MeEn 537 Homework #3 Solution

4-2

$$S(a)p = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} -a_z p_y + a_y p_z \\ a_z p_x - a_x p_z \\ -a_y p_x + a_x p_y \end{bmatrix}$$

$$a \times p = [a_x \ a_y \ a_z] \times [p_x \ p_y \ p_z] \Rightarrow$$

$i \to j \to k \to i$ ← use this for + or −

$$= \hat{i}(a_y p_z - a_z p_y) + \hat{j}(-a_x p_z + a_z p_x) + \hat{k}(a_x p_y - a_y p_x)$$

these are equal.

4-10

given:

A is square

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots$$

$$e^A e^B = e^{A+B} \quad \text{if } A \,\&\, B \text{ commute}$$

$$\det(e^A) = e^{Tr(A)}$$

show:

that if $S \in so(3)$, then $e^S \in SO(3)$

if so, then e^S must satisfy the following:

$$(e^S)^T = (e^S)^{-1} \quad ①$$

This is a known property of matrix exponentials

$$e^{S^T} = e^{-S}$$

which means $e^S(e^S)^T = I$

and

$$\det(e^S) = 1 \quad ②$$

since $(S)(S^T) = (S^T)(S) \Rightarrow$

can show that $e^S(e^S)^T = I$ as follows $\Rightarrow$

$$\underbrace{e^S(e^S)^T = e^S e^{-S}}_{\text{this step shown as being true on next page}} = e^{S-S} = e^0 = I \quad ✓$$

also

$$\det(e^S) = e^{Tr(S)} = e^0 = 1 \quad ✓$$

so $e^S \in SO(3)$

showing that $e^{-S} = e^{S^T} \Rightarrow$

$$\underline{I} - S + \frac{1}{2}S^2 - \frac{1}{3!}S^3 \ldots\ldots = \underline{I} + S^T + \frac{1}{2}(S^T)^2 + \frac{1}{3!}(S^T)^3 + \ldots$$

on a term by term basis $\Rightarrow$

$-S = S^T$ ✓

$S^2 = (S^T)^2$ ✓

$-S^3 = (S^T)^3$ ✓

etc.

4.13

Show: $\frac{dR}{dt} = S(\omega)R$ where $\omega = \{c_\psi s_\theta \dot{\phi} - s_\psi \dot{\theta}\}i + \{s_\psi s_\theta \dot{\phi} + c_\psi \dot{\theta}\}j + \{\dot{\psi} + c_\theta \dot{\phi}\}k$

start with product rule $\Rightarrow$

$$\dot{R} = \dot{R}_{z,\psi} R_{y,\theta} R_{z,\phi} + R_{z,\psi} \dot{R}_{y,\theta} R_{z,\phi} + R_{z,\psi} R_{y,\theta} \dot{R}_{z,\phi} \quad (1)$$

now using the chain rule we can replace the following:

$$\left.\begin{aligned} \dot{R}_{z,\psi} &= \frac{dR_{z,\psi}}{d\psi}\dot{\psi} = S(k) R_{z,\psi} \dot{\psi} \\ \dot{R}_{y,\theta} &= \frac{dR_{y,\theta}}{d\theta}\dot{\theta} = S(j) R_{y,\theta} \dot{\theta} \\ \dot{R}_{z,\phi} &= \frac{dR_{z,\phi}}{d\phi}\dot{\phi} = S(k) R_{z,\phi} \dot{\phi} \end{aligned}\right\} - (2)$$

can now proceed in 2 different ways

1) evaluate (1) symbolically & compare to $S(\omega)R$
   - see MATLAB code for this
2) continue to manipulate (1) & (2) to find ω $\Rightarrow$

in the next part will use eqn 4.8 & the following identity $\Rightarrow$

$R^T S(Ra) R = R^T R S(a) R^T R = S(a)$ $\Rightarrow$ so moving an R inside & pre & post multiplying by R^T & R has no effect $\Rightarrow$

$$\dot{R} = \underbrace{[S(\dot{\psi}k)R_{z,\psi}]}_{\dot{R}_{z,\psi}} R_{y,\theta} R_{z,\phi} + R_{z,\psi} \underbrace{[S(\dot{\theta}j)R_{y,\theta}]}_{\dot{R}_{y,\theta}} R_{z,\phi} + R_{z,\psi} R_{y,\theta} \underbrace{[S(\dot{\phi}k)R_{z,\phi}]}_{\dot{R}_{z,\phi}}$$

$$\dot{R} = S(\dot{\psi}k)\underbrace{R_{z,\psi} R_{y,\theta} R_{z,\phi}}_{R} + R_{z,\psi}[R^T_{z,\psi} S(R_{z,\psi}\dot{\theta}j)\overbrace{R_{z,\psi} R_{y,\theta}]R_{z,\phi}}^{R} +$$

$$R_{z,\psi} R_{y,\theta}[R^T_{y,\theta} R^T_{z,\psi} S(R_{z,\psi} R_{y,\theta} \dot{\phi}k)\overbrace{R_{z,\psi} R_{y,\theta} R_{z,\phi}}^{R}] \Rightarrow$$

$$\dot{R} = \left[S(\dot{\psi} k) + S(R_{z,\psi}\dot{\theta} j) + S(R_{z,\psi}R_{y,\theta}\dot{\phi} k) \right] R \Rightarrow$$

$$= S(\omega) R$$

where

$$\omega = \dot{\psi} k + R_z \dot{\theta} j + R_z R_y \dot{\phi} k \Rightarrow$$

$$= (c_\psi s_\theta \dot{\phi} - s_\psi \dot{\theta}) i + (s_\psi s_\theta \dot{\phi} + c_\psi \dot{\theta}) j + (\dot{\psi} + c_\theta \dot{\phi}) k \quad \checkmark$$

$\boxed{4.15}$

find: $V_1(t)$ in frame O_0 $\Rightarrow$

remembering that

$$H_1^0 = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \Rightarrow$$

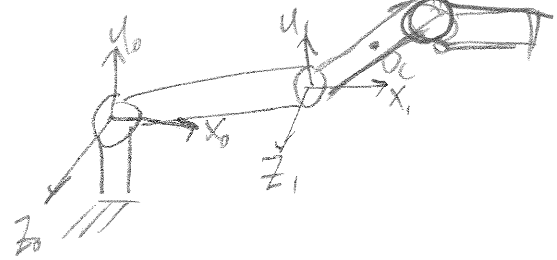
$$p^0 = R_1^0 p^1 + d_1^0 \quad \Rightarrow \text{ take time derivative } \Rightarrow$$

$$\dot{p}^0 = R_1^0 \dot{p}^1 + 0 \quad \Rightarrow$$

$$\dot{p}^0 = V_1(t) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}}$$

4.16



let first link length be a_1 & distance from O_1 to O_c be a_c $\Rightarrow$

<u>find:</u>

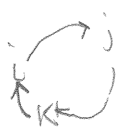
O_c & ${}_0J^c(q)$ $\Rightarrow$

$$\boxed{O_c = \begin{bmatrix} x_c = a_1 C_{\theta_1} + a_c C_{\theta_1+\theta_2} \\ y_c = a_1 S_{\theta_1} + a_c S_{\theta_1+\theta_2} \\ Z_c = 0 \end{bmatrix}}$$

done by inspection, could also use forward kinematics

$${}_0J^c(q) = \begin{bmatrix} [Z_0^0] \times [O_c^0 - O_0^0] & [Z_1^0] \times [O_c^0 - O_1^0] & 0 \\ Z_0^0 & Z_1^0 & 0 \end{bmatrix}$$

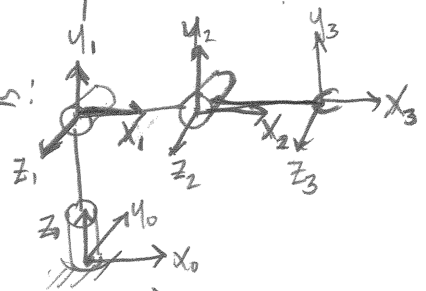
$${}_0J^c(q) = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_1 C_{\theta_1} + a_c C_{\theta_1+\theta_2} \\ a_1 S_{\theta_1} + a_c S_{\theta_1+\theta_2} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_c C_{\theta_1+\theta_2} \\ a_c S_{\theta_1+\theta_2} \\ 0 \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \end{bmatrix} \Rightarrow$$



$$\boxed{{}_0J^c(q) = \begin{bmatrix} -a_1 S_{\theta_1} - a_c S_{\theta_1+\theta_2} & -a_c S_{\theta_1+\theta_2} & 0 \\ a_1 C_{\theta_1} + a_c C_{\theta_1+\theta_2} & a_c C_{\theta_1+\theta_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

4-17 find: J_{11} (upper left part of Jacobian) for example 4.9 & calculate the determinant

given:



this is rotated by $\frac{\pi}{2}$ about z_0 (wrong in book)

$$J_{11} = \begin{bmatrix} z_0^0 \times (o_3^0 - o_0^0) & z_1^0 \times (o_3^0 - o_1^0) & z_2^0 \times (o_3^0 - o_2^0) \end{bmatrix}$$

using code (see p4_17.m)

J_{11} = same as book

$\det(J_{11})$ = evaluates to same determinant given in the book.

Problem 2 – (see hw3_problem2.m)

a)

i)

$$ {}_nJ^n(q) = \begin{bmatrix} Z_0^2 \times (O_2^2 - O_0^2) & Z_1^2 \times (O_2^2 - O_1^2) \\ Z_0^2 & Z_1^2 \end{bmatrix} $$

can get these values by finding

$$ T_0^2 = \begin{bmatrix} R_0^2 & t_0^2 \\ 0 & 1 \end{bmatrix} \quad (Z_0^2 \text{ is } 3^{rd} \text{ column of } R_0^2 \text{ \& } t_0^2 = O_0^2) $$

$$ T_1^2 = \begin{bmatrix} R_1^2 & t_1^2 \\ 0 & 1 \end{bmatrix} $$

using MATLAB toolbox ⇒

at $q = [0,0]$

$$ {}_nJ^n(q) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} $$

ii)

$$ {}_nJ^n(q) = \begin{bmatrix} R_0^n & 0 \\ 0 & R_0^n \end{bmatrix} {}_nJ^0(q) $$ ⇒ using MATLAB this is equal to part i)

part b)

from MATLAB evaluation

i) $\tau = \begin{bmatrix} 0 \\ 0.707 \end{bmatrix}$

ii) $\tau = \begin{bmatrix} 0 \\ 1.0 \end{bmatrix}$

in both cases force line of action passes through joint 1's axes causing no torque

iii) $\tau = \begin{bmatrix} 0 \\ 1.0 \end{bmatrix}$ — affect of x & y forces on 1st joint cancel in this configuration.

iv) $\tau = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

v) $\tau = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ — in this configuration, the force in the x-direction is taken entirely by the structure of the robot.

in general if the sign of F changes uniformly (in every axis) because it is a reaction force, the torques will also simply change sign to have the arm remain in static equilibrium.

c) & d) were removed.