Quiz 2 – Linear Algebra (Mar 31, 2022)

本人郑重承诺将秉承诚信原则，自觉遵守考场纪律，并承担违纪或作弊带来的后果。

学号：________________ 姓名：________________

题号	1	2	3	4	5	合计
分值	10 分	10 分	10 分	10 分	10 分	50 分
得分						

This 50-minute long test includes 5 questions. Write ***all your answers*** on this examination paper.

1. (1) $\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}^6 = \underline{\begin{pmatrix} 2^6 & \\ & 2^6 \end{pmatrix}}$.

$A=\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = 2B.$

$B=\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\theta=\pi/3$, $B^6=\begin{pmatrix} \cos 6\theta & -\sin 6\theta \\ \sin 6\theta & \cos 6\theta \end{pmatrix}=I$,

则 $A^6 = 2^6 B^6 = 2^6 I.$

(2) If $(1,2,t)^T$ can be linearly represented by the vectors $(2,1,1)^T, (-1,2,7)^T, (1,-1,-4)^T$, then t=__5__.

即: $x_1\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} -1 \\ 2 \\ 7 \end{pmatrix} + x_3\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ t \end{pmatrix}$ 有解.

$\begin{pmatrix} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ 1 & 7 & -4 & t \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 5 & 3 & t-2 \\ 0 & 0 & 0 & t-5 \end{pmatrix} \Rightarrow t-5=0 \Rightarrow t=5$

(3) True or False? If $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_s \in \mathbf{R}^n$ and $\boldsymbol{A}$ is an $n \times n$ matrix, then $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_s$ are linearly dependent if and only if $\boldsymbol{A\alpha}_1, \boldsymbol{A\alpha}_2, \cdots, \boldsymbol{A\alpha}_s$ are linearly dependent. (F)

反例: $A=0.$

(4) True or False? For an $m \times n$ matrix $\boldsymbol{A}$, the system $\boldsymbol{Ax}=\boldsymbol{b}$ has infinitely many solutions if and only if $\operatorname{rank}(\boldsymbol{A})=n$. (F)

反例: $A=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $b=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(5) True or False? If the matrix $\boldsymbol{A}$ has full row rank, then it has a left-inverse. (F)

反例: $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. $r(A)=2$.

若 $BA=I$, 则 I 是 3x3 的, 且 $r(I)=3$.

则有 $r(BA)=r(I)=3 > r(A)=2$.

与 $r(BA) \le r(A)$ 矛盾, 则 B 不存在.

2. Let $\boldsymbol{\alpha}_1 = (0,2,2)^T, \boldsymbol{\alpha}_2 = (-2,0,-2)^T, \boldsymbol{\alpha}_3 = (2,1,4)^T, \boldsymbol{\alpha}_4 = (3,5,8)^T$.

(1) Check the linear dependence among $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4$. Why?

(3) Can $\boldsymbol{\alpha}_4$ be written as a linear combination by the vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$? If so, write the combination.

(1) $(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4) = \begin{pmatrix} 0 & -2 & 2 & 3 \\ 2 & 0 & 1 & 5 \\ 2 & -2 & 4 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 1 & 5 \\ 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

则 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 线性相关.

(2). 由(1)得 α_4 可以被 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 线性表出,

令 $x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = \alpha_4$, 则有:

$$\begin{pmatrix} 0 & -2 & 2 & 3 \\ 2 & 0 & 1 & 5 \\ 2 & -2 & 4 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 5/2 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5/2 \\ -3/2 \\ 0 \end{pmatrix},$$

则: $\alpha_4 = 5/2\,\alpha_1 - 3/2\,\alpha_2$.

3. Check if the following subsets are subspaces. If the subset is a subspace, find its basis and dimension. If not, explain why.

(1) $\{(x,y,z,w,t) \in \mathbf{R}^5 | x - 3z - 5w + 4t = 0\}$.

(2) All vectors in $\mathbf{R}^2$ whose components are positive or zero.

(1) 是子空间: 有:

$$\begin{pmatrix} x \\ y \\ z \\ w \\ t \end{pmatrix} = k_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 5 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + k_4 \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad k_1, k_2, k_3, k_4 \in \mathbb{R},$$

则基底为 $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, 维数为 4.

(2). 不是.

设集合为 $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 \geq 0, x_2 \geq 0 \right\}$.

有 $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in V$, 但 $-2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \notin V$.

4. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 5 & 1 & 2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

(1) Under what condition on $\boldsymbol{b}$ is the system $\boldsymbol{Ax} = \boldsymbol{b}$ solvable? Find all the solutions when $\boldsymbol{b} = [1 \quad 1 \quad 1]^T$.

(2) Find the dimension and a basis for the four fundamental subspaces of $\boldsymbol{A}$.

(1). $(A \quad b) \longrightarrow \begin{pmatrix} 1 & 0 & -2 & 1 & b_1 - 3b_2 \\ 0 & 1 & 1 & 0 & b_2 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{pmatrix}$,

则只有 $b_3 + b_2 - 2b_1 = 0$ 时 $Ax = b$ 有解.

当 $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ 时, $(A \quad b) \longrightarrow \begin{pmatrix} 1 & 0 & -2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,

则有 $x = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k_1 \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $k_1, k_2 \in \mathbb{R}$.

(2). 由(1)得 $A \longrightarrow \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

则有 $C(A)$ 基底为: $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$; $C(A^T)$ 基底为 $\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$,

$N(A)$ 基底为: $\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

$A^T = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 5 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

则 $N(A^T)$ 基底为: $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$.

$\dim C(A) = \dim C(A^T) = \dim N(A) = 2$, $\dim N(A^T) = 1$.

5. Prove that: $\text{rank}(\boldsymbol{A}+\boldsymbol{B}) \leq \text{rank}(\boldsymbol{A}) + \text{rank}(\boldsymbol{B})$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices of the same size.

设 A, B 均为 $m\times n$ 的矩阵，$\text{rank}(A)=r_A$, $\text{rank}(B)=r_B$,

$\alpha_1, \alpha_2, \cdots, \alpha_{r_A}$ 是 A 的主元所在的列，

$\beta_1, \beta_2, \cdots, \beta_{r_B}$ 是 B 的主元所在的列，

则有 A 中的列 $a_1, a_2, \cdots, a_n$ 均可由 $\alpha_1, \alpha_2, \cdots, \alpha_{r_A}$ 线性表出，

B 中的列 $b_1, b_2, \cdots, b_n$ 均可由 $\beta_1, \beta_2, \cdots, \beta_{r_B}$ 线性表出，

则有 $A+B$ 中的列 $a_1+b_1, a_2+b_2, \cdots, a_n+b_n$ 可以

由 $\alpha_1, \alpha_2, \cdots, \alpha_{r_A}, \beta_1, \beta_2, \cdots, \beta_{r_B}$ 线性表出，

则 $A+B$ 中至多有 r_A+r_B 列线性无关，

因而有 $\text{rank}(A+B) \leq r_A + r_B = \text{rank}(A) + \text{rank}(B)$.