Linear Algebra and Its Applications

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1 Matrices and Gaussian Elimination

√ 1.1 Introduction . . . .

√ 1.2 The Geometry of Linear Equations . . . . . .

√ 1.3 An Example of Gaussian Elimination . . . .

√ 1.4 Matrix Notation and Matrix Multiplication . .

1.7 1.5 Triangular Factors and Row Exchanges . . .

1.5 1.6 Inverses and Transposes . . . .

skipped 1.7 Special Matrices and Applications . . . . . .

Review Exercises . . . .

1.6 Partitioned Matrices

1 Matrices and Gaussian Elimination

1.5

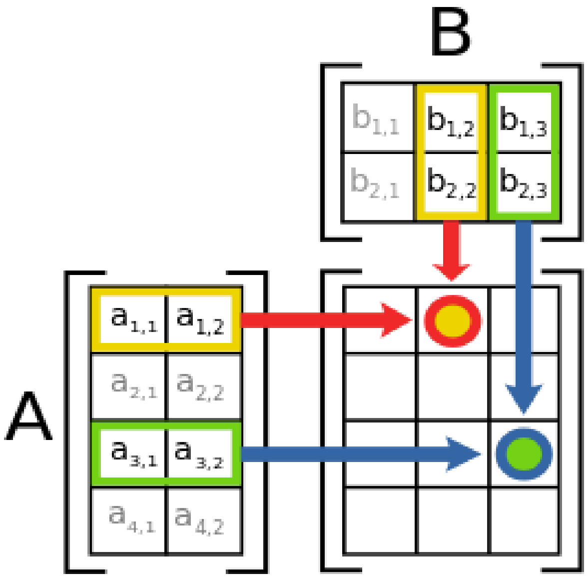
INVERSES AND TRANSPOSES

（矩阵的逆和转置）

Definitions

Properties

Algorithms



*** Textbook: Section1.6**

I. Transpose of a Matrix (转置矩阵)

1. Definition (定义)

Given an $m\times n$ matrix $\boldsymbol{A}$, the **transpose** of $\boldsymbol{A}$ is the $n\times m$ matrix, denoted by $\boldsymbol{A}^{\mathrm{T}}$, whose columns are formed from the corresponding rows of $\boldsymbol{A}$: the ith row of $\boldsymbol{A}$ becomes the ith column of $\boldsymbol{A}^{\mathrm{T}}$. (把 $m\times n$ 矩阵 $\boldsymbol{A}$ 的行换成同序数的列所得到的 $n\times m$ 矩阵称为矩阵 $\boldsymbol{A}$ 的**转置矩阵**, 记作 $\boldsymbol{A}^{\mathrm{T}}$（或 $\boldsymbol{A}'$）.)

Let $\boldsymbol{A}=[a_{ij}]_{m\times n}$, $\boldsymbol{A}^{\mathrm{T}}=[b_{ij}]_{n\times m}$, then $a_{ij}=b_{ji}$.

For example,

$$\boldsymbol{A}=\begin{bmatrix}1 & 2 & 2\\ 4 & 5 & 8\end{bmatrix},\qquad \boldsymbol{A}^{\mathrm{T}}=\begin{bmatrix}1 & 4\\ 2 & 5\\ 2 & 8\end{bmatrix};$$

$$\boldsymbol{B}=(18, 6),\qquad \boldsymbol{B}^{\mathrm{T}}=\begin{bmatrix}18\\ 6\end{bmatrix}.$$

2、Rules (转置矩阵的运算性质)

Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{A}_1,\ldots,$ and $\boldsymbol{A}_n$ denote matrices whose sizes are appropriate for the following sums and products.

(1) $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{A};$

(2) $(\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}};$

$$(\boldsymbol{A}_1+\boldsymbol{A}_2+\cdots+\boldsymbol{A}_n)^{\mathrm{T}} = \boldsymbol{A}_1^{\mathrm{T}}+\boldsymbol{A}_2^{\mathrm{T}}+\cdots+\boldsymbol{A}_n^{\mathrm{T}};$$

(3) $(k\boldsymbol{A})^{\mathrm{T}} = k\boldsymbol{A}^{\mathrm{T}};$

(4) $\underline{(\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}};}$

The transpose of a product of matrices equals the product of their transposes in the *reverse* order.

$$(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_n)^{\mathrm{T}} = \boldsymbol{A}_n^{\mathrm{T}}\cdots\boldsymbol{A}_2^{\mathrm{T}}\boldsymbol{A}_1^{\mathrm{T}}.$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 8 & 15 \end{bmatrix}, \qquad B^{\mathrm{T}}A^{\mathrm{T}} = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 7 & 15 \end{bmatrix}.$$

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}.$$

"Proof".

$$(j,i)\text{-entry of } (AB)^{\mathrm{T}} = (AB)_{ij} = (\text{row } i \text{ of } A) \text{ times } (\text{column } j \text{ of } B)$$

$$= [a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}]\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$(j,i)\text{-entry of } B^{\mathrm{T}}A^{\mathrm{T}} = \left(\text{row } j \text{ of } B^{\mathrm{T}}\right) \text{ times } \left(\text{column } i \text{ of } A^{\mathrm{T}}\right)$$

$$= [b_{1j} \quad b_{2j} \quad \cdots \quad b_{nj}]\begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} = b_{1j}a_{i1} + b_{2j}a_{i2} + \cdots + b_{nj}a_{in}$$

对称(symmetric)与反对称(skew-symmetric)

Definition Let $\boldsymbol{A}$ be a square matrix of order n. If $\boldsymbol{A}^{\mathrm{T}}=\boldsymbol{A}$,

i.e.,
$$a_{ij} = a_{ji} \quad (i, j = 1, 2, \ \ldots, n),$$
then $\boldsymbol{A}$ is called a **symmetric matrix (对称矩阵)**.

$$\boldsymbol{A} = \begin{bmatrix} 12 & 6 & 1 \\ 6 & 8 & 0 \\ 1 & 0 & 6 \end{bmatrix}. \qquad \boldsymbol{A} = \begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & -9 \\ -1 & 9 & 0 \end{bmatrix}.$$

If $\boldsymbol{A}^{\mathrm{T}}= -\boldsymbol{A}$, then $\boldsymbol{A}$ is called a **skew-symmetric matrix (反对称矩阵)**, i.e.,
$$a_{ij} = -\, a_{ji} \quad (i, j = 1, 2, \ldots, n).$$
For a skew-symmetric matrix $\boldsymbol{A}$, we have $a_{ii} = 0$.

Example 1 Suppose $\boldsymbol{A}$, $\boldsymbol{B}$ are n by n matrices, try to verify that $\boldsymbol{AB}^{\mathrm{T}} + \boldsymbol{BA}^{\mathrm{T}}$ is symmetric.

Proof Since

$$
\begin{aligned}
(\boldsymbol{AB}^{\mathrm{T}} + \boldsymbol{BA}^{\mathrm{T}})^{\mathrm{T}} &= (\boldsymbol{AB}^{\mathrm{T}})^{\mathrm{T}} + (\boldsymbol{BA}^{\mathrm{T}})^{\mathrm{T}} \\
&= (\boldsymbol{B}^{\mathrm{T}})^{\mathrm{T}} \boldsymbol{A}^{\mathrm{T}} + (\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} \boldsymbol{B}^{\mathrm{T}} \\
&= \boldsymbol{BA}^{\mathrm{T}} + \boldsymbol{AB}^{\mathrm{T}} = \boldsymbol{AB}^{\mathrm{T}} + \boldsymbol{BA}^{\mathrm{T}},
\end{aligned}
$$

therefore $\boldsymbol{AB}^{\mathrm{T}} + \boldsymbol{BA}^{\mathrm{T}}$ is symmetric.

Similary, if $\boldsymbol{A}$ is an $m \times n$ matrix, then it is easy to show that $\boldsymbol{AA}^{\mathrm{T}}$, $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ are both symmetric matrices.

e.g., $\boldsymbol{A} = [1 \ \ 2]$. What are $\boldsymbol{AA}^{\mathrm{T}}$ and $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$?

Inverse

引例 ***Coding & Decoding; Encrypting & Decrypting***

A	B	C	D	E	F	G	H	I	J	K	L	M	
1	2	3	4	5	6	7	8	9	10	11	12	13	
N	O	P	Q	R	S	T	U	V	W	X	Y	Z	Space
14	15	16	17	18	19	20	21	22	23	24	25	26	0

考虑加解密方案：明文信息经编码后分成三个一组（空格也是一种明文信息，不足3个时可加空格）. 对明文 $(p_1, p_2, p_3)^T$, 相应的密文为：

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \boldsymbol{A}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \boldsymbol{b}, \quad \text{其中 } \boldsymbol{A} = \begin{bmatrix} 0 & 0 & -2 \\ -1 & -4 & -3 \\ -1 & -3 & -4 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix}.$$

已知明文DIG,求密文； 已知密文$(-8,-103,-86)^T$,求明文.

II. Inverse of a Matrix (矩阵的逆)-- Definition

- An $n \times n$ matrix $\boldsymbol{A}$ is said to be **invertible** (可逆的) if there is an $n \times n$ matrix $\boldsymbol{C}$ such that

$$\boldsymbol{CA} = \boldsymbol{I} \quad \underline{\text{and}} \quad \boldsymbol{AC} = \boldsymbol{I},$$

where $\boldsymbol{I} = \boldsymbol{I}_n$, the $n \times n$ identity matrix.

- In this case, $\boldsymbol{C}$ is an **inverse** of $\boldsymbol{A}$ ($\boldsymbol{A}$的逆).
- In fact, $\boldsymbol{C}$ is uniquely determined by $\boldsymbol{A}$, because if $\boldsymbol{B}$ were another inverse of $\boldsymbol{A}$, then

$$\boldsymbol{B} = \boldsymbol{BI} = \boldsymbol{B}(\boldsymbol{AC}) = (\boldsymbol{BA})\boldsymbol{C} = \boldsymbol{IC} = \boldsymbol{C}.$$

- This unique inverse is denoted by $\boldsymbol{A}^{-1}$ (‘$\boldsymbol{A}$ inverse’), so that

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}.$$

For example, $\boldsymbol{A}=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$, $\boldsymbol{M}=\begin{bmatrix}1/2 & 1/2\\ -1/2 & 1/2\end{bmatrix}$,

Since $\boldsymbol{AM}=\boldsymbol{MA}=\boldsymbol{I}$, $\boldsymbol{M}$ is the inverse of $\boldsymbol{A}$.

- A matrix that is *not* invertible is also called a singular matrix (奇异矩阵), and an invertible matrix is called a nonsingular matrix (非奇异矩阵).

Not all matrices have inverses. (并非所有矩阵都可逆)

An inverse is impossible when $\boldsymbol{Ax}$ is zero and $\boldsymbol{x}$ is nonzero.

Application:

If $\boldsymbol{A}$ is invertible, the *one and only solution* to $\boldsymbol{Ax}=\boldsymbol{0}$ is $\boldsymbol{x}=\boldsymbol{0}$.

当 $\boldsymbol{A}$ 可逆(即: 非奇异)时, $\boldsymbol{Ax}=\boldsymbol{0}$ **只有零解** $\boldsymbol{x}=\boldsymbol{0}$.

- **Theorem 1** Let $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. If $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then

  $\boldsymbol{A}$ is invertible and

$$\boldsymbol{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

  If $a_{11}a_{22} - a_{12}a_{21} = 0$, then $\boldsymbol{A}$ is not invertible.

- The quantity $a_{11}a_{22} - a_{12}a_{21}$ is called the **determinant (行列式)** of $\boldsymbol{A}$, and we write $\det \boldsymbol{A} = a_{11}a_{22} - a_{12}a_{21}$.

- This theorem says that a 2×2 matrix $\boldsymbol{A}$ is invertible if and only if $\det \boldsymbol{A} \neq 0$.

Proof: （by definition, or 待定系数法）

Let $\boldsymbol{M} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, which satisfies that $\boldsymbol{AM} = \boldsymbol{I}$,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Rightarrow \begin{bmatrix} a_{11}x_1 + a_{12}x_3 & a_{11}x_2 + a_{12}x_4 \\ a_{21}x_1 + a_{22}x_3 & a_{21}x_2 + a_{22}x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_3 = 1, \\ a_{21}x_1 + a_{22}x_3 = 0, \\ a_{11}x_2 + a_{12}x_4 = 0, \\ a_{21}x_2 + a_{22}x_4 = 1, \end{cases} \Rightarrow \begin{cases} x_1 = \dfrac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}}, \\ x_2 = \dfrac{-a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \\ x_3 = \dfrac{-a_{21}}{a_{11}a_{22} - a_{12}a_{21}}, \\ x_4 = \dfrac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}}. \end{cases} \Rightarrow \quad \boldsymbol{M} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

$$(a_{11}a_{22} - a_{12}a_{21} \neq 0)$$

We can also check that $\boldsymbol{AM} = \boldsymbol{MA} = \boldsymbol{I}$. Therefore

$$\boldsymbol{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

求二阶可逆矩阵***A***的逆的“**两调一除**”方法:

先将矩阵 $\boldsymbol{A}$ 的主对角元素互换位置, 再将次对角元素反号, 最后用 $\det \boldsymbol{A}$ 去除 $\boldsymbol{A}$ 的每一个元素.

For example,

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}, \quad \boldsymbol{A}^{-1} = \frac{1}{3} \begin{bmatrix} 0 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Example 2 A diagonal matrix has an inverse provided no diagonal entries are zero. (主对角元都是非零数的对角阵是可逆的.)

$$\begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix}^{-1} = \begin{bmatrix} a_1^{-1} & & & \\ & a_2^{-1} & & \\ & & \ddots & \\ & & & a_n^{-1} \end{bmatrix}$$

Note:

If $ab \neq 0$, then $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} 0 & b^{-1} \\ a^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Note: $\boldsymbol{A}+\boldsymbol{B}$ is not necessarily invertible for invertible matrices $\boldsymbol{A}$, $\boldsymbol{B}$.

And we can easily give an example that shows

$$(\boldsymbol{A}+\boldsymbol{B})^{-1} \neq \boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}, \text{ even if } \boldsymbol{A}+\boldsymbol{B} \text{ is invertible.}$$

For example, $\boldsymbol{A} = \mathrm{diag}(2,-1)$, $\boldsymbol{B} = \boldsymbol{I}_2$, $\boldsymbol{C} = \mathrm{diag}(1,2)$.

$$\boldsymbol{A}+\boldsymbol{B} = \mathrm{diag}(3,0) : \text{ not invertible}$$

$$\boldsymbol{A}+\boldsymbol{C} = \mathrm{diag}(3,1) : \text{ invertible}$$

$$(\boldsymbol{A}+\boldsymbol{C})^{-1} = \mathrm{diag}(1/3, 1) \neq \boldsymbol{A}^{-1} + \boldsymbol{C}^{-1} = \mathrm{diag}(3/2, -1/2).$$

Example 3 Let a square matrix $\boldsymbol{B}$ be idempotent (幂等, i.e., $\boldsymbol{B}^2=\boldsymbol{B}$), and $\boldsymbol{A}=\boldsymbol{I}+\boldsymbol{B}$. Show that $\boldsymbol{A}$ is invertible, and $\boldsymbol{A}^{-1}=(3\boldsymbol{I}-\boldsymbol{A})/2$.

Proof By $\boldsymbol{B}=\boldsymbol{A}-\boldsymbol{I}$, $\boldsymbol{B}^2=(\boldsymbol{A}-\boldsymbol{I})^2=\boldsymbol{A}^2-2\boldsymbol{A}+\boldsymbol{I}$,

and $\boldsymbol{B}^2=\boldsymbol{B}$, we can get

$$\boldsymbol{A}^2-2\boldsymbol{A}+\boldsymbol{I}=\boldsymbol{A}-\boldsymbol{I},$$

$$\boldsymbol{A}^2-3\boldsymbol{A}=\boldsymbol{A}(\boldsymbol{A}-3\boldsymbol{I})=-2\boldsymbol{I},$$

i.e., $$\boldsymbol{A}[(3\boldsymbol{I}-\boldsymbol{A})/2]=\boldsymbol{I}.$$

Similarly, $$[(3\boldsymbol{I}-\boldsymbol{A})/2]\boldsymbol{A}=\boldsymbol{I}.$$

Therefore, $\boldsymbol{A}$ is invertible, and $\boldsymbol{A}^{-1}=(3\boldsymbol{I}-\boldsymbol{A})/2$.

III. Inverse -- Properties

- **Theorem 2** If $\boldsymbol{A}$ is an invertible $n \times n$ matrix, then for each $\boldsymbol{b}$ in $\mathbb{R}^n$, the equation $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ has the unique (*one and only one*) solution $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

- **Proof:** Take any $\boldsymbol{b}$ in $\mathbb{R}^n$.

A solution exists because if $\boldsymbol{A}^{-1}\boldsymbol{b}$ is substituted for $\boldsymbol{x}$, then $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{A}^{-1}\boldsymbol{b}) = (\boldsymbol{A}\boldsymbol{A}^{-1})\boldsymbol{b} = \boldsymbol{I}\,\boldsymbol{b} = \boldsymbol{b}$.

So $\boldsymbol{A}^{-1}\boldsymbol{b}$ is a solution.

To prove that the solution is unique, show that if $\boldsymbol{u}$ is any solution, then $\boldsymbol{u}$ must be $\boldsymbol{A}^{-1}\boldsymbol{b}$.

If $\boldsymbol{A}\,\boldsymbol{u} = \boldsymbol{b}$, we can multiply both sides by $\boldsymbol{A}^{-1}$ and obtain $\boldsymbol{A}^{-1}\boldsymbol{A}\,\boldsymbol{u} = \boldsymbol{A}^{-1}\,\boldsymbol{b}$, $\boldsymbol{I}\,\boldsymbol{u} = \boldsymbol{A}^{-1}\,\boldsymbol{b}$, and $\boldsymbol{u} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

- **Theorem 3**

a. If $\boldsymbol{A}$ is an invertible matrix, then $\boldsymbol{A}^{-1}$ is invertible and $(\boldsymbol{A}^{-1})^{-1}=\boldsymbol{A}$. (若 $\boldsymbol{A}$ 可逆，则 $\boldsymbol{A}^{-1}$ 亦可逆)

b. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n\times n$ invertible matrices, then so is $\boldsymbol{AB}$ (若 $\boldsymbol{A}$，$\boldsymbol{B}$ 为同阶可逆方阵，则 $\boldsymbol{AB}$ 也可逆),

and the inverse of $\boldsymbol{AB}$ is the product of the inverses of $\boldsymbol{A}$ and $\boldsymbol{B}$ in the *reverse* order. That is,

$$(\boldsymbol{AB})^{-1}=\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}.\qquad \boxed{(\boldsymbol{A}^{k})^{-1}=(\boldsymbol{A}^{-1})^{k}=\boldsymbol{A}^{-k}}$$

c. If $\boldsymbol{A}$ is an invertible matrix, then so is $\boldsymbol{A}^{\mathrm{T}}$ (若 $\boldsymbol{A}$ 可逆, 则 $\boldsymbol{A}^{\mathrm{T}}$ 亦可逆), and the inverse of $\boldsymbol{A}^{\mathrm{T}}$ is the transpose of $\boldsymbol{A}^{-1}$. That is,

$$(\boldsymbol{A}^{\mathrm{T}})^{-1}=(\boldsymbol{A}^{-1})^{\mathrm{T}}.$$

- **Proof:** To verify statement (a), find a matrix $\boldsymbol{C}$ such that

$$\boldsymbol{A}^{-1}\boldsymbol{C} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{C}\boldsymbol{A}^{-1} = \boldsymbol{I}\ .$$

  These equations are satisfied with $\boldsymbol{A}$ in place of $\boldsymbol{C}$.

  Hence $\boldsymbol{A}^{-1}$ is invertible, and $\boldsymbol{A}$ is its inverse.

- Next, to prove statement (b), compute:

$$(\boldsymbol{AB})(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) = \boldsymbol{A}(\boldsymbol{BB}^{-1})\boldsymbol{A}^{-1} = \boldsymbol{A}\,\boldsymbol{I}\,\boldsymbol{A}^{-1} = \boldsymbol{A}\,\boldsymbol{A}^{-1} = \boldsymbol{I}\ .$$

  A similar calculation shows that $(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})\,(\boldsymbol{AB}) = \boldsymbol{I}$.

- For statement (c), use Theorem of transpose matrix, read from right to left,

$$(\boldsymbol{A}^{-1})^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = (\boldsymbol{AA}^{-1})^{\mathrm{T}} = \boldsymbol{I}^{\mathrm{T}} = \boldsymbol{I}.$$

  Similarly, $\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}^{-1})^{\mathrm{T}} = \boldsymbol{I}^{\mathrm{T}} = \boldsymbol{I}.$

The inverses of elementary matrices (初等矩阵的逆矩阵)

Recall: (1)初等对换矩阵：

将单位矩阵的第 i, j 行(或列)对换

$$\boldsymbol{P}_{ij} = \begin{array}{cl}
\left[\begin{array}{ccccccccccc}
1 &&&&&&&&&& \\
& \ddots &&&&&&&&& \\
&& 1 &&&&&&&& \\
&&& 0 && \cdots && 1 &&& \\
&&&& 1 &&&&&& \\
&&& \vdots && \ddots && \vdots &&& \\
&&&&&& 1 &&&& \\
&&& 1 && \cdots && 0 &&& \\
&&&&&&&& 1 && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& 1
\end{array}\right] &
\begin{array}{l}
\\ \\ \\ \leftarrow 第\ i\ 行 \\ \\ \\ \\ \leftarrow 第\ j\ 行 \\ \\ \\ \\
\end{array} \\
\text{第 } i \text{ 列} \qquad \text{第 } j \text{ 列} &
\end{array}$$

(2)初等倍乘矩阵：

将单位矩阵第 i 行(或列)乘 $k \neq 0$

$$
\begin{array}{cc}
\boldsymbol{D}_i(k) = \begin{bmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1 & & & & \\
& & & k & & & \\
& & & & 1 & & \\
& & & & & \ddots & \\
& & & & & & 1
\end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow \text{第}\, i \,\text{行} \\ \\ \\ \\ \end{array} \\
\text{第}\, i \,\text{列} &
\end{array}
$$

(3)初等倍加矩阵：

将单位矩阵第 i 行乘 k 加到第 j 行，

或将第 j 列乘 k 加到第 i 列

$$\boldsymbol{E}_{ij}(k)=\begin{array}{c}\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & \vdots & \ddots & & & \\ & & k & \cdots & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}\begin{array}{l} \\ \\ \leftarrow \text{第 } i \text{ 行} \\ \\ \leftarrow \text{第 } j \text{ 行} \\ \\ \\ \end{array} \\ \text{第 } i \text{ 列} \qquad \text{第 } j \text{ 列} \end{array}$$

The inverses of elementary matrices (初等矩阵的逆矩阵)

变换 $r_i \leftrightarrow r_j$ 的逆变换是其本身，则

$$\boldsymbol{P}_{ij}^{-1} = \boldsymbol{P}_{ij};$$

变换 $kr_i\ (k \neq 0)$的逆变换是 $\dfrac{1}{k}r_i$，则

$$\boldsymbol{D}_i^{-1}(k) = \boldsymbol{D}_i\left(\frac{1}{k}\right);$$

变换 $r_j + kr_i$ 的逆变换是 $r_j + (-k)r_i$，则

$$\boldsymbol{E}_{ij}^{-1}(k) = \boldsymbol{E}_{ij}(-k).$$

初等矩阵的逆矩阵仍为同类型的初等矩阵.

Each elementary matrix $\boldsymbol{E}$ is invertible. The inverse of $\boldsymbol{E}$ is the elementary matrix of the same type that transforms $\boldsymbol{E}$ back into $\boldsymbol{I}$.

IV. Algorithm (初等变换法求逆矩阵)

定理4 可逆矩阵可以经过若干次初等变换化为单位矩阵.

析 任何矩阵$\boldsymbol{A}$, 都可经初等行变换将其化为行简化阶梯形矩阵.

任何方阵$\boldsymbol{A}$, 都可经初等行变换将其化为上三角形矩阵.

任何可逆矩阵$\boldsymbol{A}$, 都可经初等行变换将其化为单位矩阵$\boldsymbol{I}$.

即 $\boldsymbol{P}_s\ldots\boldsymbol{P}_2\boldsymbol{P}_1\boldsymbol{A}=\boldsymbol{I}$. ($\boldsymbol{P}_1, \ldots, \boldsymbol{P}_s$均为初等矩阵)

Hint: Suppose that $\boldsymbol{A}$ is invertible.

Then, since the equation $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$ has a solution for each $\boldsymbol{b}$ (Theorem 2), $\boldsymbol{A}$ has a pivot position in every row.

Because $\boldsymbol{A}$ is square, the n pivot positions must be on the diagonal, which implies that the reduced echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}_n$.

Then, since each step of the row reduction of $\boldsymbol{A}$ corresponds to left-multiplication by an elementary matrix, there exist elementary matrices $\boldsymbol{P}_1, \ldots,$ $\boldsymbol{P}_s$ such that $\boldsymbol{P}_s\ldots\boldsymbol{P}_2\boldsymbol{P}_1\boldsymbol{A}=\boldsymbol{I}$.

由$\underline{P_s \dots P_2 P_1} A = I$ 得 $A = P_1^{-1} P_2^{-1} \cdots P_s^{-1} I$ 和

$A^{-1} = P_s \dots P_2 P_1 = \underline{P_s \dots P_2 P_1} I$

初等矩阵的逆矩阵仍然是初等矩阵

(A^{-1} results from applying P_1, ..., P_s successively to I.)

This is the same sequence that reduced A to I.

结论： (1) 可逆矩阵可以表示为若干初等矩阵的乘积；

(2) 对 A 作若干初等变换, 将 A 化为单位矩阵 I 时,

同样的这些初等变换将单位矩阵 I 化为 A^{-1}.

$$P_s \cdots P_2 P_1 [A \quad I] = [I \quad A^{-1}]$$

Row reduce the matrix $[A \quad I]$. If A is row equivalent to I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$.
Otherwise, A does not have an inverse.

用初等行变换求 A 的逆矩阵
(the ***Gauss-Jordan method*** for calculating A^{-1})

即对 $n \times 2n$ 矩阵 $[A \quad I_n]$ 实施一系列初等行变换，把矩阵 A 变成 I_n 时，原来的 I_n 就变成了 A^{-1}.

$$[A, I_n] \xrightarrow{\text{ERO}} [I_n, A^{-1}]$$

- **Theorem 5** An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Example 4 Write the matrix $\boldsymbol{A}$ as the product of elementary matrices, where

$$\boldsymbol{A}=\begin{bmatrix}1 & 0 & 0\\ 2 & 0 & -1\\ 0 & -1 & 0\end{bmatrix}.$$

Solution The matrix $\boldsymbol{A}$ can be obtained from the 3 by 3 identity matrix $\boldsymbol{I}$ by 4 elementary operations

$$r_2 \leftrightarrow r_3, \quad c_1 + 2c_3, \quad (-1)r_3, \quad (-1)c_3$$

therefore $\boldsymbol{A} = \boldsymbol{P}_3\boldsymbol{P}_1\boldsymbol{I}\boldsymbol{P}_2\boldsymbol{P}_4 = \boldsymbol{P}_3\boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{P}_4,$

where

$$\boldsymbol{P}_1=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad \boldsymbol{P}_2=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & 1\end{pmatrix}, \quad \boldsymbol{P}_3=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}, \quad \boldsymbol{P}_4=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Example 5 Use ERO to find the inverse of $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}$.

Solution

$$[A, I] = \left[\begin{array}{ccc:ccc} 0 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r1 \leftrightarrow r2} \left[\begin{array}{ccc:ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r3+r1} \left[\begin{array}{ccc:ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow[r2+r3]{r1+r3\times(-2)} \left[\begin{array}{ccc:ccc} 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[r1+r2\times(-1)]{r2\times\frac{1}{2}} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}.$$

Remark: 1. *Can we use* ***elementary column operations*** *on* ***A*** *to find its inverse?*

也可以对 $\begin{bmatrix} A \\ I \end{bmatrix}$ 施行初等列变换，当 A 变成单位矩阵时，I 被化为了 A^{-1}.

Remark: 2. *Can we use elementary operations to solve system of linear equations?*

初等行变换求逆矩阵的方法，还可用于求矩阵 $A^{-1}b$.

$$A^{-1}[A \quad b] = [I \quad A^{-1}b]$$

$$\begin{array}{cc} [A & b] \\ \downarrow & \downarrow \text{初等行变换} \\ I & A^{-1}b \end{array}$$

Example 6 Find the matrix $\boldsymbol{X}$, such that $\boldsymbol{AX} = \boldsymbol{B}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 2 & 5 \\ 3 & 1 \\ 4 & 3 \end{bmatrix}.$$

Solution If $\boldsymbol{A}$ is invertible, then $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$.

$$[\boldsymbol{A} \;\; \boldsymbol{B}] = \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 2 & 5 \\ 2 & 2 & 1 & 3 & 1 \\ 3 & 4 & 3 & 4 & 3 \end{array}\right] \to \begin{bmatrix} 1 & 2 & 3 & 2 & 5 \\ 0 & -2 & -5 & -1 & -9 \\ 0 & -2 & -6 & -2 & -12 \end{bmatrix}$$

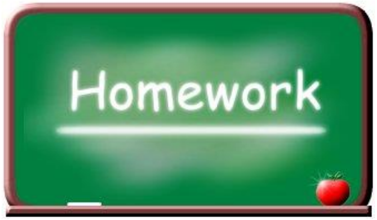
$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 & 5 \\ 0 & -2 & -5 & -1 & -9 \\ 0 & -2 & -6 & -2 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & -4 \\ 0 & -2 & -5 & -1 & -9 \\ 0 & 0 & -1 & -1 & -3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & -2 & 0 & 4 & 6 \\ 0 & 0 & -1 & -1 & -3 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right],$$

$$\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B} = \begin{bmatrix} 3 & 2 \\ -2 & -3 \\ 1 & 3 \end{bmatrix}.$$

What if -- we want to solve *XA*=*B* for *X* ?

Homework



- See Blackboard announcement
- ***Hardcover* textbook + Supplementary problems**

Deadline (DDL):

- Next tutorial class