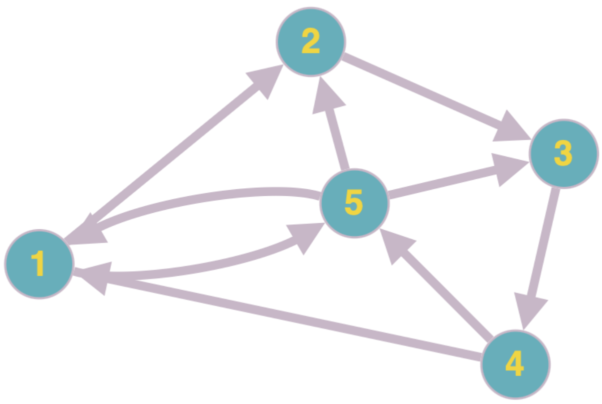
1. Determine if the following matching is stable, and explain why or why not.

$$
\begin{array}{ll}
d_a : h_x \underline{h_w} & h_w(2) : \underline{d_a} d_c d_b d_e \underline{d_f} \\
d_b : h_w \underline{h_x} & h_x(2) : \underline{d_b} d_f d_a d_d \underline{d_e} \\
d_c : h_w \underline{h_y} & h_y(2) : d_d \underline{d_c} \\
d_d : h_x h_y & \\
d_e : \underline{h_x} h_w & \\
d_f : \underline{h_w} h_x &
\end{array}
$$

2. Determine if the following matching is stable, and explain why or why not.

$$
\begin{array}{ll}
d_a : \underline{h_x} h_w & h_w(2) : d_a \underline{d_c} \underline{d_b} d_e d_f \\
d_b : \underline{h_w} h_x & h_x(2) : d_b \underline{d_f} \underline{d_a} d_d d_e \\
d_c : \underline{h_w} h_y & h_y(2) : \underline{d_d} d_c \\
d_d : h_x \underline{h_y} & \\
d_e : h_x h_w & \\
d_f : h_w \underline{h_x} &
\end{array}
$$

3. Write down Cycle Formulation for Kidney Exchange Problem with at most length-3 cycles given the graph below:



4. Edge Formulation seems to be worse than Cycle Formulation in practice; however, there is a variant of Kidney Exchange Problem which is intractable to model with Cycle Formulation and trivial to model with Edge Formulation. What is the variant we are talking about?

5. One of the weaknesses of the Edge Formulation is the number of path constraints. If we have a graph with n vertices and we are looking for cycles up to length $K, K \ll n$, there can be at most $n(n-1)\dots(n-k)$ paths of length K (paths that have K edges and $K+1$ vertices). It has been shown that we do not really need paths of length K, and it is enough to enumerate all the paths of length $K-1$. Can you modify the path constraints of the Edge Formulation to only enumerate paths of length $K-1$?

6. In the lecture this morning, we have seen that the LP relaxation of Cycle Formulation is at least as strong as the LP relaxation of Edge formulation. Describe how to construct an instance of Kidney Exchange Problem with cycles of length K such that when using the Edge Formulation, the LP relaxation is c units worse than LP relaxation of Cycle Formulation, for any c where $c \gg K$.

7. Given the following variables, complete the partial integer linear programme by adding any required constraint(s) for **super stability**.

   Definition: A matching is super stable if there is no pair d, h such that

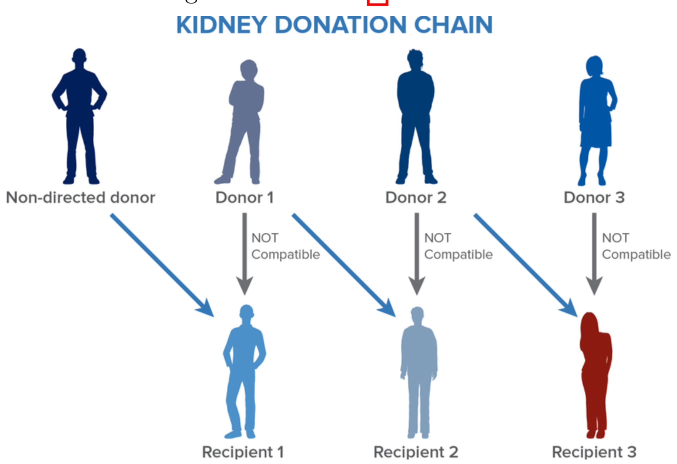
   (a) h is either undersubscribed, or prefers d to one of their assigned doctors, or is indifferent between d and at least one of their assigned doctors, and

   (b) d is either unmatched, or prefers h to their assigned hospital, or is indifferent between h and their assigned hospital.

n_d	Number of doctors
n_h	Number of hospitals
$D(j)$	The set of doctors that hospital j finds acceptable
$H(i)$	The set of hospitals that doctor i finds acceptable
$D_i^{<}(j)$	The set of doctors that hospital j thinks are strictly better than doctor i
$D_i^{=}(j)$	The set of doctors that hospital j thinks are equally as good as doctor i
$D_i^{\leq}(j)$	The set of doctors that hospital j thinks are at least as good as doctor i
$H_j^{<}(i)$	The set of hospitals that doctor i thinks are strictly better than hospital j
$H_j^{=}(i)$	The set of hospitals that doctor i thinks are equally as good as hospital j
$H_j^{\leq}(i)$	The set of hospitals that doctor i thinks are at least as good as hospital j

$$\begin{aligned}
\max \sum_{i=1}^{n_d} \sum_{j \in H(i)} & x_{ij} \\
\text{s.t.} \sum_{j \in H(i)} x_{ij} &\leq 1, && i \in \{1, \ldots, n_d\}, \\
\sum_{i \in D(j)} x_{ij} &\leq c_j, && j \in \{1, \ldots, n_h\}, \\
x_{ij} &\in \{0, 1\} && i \in \{1, \ldots, n_d\}, j \in H(i)
\end{aligned}$$

8. A common extension to Kidney Exchange Problem is called **Kidney Exchange Problem with Non-directed Donors**. In this variant of the problem, the set of vertices in the compatibility graph V is made up of a set of recipient-donor pairs R and a set of non-directed donors N, so $V = R \cup N$. Non-directed donors, also known as altruistic donors, are donors who do not have any specific recipient attached to them, they just want to donate their kidney to the pool. Since they have no corresponding recipient, they form chains (paths) instead of cycles. A chain of length l has l arcs. [1]



   Furthermore, in addition to the bound of length for cycles K, there is a bound for length of chains L.

[1] Image taken from https://www.pennmedicine.org/physicians-hub/clinical-briefing/end-chain-kidney-paired-donation

(a) Can the Cycle Formulation model chains? If so, how?

(b) Can the Edge formulation model chains if $L \leq K$?

(c) What happens with the Edge Formulation if $L > K$?