

Application to Coupled Flow Problems



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The Full Set of Equations

Velocity and Pressure

$$\begin{aligned}\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p + 2\boldsymbol{\omega} \times \mathbf{u} &= \mathbf{f}_u - \beta \theta \mathbf{g} + (\nabla \times \mathbf{b}) \times \mathbf{b}, \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

Magnetic Field

$$\begin{aligned}\partial_t \mathbf{b} + \lambda \nabla \times (\nabla \times \mathbf{b}) - \nabla \times (\mathbf{u} \times \mathbf{b}) &= \mathbf{f}_b, \\ \nabla \cdot \mathbf{b} &= 0\end{aligned}$$

Temperature

$$\partial_t \theta - \alpha \Delta \theta + (\mathbf{u} \cdot \nabla) \theta = f_\theta$$

Local Projection Stabilization

Idea

- Separate discrete function spaces into small and large scales
- Add stabilization terms only on small scales.

Notations and prerequisites

- Family of shape-regular macro decompositions $\{\mathcal{M}_h\}$
- Let $D_M \subset [L^\infty(M)]^d$ denote a FE space on $M \in \mathcal{M}_h$.
- For each $M \in \mathcal{M}_h$, let $\pi_M: [L^2(M)]^d \rightarrow D_M$ be the orthogonal L^2 -projection.
- $\kappa_M = Id - \pi_M$ fluctuation operator
- Averaged streamline direction $\mathbf{u}_M \in \mathbb{R}^d$:
$$|\mathbf{u}_M| \leq C\|\mathbf{u}\|_{L^\infty(M)}, \quad \|\mathbf{u} - \mathbf{u}_M\|_{L^\infty(M)} \leq Ch_M |\mathbf{u}|_{W^{1,\infty}(M)}$$

Assumptions

Assumption - Approximation

It holds for all $w \in W^{l,2}(M)$, $M \in \mathcal{M}_h$ and $l \leq s \leq k$

$$\|\kappa_M w\|_{L^2(M)} \leq Ch_M^l \|w\|_{W^{l,2}(M)}$$

Assumption - Inf-Sup Stability

Consider FE spaces (V_h, Q_h) satisfying a discrete inf-sup-condition:

$$\inf_{q \in Q_h \setminus \{0\}} \sup_{v \in V_h \setminus \{0\}} \frac{(\nabla \cdot v, q)}{\|\nabla v\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}} \geq \beta > 0$$

$$\Rightarrow \quad \mathbf{V}_h^{div} := \{v_h \in V_h \mid (\nabla \cdot v_h, q_h) = 0 \quad \forall q_h \in Q_h\} \neq \{0\}$$

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Rotating Frames of Reference

Navier Stokes Equations in an Inertial Frame of Reference

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \times (0, T) \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \times (0, T) \end{aligned}$$

$\Omega \subset \mathbb{R}^d$ bounded polyhedral domain

Navier Stokes Equations in a Rotating Frame of Reference

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu \Delta \mathbf{v} + 2\boldsymbol{\omega} \times \mathbf{v} + \nabla \tilde{p} &= \mathbf{f} && \text{in } \Omega \times (0, T) \\ \nabla \cdot \mathbf{v} &= 0 && \text{in } \Omega \times (0, T) \end{aligned}$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\frac{1}{2} \nabla (\boldsymbol{\omega} \times \mathbf{r})^2 \quad \tilde{p} = p - \frac{1}{2} (\boldsymbol{\omega} \times \mathbf{r})^2$$

Weak Formulation

Find $\mathcal{U}_h = (\mathbf{u}_h, p_h) : (0, T) \rightarrow \mathbf{V}_h \times Q_h$, such that

$$(\partial_t \mathbf{u}_h, \mathbf{v}_h) + \mathcal{A}_G(\mathbf{u}_h, \mathcal{U}_h, \mathcal{V}_h) + (2\boldsymbol{\omega} \times \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)$$

for all $\mathcal{V}_h = (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$

where

$$\mathcal{A}_G(\mathbf{w}; \mathcal{U}, \mathcal{V}) := a_G(\mathcal{U}, \mathcal{V}) + c(\mathbf{w}; \mathbf{u}, \mathbf{v})$$

$$a_G(\mathcal{U}, \mathcal{V}) := \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u})$$

$$c(\mathbf{w}, \mathbf{u}, \mathbf{v}) := \frac{((\mathbf{w} \cdot \nabla) \mathbf{u}, \mathbf{v}) - ((\mathbf{w} \cdot \nabla) \mathbf{v}, \mathbf{u})}{2}$$

Stabilization Terms

- LPS Streamline upwind Petrov-Galerkin (SUPG)

$$s_u(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) := \sum_{M \in \mathcal{M}_h} \tau_M(\mathbf{w}_M)(\kappa_M((\mathbf{w}_M \cdot \nabla) \mathbf{u}_h), \kappa_M((\mathbf{w}_M \cdot \nabla) \mathbf{v}_h))_M$$

- grad-div

$$t_h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) := \sum_{M \in \mathcal{M}_h} \gamma_M(\mathbf{w}_M)(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h)_M$$

- LPS Coriolis stabilization

$$a_h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) := \sum_{M \in \mathcal{M}_h} \alpha_M(\mathbf{w}_M)(\kappa_M(\omega_M \times \mathbf{u}_h), \kappa_M(\omega_M \times \mathbf{v}_h))_M$$

Convergence Result

Theorem

For a sufficiently smooth solution we obtain for $\mathbf{e}_h = \mathbf{u}_h - j_u \mathbf{u}$:

$$\|\mathbf{e}_h\|_{L^\infty(0,t;[L^2(\Omega)]^d)}^2 + \int_0^t \|\mathbf{e}_h(\tau)\|_{LPS}^2 d\tau \leq C \exp(C_G t) h^{2k}$$

with a Gronwall constant

$$C_G(\mathbf{u}) = 1 + C|\mathbf{u}|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + Ch\|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}^2$$

The parameters have to satisfy ($1 \leq s \leq k$):

$$h_M \leq C \frac{\sqrt{\nu}}{\|\mathbf{u}_h\|_{L^\infty(M)}}$$

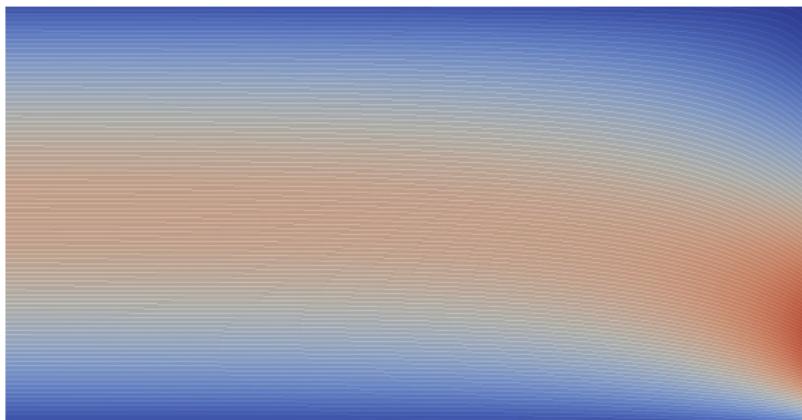
$$\tau_M \leq \tau_0 \frac{h_M^{2(k-s)}}{|\mathbf{u}_M|^2}$$

$$\gamma_M = \gamma_0$$

$$\alpha_M \leq \alpha_0 \frac{h_M^{2(k-s-1)}}{\|\boldsymbol{\omega}\|_{L^\infty(M)}^2}$$

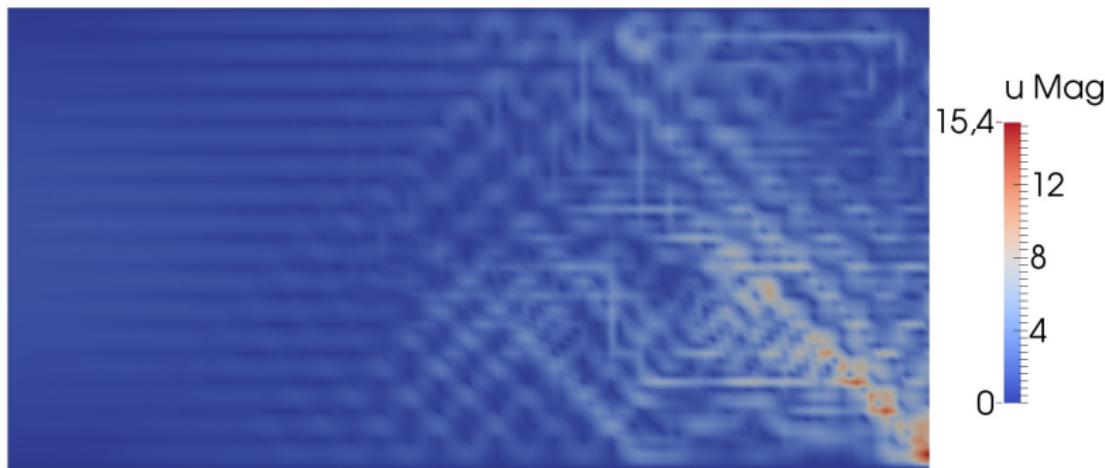
Numerical Results, Rotating Poiseuille Flow

- $\Omega = [-2, 2] \times [-1, 1]$
- $\mathbf{u}(x, y) = \begin{cases} (1 - y^2, 0)^T, & x = -2 \\ (0, 0)^T, & |y| = 1 \end{cases}, \quad (\nabla \mathbf{u} \cdot \mathbf{n})(x = 2, y) = 0$
- $\mathbf{u}_0 = 0, \quad p_0 = 0, \quad \mathbf{f} = 0 \quad \boldsymbol{\omega} = (0, 0, 100), \quad \nu = 10^{-3}$

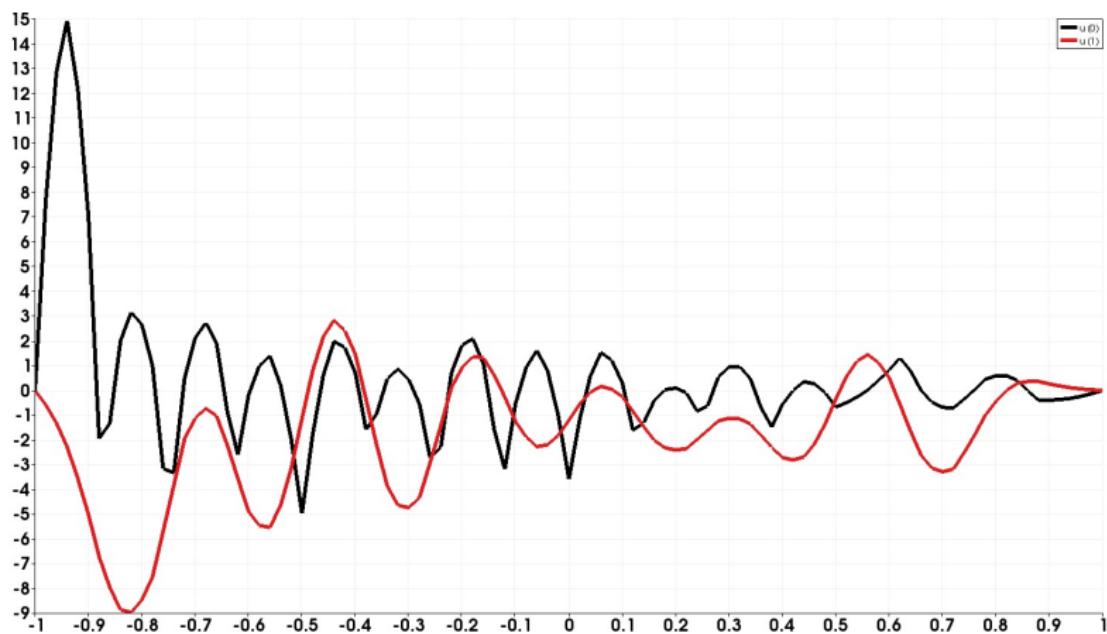


Flow for the parameters $\boldsymbol{\omega} = (0, 0, 1)$, $\nu = 10^{-1}$

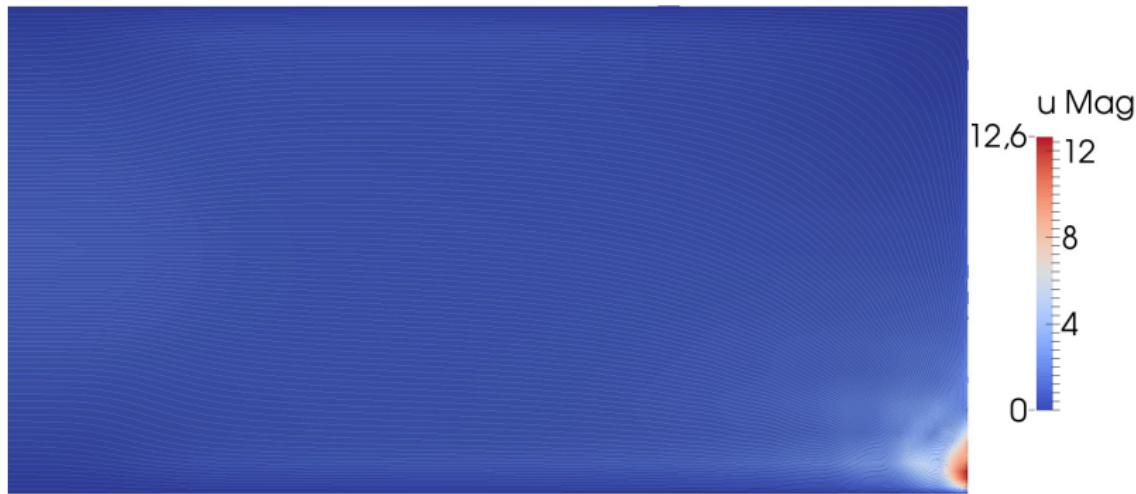
Rotating Poiseuille Flow, grad-div



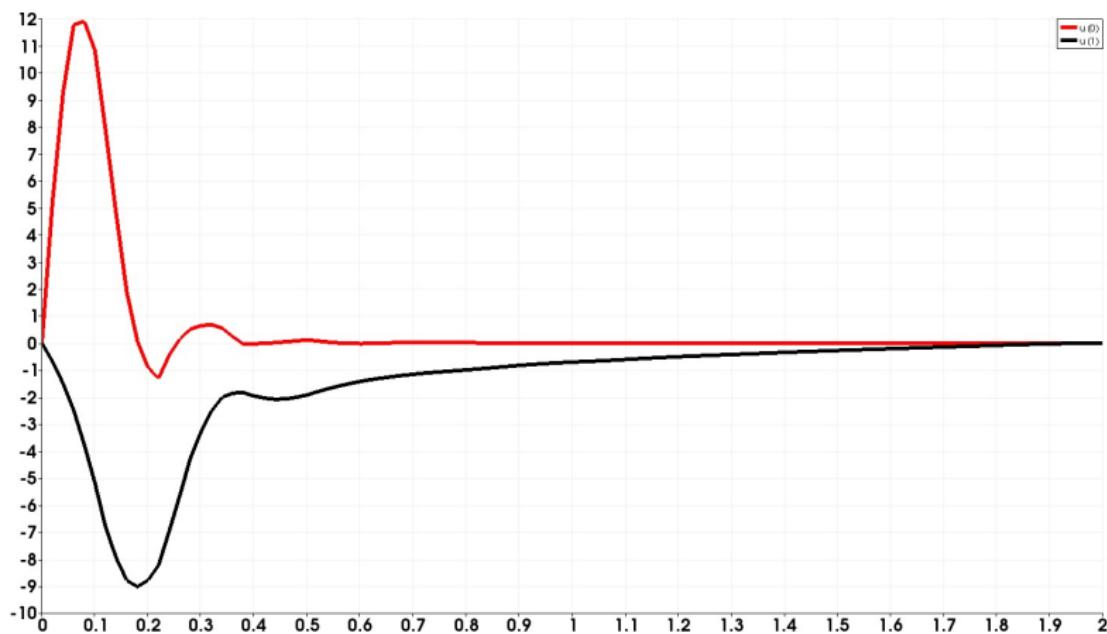
Rotating Poiseuille Flow, grad-div



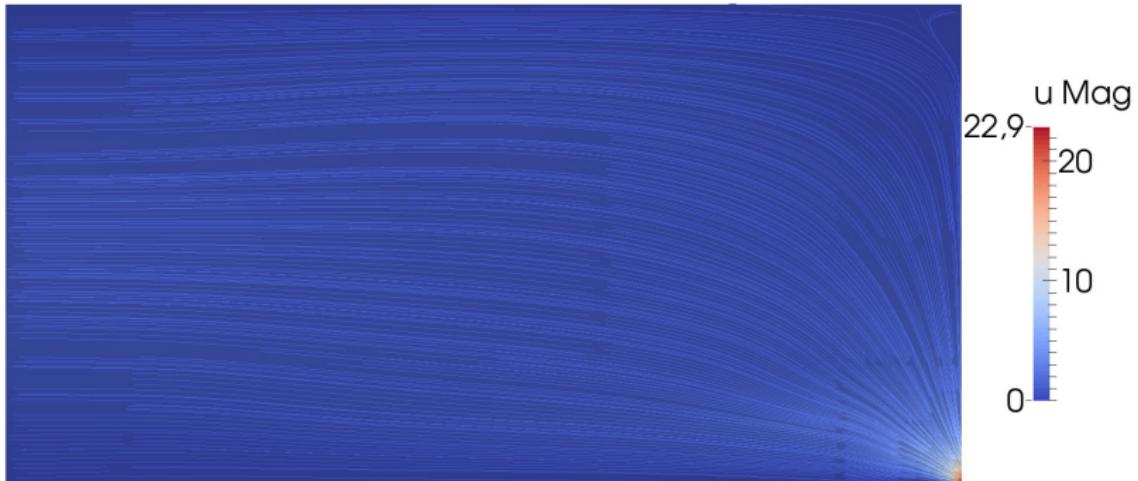
Rotating Poiseuille Flow, SUPG Coriolis



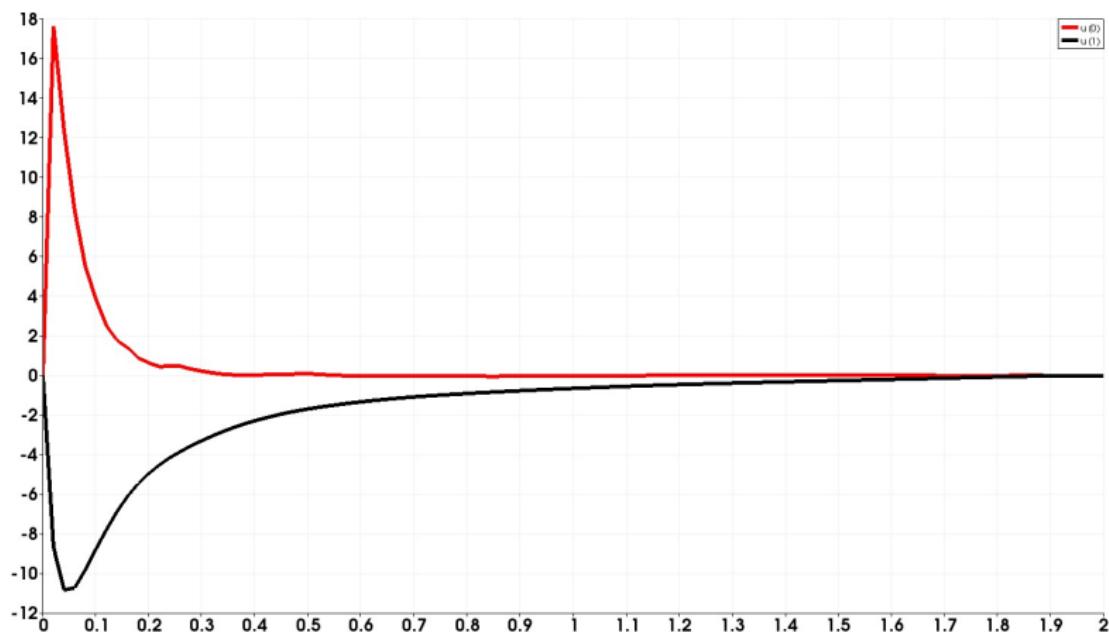
Rotating Poiseuille Flow, SUPG Coriolis



Rotating Poiseuille Flow, SUPG Coriolis Adaptive



Rotating Poiseuille Flow, SUPG Coriolis Adaptive



Taylor-Proudman, Stewartson Layer

Navier Stokes Equations in a Rotating Frame of Reference

$$\frac{\partial \mathbf{u}}{\partial t} + Ro(\mathbf{u} \cdot \nabla) \mathbf{u} + 2\hat{\mathbf{e}}_z \times \mathbf{u} = Ek\Delta \mathbf{u} - \nabla p$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u} = r \sin \theta \hat{\mathbf{e}}_\phi \quad \text{at } r = r_i$$

$$\mathbf{u} = \mathbf{0} \quad \text{at } r = r_o$$

$$r_i = 1/2$$

$$r_o = 3/2$$

$$Ro := \Delta\Omega/\Omega$$

$$Ek := \frac{\nu}{\Omega(r_o - r_i)^2}$$

Taylor-Proudman, Stewartson Layer

Fluid structure between two rotating spheres

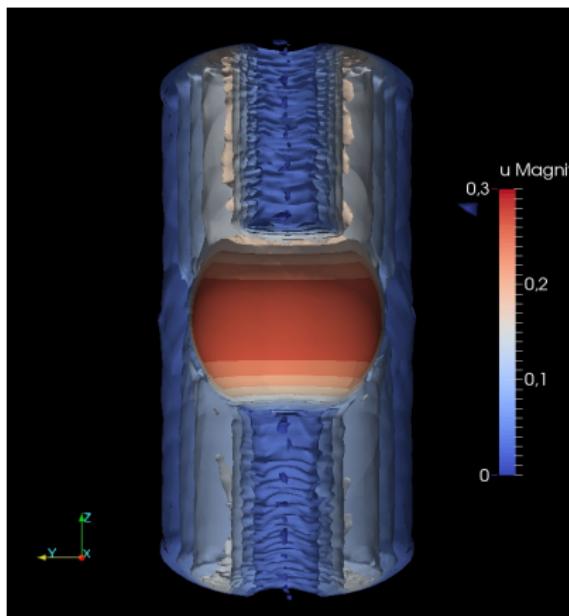


Figure: $Ek = 10^{-6}$

Taylor-Proudman, Stewartson Layer

Fluid structure between two rotating spheres

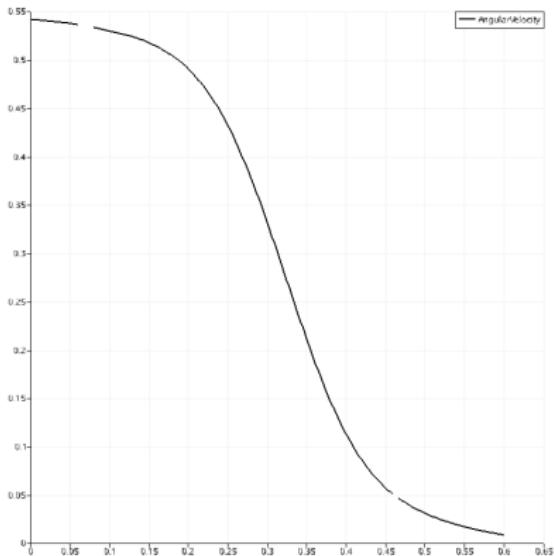
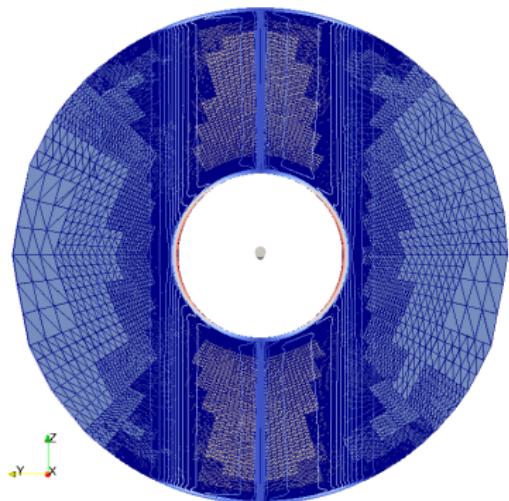


Figure: $Ek = 10^{-4} Ro = -0.5$

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Incompressible Nodal-based MHD - Joint work with Benjamin Wacker

Stationary Linearized Incompressible Nodal-based MHD model

$$\begin{aligned} -\nu \Delta \mathbf{u} + (\mathbf{a} \cdot \nabla) \mathbf{u} + \nabla p - (\nabla \times \mathbf{b}) \times \mathbf{d} &= \mathbf{f}_u, \\ \nabla \cdot \mathbf{u} &= 0, \\ \lambda \nabla \times (\nabla \times \mathbf{b}) + \nabla r - \nabla \times (\mathbf{u} \times \mathbf{d}) &= \mathbf{f}_b, \\ \nabla \cdot \mathbf{b} &= 0 \end{aligned}$$

- \mathbf{u} velocity field, p kinematic pressure
- \mathbf{a} extrapolation for \mathbf{u}
- \mathbf{b} induced magnetic field, r magnetic pseudo pressure
- \mathbf{d} extrapolation for \mathbf{b}
- ν kinematic viscosity, λ magnetic diffusivity

Weak Formulation

Find $\mathcal{U}_h := (\mathbf{u}_h, \mathbf{b}_h, p_h, r_h) \in \mathbf{V}_h \times \mathbf{C}_h \times Q_h \times S_h$ such that

$$\mathcal{A}_{G,u}(\mathcal{U}_h, \mathcal{V}_h) + \mathcal{A}_{G,b}(\mathcal{U}_h, \mathcal{V}_h) = \langle \mathbf{f}_u, \mathbf{v} \rangle + \langle \mathbf{f}_b, \mathbf{c} \rangle,$$

for all $\mathcal{V}_h := (\mathbf{v}_h, \mathbf{c}_h, q_h, s_h) \in \mathbf{V}_h \times \mathbf{C}_h \times Q_h \times S_h$.

$$\begin{aligned} \mathcal{A}_{G,u}(\mathbf{U}, \mathbf{V}) &= \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + \langle \mathbf{a} \cdot \nabla \mathbf{u}, \mathbf{v} \rangle - \langle (\nabla \times \mathbf{b}) \times \mathbf{d}, \mathbf{v} \rangle \\ &\quad - (p, \nabla \cdot \mathbf{v}) + (\nabla \cdot \mathbf{u}, q) \end{aligned}$$

$$\begin{aligned} \mathcal{A}_{G,b}(\mathbf{U}, \mathbf{V}) &= \lambda(\nabla \times \mathbf{b}, \nabla \times \mathbf{c}) - \langle \nabla \times (\mathbf{u} \times \mathbf{d}), \mathbf{c} \rangle \\ &\quad + (\nabla r, \mathbf{c}) - (\mathbf{b}, \nabla s) \end{aligned}$$

Stabilization

- LPS-SU

$$s_1(\mathbf{u}_h, \mathbf{v}_h)_M := \tau_{SU}(\kappa_M((\mathbf{a}_M \cdot \nabla) \mathbf{u}_h), \kappa_M((\mathbf{a}_M \cdot \nabla) \mathbf{v}_h))_M$$

- grad-div velocity

$$s_2(\mathbf{u}_h, \mathbf{v}_h)_M := \tau_{gd,u}(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h)_M$$

- LPS-Lorentz

$$s_3(\mathbf{b}_h, \mathbf{c}_h)_M := \tau_{Lor}(\kappa_M((\nabla \times \mathbf{b}_h) \times \mathbf{d}_M), \kappa_M((\nabla \times \mathbf{c}_h) \times \mathbf{d}_M))_M$$

- LPS-Induction

$$s_4(\mathbf{u}_h, \mathbf{v}_h)_M := \tau_{Ind}(\kappa_M(\nabla \times (\mathbf{u}_h \times \mathbf{d}_M)), \kappa_M(\nabla \times (\mathbf{v}_h \times \mathbf{d}_M)))_M$$

- PSPG

$$s_5(r_h, s_h)_M := \tau_{PSPG}(\nabla r_h, \nabla s_h)_M$$

- grad-div magnetic

$$s_6(\mathbf{b}_h, \mathbf{c}_h)_M := \tau_{gd,b}(\nabla \cdot \mathbf{b}_h, \nabla \cdot \mathbf{c}_h)_M$$

Convergence Result

Theorem

For sufficiently smooth solutions we obtain:

$$\begin{aligned} & \| \mathbf{U}_h - \mathbf{U} \|_{\text{G}}^2 + \| \mathbf{U}_h - \mathbf{U} \|_{\text{LPS}}^2 \\ & \leq C \sum_M h_M^{2k} \left(|\mathbf{u}|_{k+1, \omega_M}^2 + |\mathbf{b}|_{k+1, \omega_M}^2 + |p|_{k, \omega_M}^2 \right) \end{aligned}$$

with a parameter choice according to

$$h_M \leq C \min \left\{ \frac{\sqrt{\nu}}{\|\mathbf{a}\|_{\infty, M}}, \frac{\sqrt{\lambda}}{\|\mathbf{d}\|_{\infty, M}}, \frac{\sqrt{\nu}}{\|\mathbf{d}\|_{\infty, M} + \|\nabla \mathbf{d}\|_{\infty, M}} \right\}$$

$$\tau_{SU} \leq C \frac{h_M^{2(k-s)}}{|\mathbf{a}_M|^2} \quad \tau_{gd,u} \sim \gamma_0 \quad \tau_{Lor}, \tau_{Ind} \leq C \frac{h_M^{2(k-s)}}{|\mathbf{d}_M|^2}$$

$$\tau_{TPSPG} \sim \frac{L_{0,m}^2}{\lambda} \quad \tau_{gd,b} \sim \frac{h_M^2 \lambda}{L_{0,m}^2}$$

Analytical 2D Problem

Consider on $\Omega = (-1, 1) \times (-1, 1)$ for $t \in [0, 0.1]$ the solution:

$$\begin{aligned}\mathbf{u}(t, x, y) &:= \mathbf{b}(t, x, y) \\ &:= \exp\left(\frac{t}{25}\right) \left(x^4 \sin\left(\frac{\pi t}{10}\right), -4x^3y \sin\left(\frac{\pi t}{10}\right) \right)^T, \\ p(x, y) &:= x^2 + y^2, \quad r(x, y) := 0\end{aligned}$$

Boundary conditions: $\mathbf{u}|_{\partial\Omega} = \mathbf{u}_b$ $\mathbf{n} \times \mathbf{b} = \mathbf{n} \times \mathbf{b}_D$

Parameters:

$$\begin{aligned}\Delta t &= 10^{-4}, & \nu &= 10^{-6}, & \lambda &= 10^{-6} \\ \tau_2 &= 1, & \tau_5 &= \frac{h^2 \lambda}{L_0^2}, & \tau_6 &= \frac{L_0^2}{\lambda}, & L_0 &= 2\end{aligned}$$

Ansatz spaces: $V_h/Q_h = C_h/S_h = [\mathbb{Q}_2]^2 / \mathbb{Q}_1$

Convergence Results

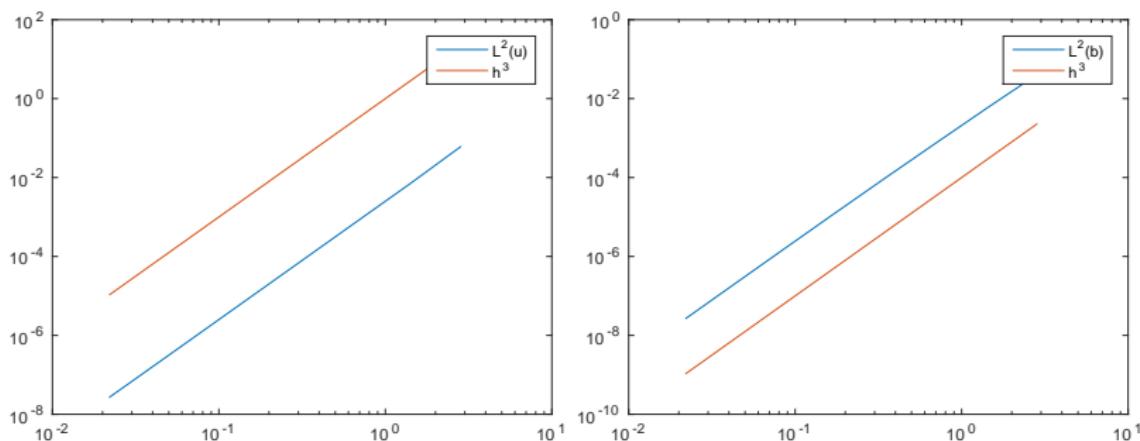


Figure: $L^2(\Omega)$ Errors for the 2D-time-dependent analytical problem

Convergence Results

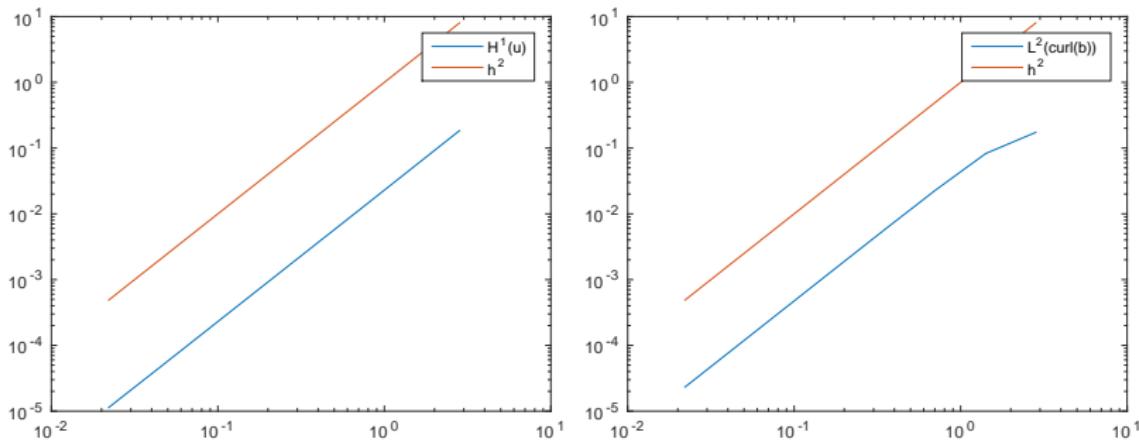


Figure: Errors for the 2D-time-dependent analytical problem

Convergence Results

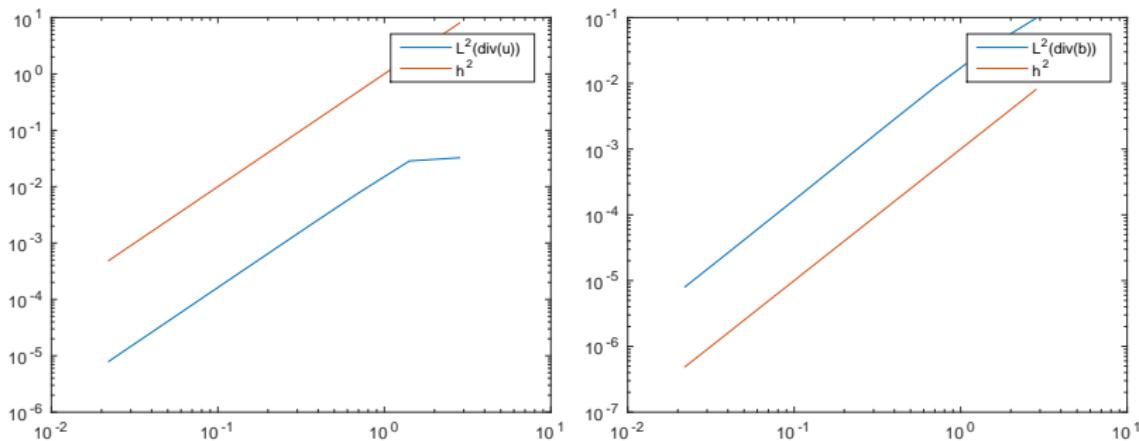


Figure: Divergence Errors for the 2D-time-dependent analytical problem

Convergence Results

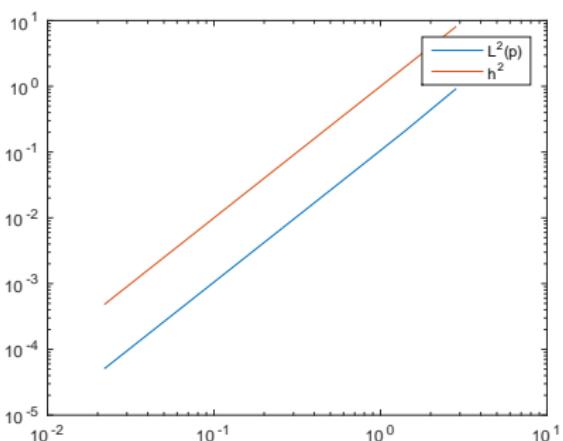


Figure: Pressure Errors for the 2D-time-dependent analytical problem

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Non-Isothermal Flow - Joint work with Helene Dallmann

Oberbeck-Boussinesq-Model

$$\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f}_u - \beta \theta \mathbf{g}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t \theta - \alpha \Delta \theta + (\mathbf{u} \cdot \nabla) \theta = f_\theta$$

- velocity \mathbf{u} , kinematic pressure p , temperature θ
- thermal diffusivity α , thermal expansion coefficient β , gravity \mathbf{g}
- small temperature differences \Rightarrow density $\varrho \approx \text{const.}$

Weak Formulation

Find $(\mathcal{U}_h, \theta_h) = (\mathbf{u}_h, p_h, \theta_h) : (0, T) \rightarrow \mathbf{V}_h^{div} \times Q_h \times \Theta_h$ such that

$$(\partial_t \mathbf{u}_h, \mathbf{v}_h) + \mathcal{A}_{G,u}(\mathbf{u}_h, \mathcal{U}_h, \mathcal{V}_h) = (f, \mathbf{v}_h) - (\beta \theta_h g, \mathbf{v}_h)$$

$$(\partial_t \theta_h, \psi_h) + \mathcal{A}_{G,\theta}(\mathbf{u}_h, \theta_h, \psi_h) = (f_\theta, \psi_h)$$

for all $(\mathbf{V}_h, \psi_h) = (\mathbf{v}_h, q_h, \psi_h) \in \mathbf{V}_h \times Q_h \times \Theta_h$

where

$$\mathcal{A}_{G,u}(\mathbf{w}; \mathcal{U}, \mathcal{V}) := a_G(\mathcal{U}, \mathcal{V}) + c(\mathbf{w}; \mathbf{u}, \mathbf{v})$$

$$a_G(\mathcal{U}, \mathcal{V}) := \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u})$$

$$\mathcal{A}_{G,\theta}(\mathbf{u}, \theta, \psi) := \alpha(\nabla \theta, \nabla \psi) + c(\mathbf{u}; \theta, \psi)$$

$$c(\mathbf{w}, \mathbf{u}, \mathbf{v}) := \frac{((\mathbf{w} \cdot \nabla) \mathbf{u}, \mathbf{v}) - ((\mathbf{w} \cdot \nabla) \mathbf{v}, \mathbf{u})}{2}$$

Stabilization terms

- LPS-SUPG for the velocity

$$s_u(\mathbf{u}_h; \mathbf{w}_h, \mathbf{v}_h) := \sum_{M \in \mathcal{M}_h} \tau_M^u(\mathbf{u}_h) (\kappa_M^u(\mathbf{u}_M \cdot \nabla \mathbf{w}_h), \kappa_M^u(\mathbf{u}_M \cdot \nabla \mathbf{v}_h))_M$$

- grad-div

$$t_h(\mathbf{u}_h; \theta_h, \psi_h) := \sum_{M \in \mathcal{M}_h} \gamma_M(\mathbf{u}_h) (\nabla \cdot \mathbf{w}_h, \nabla \cdot \mathbf{v}_h)_M$$

- LPS-SUPG for the temperature

$$s_\theta(\mathbf{u}_h; \theta_h, \psi_h) := \sum_{M \in \mathcal{M}_h} \tau_M^\theta(\mathbf{u}_M) (\kappa_M^\theta(\mathbf{u}_M \cdot \nabla \theta_h), \kappa_M^\theta(\mathbf{u}_M \cdot \nabla \psi_h))_M$$

Convergence Result

Theorem

For sufficiently smooth solutions we obtain for $\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h$ and $e_\theta = \theta - \theta_h$:

$$\begin{aligned} & \|\mathbf{e}_h\|_{L^\infty(0,t;L^2(\Omega))}^2 + \int_0^t \|(\mathbf{e}_h(\tau), 0)\|_{LPS}^2 d\tau \\ & + \|e_\theta\|_{L^\infty(0,t;L^2(\Omega))}^2 + \int_0^t |[e_\theta(\tau)]|_{LPS}^2 d\tau \leq \mathcal{O}(h^{2k}) \end{aligned}$$

with a parameter choice according to

$$h_M \leq C \frac{\sqrt{\nu}}{\|\mathbf{u}\|_{L^\infty(M)}}$$

$$\tau_M \leq \tau_0 \frac{h_M^{2(k-s)}}{|\mathbf{u}_M|^2}$$

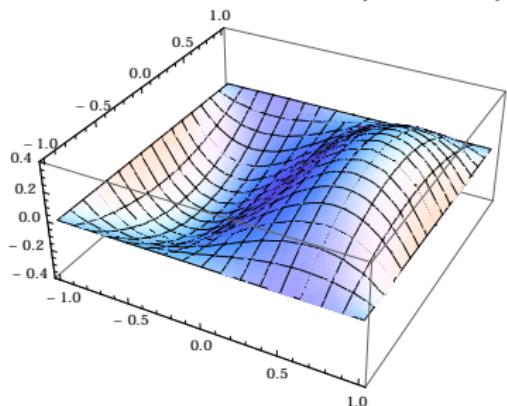
$$\gamma_M = \gamma_0$$

$$h_M \leq C \frac{\sqrt{\alpha}}{\|\mathbf{u}\|_{L^\infty(M)}}$$

$$\tau_M^\theta \leq \tau_{0,\theta} \frac{h_M^{2(k-s_\theta)}}{|\mathbf{u}_M|^2}$$

Bubble Enriched Ansatz Spaces

Enrich the tensor product-polynomial space \mathbb{Q}_k to obtain



$$\mathbb{Q}_k^+(\hat{T}) := \mathbb{Q}_k(\hat{T}) + \psi \cdot \text{span}\{\hat{x}_i^{k-1}, i = 1, \dots, d\}$$

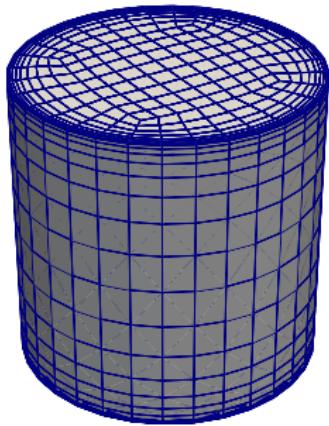
with bubble functions

$$\psi(\hat{\mathbf{x}}) := \prod_{i=1}^d (1 - \hat{x}_i^2)$$

Used spaces

- Taylor-Hood elements (Q_2/Q_1) for velocity and pressure
 - Q_2 or Q_2^+ for the temperature
 - Q_1 for the projection spaces

Rayleigh–Bénard Convection



Flow driven by temperature difference

Dirichlet BC's for bottom and top plate:

$$\theta_{bottom} = 0.5, \quad \theta_{top} = -0.5,$$

and isolating hull:

$$\mathbf{n} \cdot \nabla \theta|_{r=0.5} = 0$$

$$R = .5 \quad H = 1 \quad \beta = 1$$

No-slip boundary conditions for the velocity:

$$\nu = Pr^{-0.5} \cdot Ra^{0.5}$$

$$\mathbf{u} = \mathbf{0}|_{\partial\Omega}$$

$$\alpha = Ra^{-0.5} \cdot Pr^{-0.5}$$

Isosurfaces of the Temperature, $Pr = 0.786$

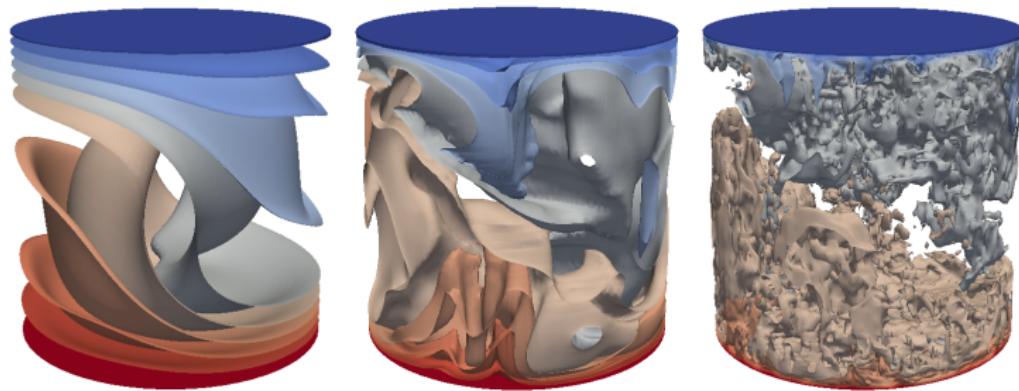


Figure: Isosurfaces of the Temperature, $T = 1000$, $Pr = 0.786$,
 $Ra = 10^5$, $Ra = 10^7$, $Ra = 10^9$, $N = 10 \cdot 16^3$, $\gamma_M = 0.1$

Benchmark quantity - Nusselt number

- Heat flux: bottom (warm) \Rightarrow top (cold)

$$q_z := u_z \theta - \alpha \partial_z \theta$$

- Nusselt number

$$Nu(z_0, t) := \frac{L}{\alpha A \Delta \theta} \langle q_z \rangle_{z=z_0}(z_0, t)$$

measure for $\frac{\text{convective heat transfer}}{\text{conductive heat transfer}}$

- $Nu(z_0, t)$ independent of z_0 for a stationary temperature field.

Time-development

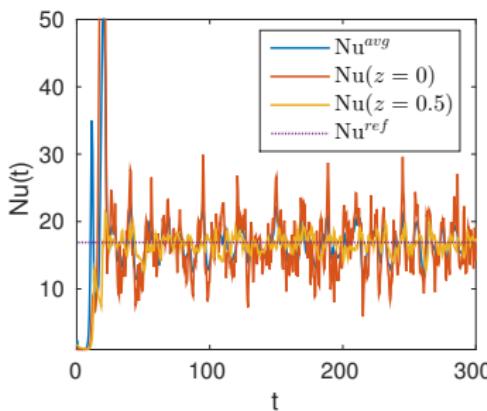
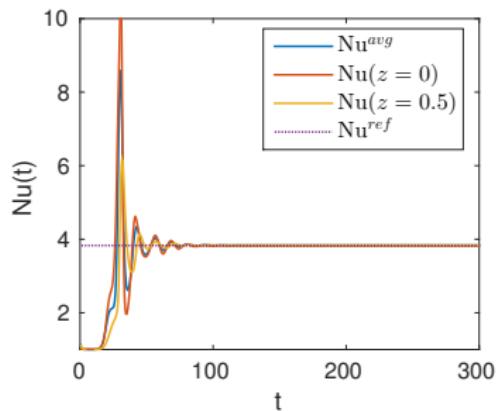


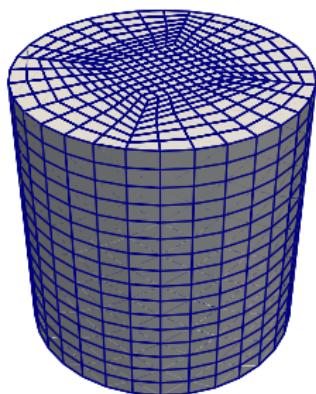
Figure: Time-development of Nu for $t \in [0, 300]$, $\gamma_M = 0.1$ $Pr = 0.786$, $Ra = 10^5$ und $Ra = 10^7$ ($N = 10 \cdot 8^3$)

Dependence on Ra ($N = 10 \cdot 8^3$ cells)

Ra	γ_M	Nu^{avg}	σ	$Nu^{\text{ref}} [\text{DNS}]$
10^5	0	3.8396	0.0356	3.83
	1	3.8364	0.0307	
	0.1	3.8372	0.0303	
10^6	0	8.6457	0.3378	8.6
	1	8.5148	0.0542	
	0.1	8.6475	0.0190	
10^7	0	16.4143	1.8302	16.9
	1	16.7361	0.1569	
	0.1	16.8767	0.1068	
10^8	0	37.7301	29.4731	31.9
	1	30.7236	0.7044	
	0.1	31.2902	0.6957	

Isotropic Mesh, $Ra = 10^9$ ($N = 10 \cdot 8^3$)

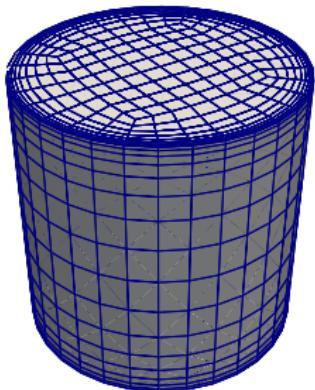
γ_M	τ_M^u	τ_L^θ	$Nu_{Id,th}^{\text{avg}}$	$\sigma_{Id,th}$	$Nu_{Id,bb}^{\text{avg}}$	$\sigma_{Id,bb}$	Nu^{ref}
0.01	0	0	41.4584	40.1989	47.5335	23.4029	63.1
0.01	hu1	0	38.7093	43.0326	44.2998	24.7851	
0.01	0	hu1	37.6081	10.8360	54.2603	16.5349	
0.01	hu1	hu1	37.0516	10.3065	49.1255	12.9235	



$$\begin{aligned} th: & (\mathbb{Q}_2/\mathbb{Q}_1) \wedge \mathbb{Q}_1 \wedge (\mathbb{Q}_2/\mathbb{Q}_1), \\ bb: & (\mathbb{Q}_2^+/\mathbb{Q}_1) \wedge \mathbb{Q}_1 \wedge (\mathbb{Q}_2^+/\mathbb{Q}_1) \\ hu1: & \tau_{M/L}^{u/\theta} = \tfrac{1}{2}h/\|\boldsymbol{u}_h\|_{\infty,M/L} \end{aligned}$$

Anisotropic mesh, $Ra = 10^9$ ($N = 10 \cdot 8^3$) cells

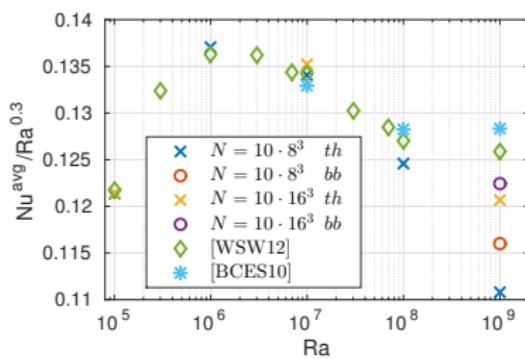
γ_M	τ_M^u	τ_L^θ	Nu_{th}^{avg}	σ_{th}	Nu_{bb}^{avg}	σ_{bb}	Nu^{ref}
0	0	0	118.7932	137.5588			63.1
0.01	0	0	55.5231	1.3464	58.1419	1.4833	
0.01	hu1	0	53.8371	1.4130	58.2691	1.4702	
0.01	0	hu1	52.4530	3.4847	56.5274	3.0578	
0.01	hu1	hu1	51.8141	3.4344	54.0410	3.3333	



th : $(Q_2/Q_1) \wedge Q_1 \wedge (Q_2/Q_1)$,
 bb : $(Q_2^+/Q_1) \wedge Q_1 \wedge (Q_2^+/Q_1)$
 $hu1$: $\tau_{M/L}^{u/\theta} = \frac{1}{2}h/\|\mathbf{u}_h\|_{\infty,M/L}$

Vary Discretization

	#cells	#DoF(u)	#DoF(p)	#DoF(θ)	$Nu_{\gamma_M=0.01}^{\text{avg}}$	$\sigma_{\gamma_M=0.01}$
th	$10 \cdot 8^3$	129,987	5,729	43,329	55.5231	1.3464
bb	$10 \cdot 8^3$	176,067	5,729	58,689	58.1419	1.4833
th	$10 \cdot 16^3$	1,011,075	43,329	337,025	60.4889	1.1574
bb	$10 \cdot 16^3$	1,379,715	43,329	459,905	61.3628	0.4668



Summary

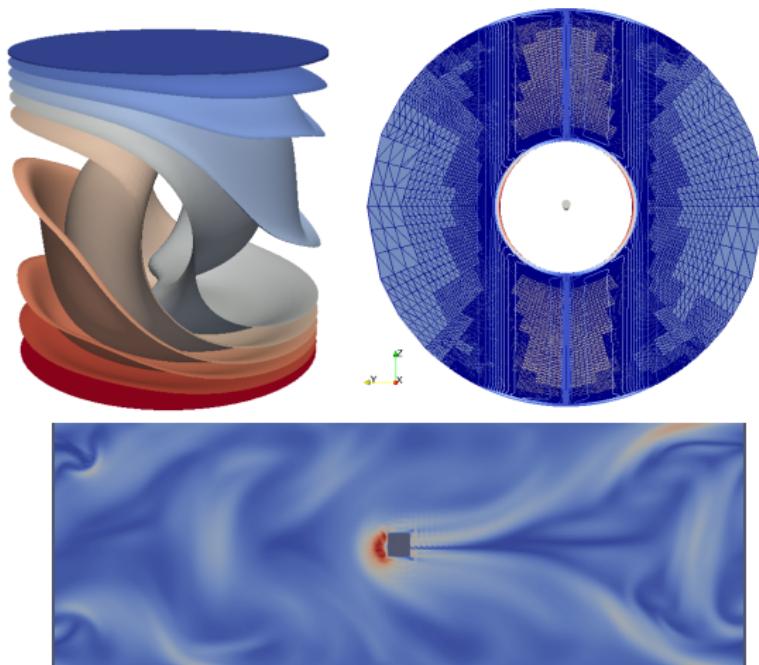
Analytical and numerical framework for LPS stabilized model of incompressible flow with

- rotating frames of references
- temperature coupling
- magnetohydrodynamic coupling

Next step: Combine all of these!

References

-  Daniel Arndt, Helene Dallmann, and Gert Lube, *Local Projection FEM Stabilization for the Time-Dependent incompressible Navier-Stokes Problem*, Numerical Methods for Partial Differential Equations **31** (2015), no. 4, 1224–1250.
-  Daniel Arndt and Gert Lube, *FEM with Local Projection Stabilization for Incompressible Flows in Rotating Frames*, Preprint.
-  Helene Dallmann and Daniel Arndt, *Stabilized Finite Element Methods for the Oberbeck-Boussinesq Model*, In Preparation.
-  Helene Dallmann, *Finite Element Methods with Local Projection Stabilization for Thermally Coupled Incompressible Flow*, Ph.D. thesis, Georg-August-Universität Göttingen, 2015.
-  Benjamin Wacker, Daniel Arndt, and Gert Lube, *Nodal-Based Finite Element Methods with Local Projection Stabilization for Linearized Incompressible Magnetohydrodynamics*, CMAME (2015).



Thanks for your attention!