 \end{bmatrix}$$

$$R^2_0 = \begin{bmatrix} C_1 C_2 & S_1 C_2 & -S_2 \\ -S_2 C_1 & -S_1 S_2 & -C_2 \\ -S_1 & C_1 & 0 \end{bmatrix}$$

$$J^2(q) = \begin{bmatrix} \begin{bmatrix} -S_2 \\ -C_2 \\ 0 \end{bmatrix} \times \left(-R^2_0 t\right) & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{pmatrix} -a_2 \\ 2a_2 C_2 S_2 \\ 0 \end{pmatrix} \\ \begin{bmatrix} -S_2 \\ C_2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

(b) $\tau = J^T(q)\,F = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -a_2 s_1 c_2 - a_1 s_1 & a_2 c_1 c_2 + a_1 c_1 & 0 & 0 & 0 & 1 \\ -a_2 s_2 c_1 & -a_2 s_1 s_2 & -a_2 s_1 s_1 c_2 - a_2 c_1 c_1 c_2 & -s_1 & c_1 & 0 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix}$

F^0

i) $q = [0, \frac{\pi}{4}]$, $F = [-1, 0, 0]$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} 0 & a_2 \frac{\sqrt{2}}{2} + a_1 & 0 & 0 & 0 & 1 \\ -\frac{\sqrt{2}}{2} a_2 & 0 & -\frac{\sqrt{2}}{2} a_2 & 0 & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} a_2 \end{bmatrix}$$

ii) $q = [0, \frac{\pi}{2}]$, $F = [-1, 0, 0]$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} 0 & a_1 & 0 & 0 & 0 & 1 \\ -a_2 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a_2 \end{bmatrix}$$

iii) $q = [\frac{\pi}{4}, \frac{\pi}{4}]$, $F = [-1, -1, 0]$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -\frac{a_2}{2} - a_1 \frac{\sqrt{2}}{2} & +\frac{a_2}{2} + \frac{\sqrt{2}}{2} a_1 & 0 & 0 & 0 & 1 \\ -\frac{a_2}{2} & -\frac{a_2}{2} & -a_2 \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ a_2 \end{bmatrix}$$

iv) $q = [0, 0]$, $F = [0, 0, 1]$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} 0 & a_1 + a_2 & 0 & 0 & 0 & 1 \\ 0 & 0 & -a_2 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -a_2 \end{bmatrix}$$

v) $q = [0, 0]$, $F = [1, 0, 0]$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} 0 & a_1 + a_2 & 0 & 0 & 0 & 1 \\ 0 & 0 & -a_2 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

~~how to find torque vector?~~

no force can be obtained along x^0-frame at this configuration