Quiz 1– Linear Algebra (Mar 10, 2022)

1. (1) True or False: If $\boldsymbol{A}$ is an $n \times n$ matrix, then $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{A}^T\boldsymbol{A}$. (F)

$$A=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad A^T=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad AA^T=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad A^TA=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

(2) True or False: If $\boldsymbol{A}$ is an $n \times n$ symmetric matrix, and $\boldsymbol{B}$ is an $n \times n$ skew-symmetric matrix, then $\boldsymbol{AB}+\boldsymbol{BA}$ is a skew-symmetric matrix. (T)

$$(AB+BA)^T=(AB)^T+(BA)^T=B^TA^T+A^TB^T=-BA-AB$$

(3) True or False: If two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy the equation $\boldsymbol{AB}=\boldsymbol{I}$, then $\boldsymbol{A}$ is invertible, and $\boldsymbol{A}^{-1}=\boldsymbol{B}$. (F) $\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

(4) If $\boldsymbol{A},\ \boldsymbol{I}-\boldsymbol{A},\ \boldsymbol{I}-\boldsymbol{A}^{-1}$ are invertible, then

$$(\boldsymbol{I}-\boldsymbol{A})^{-1}+(\boldsymbol{I}-\boldsymbol{A}^{-1})^{-1}=\underline{\quad I \quad}.$$

原式 $=(A(A^{-1}-I))^{-1}+(I-A^{-1})^{-1}=-(I-A^{-1})^{-1}A^{-1}+(I-A^{-1})^{-1}$ $=(I-A^{-1})*(I-A^{-1})^{-1}$ $=I.$

(5) If $\boldsymbol{A}$ is invertible and $(\boldsymbol{AB})^{\mathrm{T}}=\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$ then $\boldsymbol{B}=\underline{I+A^T}$.

$$((AB)^T)^T=(A^T+AA^T)^T \Rightarrow AB=A+AA^T$$
$$\Rightarrow B=I+A^T$$

2. Let

$$\boldsymbol{A}=\begin{bmatrix}2&4&4\\2&6&8\\2&6&10\end{bmatrix},\boldsymbol{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix},\boldsymbol{b}=\begin{bmatrix}6\\6\\4\end{bmatrix}.$$

(1) Find an $\boldsymbol{LU}$ and $\boldsymbol{LDU}$ factorization for $\boldsymbol{A}$.

(2) Solve the linear system $\boldsymbol{Ax}=\boldsymbol{b}$.

(1) $A \xrightarrow{r_2-r_1} \begin{pmatrix}2&4&4\\0&2&4\\2&6&10\end{pmatrix} \xrightarrow{r_3-r_1} \begin{pmatrix}2&4&4\\0&2&4\\0&2&6\end{pmatrix} \xrightarrow{r_3-r_2} \begin{pmatrix}2&4&4\\0&2&4\\0&0&2\end{pmatrix}$

$L_1=\begin{pmatrix}1&&\\-1&1&\\&&1\end{pmatrix},\quad L_2=\begin{pmatrix}1&&\\&1&\\-1&&1\end{pmatrix},\quad L_3=\begin{pmatrix}1&&\\&1&\\&-1&1\end{pmatrix}.$

$L_3L_2L_1A=U \Rightarrow L=L_1^{-1}L_2^{-1}L_3^{-1}=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}.$

LU分解: $L=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix},\quad U=\begin{pmatrix}2&4&4\\0&2&4\\0&0&2\end{pmatrix}.$

LDU分解: $L=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix},\quad D=\begin{pmatrix}2&&\\&2&\\&&2\end{pmatrix},\quad U=\begin{pmatrix}1&2&2\\0&1&2\\0&0&1\end{pmatrix}.$

(2) 令 $Ux=y$, 则 $Ly=b$, 则 $y=\begin{pmatrix}6\\0\\-2\end{pmatrix}$, $x=\begin{pmatrix}1\\2\\-1\end{pmatrix}.$

3. (1) $\boldsymbol{A} = \begin{bmatrix} 2 & 1 & -1 \\ 6 & 3 & -3 \\ -4 & -2 & 2 \end{bmatrix}$. Find $\boldsymbol{A}^n$, where n is a positive integer.

(2) Calculate $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}^8$.

(1) $A = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \end{pmatrix}$, 令 $\alpha = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$, $\beta = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$,

则 $A^n = (\alpha\beta^T)^n = \alpha\beta^T\alpha\beta^T \cdots \alpha\beta^T = \alpha(\beta^T\alpha)^{n-1}\beta^T$

$= \alpha 7^{n-1}\beta^T = 7^{n-1}\alpha\beta^T = 7^{n-1}A$.

(2). 令 $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$, 则:

$A^8 = (2B)^8 = 2^8 B^8 = 2^8 \begin{pmatrix} 1 & 8\cdot\frac{1}{2} \\ 0 & 1 \end{pmatrix} = 2^8 \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2^8 & 2^{10} \\ & 2^8 \end{pmatrix}$.

4. Suppose that $\boldsymbol{AX} = \boldsymbol{A} + 2\boldsymbol{X}$, where $\boldsymbol{A} = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$. Find $\boldsymbol{X}$.

$AX = A + 2X \Rightarrow (A - 2I)X = A$.

$A - 2I = \begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$,

则 $A-2I$ 可逆, 则有 $X = (A-2I)^{-1}A$.

$\begin{pmatrix} A-2I & A \end{pmatrix} = \begin{pmatrix} 2 & 2 & 3 & 4 & 2 & 3 \\ 1 & -1 & 0 & 1 & 1 & 0 \\ -1 & 2 & 1 & -1 & 2 & 3 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 3 & -8 & -6 \\ 0 & 1 & 0 & 2 & -9 & -6 \\ 0 & 0 & 1 & -2 & 12 & 9 \end{pmatrix}$,

则 $X = \begin{pmatrix} 3 & -8 & -6 \\ 2 & -9 & -6 \\ -2 & 12 & 9 \end{pmatrix}$.

另有: $(A-2I)^{-1} = \begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix}$.

5. Let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix}$ and $\boldsymbol{B} = [\boldsymbol{C} \quad \boldsymbol{D}]$, where $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{C}$ and $\boldsymbol{D}$ are all 2×2 matrices. If

$$\boldsymbol{AB} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

Find $\boldsymbol{BA}$.

$$AB = \begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix} = \begin{pmatrix} XC & XD \\ YC & YD \end{pmatrix}.$$

則 $XC = I$, $YD = I$.

則 $CX = I$, $DY = I$,

則 $BA = \begin{pmatrix} C & D \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$

$= CX + DY = 2I.$