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Vector Spaces (向量空间)

2.2

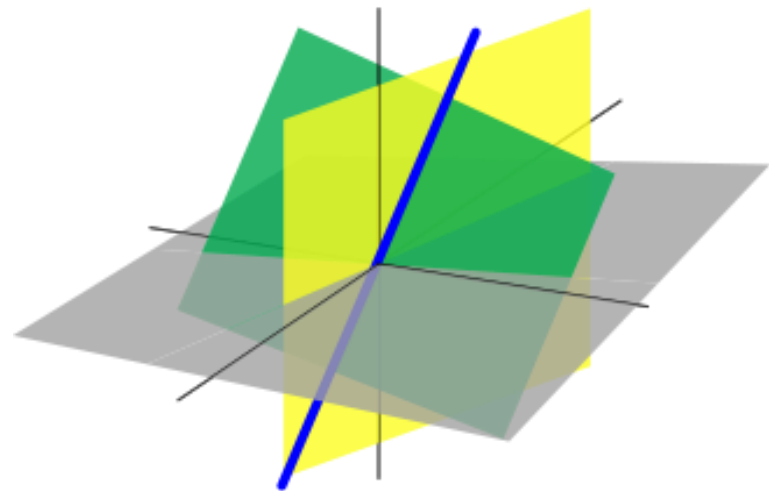
SOLVING $Ax=0$ AND $Ax=b$

(线性方程组的解)

Solving $Ax=0$

Solving $Ax=b$

Rank (秩)



Introduction:

A system of linear equations


$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots \quad \quad \quad \dots \quad \quad \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

一般线性方程组的矩阵表示

$$Ax = b.$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

 **Solution**

齐次线性方程组的矩阵表示

homogeneous $Ax = 0.$

解的结构:

在多解情况下, 讨论
解与解之间的关系

Introduction: Simple cases and more ...

- For a square invertible matrix A , there is only one solution to $A\mathbf{x} = \mathbf{b}$, and it is $\mathbf{x} = A^{-1}\mathbf{b}$.
- For a rectangular matrix $A_{m \times n}$ ($m \neq n$) or a square matrix without an inverse, there are new possibilities
 - ($[A \ \mathbf{b}] \rightarrow$ echelon form $[U \ \mathbf{c}]$
 \rightarrow reduced echelon form $[R \ \mathbf{d}]$)

Introduction: Simple cases and more ...

- For a square invertible matrix A :
 - The nullspace contains only $\mathbf{x} = \mathbf{0}$; This zero solution is usually called the **trivial solution** (平凡解).
(multiply $A\mathbf{x} = \mathbf{0}$ by A^{-1})
 - The column space is the whole space.
($A\mathbf{x} = \mathbf{b}$ has a solution for every \mathbf{b})
- The new questions appear when the nullspace contains *more than the zero vector* (i.e., there exists a **nontrivial solution**) and/or the column space contains *less than all vectors*.

For example,

The matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not invertible.

$y + z = b_1$ and $2y + 2z = b_2$ usually have no solution.

There is *no solution* unless $b_2 = 2b_1$. The column space of A contains only those \mathbf{b} 's, the multiples of $(1, 2)^T$.

When $b_2 = 2b_1$ there are *infinitely many solutions*.

A particular solution to $y + z = 2$ and $2y + 2z = 4$ is $\mathbf{x}_p = (1, 1)^T$.

The nullspace of A contains $(-1, 1)^T$ and all its multiples $\mathbf{x}_n = (-c, c)^T$, $c \in \mathbf{R}$.

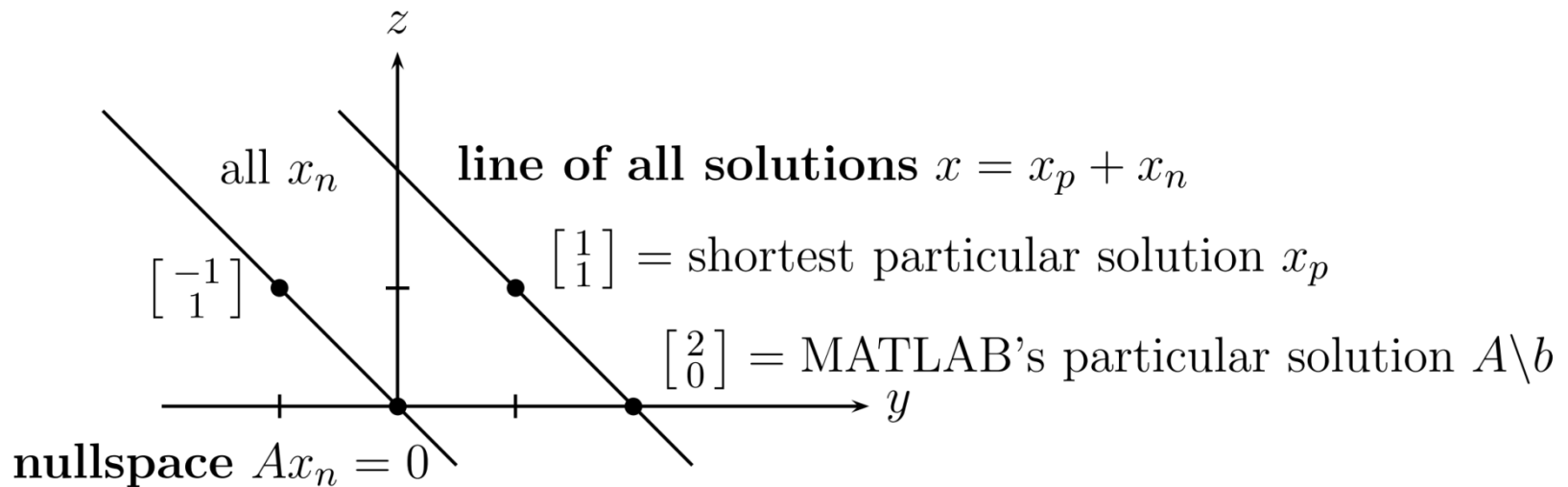
Complete**solution to**

$$y + z = 2$$

$$2y + 2z = 4$$

is solved by $\mathbf{x}_p + \mathbf{x}_n = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - c \\ 1 + c \end{bmatrix}$

Complete solution: $A\mathbf{x}_p = \mathbf{b}$ and $A\mathbf{x}_n = \mathbf{0}$ produce $A(\mathbf{x}_p + \mathbf{x}_n) = \mathbf{b}$.



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I. Solving $Ax = 0$

Example 1 Find a spanning set for the nullspace of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution: The first step is to find the general solution of $Ax=0$ in terms of free variables.

Row reduce the matrix A to *reduced echelon form* in order to write the basic variables in terms of the free variables:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \quad \begin{array}{l} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \\ 0 = 0 \end{array}$$

The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, with x_2 , x_4 , and x_5 free. (x_1, x_3 : **basic variables**, also called **pivot variables**)

The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, with x_2 , x_4 , and x_5 free (called **free variables**).

Next, decompose the vector giving the general solution into a linear combination of *vectors where the weights are the free variables*. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = x_2 \mathbf{u} + x_4 \mathbf{v} + x_5 \mathbf{w}$$

$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Every linear combination of \mathbf{u} , \mathbf{v} , and \mathbf{w} is an element of $N(\mathbf{A})$. Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for $N(\mathbf{A})$.

$\mathbf{u}, \mathbf{v}, \mathbf{w}$: special solutions

The *nullspace* of \mathbf{A} can be spanned by a few *special solutions*, where a solution of $\mathbf{Ax} = \mathbf{0}$ is called *special* if it is a solution for which each free variable takes value 1 or 0.

The best way to find all solutions to $\mathbf{Ax} = \mathbf{0}$ is from the special solutions:

Step 1. After reaching $\mathbf{Rx} = \mathbf{0}$, identify the pivot variables (i.e., basic variables) and free variables. (\mathbf{R} : reduced echelon form of \mathbf{A})

Step 2. Give one free variable the value 1, set the other free variables to 0, and solve $\mathbf{Rx} = \mathbf{0}$ for the basic variables. This \mathbf{x} is a special solution.

Step 3. Every free variable produces its own “special solution” by step 2. The combinations of special solutions form the nullspace—all solutions to $\mathbf{Ax} = \mathbf{0}$.

II. Solving $Ax = b$

- As observed above, all solutions of a homogeneous system of linear equations form a vector space (齐次线性方程组的解集构成一个向量空间). This enables us to write down the solutions in a nice way.
- However, solutions of a non-homogeneous system do not have such a nice property (非齐次线性方程组的解集不能构成向量空间) .
- A natural question is what we can say about solutions of a general system of linear equations.
- Consider the system of linear equations $Ax = b$,
and the homogeneous system $Ax = 0$.

Lemma (引理) *If u and w are two solutions of $Ax = b$, then $u - w$ is a solution of $Ax = 0$.*

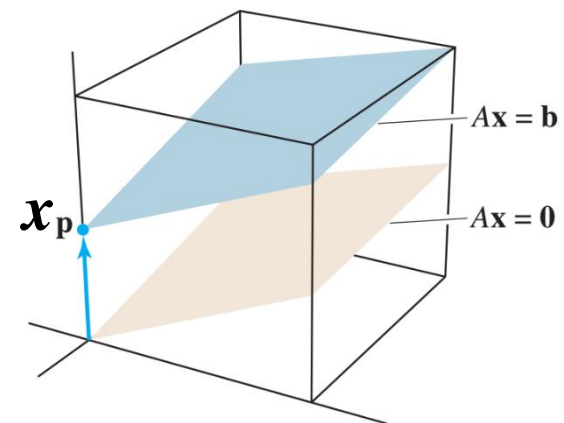
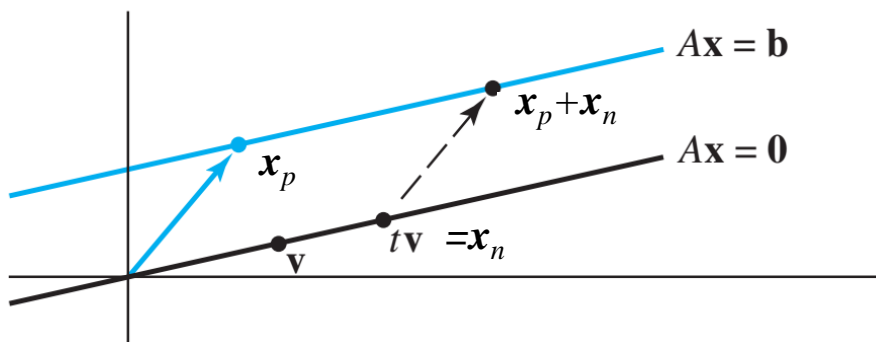
Let x_p be a **particular solution** of $Ax = b$. Then any solution x of $Ax = b$ has the form

$$x_{\text{complete}} = x_{\text{nullspace}} + x_{\text{particular}} .$$

Theorem The solutions of a homogenous system $A\mathbf{x} = \mathbf{0}$ form a subspace $N(A)$, and each solution \mathbf{x} of $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{x} = \mathbf{x}_n + \mathbf{x}_p$$

where \mathbf{x}_p is a particular solution of $A\mathbf{x} = \mathbf{b}$, and $\mathbf{x}_n \in N(A)$.



parallel solution sets of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$
 (Left: 1 free variable; Right: 2 free variables)

Example 2 For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$,

the nullspace of A is the space spanned by the solutions of $A\mathbf{x} = \mathbf{0}$, which is

$$x = z, y = -2z.$$

So a solution vector is of the form

$$(x, y, z)^T = (z, -2z, z)^T = (1, -2, 1)^T z,$$

where $z \in \mathbf{R}$.

Consider the system of linear equations $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

We observe that $A\mathbf{x} = \mathbf{b}$ has a *particular solution* $(0, 1, 0)^T$.

Therefore, the solution set for $A\mathbf{x} = \mathbf{b}$ is

$$\{(0, 1, 0)^T + (1, -2, 1)^T z \mid z \in \mathbf{R}\}.$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 1 & 2 & 3 & | & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 2 & | & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 2 & | & 1 \end{bmatrix}$$

$$[A : \mathbf{b}] \rightarrow [U : \mathbf{c}] \rightarrow [R : \mathbf{d}]$$

$$\begin{array}{rcl} x - z & = & 0 \\ y + 2z & = & 1 \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The best way to write all solutions to $A\mathbf{x} = \mathbf{b}$:

Step 1. Row reduce $A\mathbf{x} = \mathbf{b}$ to $U\mathbf{x} = \mathbf{c}$ or $R\mathbf{x} = \mathbf{d}$.

Step 2. With free variables $= 0$, find a particular solution to $A\mathbf{x}_p = \mathbf{b}$ (or $U\mathbf{x}_p = \mathbf{c}$ or $R\mathbf{x}_p = \mathbf{d}$).

Step 3. Find the special solutions to $A\mathbf{x} = \mathbf{0}$ (or $U\mathbf{x} = \mathbf{0}$ or $R\mathbf{x} = \mathbf{0}$). Each free variable, in turn, is 1. Then

$$\mathbf{x} = \mathbf{x}_p + (\text{any combination } \mathbf{x}_n \text{ of special solutions}).$$

A : coefficient matrix

$[A \ \mathbf{b}]$: augmented matrix

U : the echelon form of A

R : the reduced echelon form of A

Example 3 Find the condition on b_1, b_2, b_3 to have a solution;
Solve $A\mathbf{x} = \mathbf{b}$ when $\mathbf{b} = (0,6,-6)^T$.

$$1x_1 + 2x_2 + 3x_3 + 5x_4 = b_1$$

$$2x_1 + 4x_2 + 8x_3 + 12x_4 = b_2$$

$$3x_1 + 6x_2 + 7x_3 + 13x_4 = b_3$$

Solution

$$\begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3b_1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix}$$

The last row shows the **solvability condition** $b_3 + b_2 - 5b_1 = 0$. (有解条件)

For $\mathbf{b} = (0,6,-6)^T$,

$$\begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} U & c \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \longrightarrow \begin{bmatrix} R & d \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & -9 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

pivot columns

Everything is revealed by $R\mathbf{x} = \mathbf{d}$

The **special solutions to $A\mathbf{x}=\mathbf{0}$** have free variables $x_2 = 1, x_4 = 0$ and $x_2 = 0, x_4 = 1$.

The **particular solution to $A\mathbf{x}=\mathbf{b}$** has free variables $x_2 = 0, x_4 = 0$.

The matrix with columns being special solutions is called the *nullspace matrix*:

$$N = \begin{bmatrix} -2 & -2 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$$

Particular solution to $A\mathbf{x}_p = \mathbf{b}$:

$$\mathbf{x}_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

The complete solution to $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} \mathbf{x} &= \mathbf{x}_p + \mathbf{x}_n \\ &= \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \end{aligned}$$

where $k_1, k_2 \in \mathbf{R}$.

Example 4 设线性方程组
$$\begin{cases} ax_1 + x_2 + x_3 = 4 \\ x_1 + bx_2 + x_3 = 3 \\ x_1 + 2bx_2 + x_3 = 4 \end{cases}$$

就参数 a, b 讨论方程组的解的情况, 有解时并求出解.

解 用初等行变换将增广矩阵化为阶梯阵.

$$\begin{aligned} & \left[\begin{array}{ccc|c} a & 1 & 1 & 4 \\ 1 & b & 1 & 3 \\ 1 & 2b & 1 & 4 \end{array} \right] \xrightarrow{\text{row exchange}} \left[\begin{array}{ccc|c} 1 & b & 1 & 3 \\ 1 & 2b & 1 & 4 \\ a & 1 & 1 & 4 \end{array} \right] \xrightarrow[r_3 - ar_1]{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & b & 1 & 3 \\ 0 & b & 0 & 1 \\ 0 & 1 - ab & 1 - a & 4 - 3a \end{array} \right] \\ & \xrightarrow{r_3 + ar_2} \left[\begin{array}{ccc|c} 1 & b & 1 & 3 \\ 0 & b & 0 & 1 \\ 0 & 1 & 1 - a & 4 - 2a \end{array} \right] \xrightarrow[r_2 \leftrightarrow r_3]{r_2 - br_3} \left[\begin{array}{ccc|c} 1 & b & 1 & 3 \\ 0 & 1 & 1 - a & 4 - 2a \\ 0 & 0 & (a - 1)b & 1 - 4b + 2ab \end{array} \right] \end{aligned}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & b & 1 & 3 \\ 0 & 1 & 1-a & 4-2a \\ 0 & 0 & (a-1)b & 1-4b+2ab \end{array} \right]$$

(1) 当 $(a-1)b \neq 0$ 时, 有唯一解

$$x_1 = \frac{2b-1}{(a-1)b}, \quad x_2 = \frac{1}{b}, \quad x_3 = \frac{1-4b+2ab}{(a-1)b}$$

(2) 当 $a=1$, 且 $1-4b+2ab=1-2b=0$, 即 $b=1/2$ 时, 有无穷多解.

化为 $\left[\begin{array}{ccc|c} 1 & 1/2 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$ 于是方程组的一般解为

$$\mathbf{x} = (2, 2, 0)^T + k(-1, 0, 1)^T$$

(k 为任意常数) .

(3) 当 $a=1, b \neq 1/2$ 时, $1-4b+2ab \neq 0$, 方程组无解.

(4) 当 $b=0$ 时, $1-4b+2ab = 1 \neq 0$ 时, 方程组无解.

(原方程组中后两个方程是矛盾方程)

III. The Rank of a Matrix (矩阵的秩)

Recall the method for solving systems of linear equations $A\mathbf{x} = \mathbf{b}$ we learnt before:

Covert the *augmented matrix* $[A \mid \mathbf{b}]$

into *row echelon form* $[U \mid \mathbf{c}]$

or further convert it into *reduced row echelon form* $[R \mid \mathbf{d}]$

Since elementary row operations do not change the solutions of the system (初等行变换不改变方程组的解), the three systems of linear equations

$$A\mathbf{x} = \mathbf{b}, U\mathbf{x} = \mathbf{c}, R\mathbf{x} = \mathbf{d}$$

have same solutions (同解).

We have also noticed that the number of free variables for $A\mathbf{x} = \mathbf{b}$ depends on the number of non-zero rows of the row echelon form U .

This leads to an important parameter (参数) for a matrix.

Definition 1 For a matrix A , let U be the row echelon form. Then the **rank of A** (**A 的秩**), denoted by $\text{rank}(A)$, is the number of non-zero rows of U (行阶梯形矩阵 U 的非零行的行数).

Obviously, the following properties hold.

- The rank is not bigger than the number of rows.
- Suppose elimination reduces $A\mathbf{x} = \mathbf{b}$ to $U\mathbf{x} = \mathbf{c}$ and $R\mathbf{x} = \mathbf{d}$, with r pivot rows and r pivot columns. **The rank of those matrices is r .**

The last $m-r$ rows of U and R are zero, so there is a solution only if the last $m-r$ entries of \mathbf{c} and \mathbf{d} are also zero.

- The rank r is crucial. It counts the pivot rows in the “row space” and the pivot columns in the “column space”.

There are $n - r$ special solutions in the nullspace.

There are $m - r$ solvability conditions on \mathbf{b} or \mathbf{c} or \mathbf{d} .

Therefore, determining the rank of a given matrix is interesting for understanding the matrix.

To do this, we only need to *use elementary row operations to convert the matrix into row echelon form*.

Example 5 Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}.$$

To find the rank of \mathbf{A} , we convert \mathbf{A} into row echelon form

$$\mathbf{U} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -4 \end{bmatrix}.$$

Thus \mathbf{A} has rank 3, called **full rank** (满秩).

It follows that for any \mathbf{b} , the system of linear equations $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

对于线性方程组 $Ax=b$, 下列命题等价:

- (1) 方程组有解(或相容);
- (2) b 可由 A 的列向量组线性表示, 即 $b \in C(A)$;
- (3) $\text{rank}([A \mid b]) = \text{rank}(A)$, 即增广矩阵的秩等于系数矩阵的秩.

Key words: *free variables, basic variables (pivot variables), special solutions, particular solution, complete solution, rank*

Homework

See Blackboard

