Quiz 2 – Linear Algebra (Mar 29, 2022)

本人郑重承诺将秉承诚信原则，自觉遵守考场纪律，并承担违纪或作弊带来的后果。

学号：________________ 姓名：________________

题号	1	2	3	4	5	合计
分值	10 分	10 分	10 分	10 分	10 分	50 分
得分						

This 50-minute long test includes 5 questions. Write ***all your answers*** on this examination paper.

1. (1) If $\boldsymbol{A}=\begin{bmatrix}2&-1&1&1\\2&-1&1&1\\-2&1&-1&-1\\4&-2&2&2\end{bmatrix}$, then $\boldsymbol{A}^n$= $2^{n-1}A$, where n is a positive integer.

$A=\alpha\beta^T$, $\alpha=\begin{pmatrix}1\\1\\-1\\2\end{pmatrix}$, $\beta=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, 则 $A^n=(\alpha\beta^T)^n=\alpha\beta^T\alpha\beta^T\cdots\alpha\beta^T$

$=\alpha(\beta^T\alpha)^{n-1}\beta^T$

$=2^{n-1}\alpha\beta^T=2^{n-1}A$

(2) Let $\boldsymbol{A}=\begin{bmatrix}1&a&-1&2\\2&-1&a&5\\1&10&-6&1\end{bmatrix}$. If $\operatorname{rank}(\boldsymbol{A})=2$, then a= 3.

$A\to\begin{pmatrix}1&2&1\\0&1&-1\\0&0&9-3a\\0&0&a-3\end{pmatrix}$, 则 $\operatorname{rank}(A)=2$ 可得 $\begin{cases}9-3a=0\\a-3=0\end{cases}\Rightarrow a=3.$

(3) True or False? For an $m\times n$ matrix $\boldsymbol{A}$, the system $\boldsymbol{Ax}=\boldsymbol{b}$ is solvable if and only if $\operatorname{rank}(\boldsymbol{A})=n$. (F)

$\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$, $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $b=\begin{pmatrix}1\\0\end{pmatrix}$, 有解.

(4) True or False? $\operatorname{rank}\left(\begin{bmatrix}\boldsymbol{A}&\boldsymbol{C}\\\boldsymbol{0}&\boldsymbol{B}\end{bmatrix}\right)\ge\operatorname{rank}(\boldsymbol{A})+\operatorname{rank}(\boldsymbol{B})$. (T)

设 A, B 高斯消元后对应的上三角矩阵为 U_A, U_B, 将同样的消元过程作用于 $\begin{pmatrix}A&C\\0&B\end{pmatrix}$ 得 $\begin{pmatrix}U_A&U_C\\0&U_B\end{pmatrix}$, 有 $r\left(\begin{bmatrix}A&C\\0&B\end{bmatrix}\right)=r\left(\begin{bmatrix}U_A&U_C\\0&U_B\end{bmatrix}\right)\geq r\left(\begin{bmatrix}U_A&0\\0&U_B\end{bmatrix}\right)$

$=r(U_A)+r(U_B)$

$=r(A)+r(B)$.

(5) True or False? If n-dimensional vectors $\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n$ are linearly independent, then $\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2-\boldsymbol{\alpha}_3,\dots,\boldsymbol{\alpha}_{n-1}-\boldsymbol{\alpha}_n$ are also linearly independent. (T)

令 $k_1(\alpha_1-\alpha_2)+k_2(\alpha_2-\alpha_3)+\cdots+k_{n-1}(\alpha_{n-1}-\alpha_n)=0$,

则有 $k_1\alpha_1+(k_2-k_1)\alpha_2+(k_3-k_2)\alpha_3+\cdots+(k_{n-1}-k_{n-2})\alpha_{n-1}-k_{n-1}\alpha_n=0$,

由于 $\alpha_1,\alpha_2,\cdots,\alpha_n$ 线性无关, 则 $k_1=0$, $k_2-k_1=0$, $\cdots$, $k_{n-1}-k_{n-2}=0$, $k_{n-1}=0$,

则有 $k_1=k_2=\cdots=k_{n-1}=0$. 则 $\alpha_1-\alpha_2,\alpha_2-\alpha_3,\cdots,\alpha_{n-1}-\alpha_n$ 线性无关.

2. Let $\boldsymbol{\alpha}_1 = (1,2,0)^T, \boldsymbol{\alpha}_2 = (1, a+2, -3a)^T, \boldsymbol{\alpha}_3 = (-1, -a-2, 3a)^T, \boldsymbol{\beta} = (1,3,-3)^T$.

(1) If $\boldsymbol{\beta}$ <u>cannot</u> be linearly represented by $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$, what is the value of a?

(2) If $\boldsymbol{\beta}$ <u>can</u> be linearly represented by $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$, what is the value of a? And give the linear combination.

(1) 令 $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = \beta$, 则有:

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & a+2 & -a-2 & 3 \\ 0 & -3a & 3a & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & a & -a & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

若β不能被$\alpha_1,\alpha_2,\alpha_3$线性表出, 则 $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = \beta$ 无解,

则有 $a=0$.

(2). 由(1)得此时 $a \neq 0$. 则有:

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & a & -a & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1-1/a \\ 0 & 1 & -1 & 1/a \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{则有} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1-\frac{1}{a} \\ \frac{1}{a} \\ 0 \end{pmatrix} + k\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad k \in \mathbb{R},$$

则有 $\beta = (1-\frac{1}{a})\alpha_1 + (\frac{1}{a} + k)\alpha_2 + k\alpha_3, \quad k \in \mathbb{R}$.

3. Check if the following subsets are subspaces. If the subset is a subspace, find its basis and dimension. If not, explain why.

(1) All 3 by 3 skew-symmetric matrices $(\boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A})$.

(2) $\{(x,y,z,w) \in \mathbf{R}^4 | x + 2y - 3z - w = 1\}$.

(1) 是子空间.

基底: $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. 维数: 3.

(2). 不是. $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ 不属于该集合.

4. Let $\boldsymbol{A}=\begin{bmatrix}1 & 3 & 1 & 1\\ 0 & 1 & 1 & 0\\ 1 & 2 & 0 & 1\end{bmatrix}, \boldsymbol{b}=\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$

(1) Find the dimension and a basis for the four fundamental subspaces of $\boldsymbol{A}$.

(2) Under what condition on $\boldsymbol{b}$ is the system $\boldsymbol{Ax}=\boldsymbol{b}$ solvable? Find all the solutions when $\boldsymbol{b}=[2 \quad 1 \quad 1]^T$.

(1) $A \longrightarrow \begin{pmatrix}1 & 0 & -2 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$, 则有:

$C(A)$ 基底为 $\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}3\\1\\2\end{pmatrix}$; $C(A^T)$ 基底为: $\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$.

$N(A)$ 基底为 $\begin{pmatrix}2\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$.

$A^T \longrightarrow \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, 则有 $N(A^T)$ 基底为: $\begin{pmatrix}-1\\1\\1\end{pmatrix}$.

$\dim C(A)=2$, $\dim C(A^T)=2$, $\dim N(A)=2$, $\dim N(A^T)=1$.

(2). $(A \quad b) \longrightarrow \begin{pmatrix}1 & 0 & -2 & 1 & b_1-3b_2\\ 0 & 1 & 1 & 0 & b_2\\ 0 & 0 & 0 & 0 & b_3+b_2-b_1\end{pmatrix}$,

则 $b_3+b_2=b_1$ 时, 方程组有解.

当 $b=\begin{pmatrix}2\\1\\1\end{pmatrix}$ 时, $(A \quad b) \longrightarrow \begin{pmatrix}1 & 0 & -2 & 1 & -1\\ 0 & 1 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$,

则 $b=\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}+k_1\begin{pmatrix}2\\-1\\1\\0\end{pmatrix}+k_2\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$, $k_1, k_2 \in \mathbb{R}$.

5. Suppose that $\boldsymbol{A}$ is a matrix with full column rank, and $\boldsymbol{AB}=\boldsymbol{C}$. Show that $\boldsymbol{Bx}=\mathbf{0}$ has the same solution set as $\boldsymbol{Cx}=\mathbf{0}$.

1° 当 $Bx=0$ 时，有 $ABx = A0 = 0$，

则有 $Cx=0$.

2° 当 $Cx=0$ 时，有 $ABx=0$，

由于 A 列满秩，则 $\dim N(A)=0$，

则 $N(A)=\{0\}$，

由 $ABx=0$，则 $Bx \in N(A)$，

则 $Bx=0$.

由1°，2°得 $Bx=0$ 和 $Cx=0$ 解集相同.