1. (4). $C_{11}+C_{12}+\cdots+C_{1n} = \begin{vmatrix}1&1&\cdots&1\\0&2&\cdots&2\\\vdots&\vdots&&\vdots\\0&0&\cdots&n\end{vmatrix} = |A|$, $C_{21}+C_{22}+\cdots+C_{2n} = \begin{vmatrix}1&1&1&\cdots&1\\1&1&1&\cdots&1\\0&0&3&\cdots&3\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&n\end{vmatrix}=0$,

$C_{k1}+C_{k2}+\cdots+C_{kn} = \begin{vmatrix}1&1&\cdots&1\\0&2&\cdots&2\\\vdots&\vdots&&\vdots\\1&1&\cdots&1\\\vdots&\vdots&&\vdots\\0&0&\cdots&n\end{vmatrix}=0$, $k=2,\cdots,n-1$, $C_{n1}+C_{n2}+\cdots+C_{nn} = \begin{vmatrix}1&1&\cdots&1&1\\0&2&\cdots&2&2\\\vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&n-1&n-1\\1&1&\cdots&1&1\end{vmatrix}=0$.

Quiz 3 – Linear Algebra (Apr 28, 2022)

则 $\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} = |A| = n!$

本人郑重承诺将秉承诚信原则，自觉遵守考场纪律，并承担违纪或作弊带来的后果。

学号：________　　　　　姓名：

题号	1	2	3	4	5	合计
分值	10 分	10 分	10 分	10 分	10 分	50 分
得分						

This 50-minute long test includes 5 questions. Write ***all your answers*** on this examination paper.

$\frac{b^Ta}{a^Ta}a = \frac{6}{3}a = (2,2,2)^T$

1. (1) The projection of $\boldsymbol{b} = (3,2,1)^T$ onto the line through $\boldsymbol{a} = (1,1,1)^T$ is $\underline{(2,2,2)^T}$.

(2) Let $|\boldsymbol{A}| = \begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix} = 2$, then $|\boldsymbol{B}| = \begin{vmatrix}a_2&b_2&c_2\\2a_1&2b_1&2c_1\\a_3+a_1&b_3+b_1&c_3+c_1\end{vmatrix} = \underline{-4}$.

$|B| = 2\begin{vmatrix}a_2&b_2&c_2\\a_1&b_1&c_1\\a_3+a_1&b_3+b_1&c_3+c_1\end{vmatrix} = 2\begin{vmatrix}a_2&b_2&c_2\\a_1&b_1&c_1\\a_3&b_3&c_3\end{vmatrix} = -2\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix} = -2|A^T| = -2|A| = -4$

(3) $\begin{vmatrix}0&a&b&0\\a&0&0&b\\0&c&d&0\\c&0&0&d\end{vmatrix} = \underline{-(ad-bc)^2}$.

(4) Let $D_n = \begin{vmatrix}1&1&\cdots&1\\0&2&\cdots&2\\\vdots&\vdots&&\vdots\\0&0&\cdots&n\end{vmatrix}$, then the sum of all its cofactors is $\underline{n!}$.

解析见上方

(5) Let $|\boldsymbol{A}_{n\times n}| = a$, $|\boldsymbol{B}_{m\times m}| = b$, then $\begin{vmatrix}\boldsymbol{0}&2\boldsymbol{B}\\-\boldsymbol{A}&\boldsymbol{0}\end{vmatrix} = \underline{(-1)^{(m+1)n}2^m ab}$.

$\begin{vmatrix}0&2B\\-A&0\end{vmatrix} = (-1)^{mn}\begin{vmatrix}-A&0\\0&2B\end{vmatrix} = (-1)^{mn}|-A|\,|2B|$

2. Find the QR decomposition of the following matrix:

$= (-1)^{mn}(-1)^n|A|\,2^m|B|$

$$\begin{bmatrix}1&0&1\\0&1&2\\1&0&3\end{bmatrix}.$$

$= (-1)^{(m+1)n}2^m ab$

令 $a=\begin{pmatrix}1\\0\\1\end{pmatrix}$, $b=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $c=\begin{pmatrix}1\\2\\3\end{pmatrix}$.

$b^Ta=0$，则 $b\perp a$. $D = c - \frac{c^Ta}{a^Ta}a - \frac{c^Tb}{b^Tb}b = \begin{pmatrix}-1\\0\\1\end{pmatrix}$.

$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$, $q_2 = \frac{b}{\|b\|} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $q_3 = \frac{c}{\|c\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$.

则 $Q = (q_1\ q_2\ q_3) = \begin{pmatrix}1/\sqrt{2}&0&-1/\sqrt{2}\\0&1&0\\1/\sqrt{2}&0&1/\sqrt{2}\end{pmatrix}$.

$R = Q^TA = \begin{pmatrix}\sqrt{2}&0&2\sqrt{2}\\0&1&2\\0&0&\sqrt{2}\end{pmatrix}$.

3. Calculate the determinant $D = \begin{vmatrix} a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ b+c+d & a+c+d & a+b+d & a+b+c \end{vmatrix}$.

$$|D| = \begin{vmatrix} a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ a+b+c+d & b+a+c+d & c+a+b+d & d+a+b+c \end{vmatrix}$$

$$=(a+b+c+d)\begin{vmatrix} a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -(a+b+c+d)\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$$

$$= -(a+b+c+d)(d-a)(d-b)(d-c)(c-a)(c-b)(b-a).$$

4. Is there a straight line through the three points (0,1), (3,4), (6,5)? If not, find the best line using least squares method.

令 $y=at+b$, 则有 a,b 满足方程组:

$$\begin{pmatrix} 0 & 1 \\ 3 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \quad 令\ A=\begin{pmatrix} 0 & 1 \\ 3 & 1 \\ 6 & 1 \end{pmatrix}, \quad b=\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \quad x=\begin{pmatrix} a \\ b \end{pmatrix}$$

则 $(A\ \ b) \longrightarrow \begin{pmatrix} 3 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$, 则 $Ax=b$ 不可解, 则此直线不存在.

$A^TA=\begin{pmatrix} 45 & 9 \\ 9 & 3 \end{pmatrix}$, $A^Tb=\begin{pmatrix} 42 \\ 10 \end{pmatrix}$, 则 $A^TAx=A^Tb$ 的解为 $x=\begin{pmatrix} 2/3 \\ 4/3 \end{pmatrix}$,

则最佳逼近直线为 $y=2/3\,t + 4/3$.

5. Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$ be independent 3-dimensional column vectors, and the matrix

$$\boldsymbol{A} = [\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_1].$$

Is $\boldsymbol{A}$ invertible? Please give your reason.

$$A=(\alpha_1\ \ \alpha_2\ \ \alpha_3)\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

令 $B=(\alpha_1\ \ \alpha_2\ \ \alpha_3)$, $C=\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

由 $\alpha_1, \alpha_2, \alpha_3$ 线性无关, 则 $|B| \neq 0$,

又有 $|C| = 2 \neq 0$, 则有:

$|A| = |B||C| \neq 0$. 则有 A 可逆.