Quiz 1– Linear Algebra (Mar 8, 2022)

1. (1) True or False: If $\boldsymbol{A}$ is an $n \times n$ matrix, then $(\boldsymbol{A}^2)^n = (\boldsymbol{A}^n)^2$. (T)

$(A^2)^n = A^2 A^2 \cdots A^2 = AA \cdots A = A^n A^n = (A^n)^2$

(2) True or False: If $\boldsymbol{A}$ is an $n \times n$ symmetric matrix, and $\boldsymbol{B}$ is an $n \times n$ skew-symmetric matrix, then $(\boldsymbol{AB})^2$ is a skew-symmetric matrix. (F)

$((AB)^2)^T = (ABAB)^T = B^T A^T B^T A^T = (-B)A(-B)A$
$= BABA \neq -ABAB$

(3) True or False: If $\boldsymbol{A}^2 = \boldsymbol{A}$, then $\boldsymbol{A} = \boldsymbol{0}$ or $\boldsymbol{A} = \boldsymbol{I}$. (F)

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A^2 = A$, $A \neq 0$, $A \neq I$.

(4) Let $\boldsymbol{A} = \boldsymbol{PBQ}$, where $\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -1 & 0 \\ 5 & 4 & 3 \end{bmatrix}$, $\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\boldsymbol{Q} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then $\boldsymbol{B} = \underline{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}}$

$A = PBQ \Rightarrow P^{-1} A Q^{-1} = B$

(5) If $\boldsymbol{A}$ is an $n \times n$ matrix which satisfies the equation $\boldsymbol{A}^2 - \boldsymbol{A} - 7\boldsymbol{I} = \boldsymbol{0}$, then $(\boldsymbol{A} - 3\boldsymbol{I})^{-1} = \underline{\ A+2I\ }$.

$A^2 - A - 7I = 0 \Rightarrow (A-3I)(A+2I) = A^2 - A - 6I = I$
$\Rightarrow (A-3I)^{-1} = A+2I$

2. Suppose that $\boldsymbol{A}, \boldsymbol{B}$ are square matrices of order n. Show that $\boldsymbol{I} + \boldsymbol{AB}$ is invertible if and only if $\boldsymbol{I} + \boldsymbol{BA}$ is invertible.

(1) $I + AB$ 可逆，则有：$(I+AB)^{-1}(I+AB) = I$
$\Rightarrow (I+AB)^{-1} + (I+AB)^{-1}AB = I \Rightarrow (I+AB)^{-1}A + (I+AB)^{-1}ABA = A$
$\Rightarrow (I+AB)^{-1}A(I+BA) = A \Rightarrow B(I+AB)^{-1}A(I+BA) = BA$
$\Rightarrow B(I+AB)^{-1}A(I+BA) + I = BA + I$
$\Rightarrow I = -B(I+AB)^{-1}A(I+BA) + I + BA$
$\Rightarrow I = (I - B(I+AB)^{-1}A)(I+BA)$
$\Rightarrow (I+BA)^{-1} = I - B(I+AB)^{-1}A$. $I + BA$ 可逆.

(2). $I + BA$ 可逆，则有 $(I+BA)^{-1}(I+BA) = I$
$\Rightarrow (I+BA)^{-1}B + (I+BA)^{-1}BAB = B$
$\Rightarrow A(I+BA)^{-1}B(I+AB) + I = AB + I$
$\Rightarrow I = (I - A(I+BA)^{-1}B)(I+AB) \Rightarrow I + AB$ 可逆.

3. Let $\boldsymbol{A}=\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & 1\\ 0 & 0 & -1\end{bmatrix}$, and $\boldsymbol{A}^2-\boldsymbol{AB}=\boldsymbol{I}$, where $\boldsymbol{I}$ is a 3×3 identity matrix.

Find $\boldsymbol{B}$.

$AB = A^2 - I$，由于 A 可逆，则有 $B = A - A^{-1}$.

$$A^{-1} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow B = A - A^{-1} = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

4. Let $\boldsymbol{A}=\begin{bmatrix}1 & a & a^2\\ 0 & 1 & a\\ 0 & 0 & 1\end{bmatrix}$ and $\boldsymbol{B}=\begin{bmatrix}2 & 5 & 4\\ 1 & 2 & 2\\ 2 & 4 & 5\end{bmatrix}$

(1) Find $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}^{-1}$.

(2) Let $\boldsymbol{C}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{A}\\ \boldsymbol{B} & \boldsymbol{0}\end{bmatrix}$, find $\boldsymbol{C}^{-1}$.

(1)

$$(I\ \ A) = \begin{pmatrix} 1 & 0 & 0 & 1 & a & a^2 \\ 0 & 1 & 0 & 0 & 1 & a \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -a^2 & 1 & a & 0 \\ 0 & 1 & -a & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -a & 0 & 1 & 0 & 0 \\ 0 & 1 & -a & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} 1 & -a & 0 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}$$

$$(I\ \ B) = \begin{pmatrix} 1 & 0 & 0 & 2 & 5 & 4 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 2 \\ 1 & 0 & 0 & 2 & 5 & 4 \\ 0 & 0 & 1 & 2 & 4 & 5 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 2 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 5 & -2 & 1 & 2 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} -2 & 9 & -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{pmatrix} \Rightarrow B^{-1} = \begin{pmatrix} -2 & 9 & -2 \\ 1 & -2 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

(2). $$C^{-1} = \begin{pmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{pmatrix}.$$

5. Let $\boldsymbol{X} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}$, where $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 5 & -3 \end{bmatrix}$. Find $\boldsymbol{X}$.

$$X = AX + B$$

$$\Rightarrow X - AX = B \Rightarrow (I - A)X = B.$$

$$I - A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}.$$

则 $I - A$ 可逆，则有：

$$X = (I - A)^{-1} B$$

$$(I-A \quad B) = \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 2 & 0 \\ 1 & 0 & 2 & 5 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 4 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 3 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 3 & -1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}.$$