# Quiz 3 – Linear Algebra (Apr 26, 2022)

本人郑重承诺将秉承诚信原则，自觉遵守考场纪律，并承担违纪或作弊带来的后果。

学号：________ 姓名：

| 题号 | 1 | 2 | 3 | 4 | 5 | 合计 |
|---|---|---|---|---|---|---|
| 分值 | 10 分 | 10 分 | 10 分 | 10 分 | 10 分 | 50 分 |
| 得分 | | | | | | |

This 50-minute long test includes 5 questions. Write ***all your answers*** on this examination paper.

$\frac{b^T a}{a^T a} a = \frac{6}{3} a = (2, 2, 2)^T$

1. (1) The projection of $\boldsymbol{b} = (1,2,3)^{\mathrm{T}}$ onto the line through $\boldsymbol{a} = (1,1,1)^{\mathrm{T}}$ is $\underline{(2, 2, 2)^T}$.

$C_{12} = -\begin{vmatrix} x & 1 \\ 4 & 5 \end{vmatrix} = 4 - 5x = -1$

则 $x = 1$,

则 $C_{21} = -\begin{vmatrix} 0 & 2 \\ x & 5 \end{vmatrix} = 2.$

(2) For the determinant $\begin{vmatrix} 1 & 0 & 2 \\ x & 3 & 1 \\ 4 & x & 5 \end{vmatrix}$, if the cofactor $C_{12} = -1$, then $C_{21} = \underline{2}$.

(3) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of degree $n$, and $\boldsymbol{AB} = \boldsymbol{0}$, then which of the following is true? ( B )

$|AB| = 0 \Rightarrow |A|\,|B| = 0$

$\Rightarrow |A| = 0$ 或 $|B| = 0.$

A. $\boldsymbol{A} = \boldsymbol{0}$ or $\boldsymbol{B} = \boldsymbol{0}$; B. $|\boldsymbol{A}| = 0$ or $|\boldsymbol{B}| = 0$;

C. $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{0}$; D. $|\boldsymbol{A}| + |\boldsymbol{B}| = 0$.

(4) $\begin{vmatrix} 2 & 3 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{vmatrix} = \underline{-11}$.

(5) If $\begin{vmatrix} a_1 + b_1 & a_1 - b_1 \\ a_2 + b_2 & a_2 - b_2 \end{vmatrix} = -4$, then $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \underline{2}$.

$$\begin{vmatrix} a_1 + b_1 & a_1 - b_1 \\ a_2 + b_2 & a_2 - b_2 \end{vmatrix} = \begin{vmatrix} a_1 + b_1 & 2a_1 \\ a_2 + b_2 & 2a_2 \end{vmatrix} = 2\begin{vmatrix} a_1 + b_1 & a_1 \\ a_2 + b_2 & a_2 \end{vmatrix} = 2\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}$$

$$= -2\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -4.$$

2. Find the QR decomposition of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

令 $a = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $c = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$,

$B = b - \frac{b^T a}{a^T a} a = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, 令 $B = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$,

$D = c - \frac{c^T B}{B^T B} B - \frac{c^T a}{a^T a} a = \frac{1}{3}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$,

$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $q_3 = \frac{D}{\|D\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

$$Q = (q_1\ q_2\ q_3) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}. \quad R = Q^T A = \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} & \sqrt{2} \\ 0 & \sqrt{6}/2 & \sqrt{6}/3 \\ 0 & 0 & 1/\sqrt{3} \end{pmatrix}.$$

3. Find the root of the following equation $\begin{vmatrix} a_1 & a_2 & a_3 & a_4+x \\ a_1 & a_2 & a_3+x & a_4 \\ a_1 & a_2+x & a_3 & a_4 \\ a_1+x & a_2 & a_3 & a_4 \end{vmatrix} = 0.$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4+x \\ a_1 & a_2 & a_3+x & a_4 \\ a_1 & a_2+x & a_3 & a_4 \\ a_1+x & a_2 & a_3 & a_4 \end{vmatrix} = \begin{vmatrix} a_1+a_2+a_3+a_4+x & a_2 & a_3 & a_4+x \\ a_1+a_2+a_3+a_4+x & a_2 & a_3+x & a_4 \\ a_1+a_2+a_3+a_4+x & a_2+x & a_3 & a_4 \\ a_1+a_2+a_3+a_4+x & a_2 & a_3 & a_4 \end{vmatrix}$$

$$= (a_1+a_2+a_3+a_4+x)\begin{vmatrix} 1 & a_2 & a_3 & a_4+x \\ 1 & a_2 & a_3+x & a_4 \\ 1 & a_2+x & a_3 & a_4 \\ 1 & a_2 & a_3 & a_4 \end{vmatrix} = (a_1+a_2+a_3+a_4+x)\begin{vmatrix} 1 & 0 & 0 & x \\ 1 & 0 & x & 0 \\ 1 & x & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= (a_1+a_2+a_3+a_4+x)x^3 = 0 \implies x=0 \text{ 或 } x = -(a_1+a_2+a_3+a_4).$$

4. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$.

(1) Check the consistency of the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$.

(2) If there is no solution, find the best estimate $\hat{\boldsymbol{x}}$ by least squares.

(1) $(A \quad b) \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 6 \end{pmatrix}$, 则 $Ax=b$ 不可解.

(2) $A^TA = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $A^Tb = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$,

则 $A^TA\hat{x} = A^Tb$ 有解 $\hat{x} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$.

5. Let $\boldsymbol{A}$ be an $n$ by $n$ skew-symmetric matrix, and $n$ is odd. Show that $|\boldsymbol{A}| = 0$.

$|A| = |A^T| = |-A| = (-1)^n|A| = -|A|$,

则 $2|A| = 0$, 则 $|A| = 0$.