Chapter 4

Ordinary Differential Equations

4.1 Introduction

A differential equation is an equation involving derivatives of one or more functions. If there are some partial derivatives in the equation we call the equation a **partial differential equation** (PDE), and if there are no partial derivatives we say it is an **ordinary differential equation** (ODE). The **order** of a differential equation is the largest integer n for which the equation involves the n-th derivative of one of the functions we are solving for.

Example 4.1. Determine if each of the following is an ODE or a PDE. Find the order of each equation.

(a) $\left(\dfrac{dx}{dt}\right)^2 + x\sin t = \cos x.$

(b) $\dfrac{\partial y}{\partial t}\dfrac{\partial y}{\partial s} + y\dfrac{\partial z}{\partial t} = \sin(st).$

(c) $y'' + ty' + y = \cos t.$

In this class we will only focus on ordinary differential equations. Note that the domain and range of all solutions to differential equations are assumed to be subsets $\mathbb{R}$.

The main questions that we are trying to answer are the following:

- What is the general solution of an ODE?
- Can we find solutions satisfying certain initial values?
- How many solutions are there satisfying given initial values?
- If finding an explicit formula for a solution is not possible, can we approximate the solution?
- What are all solutions that are constant?
- Are all solutions bounded? Are there any bounded solutions?

- Are all solutions periodic? Are there any periodic solutions?
- What is the long term behavior of solutions?
- How do solutions change when we change their initial values?

4.2 Explicit First Order Equations

An equation of the form $\frac{dy}{dt} = f(t)$ is called a (first order) **explicit differential equation.**

Example 4.2. Find all solutions of the differential equation $\frac{dy}{dt} = \frac{1}{t^2 - t}$.

Theorem 4.1 (Existence and Uniqueness Theorem for Explicit Equations)**.** *Suppose $f(t)$ is continuous over an open interval (a, b). Then, for every $t_0 \in (a, b)$ and every real number y_0, there is a unique solution to the initial value problem*

$$\frac{dy}{dt} = f(t), \;\; y(t_0) = y_0.$$

This solution is given by

$$y(t) = y_0 + \int_{t_0}^{t} f(s) \; \mathrm{d}s.$$

4.3 First Order Linear Equations

An n-th order linear differential equation in **normal form** (i.e. with the leading coefficient of 1) is an equation of the form:

$$y^{(n)} + a_n(t)y^{(n-1)} + \cdots + a_2(t)y' + a_1(t)y = f(t).$$

$f(t)$ is called **forcing** and $a_i(t)$'s are called **coefficients**. This equation is often written as $L[y] = f(t)$, where L is the **differential operator** given by

$$L = D^n + a_n(t)D^{n-1} + \cdots + a_2(t)D + a_1(t).$$

Here, we write D instead of $\frac{\mathrm{d}}{\mathrm{d}t}$.

We would also like to explore **initial value problems** (i.e. equations along with initial values in a specific format) or IVP's of the form:

$$\begin{cases} y^{(n)} + a_n(t)y^{(n-1)} + \cdots + a_2(t)y' + a_1(t)y = f(t). \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \\ \vdots \\ y^{(n-1)}(t_0) = y_{n-1} \end{cases}$$

Example 4.3. Solve the equation: $y' + y = e^t$.

To solve an equation of the form $\frac{dy}{dt}+a(t)y = f(t)$, we find a function $A(t)$ for which $A'(t) = a(t)$. Multiplying both sides by $e^{A(t)}$ we can rewrite the equation as

$$\frac{d}{dt}(e^{A(t)}y) = e^{A(t)}f(t).$$

Theorem 4.2 (Existence and Uniqueness Theorem for First Order Linear Equations)**.** *Suppose $a(t)$ and $f(t)$ are continuous over an open interval (a, b). Let $t_0 \in (a, b)$ and y_0 be a real number. Then, the initial value problem given below has a unique solution defined over (a, b).*

$$\frac{dy}{dt} + a(t)y = f(t), \quad y(t_0) = y_0.$$

4.4 Separable Equations

A first order equation is called **separable** if it can be written in the form

$$\frac{dy}{dt} = f(t)g(y).$$

The name "separable" refers to the fact that we can separate the variables and write the differential equation in the form

$$\frac{dy}{g(y)} = f(t)\ dt.$$

Solutions can then be obtained by simply integrating both sides.

Example 4.4. Solve the equation $\frac{dy}{dt} = 2ty^2 + 3t^2y^2$. Can you find a solution that satisfies $y(1) = 0$?

Definition 4.1. A solution to a differential equation is called **stationary** or **equilibrium** or a **fixed point** or a **critical point** if it is constant.

All stationary solutions of the separable equation $\frac{dy}{dt} = f(t)g(y)$ are found by solving $g(y) = 0$ for y.

Example 4.5. Find all solutions of $\frac{dy}{dt} = ty^2 - ty,\ y(1) = 2$.

4.5 Change of Variables

Example 4.6. Solve the equation $y' = \frac{e^{y+t} - y - t}{y + t}$.

An equation of the form $\frac{dy}{dt} = f(at + by + c)$ can be transformed into a separable equation by substituting $u = at + by + c$.

Example 4.7. Solve the equation $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y - t}{y + t}$.

To solve an equation of the form $y' = f(y/t)$ we use the substitution $u = y/t$. This yields $y = ut$, which implies

$$y' = u't + u \Rightarrow u't + u = f(u) \Rightarrow u' = \frac{f(u) - u}{t}.$$

This equation is separable that can be solved using the method discussed earlier.

One common example is equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ay+bt}{cy+dt} \qquad (*)$$

for constants a, b, c, d. For these we can use the substitution $u = y/t$.

Example 4.8. Solve the equation $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{y-t+1}{y+t-3}$.

To solve equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ay+bt+m}{cy+dt+n}$$

for constants a, b, c, d, m, n first choose $T = t + r$, $Y = y + s$ for constants r, s. Find r, s in such a way that the equation turns into one of the form $(*)$. Then, use the substitution $u = Y/T$.

4.6 Exact Equations and Integrating Factors

Suppose the solution to a differential equation is given by an implicit equation $\phi(t, y) =$ constant. This is equivalent to $\dfrac{d\phi(t,y)}{dt} = 0$. Using the chain rule we obtain $\phi_t + \phi_y \dfrac{dy}{dt} = 0$.

Definition 4.2. An equation of the form $M(t, y) + N(t, y)y' = 0$ is called **exact** over an open rectangle $R = (a, b) \times (c, d)$ in the ty-plane, if there is a function $\phi(t, y)$ for which $\phi_t = M$ and $\phi_y = N$ over R.

All solutions of an exact equation are of the form $\phi(t, y) = c$. The name *exact* refers to the fact that the left hand side is exactly the derivative of one function.

Example 4.9. Solve the equation $e^x y + 2x + (2y + e^x)\dfrac{dy}{dx} = 0$.

Remark. Sometimes equations of the form $M(t, y)+N(t, y)\dfrac{\mathrm{d}y}{\mathrm{d}t} = 0$ are written as $M(t, y)\ \mathrm{d}t+N(t, y)\ \mathrm{d}y = 0$.

Theorem 4.3. *Suppose $M(t, y)$ and $N(t, y)$ are continuous over the rectangle $R = (a, b) \times (c, d)$ in the ty-plane. If $\phi(t, y)$ is a function satisfying $\phi_t = M$ and $\phi_y = N$ over R, then the general solution to the differential equation $M(t, y) + N(t, y)y' = 0$ over R is given by $\phi(t, y) = C$, where C is a constant.*

Question. How do we know which equations are exact?

From multivariable calculus we know $\phi_{ty} = \phi_{yt}$, assuming second partials of ϕ are continuous. Therefore, in order for an equation $M + N\dfrac{dy}{dt} = 0$ to be exact we need to make sure $M_y = N_t$. The following theorem shows that under certain conditions, the converse is also true.

Theorem 4.4. *Let $M(t, y), N(t, y)$ be continuous and have continuous first partials over an open rectangle $R = (a, b) \times (c, d)$. Then, there is a function $\phi(t, y)$ defined over R for which $\phi_t = M$ and $\phi_y = N$ if and only if $M_y = N_t$.*

Example 4.10. Solve $(xy^2 + y + e^x) + (x^2y + x)\dfrac{dy}{dx} = 0$.

Example 4.11. Solve the initial value problem $3t^2y + 8ty^2 + (t^3 + 8t^2y + 12y^2)\dfrac{dy}{dt} = 0,\ y(2) = 1$.

Example 4.12. Solve the equation $2ty + (2t^2 - e^y)\dfrac{dy}{dt} = 0$.

When the equation is not exact one possible remedy is to multiply both sides by a factor $\mu(t, y)$ in such a way that the equation becomes exact. Such a factor μ is called an **integrating factor**.

Example 4.13. Solve the equation: $4xy + 3y^3 + (x^2 + 3xy^2)\dfrac{dy}{dx} = 0$.

Example 4.14. Find all functions $M(t, y)$ with continuous first partials for which t is an integrating factor of the equation

$$M(t, y) + t\frac{dy}{dt} = 0.$$

4.7 More Examples

Example 4.15. Consider the differential equation $y' = f(t)$, where $f(t) = \begin{cases} 2t-1 & \text{if } t > 0 \\ 1 & \text{if } t < 0 \end{cases}$ Find all continuous solutions y to this differential equation.

Solution. Integrating we obtain $y = t^2 - t + C_1$ for $t > 0$, and $y = t + C_2$ for $t < 0$. Since this function is continuous, we must have

$$\lim_{t\to 0^+} t^2 - t + C_1 = \lim_{t\to 0^-} t + C_2 = y(0).$$

Therefore, $C_1 = C_2 = y(0)$. This means all solutions are of the following form:

$$f(t) = \begin{cases} t^2 - t + C & \text{if } t > 0 \\ t + C & \text{if } t \le 0 \end{cases}$$

where C is a constant. □

Example 4.16. Consider the linear transformation $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ given by $T(f)(t) = f'(t)$. Find all eigenpairs of this transformation.

Solution. $(\lambda, f(t))$ is an eigenpair iff $T(f)(t) = \lambda f(t)$, which means $f'(t) = \lambda f(t)$. Note that since f and f' are real valued functions and $f(t) \neq 0$, the scalar λ must be real. The equation $f'(t) - \lambda f(t) = 0$ is a first order differential equation with integrating factor $e^{-\lambda t}$. This yields $e^{-\lambda t} f(t) = c$ is a constant. Therefore, $f(t) = ce^{-\lambda t}$ yields all eigenvector of T, where $c \neq 0$ is a constant. This means all eigenpairs of T are of the form

$$(\lambda, ce^{\lambda t}),$$

where $c, \lambda \in \mathbb{R}$ are constants and $c \neq 0$. □

Example 4.17. Solve each of the following differential equations.

(a) $y' + 2ty = 0$.

(b) $ty' + y = \sin t$.

(c) $\dfrac{y'}{\cos t} + y = 1$.

(d) $y'' + y' = 0$

Solution. (a) Integrating $2t$ we obtain t^2. Thus, one integrating factor is e^{t^2}. This yields

$$\frac{d}{dt}\left(e^{t^2} y\right) = 0 \Rightarrow e^{t^2} y = C \Rightarrow y = Ce^{-t^2}.$$

(b) The left hand side is already the derivative of ty, so we can rewrite the equation as

$$\frac{d(yt)}{dt} = \sin t \Rightarrow yt = -\cos t + C \Rightarrow y = -\frac{\cos t - C}{t}.$$

(c) Multiplying by $\cos t$ we obtain $y' + y\cos t = \cos t$. An integrating factor is $e^{\sin t}$. This yields the equation

$$\frac{d}{dt}\left(e^{\sin t} y\right) = e^{\sin t}\cos t \Rightarrow e^{\sin t} y = e^{\sin t} + C \Rightarrow y = 1 + Ce^{-\sin t}.$$

(d) This is not a first order equation, but if you think of $z = y'$ as a new function, then it becomes a first order linear equation. An integrating factor is e^t. This yields

$$\frac{d}{dt}\left(e^t y'\right) = 0 \Rightarrow e^t y' = C \Rightarrow y' = Ce^{-t} \Rightarrow y = -Ce^{-t} + D,$$

where C, D are two constants. □

Example 4.18. Discuss the long term behavior of solutions, i.e. the limit of each solution as $t \to \infty$.

(a) $y' + \alpha y = \alpha$, where α is a constant.

(b) $y' + 2ty = 1$.

Solution. (a) The integrating factor is $e^{\alpha t}$. Therefore, we can rewrite the equation as

$$\frac{d}{dt}\left(e^{\alpha t} y\right) = \alpha e^{\alpha t} \Rightarrow e^{\alpha t} y = e^{\alpha t} + C \Rightarrow y = 1 + Ce^{-\alpha t},$$

where $C = y(0) - 1$.

When $\alpha = 0$, we have $y = 1 + C$ is a constant.
When $\alpha > 0$, we see that $e^{-\alpha t} \to 0$ as $t \to \infty$. Therefore, $y \to 1$.
When $\alpha < 0$, $e^{-\alpha t} \to \infty$ as $t \to \infty$. Therefore, depending on if $C = 0$, $C < 0$ or $C > 0$, the solution stays at 1, tends to $-\infty$, or tends to ∞ as $t \to \infty$.

(b) The integrating factor is e^{t^2}. The equation then becomes

$$\frac{d}{dt}\left(e^{t^2}y\right) = e^{t^2} \Rightarrow e^{t^2}y = C + \int_0^t e^{s^2}\ \mathrm{d}s \Rightarrow y = Ce^{-t^2} + \int_0^t e^{s^2-t^2}\ \mathrm{d}s.$$

As $t \to \infty$, we have $t^2 \to -\infty$ and thus $e^{-t^2} \to 0$. The integral above cannot be evaluated, but we can estimate this integral. Note that since we are looking for the limit of y as $t \to \infty$ we may assume $0 \leq s \leq t$. Note also that when $x \geq 0$, the Taylor series for e^x yields

$$e^x = 1 + x + \cdots \geq 1 + x \Rightarrow e^{-x} \leq \frac{1}{1+x} \Rightarrow e^{s^2-t^2} \leq \frac{1}{1+t^2-s^2} \Rightarrow \int_0^t e^{s^2-t^2}\ \mathrm{d}s \leq \int_0^t \frac{1}{1+t^2-s^2}\ \mathrm{d}s.$$

The integral can now be evaluated using the method of partial fractions. For simplicity set $a = \sqrt{1+t^2}$. Note that $a > t \geq s \geq 0$.

$$\int_0^t \frac{1}{a^2-s^2}\ ds = \frac{1}{2a}\int_0^t \frac{1}{a+s} + \frac{1}{a-s}\ ds = \frac{1}{2a}\ln\left(\frac{a+s}{a-s}\right)\Big]_{s=0}^{s=t} = \frac{1}{2a}\left(\ln\left(\frac{a+t}{a-t}\right) - \ln 1\right) = \frac{1}{2a}\ln\left(\frac{a+t}{a-t}\right).$$

Substituting $a = \sqrt{1+t^2}$ we see the following:

$$\frac{a+t}{a-t} = \frac{(a+t)^2}{a^2-t^2} = (a+t)^2 \Rightarrow \ln\left(\frac{a+t}{a-t}\right) = 2\ln(a+t) \leq 2\ln(2a).$$

Therefore,

$$\int_0^t \frac{1}{a^2-s^2}\ ds \leq \frac{2\ln(2\sqrt{1+t^2})}{2\sqrt{1+t^2}} = \frac{\ln(2\sqrt{1+t^2})}{\sqrt{1+t^2}}.$$

As $t \to \infty$, so does $\sqrt{1+t^2}$. Since $\ln x$ grows slower than x, we have $\dfrac{\ln(2\sqrt{1+t^2})}{\sqrt{1+t^2}} \to 0$ as $t \to \infty$. Therefore, by the squeeze theorem applies to

$$0 \leq \int_0^t e^{s^2-t^2}\ ds \leq \frac{\ln(2\sqrt{1+t^2})}{\sqrt{1+t^2}},$$

we conclude that $y \to 0$ as $t \to \infty$. $\square$

Example 4.19. Consider the differential equation $y' + \alpha y = t$. For which constants α does this equation have at least one periodic solution?

Solution. Suppose y is a periodic solution. This means there is a positive constant p for which $y(t+p) = y(t)$ for all $t \in \mathbb{R}$. This means $y'(t+p) = y'(t)$. Therefore,

$$y'(t+p) + \alpha y(t+p) = y'(t) + \alpha y(t).$$

Since $y(t)$ is a solution to the given differential equation, the left hand side is $t+p$, while the right hand side is t. Therefore, $p = 0$, which implies no such constant α exists. $\square$

Example 4.20. Prove that the equation $y' + y = 2\sin t$ has a unique periodic solution.

Solution. The integrating factor is e^t. This yields the equation

$$\frac{d}{\mathrm{d}t}\left(e^t y\right) = 2e^t \sin t \Rightarrow e^t y = e^t(\sin t - \cos t) + C \Rightarrow y = \sin t - \cos t + Ce^{-t}.$$

Note that by the Extreme Value Theorem, a periodic continuous function must also be bounded. As $t \to -\infty$, e^{-t} approaches infinity. Therefore, if C is nonzero, the function would be unbounded and thus not periodic. Therefore, the only solution where y is peridic is $y = \sin t - \cos t$, which is clearly periodic with period 2π. □

Example 4.21. Solve each differential equation.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{te^y}{1+t^2}$.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \cos(y+2t) - 2$.

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{ty+t}{y^2+ty^2}$.

(d) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = ty^2 - t + 2y^2 - 2$.

(e) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{y+2t}{2y+t}$.

(f) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{y+2t+1}{2y+t-1}$.

Solution. (a) This is a separable equation. First, note that e^y cannot be zero and thus, there are no stationary solutions.

Rearranging we have

$$\int e^{-y}\ \mathrm{d}y = \int \frac{t}{1+t^2}\ \mathrm{d}t \Rightarrow -e^{-y} = \frac{1}{2}\ln(1+t^2) + C \Rightarrow -2e^{-y} = \ln(1+t^2) + C.$$

(b) We will use the change of variable $u = y + 2t$. This yields

$$u' = y' + 2 = \cos u \Rightarrow \sec u\ \mathrm{d}u = \ \mathrm{d}t \Rightarrow \ln|\sec u + \tan u| = t + C \Rightarrow |\sec u + \tan u| = e^C e^t.$$

Since e^C can be any positive constant we can drop the absolute value and write $\sec(y+2t)+\tan(y+2t) = Ce^t$ with $C \neq 0$.

The stationary solutions of the equation $u' = \cos u$ are those satisfying $\cos u = 0$ or $u = \pi k + \dfrac{\pi}{2}$. Therefore, the solutions are

$$\sec(y+2t) + \tan(y+2t) = Ce^t \text{ with } C \neq 0, \text{ and } y + 2t = \pi k + \frac{\pi}{2}, \text{ with } k \in \mathbb{Z}.$$

(c) The equation is separable and can be written as $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{t}{1+t}\dfrac{y+1}{y^2}$. This means the only stationary solution is $y = -1$.

Rearranging and using long division we obtain:

$$\frac{y^2 \, \mathrm{d}y}{y+1} = \frac{t}{1+t} \, \mathrm{d}t \Rightarrow \int \left(y - 1 + \frac{1}{y+1} \right) \, \mathrm{d}y = \int \left(1 - \frac{1}{1+t} \right) \, \mathrm{d}t \Rightarrow \frac{y^2}{2} - y + \ln|y+1| = t - \ln|1+t| + C.$$

Therefore, the solutions are

$$y = -1, \text{ and } y + t + C = \frac{y^2}{2} + \ln\left|\frac{y+1}{1+t}\right|.$$

(d) The equation can be written as $\frac{\mathrm{d}y}{\mathrm{d}t} = (t+2)(y^2-1)$. This means it is a separable equation. The stationary solutions must satisfy $y^2 = 1$. Thus, there are two stationary solutions $y = \pm 1$.

Separating the variables we find the rest of the solutions:

$$\frac{\mathrm{d}y}{y^2-1} = (t+2) \, \mathrm{d}t \Rightarrow \frac{1}{2} \int \left(\frac{1}{y-1} - \frac{1}{y+1} \right) \, \mathrm{d}y = \frac{t^2}{2} + 2t \Rightarrow \frac{1}{2} \ln\left|\frac{y-1}{y+1}\right| = \frac{t^2}{2} + 2t + C.$$

Here, we used partial fractions to integrate $\frac{1}{y^2-1}$.

(e) This is a function of y/t since

$$\frac{y+2t}{2y+t} = \frac{y/t+2}{2y/t+1}.$$

Setting $u = y/t$ we obtain $ut = y$. Differentiating we have

$$u + u't = y' = \frac{u+2}{2u+1} \Rightarrow u' = \frac{2-2u^2}{t(2u+1)}.$$

This is a separable equation. Its stationary solutions are obtained by solving $2 - 2u^2 = 0$, which yields $u = \pm 1$. The nonstationary solutions are obtained as follows:

$$\frac{(2u+1) \, \mathrm{d}u}{1-u^2} = \frac{2 \, \mathrm{d}t}{t} \Rightarrow -\frac{3}{2} \ln|1-u| - \frac{1}{2} \ln|1+u| = 2\ln|t| + C.$$

The integration on the left is obtained using partial fractions.

(f) Setting $Y = y + r$, $T = t + s$ we have $\frac{\mathrm{d}Y}{\mathrm{d}T} = \frac{\mathrm{d}y}{\mathrm{d}t}$, $y = Y - r$, $t = T - s$. Substituting we obtain the following:

$$\frac{\mathrm{d}Y}{\mathrm{d}T} = \frac{Y - r + 2T - 2s + 1}{2Y - 2r + T - s - 1}.$$

In order to homogenize this we need $r + 2s = 1$ and $2r + s = -1$. This yields $r = -1, s = 1$. This yields the equation

$$\frac{\mathrm{d}Y}{\mathrm{d}T} = \frac{Y + 2T}{2Y + T}.$$

By the previous part its solutions are

$$-\frac{3}{2} \ln|1 - Y/T| - \frac{1}{2} \ln|1 + Y/T| = 2\ln|T| + C, \text{ and } Y/T = \pm 1.$$

Substituting back Y, T in terms of y, t we obtain the solutions. □

Example 4.22. Solve each of the following equations:

(a) $(t^2 + y^2 + 2t)\ \mathrm{d}t + 2ty\ \mathrm{d}y = 0$.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -\dfrac{y\sin(ty)}{t\sin(ty) + y}$.

(c) $\sin^2 y \cos y\ \mathrm{d}y + \tan^2 x\ \mathrm{d}x = 0$.

Solution. (a) $(t^2 + y^2 + 2t)_y = 2y$, and $(2ty)_t = 2y$. Since these are the same, the equation is exact. We can find the solutions by solving the system

$$\begin{cases} \phi_t = t^2 + y^2 + 2t \\ \phi_y = 2ty \end{cases}$$

The first equation yields $\phi = \dfrac{t^3}{3} + ty^2 + t^2 + f(y)$. Substiuting this into the second equation we obtain $2ty + f'(y) = 2ty$. Thus $f(y) = 0$ works. Therefore, the general solution is

$$\frac{t^3}{3} + ty^2 + t^2 = C.$$

(b) This equation can be written as

$$(t\sin(ty) + y)\frac{\mathrm{d}y}{\mathrm{d}t} + y\sin(ty) = 0 \qquad (*)$$

We see $(t\sin(ty) + y)_t = \sin(ty) + ty\cos(ty)$ and $(y\sin(ty))_y = \sin(ty) + yt\cos(ty)$. Therefore, the equation $(*)$ is exact. Its general solution may be obtained by solving the system:

$$\begin{cases} \phi_y = t\sin(ty) + y \\ \phi_t = y\sin(ty) \end{cases}$$

The first equation yields $\phi = -\cos(ty) + y^2/2 + f(t)$. Substituting into the second equation we obtain $y\sin(ty) + f'(t) = y\sin(ty)$. Thus, $f(t) = 0$ is a solution. The general solution is:

$$-\cos(ty) + \frac{y^2}{2} = C.$$

(c) This equation is both exact and separable and can be solved using either method. □

Example 4.23. Show the following equations are not exact. In each case find an integrating factor and solve. When necessary, the form of an integrating factor is given.

(a) $(1 + 3t^2 \sin y)\ \mathrm{d}t - t\cot y\ \mathrm{d}y = 0$.

(b) $(y + ty^2)\ \mathrm{d}t - t\ \mathrm{d}y = 0$.

(c) $(t^3y^2 + y)\ \mathrm{d}t + (t^2y^3 + t)\ \mathrm{d}y = 0$; $\mu = \omega(ty)$.

(d) $(2\sin t + (t + y)\cos t)\ \mathrm{d}t + 2\sin t\ \mathrm{d}y = 0$; $\mu = \omega(t + y)$.

Solution. (a) $(1+3t^2\sin y)_y = 3t^2\cos y \neq (-t\cot y)_t = -\cot y$. Thus, the equation is not exact.

Let μ be an integrating factor. We must have

$$(\mu + 3\mu t^2\sin y)_y = (-\mu t\cot y)_t \Rightarrow \mu_y + 3\mu_y t^2\sin y + 3\mu t^2\cos y = -\mu_t t\cot y - \mu\cot y.$$

Setting $\mu_y = 0$ we obtain the following:

$$3\mu t^2\cos y = -\mu' t\cot y - \mu\cot y \Rightarrow \mu(3t^2\cos y + \cot y) = -\mu' t\cot y$$

This implies

$$\frac{\mu}{\mu'} = \frac{-t\cot y}{3t^2\cos y + \cot y}.$$

This is impossible since the right hand side is a function of both t and y but the left hand side is a function of t, only.

Setting $\mu_t = 0$, thus assuming μ is a function of y, only, we obtain the following:

$$\mu'(1+3t^2\sin y) = -\mu(\cot y + 3t^2\cos y) \Rightarrow \frac{\mu'}{\mu} = -\frac{\cot y + 3t^2\cos y}{1+3t^2\sin y} = -\frac{\cos y + 3t^2\cos y\sin y}{\sin y(1+3t^2\sin y)} = \frac{-\cos y}{\sin y}.$$

Integrating we obtain $\ln|\mu| = -\ln|\sin y| = \ln|\csc y|$. Therefore, $\mu = \csc y$ is one integrating factor.

(b) $(y+ty^2)_y = 1 + 2ty \neq (-t)_t = -1$. Thus, the equation is not exact.

Let μ be an integrating factor. We must have

$$(y\mu + ty^2\mu)_y = (-t\mu)_t \Rightarrow \mu + y\mu_y + 2ty\mu + ty^2\mu_y = -\mu - t\mu_t.$$

Setting $\mu_y = 0$ we obtain the following:

$$\mu + 2ty\mu = -\mu - t\mu' \Rightarrow \mu(2+2ty) = -t\mu'.$$

The left is a function of both t and y, while the right side is a function of t, only. So, this is impossible. We will now try setting $\mu_t = 0$. This yields:

$$\mu + y\mu_y + 2ty\mu + ty^2\mu_y = -\mu \Rightarrow \mu(2+2ty) + y\mu'(1+ty) = 0 \Rightarrow 2\mu + y\mu' = 0.$$

This equation is separable and yields $\ln|\mu| = -2\ln|y| = \ln|y^{-2}|$. Therefore, $\mu = 1/y^2$ is one integrating factor. Therefore, the following equation is exact.

$$\left(\frac{1}{y} + t\right)\, dt - \frac{t}{y^2}\, dy = 0.$$

The general solution is obtained by solving the system:

$$\begin{cases} \phi_t = \dfrac{1}{y} + t \Rightarrow \phi = \dfrac{t}{y} + \dfrac{t^2}{2} + f(y) \\ \phi_y = -\dfrac{t}{y^2} \end{cases}$$

Substituting into the second equation we obtain:

$$-\frac{t}{y^2} + f'(y) = -\frac{t}{y^2} \Rightarrow f(y) = 0 \text{ works}$$

The general solution, therefore, is

$$\frac{t}{y} + \frac{t^2}{2} = C.$$

(c) $(t^3y^2 + y)_y = 2t^3y \neq (t^2y^3 + t)_t = -2ty^3 - 1$. Thus, the equation is not exact.

Let $\mu = \omega(ty)$ be an integrating factor. By the Chain Rule, we have $\mu_t = y\omega'(ty)$ and $\mu_y = t\omega'(ty)$. We also have

$$(t^3y^2\mu + y\mu)_y = (t^2y^3\mu + t\mu)_t \Rightarrow 2t^3y\mu + t^3y^2\mu_y + \mu + y\mu_y = 2ty^3\mu + t^2y^3\mu_t + \mu + t\mu_t.$$

Substituting what we found above, we obtain the following:

$$(2t^3y - 2ty^3)\omega = (-t^4y^2 - ty + t^2y^4 + ty)\omega' \Rightarrow 2ty(t^2 - y^2)\omega = -t^2y^2(t^2 - y^2)\omega' \Rightarrow 2\omega(ty) = -ty\omega'(ty).$$

This means we need to solve $2\omega(x) = -x\omega'(x)$. This separable equation has a solution $\omega(x) = x^{-2}$. Therefore, an integrating factor is $\mu = (ty)^{-2}$. Therefore, the equation below is exact:

$$\left(t + \frac{1}{t^2y}\right)\, \mathrm{d}t + \left(y + \frac{1}{ty^2}\right)\, \mathrm{d}y = 0.$$

The solution satisfies

$$\begin{cases} \phi_t = t + \dfrac{1}{t^2y} \Rightarrow \phi = \dfrac{t^2}{2} - \dfrac{1}{ty} + f(y) \\ \phi_y = y + \dfrac{1}{ty^2} \end{cases}$$

Substituting into the second equation we obtain

$$\frac{1}{ty^2} + f'(y) = y + \frac{1}{ty^2} \Rightarrow f(y) = \frac{y^2}{2} \text{ is one solution.}$$

The general solution is, therefore,

$$\frac{t^2}{2} - \frac{1}{ty} + \frac{y^2}{2} = C.$$

(d) $(2\sin t + (t + y)\cos t)_y = \cos t \neq (2\sin t)_t = 2\cos t$. Thus, the equation is not exact.

Let $\mu = \omega(t + y)$ be an integrating factor. By the Chain Rule we have $\mu_t = \omega'(t + y)$ and $\mu_y = \omega'(t + y)$. We have the following:

$$(2\mu\sin t + (t + y)\mu\cos t)_y = (2\mu\sin t)_t \Rightarrow 2\mu_y\sin t + \mu\cos t + (t + y)\mu_y\cos t = 2\mu_t\sin t + 2\mu\cos t.$$

Using $\mu_t = \mu_y = \omega'$ we will obtain the following:

$$(t+y)\omega' \cos t = \omega \cos t \Rightarrow (t+y)\omega' = \omega \Rightarrow x\omega'(x) = \omega(x) \Rightarrow \omega(x) = x \text{ works.}$$

Therefore, $t+y$ is an integrating factor, which means the following equation is exact:

$$\left(2(t+y)\sin t + (t+y)^2 \cos t\right)\ \mathrm{d}t + 2(t+y)\sin t\ \mathrm{d}y = 0$$

The solution can be obtained by solving the system below:

$$\begin{cases} \phi_t = 2(t+y)\sin t + (t+y)^2 \cos t \\ \phi_y = 2(t+y)\sin t \Rightarrow \phi = \dfrac{(t+y)^2 \sin t}{2} + f(t) \end{cases}$$

Substituting in the first equation we obtain

$$2(t+y)\sin t + (t+y)^2\cos t + f'(t) = 2(t+y)\sin t + (t+y)^2 \cos t \Rightarrow f(t) = 0 \text{ works.}$$

Therefore, the solution is given by $\dfrac{(t+y)^2 \sin t}{2} = C$ or $(t+y)^2 \sin t = C$. □

Example 4.24. Suppose $\phi(t,y)$ has first partial derivatives over a rectangle $(a,b)\times(c,d)$ in the ty-plane. Prove that $\phi(t,y) = f(t) + g(y)$ for two differential functions f and g if and only if $\phi_{ty} = 0$.

Solution. First, assume $\phi(t,y) = f(t)+g(y)$. We have $\phi_t = f'(t)$, and thus $\phi_{ty} = 0$.

Now, assume $\phi_{ty} = 0$. The equality $\phi_{ty} = 0$ implies $\phi_t = f(t)$ is independent of y for all $y \in (c,d)$, and hence a function of t, only. By integrating again we obtain $\phi(t,y) = \displaystyle\int f(t)\ \mathrm{d}t + g(y)$ for some function g for all $t \in (c,d)$, as desired. □

Example 4.25. Show that every equation of the form $f(t) + g(y)\dfrac{dy}{dt} = 0$ is exact.

Note: This means all separable equations can be written in the form of an exact equation.

Solution. We note that $\dfrac{\partial f(t)}{dy} = \dfrac{\partial g(y)}{dt} = 0$, and thus this equation is exact. □

Example 4.26. Show that a first order linear equation $\dfrac{dy}{dt} + a(t)y - f(t) = 0$, where $a(t), f(t)$ are continuous, is exact if and only if $a(t) = 0$. Show that there is always an integrating factor that turns this equation into an exact equation.

Solution. Suppose $\dfrac{dy}{dt} + a(t)y - f(t) = 0$ is exact. We need to have

$$\frac{\partial 1}{\partial t} = 0 = \frac{\partial(a(t)y - f(t))}{\partial y} = a(t).$$

Now, suppose μ is an integrating factor. We need to have $\mu_t = a(t)\mu + a(t)y\mu_y - f(t)\mu_y$. Taking $\mu_y = 0$ we obtain $\mu_t = a(t)\mu$. We realize that $\mu = e^{A(t)}$ is a solution if $A'(t) = a(t)$. □

Example 4.27. Find all constants c for which the equation

$$2t \; \mathrm{d}t + (t + cy) \; \mathrm{d}y = 0$$

has an integrating factor of the form $\mu = \omega‘(t + y)$. For each of these constants solve the equation.

Solution. Suppose $\mu = \mu(t + y)$ is an integrating factor. We must have the following:

$$(2t\mu)_y = (t\mu + cy\mu)_t \Rightarrow 2t\mu_y = \mu + t\mu_t + cy\mu_t.$$

By the chain rule, we have $\mu_t = \mu_y = \omega'(t + y)$. This yields the following:

$$2t\omega' = \omega + t\omega' + cy\omega' \Rightarrow (t - cy)\omega' = \omega.$$

Since ω and thus ω' are functions of $t + y$, the function $t - cy$ must also be a function of $t + y$. This function can be written as $t - cy = t + y - (1 + c)y$. It is a function of $t + y$ if and only if $c = -1$. When $c = -1$ we have $(t + y)\omega' = \omega$. One solution is $\mu = t + y$. This yields

$$\begin{cases} 2t(t+y) = \phi_t \\ (t-y)(t+y) = \phi_y \end{cases}$$

The first equation yields $\phi = \dfrac{2t^3}{3} + t^2y + f(y)$. Substituting this into the second equation we obtain

$$t^2 - y^2 = t^2 + f'(y) \Rightarrow f(y) = -\frac{y^3}{3} \text{ is one solution.}$$

Therefore, the general solution is $\dfrac{2t^3 - y^3}{3} + t^2y = c.$ □

Example 4.28. Suppose $M(t, y)$ and $N(t, y)$ are continuous over a rectangle $R = (a, b) \times (c, d)$ and they have continuous partials over R. Assume, further that $M^2 + N^2 \neq 0$ over R. Prove $1/(M^2 + N^2)$ is an integrating factor of $M \; \mathrm{d}t + N \; \mathrm{d}y = 0$ if $M_t = N_y$ and $M_y = -N_t$.

Solution. By definition, for $1/(M^2 + N^2)$ to be an integrating factor the equation

$$\frac{M}{M^2 + N^2} \; \mathrm{d}t + \frac{N}{M^2 + N^2} \; \mathrm{d}y = 0$$

must be exact. By Theorem 4.4 this equation is exact if and only if

$$\left(\frac{M}{M^2 + N^2}\right)_y = \left(\frac{N}{M^2 + N^2}\right)_t.$$

By the quotient rule this is equivalent to

$$\frac{M_y(M^2 + N^2) - (2MM_y + 2NN_y)M}{(M^2 + N^2)^2} = \frac{N_t(M^2 + N^2) - (2MM_t + 2NN_t)N}{(M^2 + N^2)^2}.$$

Eliminating the denominator and combining like terms, this equality is equivalent to

$$M_y(N^2 - M^2) - 2MNN_y = N_t(M^2 - N^2) - 2MNM_t.$$

Since by assumption $M_y = -N_t$ and $N_y = M_t$ the result follows. $\square$

4.8 Exercises

Solutions to some differential equations may be implicit.

Exercise 4.1. *Draw a Venn Diagram for the following "sets":*

1. *All ODE's.*
2. *Separable Equations.*
3. *Linear Equations.*
4. *Explicit Equations.*
5. *Autonomous Equations.*
6. *Exact Equations.*
7. *Equations with Integrating Factors.*

Exercise 4.2. *Prove each function is a solution to the corresponding differential equation:*

(a) $y = 2t^{3/2}$ *with* $t > 0$; $2t^{1/2}y'' + t^{-1/2}y' - 6 = 0$.

(b) $y = \sin(t^2)$; $ty'' - y' + 4t^3y = 0$.

(c) $y = t^4 + 17t^3 + 14t$; $y^{(5)} = 0$.

Exercise 4.3. *Find all solutions of the following differential equations:*

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \cos^3 t \sin t$.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \tan^2 t$.

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{2}{t^2 - 1}$.

(d) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{2t}{t^4 + 1}$.

(e) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{1}{\sqrt{1 - t^2}}$.

(f) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = e^{2t - e^t}$.

Exercise 4.4. *Find a continuous solution $y : \mathbb{R} \to \mathbb{R}$ to the initial value problem and prove this solution is unique.*

$$y' = \begin{cases} (t-1)y & \text{if } t > 0 \\ (1-t)y & \text{if } t < 0 \end{cases} \qquad y(0) = 2.$$

Exercise 4.5. *Let $p, f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Prove that for every $y_0 \in \mathbb{R}$ there is a unique continuous solution $y : \mathbb{R} \to \mathbb{R}$ to the initial value problem*

$$y' + p(t)y = \begin{cases} f(t) & \text{if } t > 0 \\ g(t) & \text{if } t < 0 \end{cases} \qquad y(0) = y_0$$

Exercise 4.6. *Solve each of the following initial value problems:*

(a) $\dfrac{dy}{dt} = \dfrac{t^2+1}{t^3-t}, y(2) = 1.$

(b) $\dfrac{dy}{dt} = \sin^4 t, y(0) = 1.$

(c) $\dfrac{dy}{dt} = \tan t, y(\pi) = 1.$

(d) $\dfrac{dy}{dt} = \sqrt{3t-1}, y(1) = 2,\ t > 1/3.$

(e) $\dfrac{dy}{dt} = te^t, y(0) = 1.$

Exercise 4.7. *Let y be the solution to the initial value problem $\dfrac{dy}{dt} = \sin(t^3) + 2,\ y(-1) = 5$. Evaluate $y(1)$.*

Exercise 4.8. *Solve each differential equation:*

(a) $ty' - 2y = 1/t$, *with* $t < 0$.

(b) $y' \cos t + y = \sin t$, *with* $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Exercise 4.9. *Find all real constants c or show no such constant c exists, for which the differential equation $y' + cy = t$ has at least one solution that satisfies $y(0) = 1, y(1) = -1$.*

Exercise 4.10. *Find all bounded solutions of each equation:*

(a) $(t+1)y' - y + 1 = 0.$

(b) $y' = 2t + 2ty.$

Exercise 4.11. *Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = 2 + \int_2^x (t - tf(t))dt$, for all $x \in \mathbb{R}$.*

Exercise 4.12. *Find the general solution of each equation:*

(a) $ty' + \sec y = 0.$

(b) $(1+t^2)y' + (1-y^2)t = 0.$

(c) $y' = \cos y \sin^2 t.$

(d) $y' = y^2 - (a+b)y + ab$, *where* $a, b \in \mathbb{R}$ *are constants.*

Exercise 4.13. *Solve each of the following equations:*

(a) $\dfrac{dy}{dt} = (y-t)^2$

(b) $\dfrac{dy}{dt} = \dfrac{e^{t+y}}{t+y} - 1.$

(c) $y' - 4t^2 = 4yt + y^2.$

(d) $y' = \dfrac{y+t}{y+t+1}.$

Exercise 4.14. *Find an integrating factor for the following equation, given the integrating factor is of the form* $\mu = t^m y^n$.

$$(y - y^2) + ty' = 0.$$

Exercise 4.15. *Find all stationary and nonstationary solutions of the equation* $\dfrac{dy}{dt} = yt - y - t + 1.$

Exercise 4.16. *Solve the initial value problem* $(t^2+1)y' + y^2 + 1 = 0,\ y(3) = 2.$ *Your final answer must be explicit and simplified.*

Exercise 4.17. *Prove that if* y_1, y_2 *are solutions to* $y' + a(t)y = f(t)$, *then* $y_1 - y_2$ *is a solution to* $y' + a(t)y = 0.$

Exercise 4.18. *Find all solutions to each equation satisfying the given condition:*

(a) $t^2 y' \sin y = 1, \lim_{t\to\infty} y(t) = \pi.$

(b) $y' + 2y = 5\cos t$, *and* y *is periodic.*

(c) $y' - 2ty = 0$, *and* y *is bounded.*

(d) $y' = \dfrac{y+t}{y+t+1}$, *and* $y(0) = 1.$

Exercise 4.19. *Determine* $\lim_{t\to\infty} y(t)$, *for all solutions of the differential equation* $y' + y\cos t = \cos t$. *Find your answer in terms of* $y(0)$.

Exercise 4.20. *Solve the initial value problem* $(t^2+y^2)\dfrac{dy}{dt} + (3t^2 y + 2ty + y^3) = 0, y(0) = 1.$

Exercise 4.21. *Find the general solution to each equation:*

(a) $(3ty^2 + 2y)\ \mathrm{d}t + (2t^2 y + t)\ \mathrm{d}y = 0.$

(b) $y\cos t\ \mathrm{d}t + (y\sin t + \sin t + 1)\ \mathrm{d}y = 0.$

(c) $(y\cos t + y^2)\ \mathrm{d}t + (3\sin t + 4yt)\ \mathrm{d}y = 0.$

(d) $(7y + 8ty^3)\ \mathrm{d}t + (t + 3t^2 y^2)\ \mathrm{d}y = 0.$

(e) $(t^2 y + y + 1)\ \mathrm{d}t + (t + t^3)\ \mathrm{d}y = 0.$

Exercise 4.22. *Let $f(t)$ and $g(y)$ be continuous functions. Show that the equation*

$$\frac{f(t)}{y} + 1 + (g(y) + t/y)\frac{dy}{dt} = 0$$

is not generally exact. Find an integrating factor and use that to find a general solution for this equation.

Exercise 4.23. *Determine all constants c, for which the differential equation $(t^2 + y^2) + \dfrac{ct^3 + t^2}{y}\dfrac{dy}{dt} = 0$ has an integrating factor $\mu = \dfrac{1}{t^2y^2}$. For all such constants c, solve the resulting equation.*

Exercise 4.24. *Find all curves of the form $y = f(x)$ on the xy-plane that intersect the x-axis at an angle of $\dfrac{\pi}{4}$ and satisfy the differential equation $xy' + y = 2$.*

Exercise 4.25. *Prove that the IVP*

$$y'' = e^{t^2},\ y(0) = 1,\ y'(1) = -1$$

has a unique solution.

Exercise 4.26. *Let $f : I \to \mathbb{R}$ be a continuous function, where I is an open interval, $t_0, t_1 \in I$, and $y_0, y_1 \in \mathbb{R}$. Prove that there is a unique function y defined over I for which*

$$y'' = f(t),\ y(t_0) = y_0,\ y'(t_1) = y_1.$$

Exercise 4.27. *Let $f : I \to \mathbb{R}$ be a continuous function, where I is an open interval, $t_0, t_1 \in I$ be distinct real numbers, and y_0, y_1 be two real numbers. Prove that there is a unique function y defined over I for which*

$$y'' = f(t),\ y(t_0) = y_0,\ y(t_1) = y_1.$$

Exercise 4.28. *Solve each second order IVP.*

(a) $y'' = t^2 + \sin t,\ y(0) = 1,\ y'(0) = 0.$

(b) $y'' = y',\ y(0) = y'(0) = 2.$

(c) $y'' + 1 = (y' + t)^2,\ y(0) = 1, y'(0) = 2.$

(d) $y'' = (y')^2, y(0) = y'(0) = 1.$

Hint: Substitute $z = y'$.

Exercise 4.29. *Find all real constants y_0, y_1 for which the equation*

$$y'' = e^{t^2}, y(0) = y_0, y(1) = y_1$$

has a unique solution defined over $\mathbb{R}$.

Exercise 4.30. *Suppose $M(t, y)$ and $N(t, y)$ have continuous first partials over a rectangle R. Prove that the equation $M(t, y)\ \mathrm{d}t + N(t, y)\ \mathrm{d}y = 0$ has a C^1 integrating factor of the form $\mu(y)$ if and only if $\dfrac{N_t - M_y}{M}$ only depends on y.*

Exercise 4.31. *Suppose $M(t,y)$ and $N(t,y)$ have continuous first partials over a rectangle R. Prove that the function $\mu(t,y)$ with continuous first partials is an integrating factor for the equation $M(t,y)\ \mathrm{d}t+N(t,y)\ \mathrm{d}y = 0$ if and only if*

$$\mu\left(M_y - N_t\right) = N\mu_t - M\mu_y$$

on R.

In the next exercise we will prove that each first order IVP can be turned into one with initial time $t_0 = 0$.

Exercise 4.32. *Consider the IVP*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y),\ \ y(t_0) = y_0.$$

Set $z(s) = y(s+t_0)$. Prove that the above IVP is equivalent to the following IVP

$$\frac{\mathrm{d}z}{\mathrm{d}s} = f(s+t_0,z),\ \ z(0) = y_0.$$

Exercise 4.33. *Suppose $M(t,y)$ and $N(t,y)$ are continuous and have continuous first partials on the rectangle R given by $|t-t_0| < a, |y-y_0| < b$. Assume $M_y = N_t$ on R. Prove that the solution to the equation*

$$M(t,y) + N(t,y)y' = 0$$

is given by

$$\int_{y_0}^{y} N(t,u)\ \mathrm{d}u + \int_{t_0}^{t} M(u,y_0)\ \mathrm{d}u = C,$$

where C is a constant.

Exercise 4.34. *Suppose the differential equation*

$$M(t,y)\ \mathrm{d}t + N(t,y)\ \mathrm{d}y = 0 \qquad (*)$$

is exact and has a nonconstant integrating factor $\mu(t,y)$. Prove that $\mu(t,y) = C$ is a solution to $(*)$.

4.9 Challenge Problems

Exercise 4.35. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Prove that all solutions of the differential equation $y' = f(t)$ are periodic with period $L > 0$ if and only if f is periodic with period L and $\displaystyle\int_0^L f(t)\ \mathrm{d}t = 0$.*

Exercise 4.36. *Solve the initial value problem $y^2 + 2yy' + 2t + 2 = 2e^t$, $y(0) = 2$.*

Exercise 4.37. *Suppose $a, f : \mathbb{R} \to \mathbb{R}$ are continuous functions and c is a positive constant for which*

$$\lim_{t\to\infty} f(t) = 0,\ \ \text{and}\ \forall\ t \in \mathbb{R}\ a(t) \geq c.$$

Let $y(t)$ be a solution to the differential equation $y' + a(t)y = f(t)$. Prove that

$$\lim_{t\to\infty} y(t) = 0.$$

Exercise 4.38. *Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous function. prove that all solutions of the equation $y' + a(t)y = 0$ are periodic with period L if and only if $a(t)$ is periodic with period L and that $\int_0^L a(t)\ \mathrm{d}t = 0$.*

Exercise 4.39. *Solve the equation $\dfrac{dy}{dt} = -\dfrac{2y + 3ty^2}{2t + 4t^2y^2}$.*

Definition 4.3. Let k be a positive integer. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be **homogeneous** of degree k, if

$$f(tx_1, tx_2, \ldots, tx_n) = t^k f(x_1, x_2, \ldots, x_n),$$

for all $t, x_1, \ldots, x_n \in \mathbb{R}$.

Exercise 4.40. *Suppose $P(x, y)$ and $Q(x, y)$ are homogeneous functions of the same degree with continuous partial derivatives. Prove that $\frac{1}{xP+yQ}$ is an integrating factor for the equation*

$$P(x, y)\ \mathrm{d}x + Q(x, y)\ \mathrm{d}y = 0.$$

Exercise 4.41. *Solve each of the following:*

(a) $(t - ty)\ \mathrm{d}t + (t^2 + y)\ \mathrm{d}y = 0.$

(b) $(t^2 + y^2 + 1)\ \mathrm{d}t - 2ty\ \mathrm{d}y = 0.$

(c) $t^2y'y + ty' + ty^2 + y - ty = 0.$

Exercise 4.42. *Suppose $M(t, y)$ and $N(t, y)$ have continuous partials over a rectangle R. Assume both t and y are integrating factors for the equation*

$$M(t, y)\ \mathrm{d}t + N(t, y)\ \mathrm{d}y = 0.$$

Prove that all solutions of this equation are either lines of the form $y = Ct$ for a constant C, or satisfy $tN(t, y) = 0$.

Exercise 4.43. *Consider the differential equation*

$$y' = f(t, y) \qquad (*)$$

over a rectangle $R = (a, b) \times (c, d)$ in the ty-plane, where $f, f_t, f_y, f_{ty} = f_{yt}$ are all continuous. Assume $f \neq 0$ on R. Prove that the equation $(*)$ is separable if and only if $ff_{ty} = f_t f_y$.

Exercise 4.44. *Solve the initial value problem*

$$y'' + (\cos t)y' - (\sin t)y = -\sin t,\ \ y(0) = 1,\ \ y'(0) = 1.$$

4.10 Summary

- An explicit IVP $\dfrac{dy}{dt} = f(t),\ y(t_0) = y_0$ has a unique solution as long as $f(t)$ is continuous. The solution can be found by integrating both sides from t_0 to t and using the initial condition y_0 as the constant.

- To solve a linear equation $\frac{dy}{dt} + a(t)y = f(t)$:
  - Keep in mind that the goal is to write the left hand side as the derivative of one function.
  - Find $A(t)$ for which $A'(t) = a(t)$.
  - Rewrite the equation as $\frac{d}{dt}\left(e^{A(t)}y\right) = e^{A(t)}f(t)$. Then integrate both sides.
- Existence and Uniqueness Theorem for linear first order equations requires the coefficient $a(t)$ and the forcing $f(t)$ to be continuous.
- To solve a separable equation of the form $\frac{dy}{dt} = f(t)g(y)$:
  - Find all stationary solutions by solving $g(y) = 0$.
  - For nonstationary solutions: separate the variables and rewrite the equation as $\frac{dy}{g(y)} = f(t)dt$. Then integrate both sides.
- There are three common types of equations that require change of variables:
  1. Equations of the form $y' = f(ay + bt + c)$ can be solved by the change of variable $u = ay + bt + c$.
  2. Equations of the form $y' = (ay + bt)/(cy + dt)$ can be solved by the change of variable $u = y/t$.
  3. For equations of the form $y' = (ay+bt+m)/(cy+dt+n)$ we first do a translation $Y = y+r, T = t+s$ to determine which constants r, s change this equation into one of the form #2 above. After finding r, s we proceed with the change of variable $u = y/t$. Note that some problems that might look like #3 are actually instances of #1. So make sure you check for #1 first.
- An equation $M + N\frac{dy}{dt} = 0$ is exact if $M_y = N_t$.
- To solve an exact equation $M + N\frac{dy}{dt} = 0$ we will find $\phi(t, y)$ for which $\phi_t = M$, and $\phi_y = N$. The solutions then are given by $\phi(t, y) = c$.
- To solve equations using the integrating factor method:
  - First check if the equation is exact.
  - If it is not, multiply both sides by μ and set up the equation $(\mu M)_y = (\mu N)_t$.
  - Find an appropriate μ. Generally, finding μ is not easy and there is no method that always works. Test if $\mu_y = 0$ would yield a function of t for μ, or if $\mu_t = 0$ would yield a function of y for μ.
  - Multiply both sides of the equation by μ, and solve the resulting equation using the method for exact equations.