

## 1

# Matrices and Gaussian Elimination

## 1.4

## MATRIX OPERATIONS

(矩阵运算)

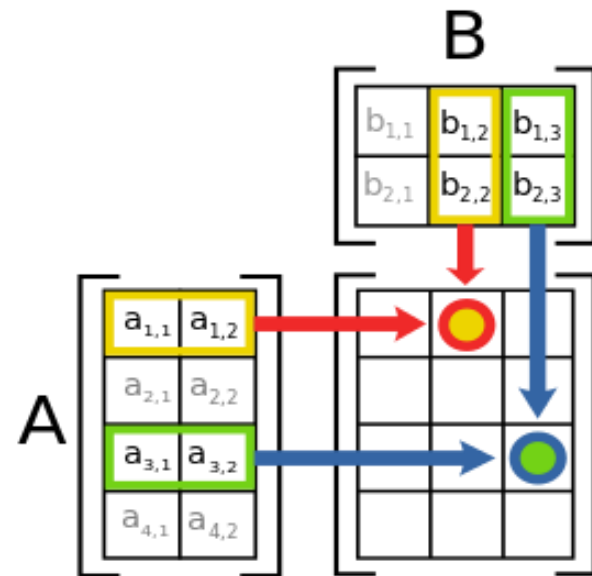
Addition

Scalar multiplication

Multiplication

Power

*Elementary Matrices*



# MATRIX NOTATION

A matrix is an arrangement of  $mn$  elements with  $m$  rows and  $n$  columns, denoted by

$$\begin{array}{c} \text{Column } j \\ \text{Row } i \end{array} \left[ \begin{array}{ccccc} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{array} \right] = A$$

$$A = [a_{ij}]_{m \times n}$$

If  $A$  is an  $m \times n$  matrix, that is, a matrix with  $m$  rows and  $n$  columns, then the scalar entry in the  $i$ th row and  $j$ th column of  $A$  is denoted by  $a_{ij}$  and is called the  $(i, j)$ -entry of  $A$ .

**THE ORDER IS IMPORTANT: rows  $\times$  columns**

- If two matrices have the same number of rows and the same number of columns, then they are called **matrices of the same size (同型矩阵)**.

For example,  $\begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 3 & 7 \end{bmatrix}$  and  $\begin{bmatrix} 14 & 3 \\ -8 & 4 \\ 3i & 9 \end{bmatrix}$ .

- If  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are matrices of the same size, and the corresponding entries are the same, i.e.,

$$a_{ij} = b_{ij} \quad (i = 1, 2, \dots, m; \quad j = 1, 2, \dots, n),$$

then  $A$  and  $B$  are **equal (相等)**, denoted by  $A = B$ .

**Attention!** *equal* vs *equivalent*      $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = B$

## Some special matrices

(1) A matrix with the same number of rows and columns are called a **square matrix(方阵)**. An  $n \times n$  matrix is also called a matrix of degree  $n$  / order  $n$  ( $n$  阶方阵).

For instance,  $\begin{bmatrix} 13 & 6 & 2i \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$  is a complex matrix of order 3 (3 阶复方阵).

(2)  $1 \times n$  matrix  $\mathbf{A} = [a_1, a_2, \dots, a_n]$  : **行矩阵 (或 行向量, row vector)**.

$m \times 1$  matrix  $\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$  : **列矩阵(或列向量, column vector)**.

(3) 
$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$
 is called a **diagonal matrix**,  
 (主) 对角线  
 (main) diagonal

(对角矩阵或对角阵)

denoted as  $\text{diag}(a_{11}, a_{22}, \dots, a_{nn})$ .

A diagonal matrix with the same entry on the diagonal is called a **scalar matrix**(数量矩阵).

The following diagonal matrix is called an **identity matrix** (单位矩阵), denoted as  $\mathbf{I}_n$  or  $\mathbf{I}$ .

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

(4) The matrix with each entry as zero is called a **zero matrix (零矩阵)**, denoted as **0**.

*Zero matrices of different sizes are treated as different matrices.*

(5)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & & & \\ b_{21} & b_{22} & & \\ \vdots & \vdots & \ddots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

**upper triangular matrix**  
**(上三角矩阵)**

**lower triangular matrix**  
**(下三角矩阵)**

对角线左下(右上)方的元素都为 0 的方阵称为  
**(upper/lower triangular matrix)**.

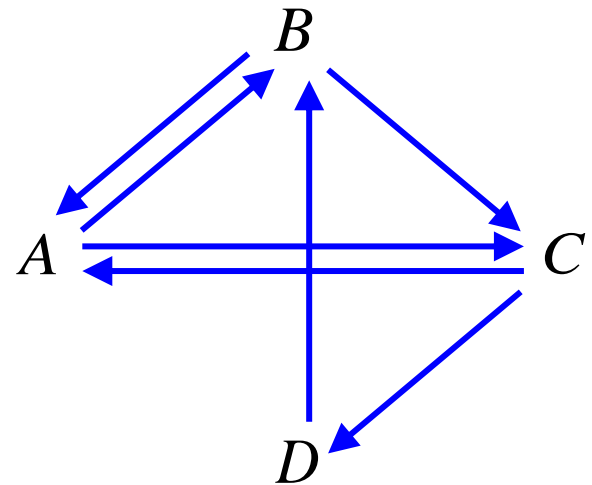
# 引例(introductory example):

## 城市间的航班图

如果从  $A$  到  $B$  有航班, 则  
用带箭头的线连接  $A$  与  $B$ .

航班图可用矩阵来表示:

		到达城市			
		A	B	C	D
出发城市	A	0	1	1	0
	B	1	0	1	0
	C	1	0	0	1
	D	0	1	0	0



## 引例 (introductory example): 城市间的航班图

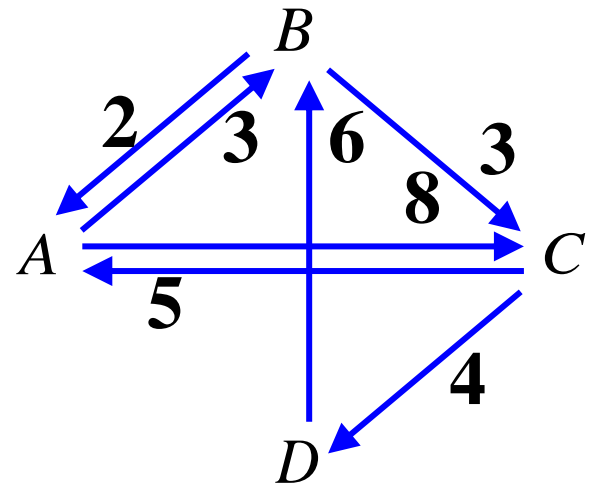
如果从  $A$  到  $B$  有航班, 则  
用带箭头的线连接  $A$  与  $B$ .

航班量也可用矩阵来表示:

$$\begin{bmatrix} 0 & 3 & 8 & 0 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

第一天

第二天





## 城市间的航班图

如果从  $A$  到  $B$  有航班, 则用带箭头的线连接  $A$  与  $B$ .

航班量也可用矩阵来表示:

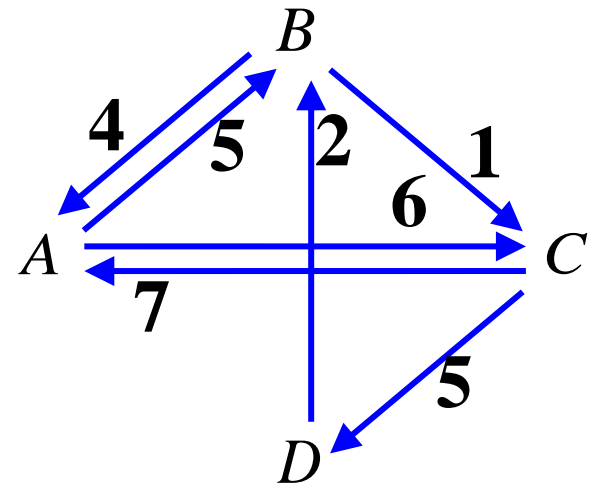
$$\begin{bmatrix} 0 & 3 & 8 & 0 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

第一天

$$\begin{bmatrix} 0 & 5 & 6 & 0 \\ 4 & 0 & 1 & 0 \\ 7 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

第二天

第三天...



问题: 各城市2天内发送的航班量?

**Matrix operations (矩阵的运算)**

# Matrix operations

## (矩阵的运算)

I. Addition (矩阵的加法)

II. Scalar multiplication (数与矩阵相乘)

III. Multiplication (矩阵的乘法)

IV. Power (方阵的幂)

Elementary Matrices (初等矩阵)

# I. 矩阵的加法(Addition)

## 1. 定义 (Definition)

each entry in  $A+B$  is the sum of the corresponding entries in  $A$  and  $B$ .

设有两个  $m \times n$  矩阵  $A = [a_{ij}]$  和  $B = [b_{ij}]$ , 那么矩阵  $A$  与  $B$  的**和(sum)**记作  $A+B$ , 规定为

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

**注** 只有两个矩阵同型时, 才能进行加法运算;

(The sum  $A+B$  is defined only when  $A$  and  $B$  are the same size.)

**For example,**

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & -9 & 0 \\ 3 & 6 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 8 & 9 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 12+1 & 3+8 & -5+9 \\ 1+6 & -9+5 & 0+4 \\ 3+3 & 6+2 & 8+1 \end{bmatrix} = \begin{bmatrix} 13 & 11 & 4 \\ 7 & -4 & 4 \\ 6 & 8 & 9 \end{bmatrix}.$$

## 2. 矩阵加法的运算规律

Let  $A$ ,  $B$ , and  $C$  be matrices of the same size, then

$$(1) \quad A + B = B + A;$$

$$(2) \quad (A + B) + C = A + (B + C);$$

$$(3) \quad -A = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{m1} & -a_{m1} & \cdots & -a_{mn} \end{bmatrix} = [-a_{ij}];$$

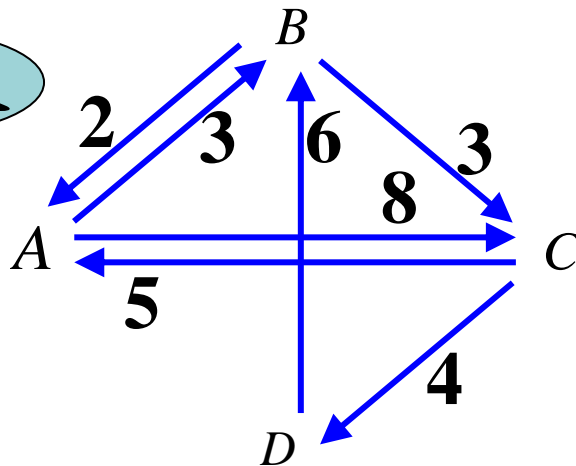
$$(4) \quad A + (-A) = \mathbf{0}, \quad A - B = A + (-B).$$

定义矩阵的减法(subtraction)

# 思考 城市间航班客流量

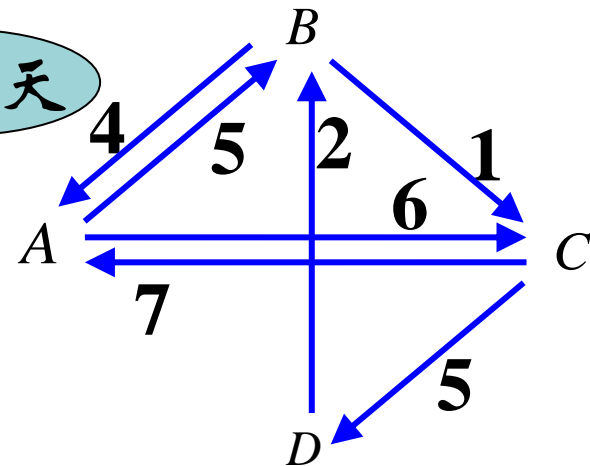


第一天



$$\begin{bmatrix} 0 & 3 & 8 & 0 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

第二天



$$\begin{bmatrix} 0 & 5 & 6 & 0 \\ 4 & 0 & 1 & 0 \\ 7 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

**问题1:** 各城市2天内发送的航班量?

**问题2:** 收取的机场建设费(航空基金)有多少?

## II. 数与矩阵相乘(Scalar multiplication)

### 1. 定义

数  $\lambda$  与矩阵  $A$  的乘积 (简称为数乘, scalar multiple)

记作  $\lambda A$  或  $A\lambda$ , 规定为

$$\lambda A = A\lambda = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda a_{m1} & \lambda a_{m1} & \cdots & \lambda a_{mn} \end{bmatrix}.$$

If  $\lambda$  is a scalar and  $A$  is a matrix, then the **scalar multiple**  $\lambda A$  is the matrix whose columns are  $\lambda$  times the corresponding columns in  $A$ .

## 2. 数乘矩阵的运算规律

Let  $A$ ,  $B$ , and  $C$  be matrices of the same size ( $m \times n$ ), and let  $\lambda$  and  $\mu$  be scalars, then

$$(1) (\lambda\mu)A = \lambda(\mu A);$$

$$(2) (\lambda + \mu)A = \lambda A + \mu A;$$

$$(3) \lambda(A + B) = \lambda A + \lambda B.$$

矩阵的加法与数乘统称为矩阵的线性运算.



**Example 1** Let  $\mathbf{A} - 3\mathbf{B} = 4\mathbf{A} - \mathbf{C}$ , where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Find  $\mathbf{C}$ .

**Solution** From  $\mathbf{A} - 3\mathbf{B} = 4\mathbf{A} - \mathbf{C}$ , we have

$$\mathbf{C} = 4\mathbf{A} - \mathbf{A} + 3\mathbf{B} = 3(\mathbf{A} + \mathbf{B}),$$

$$\text{so } \mathbf{C} = \begin{bmatrix} 3(1+1) & 3(-1-1) \\ 3(0+0) & 3(2+1) \\ 3(3-1) & 3(1+0) \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 9 \\ 6 & 3 \end{bmatrix}.$$

### III. 矩阵乘法(Multiplication)

#### 引例1 超市购物

=>Product of Matrices

同样的商品在不同的超市内的售价是不尽相同的. 这样, 在一次需要购买多种商品时, 就有到哪一家超市去买花费最少的问题.

这就要用到价格矩阵, 如

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix} \quad \begin{bmatrix} 1.7 & 1.1 & 21 & 7 \\ 1.5 & 1.4 & 26 & 9 \\ 1.8 & 1.3 & 28 & 8 \end{bmatrix}$$

可用来表示3家超市里4种商品的“价目表”  
第1行的元依次表示超市1里4种商品的售价

# III. 矩阵乘法(Multiplication)

## 引例1 超市购物

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

商品1 商品2 商品3 商品4

$$\begin{bmatrix} 1.7 & 1.1 & 21 & 7 \\ 1.5 & 1.4 & 26 & 9 \\ 1.8 & 1.3 & 28 & 8 \end{bmatrix}$$

超市1

超市2

超市3

购物者1对4种商品的需求分别为 $a_{11}, a_{21}, a_{31}, a_{41}$ ，  
则在不同超市去购买所需花费总额为？

若有 $n$ 名购物者，则可将他们的需求构成需求矩阵

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$$

那么这 $n$ 名购物者的采购  
方案可以用一个数表来表示：  
总价矩阵  $\equiv ?$

$$\text{价格矩阵} \times \text{需求矩阵} = \text{总价矩阵}$$

	商品1	商品2	商品3	商品4	购物者1	购物者2	.....	购物者 $n$	
超市1	$\begin{bmatrix} 1.7 & 1.1 & 21 & 7 \\ 1.5 & 1.4 & 26 & 9 \\ 1.8 & 1.3 & 28 & 8 \end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$	需求1 需求2 需求3 需求4						
超市2									
超市3									

	购物者1	购物者2	.....	购物者 $n$
超市1	$\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ * & * & \cdots & * \end{bmatrix}$			
超市2				
超市3				

# 矩阵乘法(Multiplication)

## 引例2 数学例子

设  $x_1, x_2, x_3$  和  $y_1, y_2$  是两组变量, 它们之间的关系为

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{aligned} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = [a_{ik}]_{2 \times 3}$$

又设  $t_1, t_2$  是另一组变量, 它们与  $x_1, x_2, x_3$  的关系为

$$\begin{aligned} x_1 &= b_{11}t_1 + b_{12}t_2 \\ x_2 &= b_{21}t_1 + b_{22}t_2 \\ x_3 &= b_{31}t_1 + b_{32}t_2 \end{aligned} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = [b_{kj}]_{3 \times 2}$$

则

$c_{11}$

$$y_1 = (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31})t_1 + (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32})t_2$$

$$y_2 = (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31})t_1 + (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32})t_2$$

矩阵  $C = [c_{ij}]_{2 \times 2}$  是矩阵  $A$  与  $B$  的一个运算, 定义为矩阵的乘积.

# 矩阵乘法(Multiplication)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix},$$

其中

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^p a_{ik}b_{kj},$$

*Row-column rule for computing AB*

上式右边 $(i, j)$ 元素  $c_{ij}$  等于左边的第一个矩阵的  
第  $i$  行与第二个矩阵的第  $j$  列对应元素乘积之和。

矩阵运算中具有的特殊规律, 主要产生于矩阵的乘法运算。

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix},$$

Each entry of  $\mathbf{AB}$  is the product (乗積) of a *row* and a *column*:  
 $(\mathbf{AB})_{ij} = (\text{row } i \text{ of } \mathbf{A}) \text{ times } (\text{column } j \text{ of } \mathbf{B})$

Each column of  $\mathbf{AB}$  is the product of a *matrix* and a *column*:  
 $\text{column } j \text{ of } \mathbf{AB} = \mathbf{A} \text{ times } (\text{column } j \text{ of } \mathbf{B})$

Each row of  $\mathbf{AB}$  is the product of a *row* and a *matrix*:  
 $\text{row } i \text{ of } \mathbf{AB} = (\text{row } i \text{ of } \mathbf{A}) \text{ times } \mathbf{B}$

# 矩阵乘法(Multiplication)

## 1. Definition

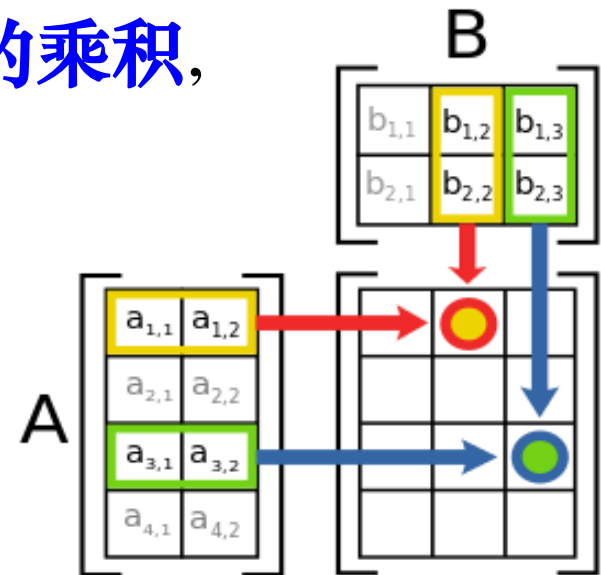
设  $A = [a_{ik}]_{m \times p}$ ,  $B = [b_{kj}]_{p \times n}$  为两个矩阵, 令

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^p a_{ik}b_{kj},$$

$$i = 1, 2, \dots, m; j = 1, 2, \dots, n,$$

称  $m \times n$  矩阵  $C = [c_{ij}]_{m \times n}$  为  **$A$  与  $B$  的乘积**,  
记为  **$C = AB$** .

**注** 只有当第一个矩阵的列数等于第二个矩阵的行数时, 两个矩阵才能相乘.





**Example 2** Find  $AB$ , where

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 3 & 0 \\ 0 & 5 & -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 2 & 1 \\ 3 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}.$$

**Solution:**

$$C = AB = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 3 & 0 \\ 0 & 5 & -1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 1 & 2 & 1 \\ 3 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 6 & 7 \\ 10 & 2 & -6 \\ -2 & 17 & 10 \end{bmatrix}.$$

$3 \times 4$        $4 \times 3$   
 $\uparrow$        $\uparrow$        $\uparrow$        $\uparrow$   
*Match*  
*Size of AB*

## Exercises

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 5 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 & 8 \\ 6 & 0 & 1 \end{bmatrix} = ?$$

**注** 只有当第一个矩阵的列数等于第二个矩阵的行数时，两个矩阵才能相乘.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = [1 \times 3 + 2 \times 2 + 3 \times 1] = 10$$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

**Example** Let  $\mathbf{A}$  and  $\mathbf{B}$  be  $n \times 1$  and  $1 \times n$  matrices, and

$$\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \mathbf{B} = [b_1 \quad b_2 \quad \cdots \quad b_n].$$


Compute  $\mathbf{AB}$  and  $\mathbf{BA}$ .

**Solution**

$$\mathbf{AB} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \cdots & a_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & a_n b_2 & \cdots & a_n b_n \end{bmatrix}.$$


$$\mathbf{BA} = [a_1 b_1 + a_2 b_2 + \cdots + a_n b_n].$$

A system of  
linear equations



$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \right.$$

The  $i$ -th equation:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$


$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$(a_{i1}, a_{i2}, \dots, a_{in}) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b_i \quad (i = 1, 2, \dots, m)$$

$$\text{Let } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$Ax = b$$

# System of Linear Equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$
  

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = b$$

*Vector Equation*

*Matrix Equation*

$$Ax = b.$$

## Coefficient Matrix


$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$= [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n]$$

## Augmented Matrix

$$(A, b) = [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n \ b]$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$


**Solution**  
**(解向量)**

# System of Linear Equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \quad \quad \cdots \quad \quad \quad \cdots \quad \quad \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$



## Coefficient Matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$= [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n]$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = b$$

*Vector Equation*



*Matrix Equation*


$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$



## Augmented Matrix

$$(\mathbf{A}, \mathbf{b}) = [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n \ \mathbf{b}]$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$


**Solution**  
**(解向量)**

## 2. Rules for Matrix Multiplication

(1) Let  $A = [a_{ij}]_{m \times n}$ ,  $B = [b_{ij}]_{n \times p}$ ,  $C = [c_{ij}]_{p \times r}$ , then

$$(AB)C = A(BC), \quad k(AB) = (kA)B = A(kB).$$

associative law of multiplication

(2) Let  $A = [a_{ij}]_{m \times p}$ ,  $B = [b_{ij}]_{p \times n}$ ,  $C = [c_{ij}]_{p \times n}$ ,  $D = [d_{ij}]_{n \times s}$ , then

$$A(B + C) = AB + AC, \quad (B + C)D = BD + CD.$$

left distributive law

right distributive law

(3) Let  $A = [a_{ij}]_{m \times n}$ ,  $I_m$ ,  $I_n$  are identity matrices of degree  $m$  and  $n$  respectively, then

$$A = I_m A = A I_n; \quad kA = (kI_m)A = A(kI_n).$$

identity for matrix multiplication

**证明:  $(AB)C=A(BC)$ .**

设  $A=(a_{ij})_{m \times n}$ ,  $B=(b_{ij})_{n \times p}$ ,  $C=(c_{ij})_{p \times r}$ , 则  $(AB)C$  与  $A(BC)$  都是  $m \times r$  矩阵.

只需证明:  $\forall i=1, \dots, m, \forall j=1, \dots, r$ , 有

$$\begin{aligned}
 [(AB)C]_{ij} &= \sum_{k=1}^p (AB)_{ik} C_{kj} = \sum_{k=1}^p \left( \sum_{l=1}^n a_{il} b_{lk} \right) c_{kj} \\
 &= \sum_{l=1}^n a_{il} \left( \sum_{k=1}^p b_{lk} c_{kj} \right) \quad \text{交换和号顺序} \\
 &= \sum_{l=1}^n a_{il} (BC)_{lj} = [A(BC)]_{ij}
 \end{aligned}$$

所以  $(AB)C=A(BC)$ .



**问题：** 矩阵乘法是否满足**交换律(commutative law)**,  
即  $AB = BA$  ?

$$\begin{bmatrix} 1 & 6 & 8 \\ 6 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 5 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

不一定同  
时有意义

不一定  
同型

不  
相等

例外：  
可交换

The product of two matrices is not commutative:

$AB$  is **not necessarily** equal to  $BA$ .

**For example,** if  $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ , then

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix},$$

therefore  $AB \neq BA$ .

## Warnings:

1. In general,  $AB \neq BA$ .
2. If a product  $AB$  is the zero matrix, you *cannot* conclude in general that either  $A = \mathbf{0}$  or  $B = \mathbf{0}$ .
3. The cancellation laws do *not* hold for matrix multiplication. That is, if  $AB = AC$ , then it is *not* true in general that  $B = C$ .

**投票：** 两个上(下)三角阵 $A$ 与 $B$ 的乘积 $AB$ 是否仍是上(下)三角阵？

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} \quad C = AB = (c_{ij})_{n \times n}$$

其主对角元 $(AB)_{ii} = ?$

**证明：** 两个上(下)三角阵 $A$ 与 $B$ 的乘积 $AB$ 仍是上(下)三角阵, 且其主对角元 $(AB)_{ii}=a_{ii}b_{ii}$ .

证 设  $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} \quad C=AB=(c_{ij})_{n \times n}$

$i > j$  时,  $c_{ij} = \sum_{k=1}^n a_{ik}b_{kj} = \sum_{k=1}^{i-1} a_{ik}b_{kj} + \sum_{k=i}^n a_{ik}b_{kj}$

$$a_{ik} = 0 \quad (k = 1, 2, \dots, i-1)$$

$$b_{kj} = 0 \quad (k = i, i+1, \dots, n)$$

即  $i > j$  时,  $c_{ij} = 0$   $C$  为上三角阵

而  $c_{ii} = \sum_{k=1}^n a_{ik}b_{ki} = \sum_{k=1}^{i-1} a_{ik}b_{ki} + \sum_{k=i}^n a_{ik}b_{ki}$   
 $= 0 + a_{ii}b_{ii} = a_{ii}b_{ii}$ .

$$a_{ik} = 0 \quad (k = 1, 2, \dots, i-1)$$

$$b_{ki} = 0 \quad (k = i+1, \dots, n)$$

## IV. 方阵的幂(Power)

### 1. 定义

设  $A$  是  $n$  阶方阵, 定义  $A$  的幂为

$$A^0 = I, \quad A^1 = A, \quad A^2 = A^1 A^1, \quad \dots, \quad A^{k+1} = A^k A^1.$$

**注** 只有方阵, 它的幂才有意义.

### 2. 矩阵的幂的运算规律

设  $k, l$  为非负整数, 则

$$(1) \quad A^{k+l} = A^k A^l;$$

$$(2) \quad (A^k)^l = A^{kl}.$$

由矩阵乘法不满足交换律, 一般地  $(AB)^k \neq A^k B^k$ .

但也有例外, 如设  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ,  $B = \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix}$ , 则有

$$AB = \begin{bmatrix} 5 & 6 \\ 9 & 14 \end{bmatrix}, BA = \begin{bmatrix} 5 & 6 \\ 9 & 14 \end{bmatrix}, \Rightarrow AB = BA.$$

**注** 当  $A$  与  $B$  可交换时, 有

$$(AB)^k = A^k B^k,$$

$$(A + B)^2 = A^2 + 2AB + B^2,$$

$$(A - B)(A + B) = A^2 - B^2.$$

设

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

为一元多项式, 其中  $a_0, a_1, \dots, a_n$  为多项式的系数.

设  $A$  为方阵, 则称

$$p(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I$$

为**方阵  $A$  的多项式**.

**(3)** 设  $f(x), g(x), h(x)$  为一元多项式,  $A$  是方阵.

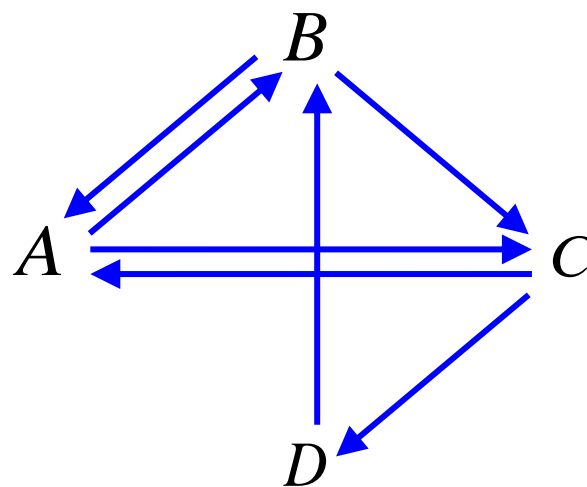
如果  $f(x) = g(x) h(x)$ , 则  $f(A) = g(A) h(A)$ .

## 城市间航班图



航班图可用矩阵来表示:

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$



## 练习

1: 计算  $M^2$  ?

2: 思考  $M^2$  中元素的实际含义.

$$M^2 = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$



**Example 3** Compute  $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^n$ .

**Solution** Let

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then  $\mathbf{A}$  is a scalar matrix, and  $\mathbf{AB} = \mathbf{BA}$ , Therefore,

$$\begin{aligned} \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^n &= (\mathbf{A} + \mathbf{B})^n = \sum_{k=0}^n \mathbf{C}_n^k \mathbf{A}^{n-k} \mathbf{B}^k \\ &= \mathbf{C}_n^0 \mathbf{A}^n \mathbf{B}^0 + \mathbf{C}_n^1 \mathbf{A}^{n-1} \mathbf{B} + \mathbf{C}_n^2 \mathbf{A}^{n-2} \mathbf{B}^2 + \cdots \mathbf{C}_n^n \mathbf{B}^n. \end{aligned}$$

Since  $\mathbf{B}^2 = \mathbf{0}$ , we have  $\mathbf{B}^2 = \mathbf{B}^3 = \dots = \mathbf{B}^n = \mathbf{0}$ . And

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^n = \mathbf{A}^n + n\mathbf{A}^{n-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix} + \begin{bmatrix} n3^{n-1} & 0 \\ 0 & n3^{n-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix} + \begin{bmatrix} 0 & n3^{n-1} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3^n & n3^{n-1} \\ 0 & 3^n \end{bmatrix}.$$

## *V. Elementary Matrices*

**定义** 由单位矩阵  $I$  经过一次初等变换得到的方阵称为**初等矩阵**. (An **elementary matrix** is one that is obtained by performing a single elementary operation on an identity matrix.)

三种初等变换对应着三种初等矩阵:

- (1) 对调两行或两列——**初等对换矩阵**;
- (2) 以数  $k \neq 0$  乘某行或某列——**初等倍乘矩阵**;
- (3) 以数  $k$  乘某行(列)加到另一行(列)上去——**初等倍加矩阵**.

# (1)初等对换矩阵:

将单位矩阵的第  $i, j$  行(或列)对换

$$P_{ij} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ \leftarrow \text{第 } i \text{ 行} & & 0 & \cdots & 1 & \\ & & \vdots & & \vdots & \\ & & & 1 & & \\ \leftarrow \text{第 } j \text{ 行} & & 1 & \cdots & 0 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}$$

第  $i$  列      第  $j$  列

## (2)初等倍乘矩阵:

将单位矩阵第  $i$  行(或列)乘  $k \neq 0$

$$D_i(k) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & k & \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{第 } i \text{ 行} \\ \text{第 } i \text{ 列} \end{array}$$

### (3)初等倍加矩阵:

将单位矩阵第  $i$  行乘  $k$  加到第  $j$  行,  
 或将第  $j$  列乘  $k$  加到第  $i$  列

$$E_{ij}(k) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & k & \cdots & 1 \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$

← 第  $i$  行

← 第  $j$  行

第  $i$  列    第  $j$  列

■ **Example 4** Let

$$\mathbf{K}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Compute  $\mathbf{K}_1\mathbf{A}$ ,  $\mathbf{K}_2\mathbf{A}$ , and  $\mathbf{K}_3\mathbf{A}$ , and describe how these products can be obtained by elementary row operations on  $\mathbf{A}$ .

**Solution:**

Addition of -4 times **row** 1 of **A** to **row** 3 produces  $\mathbf{K}_1\mathbf{A}$ .

$$\mathbf{K}_1\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}.$$

$$\mathbf{K}_2\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}.$$

An interchange of **rows** 1 and 2 of **A** produces  $\mathbf{K}_2\mathbf{A}$ .

$$\mathbf{K}_3\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

Multiplication of **row** 3 of **A** by 5 produces  $\mathbf{K}_3\mathbf{A}$ .



- **Left-multiplication** (左乘, that is, multiplication on the left) by  $K_1$  in Example 4 has the same effect on any  $3 \times n$  matrix.
- Since  $K_1 I = K_1$ , we see that  $K_1$  *itself* is produced by this same row operation on the identity. (注：由单位矩阵  $I$  经过一次初等变换得到的方阵称为**初等矩阵**)
- Example 4 illustrates the following general fact about elementary matrices.
- If an elementary row operation is performed on an  $m \times n$  matrix  $A$ , the resulting matrix can be written as  $KA$ , where the  $m \times m$  matrix  $K$  is created by performing the same row operation on  $I_m$ .

## What if -- right-multiplication?

Addition of -4 times **column 3** of **A** to **column 1** produces  $AK_1$ .

$$AK_1 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a-4c & b & c \\ d-4f & e & f \\ g-4i & h & i \end{bmatrix}.$$

$$AK_2 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b & a & c \\ e & d & f \\ h & g & i \end{bmatrix}.$$

An interchange of **columns** 1 and 2 of **A** produces  $AK_2$ .

$$AK_3 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} a & b & 5c \\ d & e & 5f \\ g & h & 5i \end{bmatrix}.$$

Multiplication of **column** 3 of **A** by 5 produces  $AK_3$ .

## 初等矩阵与矩阵的乘积

1) 用  $m$  阶初等矩阵  $P_{ij}$  **左乘** 矩阵  $A = [a_{ij}]_{m \times n}$ , 得

$$P_{ij}A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{第 } i \text{ 行} \\ \\ \leftarrow \text{第 } j \text{ 行} \\ \\ \end{matrix}$$

相当于把矩阵  $A$  第  $i$  **行** 与第  $j$  **行** 对调 ( $r_i \leftrightarrow r_j$ ).

2) 用  $m$  阶初等矩阵  $D_i(k)$  左乘矩阵  $A = [a_{ij}]_{m \times n}$ , 得

$$D_i(k)A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \leftarrow \text{第 } i \text{ 行}$$

相当于以数  $k$  乘矩阵  $A$  的第  $i$  行 ( $kr_i$ ) ( $k \neq 0$ ).

3) 用  $E_{ij}(k)$  左乘矩阵  $A = [a_{ij}]_{m \times n}$ , 得

$$E_{ij}(k)A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} + ka_{i1} & a_{j2} + ka_{i2} & \cdots & a_{jn} + ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{第 } i \text{ 行} \\ \\ \leftarrow \text{第 } j \text{ 行} \\ \\ \end{matrix}$$

相当于把  $A$  的第  $i$  行乘  $k$  加到第  $j$  行 ( $r_j + kr_i$ ).

- 4) 用  $n$  阶初等矩阵  $P_{ij}$  右乘矩阵  $A = [a_{ij}]_{m \times n}$ , 相当于把矩阵  $A$  第  $i$  列与第  $j$  列对调 ( $c_i \leftrightarrow c_j$ ).
- 5) 用  $n$  阶初等矩阵  $D_i(k)$  右乘矩阵  $A = [a_{ij}]_{m \times n}$ , 相当于以数  $k$  乘矩阵  $A$  的第  $i$  列 ( $kc_i$ ).
- 6) 用  $n$  阶初等矩阵  $E_{ij}(k)$  右乘矩阵  $A = [a_{ij}]_{m \times n}$ , 相当于将矩阵  $A$  的第  $j$  列乘数  $k$  加到第  $i$  列 ( $c_i + kc_j$ ).

三种初等矩阵左乘矩阵  $A$  是对  $A$  作相应的初等行变换  
 三种初等矩阵右乘矩阵  $B$  是对  $B$  作相应的初等列变换

**Note:**

$$\mathbf{K}_1 = \mathbf{P}_{13}, \mathbf{K}_2 = \mathbf{E}_{14}(c), \mathbf{K}_3 = \mathbf{D}_2(k)$$

**Example 5** Let

$$\mathbf{K}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ & 0 & 0 & 1 \\ c & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_3 = \begin{bmatrix} 1 & & & \\ & k & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Find  $\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3$ .**Solution** We note that  $\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3 = \mathbf{P}_{13} \mathbf{E}_{14}(c) \mathbf{D}_2(k)$ ,

so

$$\mathbf{K}_2 \mathbf{K}_3 = \mathbf{E}_{14}(c) \mathbf{D}_2(k) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ c & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & k & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & k & & \\ & & 1 & \\ c & & & 1 \end{bmatrix}$$

**Note:**

$$\mathbf{K}_1 = \mathbf{P}_{13}, \mathbf{K}_2 = \mathbf{E}_{14}(c), \mathbf{K}_3 = \mathbf{D}_2(k)$$

$$\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3 = \mathbf{P}_{13} \mathbf{K}_2 \mathbf{K}_3, \text{ and}$$

$$\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3 = \mathbf{P}_{13} \begin{bmatrix} 1 & & & \\ & k & & \\ & & 1 & \\ c & & & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & k & 0 & 0 \\ 1 & 0 & 0 & 0 \\ c & 0 & 0 & 1 \end{bmatrix}$$

**Remark** We can also use right multiplication and the corresponding elementary column operations to do the calculation. It leads to the same result.



**Example 6** (将初等矩阵概念用于消元: Elimination)

For 3 equations in 3 unknowns:

Suppose  $E$  subtracts twice the first equation from the second.

Suppose  $F$  is the matrix for the next step, *to add row 1 to row 3*.

$$E = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix}.$$

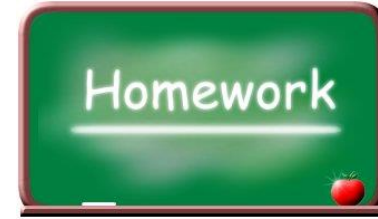
*These two matrices **do commute** and the product does both steps at once:*

$$EF = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ 1 & & 1 \end{bmatrix} = FE.$$

In either order,  $EF$  or  $FE$ , this changes rows 2 and 3 using row 1.

**What if :**  $E$  is the same but  $G$  add row 2 to row 3?  **$EG \neq GE$ .**

# Homework



- See Blackboard announcement
- ***Hardcover* textbook + Supplementary problems**
- Pay attention to the notation

In the textbook, the *elementary matrix*  $E_{ij}$  subtracts  $l$  times row  $j$  from row  $i$ .

$$E_{31} = \begin{bmatrix} 1 & & \\ & 1 & \\ -l & & 1 \end{bmatrix},$$

## Deadline (DDL):

- Next tutorial class

