Aircraft Structural Analysis

Master Course in Aerospace Engineering

Extra material II — Levy's approximate solution for rectangular plates (for comparison with FEM)

Reference material

Rectangular Plates - Approximate Solutions, Chapter 5
Levy's solution for simply supported rectangular plates
(sections 5.4, 5.5)

of the reference book: Ansel C. Ugural, "Stresses in Plates and Shells", 2nd ed., McGraw-Hill

The motivation…

- to find a solution for the **out-of-plane displacements (deflections)**, in an approximate way

- to write the approximate solution for deflections as a **single Fourier series**

- to **increase the convergence rate** of the solution, for bending moments (and stresses)

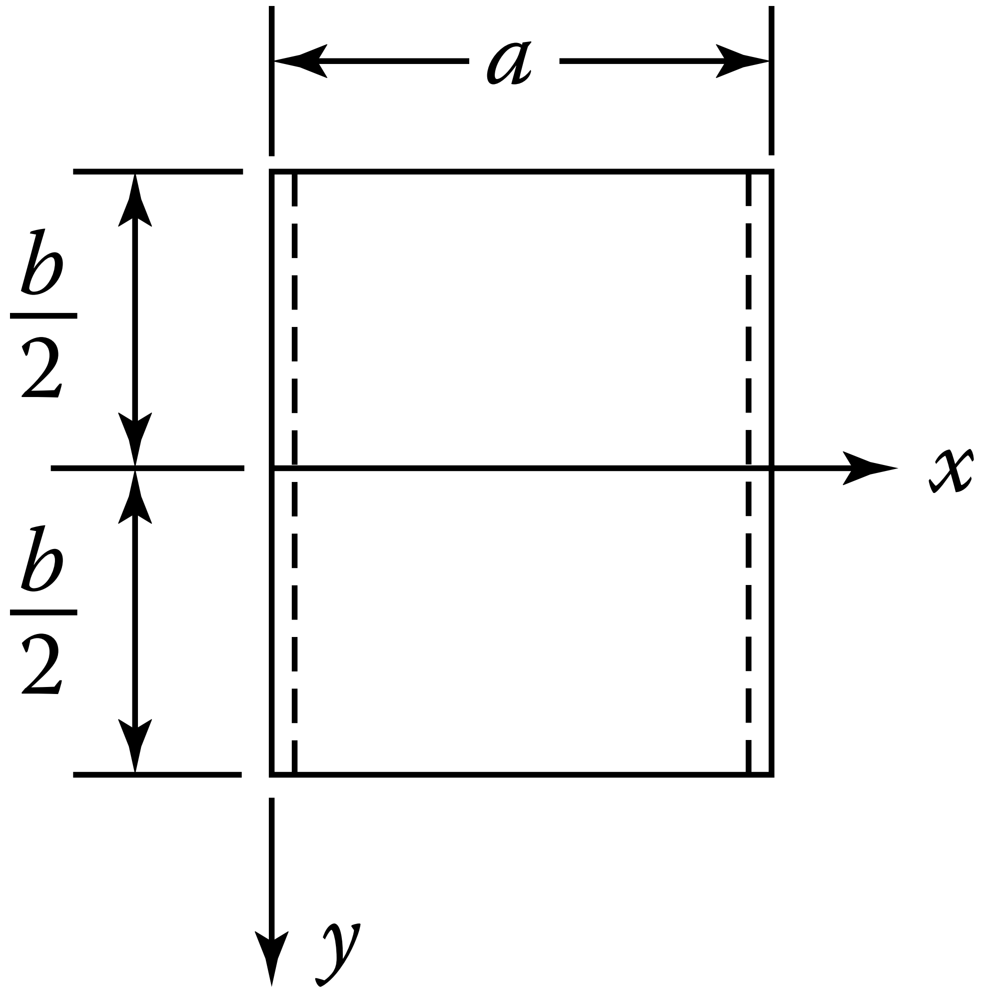
- to avoid using (directly) the fourth-order differential equation for displacements

$$w(x,y) = ???$$

The idea…

- the solution for the deflection will come from the **superposition** of two individual problems

- rectangular plates with **particular boundary conditions** on two opposite sides (simply supported, x= 0, x=a) and **arbitrary boundary conditions** on the other edges (y=+/- b/2)

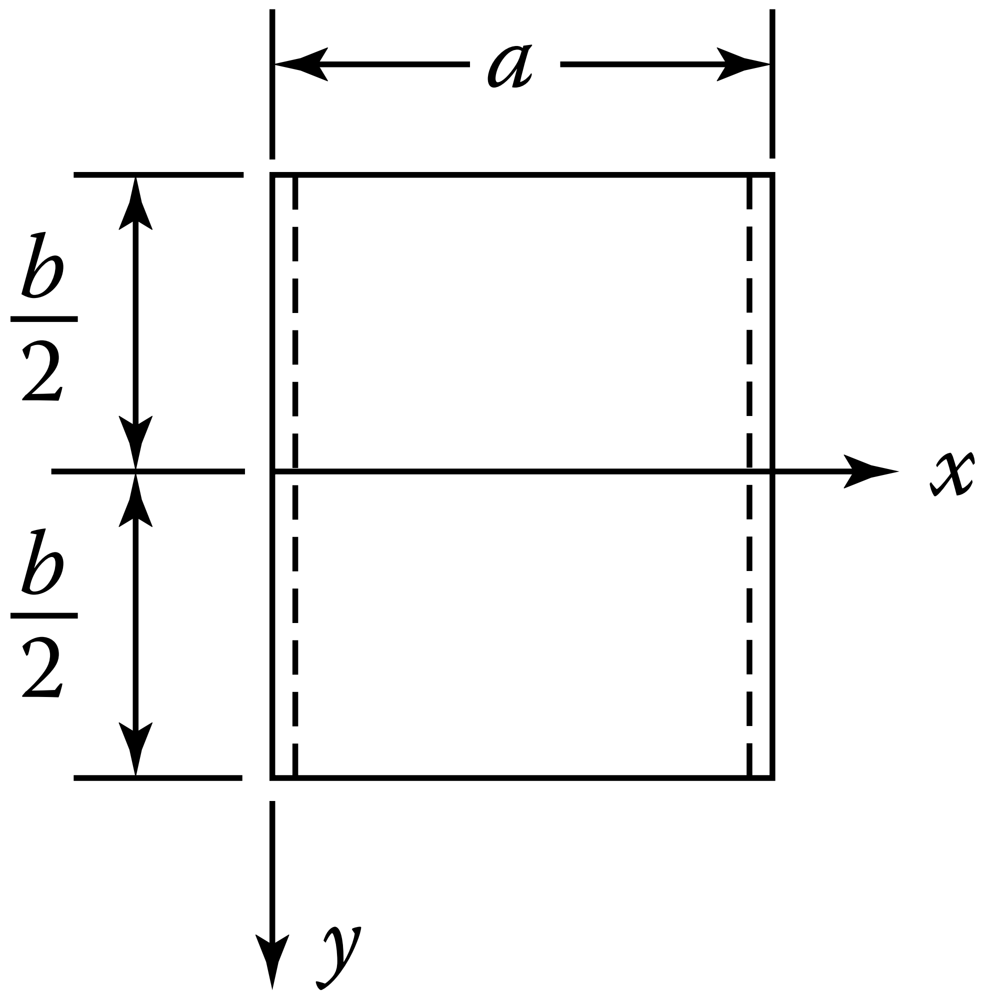


$$\begin{cases} w = 0\big|_{x=0,x=a} \\ \dfrac{\partial^2 w}{\partial x^2} = 0\Bigg|_{x=0,x=a} \end{cases}$$

The idea…

- the solution for the deflection will come from the superposition of two individual problems

$$w(x,y) = w_h(x,y) + w_p(x,y)$$



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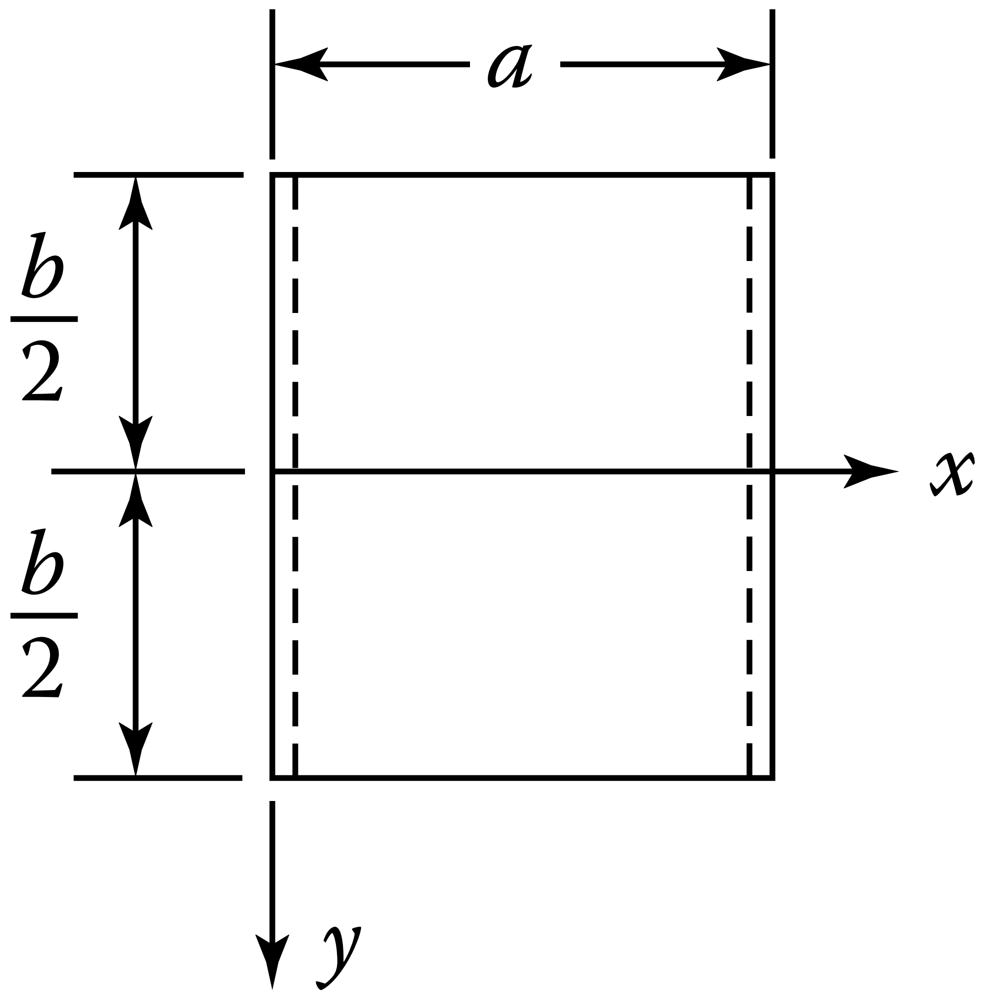


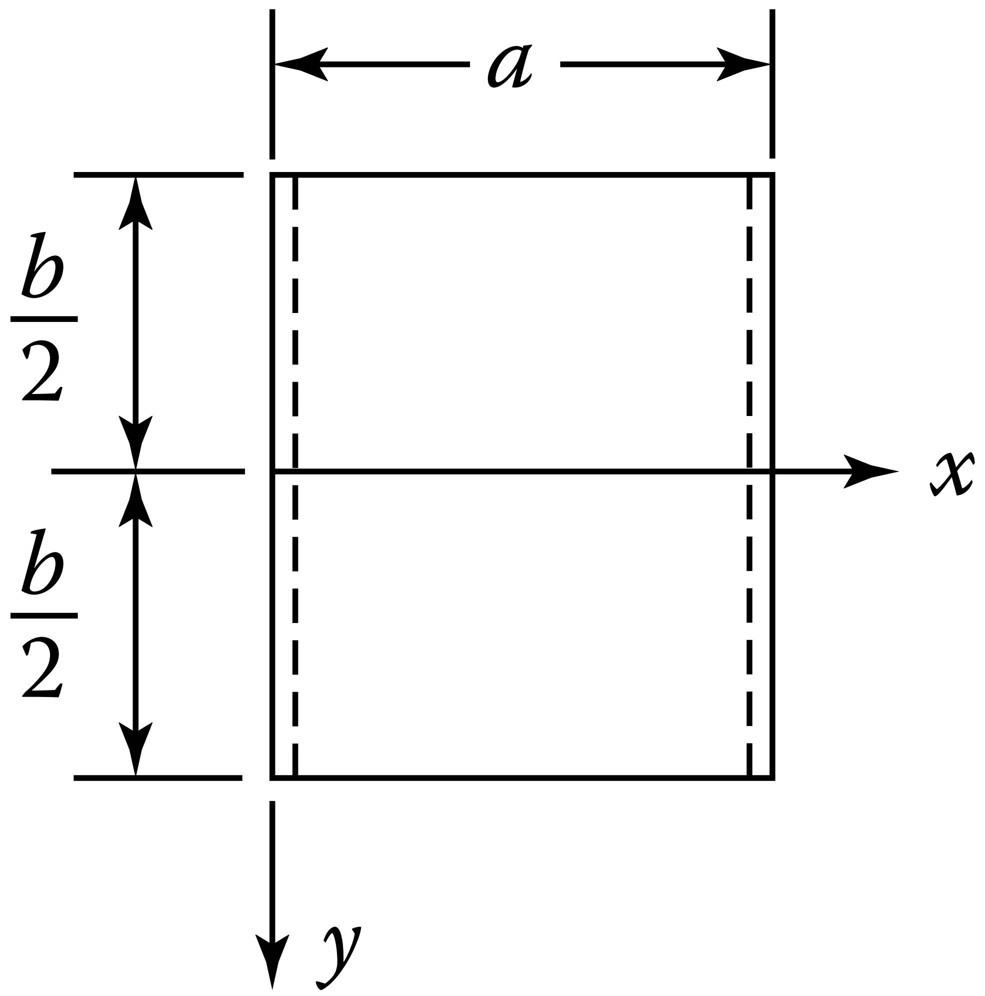
a **homogeneous solution** that depends only on the **geometry** and **boundary conditions** of the plate

$$p(x,y) = 0 \rightarrow \quad \nabla^4 w_h(x,y) = 0$$

The idea…

- the solution for the deflection will come from the superposition of two individual problems

$$w(x,y) = w_h(x,y) + w_p(x,y)$$



a **particular solution** that depends only on the **loading**

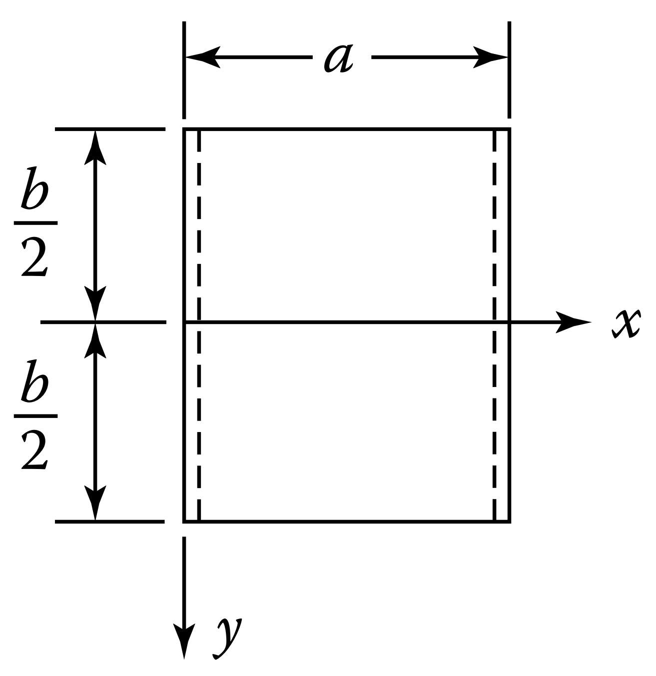
$$\nabla^4 w_p(x,y) = \frac{p(x,y)}{D}$$

a **homogeneous solution** that depends only on the **geometry** and **boundary conditions** of the plate

$$p(x,y) = 0 \rightarrow \qquad \nabla^4 w_h(x,y) = 0$$

A proposal for the homogeneous solution…

here it is, a (single) Fourier series, separating the variables…



$$f_m(y) = ???$$

$$w_h(x,y) = \sum_{m=1}^{\infty} f_m(y) \sin \frac{m\pi}{a} x \longrightarrow \nabla^4 w_h(x,y) = 0$$

$$\nabla^4 w(x,y) = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}$$

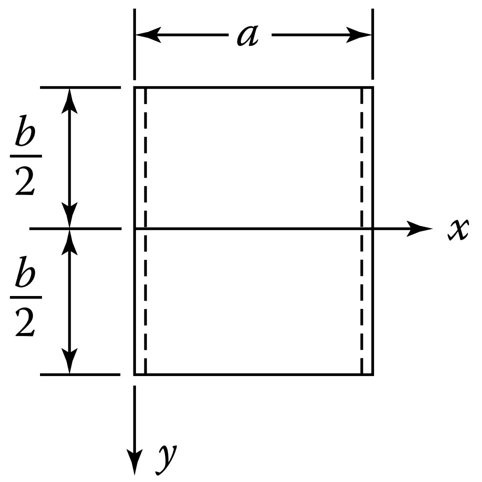
$$\nabla^4 w_h(x,y) = \frac{\partial^4 w_h}{\partial x^4} + 2\frac{\partial^4 w_h}{\partial x^2 \partial y^2} + \frac{\partial^4 w_h}{\partial y^4} = 0$$

$$\left[\left(\frac{m\pi}{a}\right)^4 f_m(y) - 2\left(\frac{m\pi}{a}\right)^2 \frac{\mathrm{d}^2 f_m(y)}{\mathrm{d}y^2} + \frac{\mathrm{d}^4 f_m(y)}{\mathrm{d}y^4}\right] \sin\frac{m\pi x}{a} = 0$$

$$\frac{\mathrm{d}^4 f_m(y)}{\mathrm{d}y^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{\mathrm{d}^2 f_m(y)}{\mathrm{d}y^2} + \left(\frac{m\pi}{a}\right)^4 f_m(y) = 0$$

$$m = 1,2,3,\cdots$$

Here comes the homogeneous part of the deflection…



$$w_h(x,y) = \sum_{m=1}^{\infty} f_m(y) \sin\frac{m\pi}{a}x \longrightarrow \nabla^4 w_h(x,y) = 0$$

$$f_m(y) = A_m \sinh\frac{m\pi y}{a} + B_m \cosh\frac{m\pi y}{a} + \frac{m\pi y}{a}\left(C_m \sinh\frac{m\pi y}{a} + D_m \cosh\frac{m\pi y}{a}\right)$$

$$\boxed{w_h = \sum_{m=1}^{\infty}\left[A_m \sinh\frac{m\pi y}{a} + B_m \cosh\frac{m\pi y}{a} + \frac{m\pi y}{a}\left(C_m \sinh\frac{m\pi y}{a} + D_m \cosh\frac{m\pi y}{a}\right)\right]\sin\frac{m\pi}{a}x}$$

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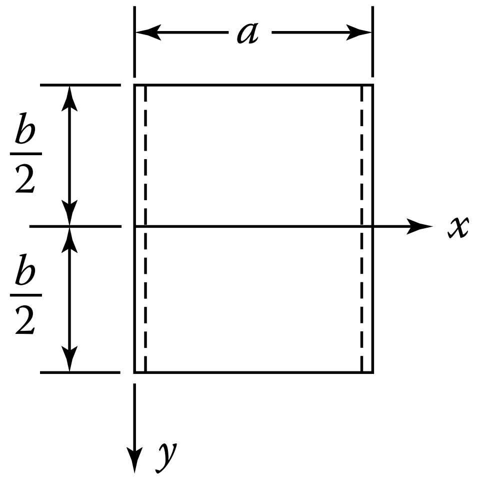


$$w(x,y) = w_h(x,y) + w_p(x,y)$$

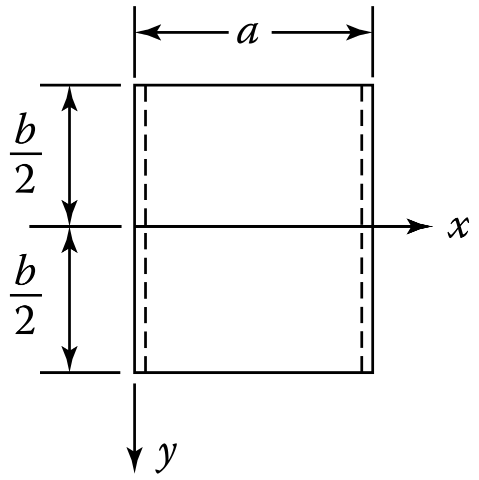
$$w_h(x,y) = \sum_{m=1}^{\infty} f_m(y) \sin\frac{m\pi}{a}x \longrightarrow \nabla^4 w_h(x,y) = 0$$

$$f_m(y) = A_m \sinh\frac{m\pi y}{a} + B_m \cosh\frac{m\pi y}{a} + \frac{m\pi y}{a}\left(C_m \sinh\frac{m\pi y}{a} + D_m \cosh\frac{m\pi y}{a}\right)$$

$$\boxed{w_h = \sum_{m=1}^{\infty}\left[A_m \sinh\frac{m\pi y}{a} + B_m \cosh\frac{m\pi y}{a} + \frac{m\pi y}{a}\left(C_m \sinh\frac{m\pi y}{a} + D_m \cosh\frac{m\pi y}{a}\right)\right]\sin\frac{m\pi}{a}x}$$

these constants will now come from the boundary conditions on y=+/- b/2

Now for the particular solution for the deflection…



$$w(x,y) = w_h(x,y) + w_p(x,y)$$

$$w_h(x,y) = \sum_{m=1}^{\infty} f_m(y) \sin \frac{m\pi}{a} x$$

- the same idea, with a single Fourier series also for the term $w_p(x,y)$

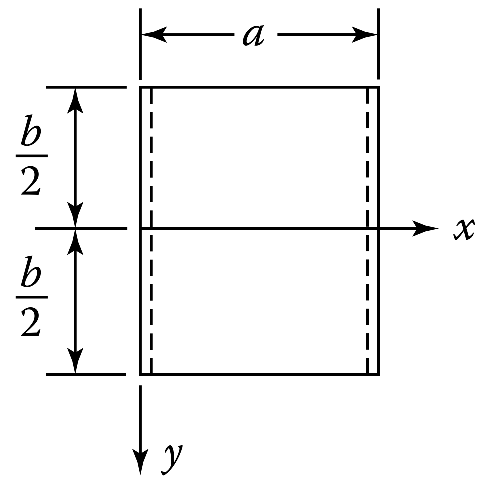
$$w_p(x,y) = \sum_{m=1}^{\infty} g_m(y) \sin \frac{m\pi}{a} x$$

- and doing exactly the same, for the general loading term…

$$p(x,y) = \sum_{m=1}^{\infty} p_m(y) \sin \frac{m\pi}{a} x$$

$$p_m(y) = \frac{2}{a} \int_0^a p(x,y) \sin \frac{m\pi}{a} x \, \mathrm{d}x$$

Next, we put everything together into the differential equation…



$$w_p(x,y)=\sum_{m=1}^{\infty} g_m(y)\sin\frac{m\pi}{a}x$$

$$p(x,y)=\sum_{m=1}^{\infty} p_m(y)\sin\frac{m\pi}{a}x$$

$$\longrightarrow \quad \nabla^4 w_p(x,y)=\frac{p(x,y)}{D}$$

$$\frac{\mathrm{d}^4 g_m(y)}{\mathrm{d}y^4}-2\left(\frac{m\pi}{a}\right)^2\frac{\mathrm{d}^2 g_m(y)}{\mathrm{d}y^2}+\left(\frac{m\pi}{a}\right)^4 g_m(y)=\frac{p_m(y)}{D}$$

$$\boxed{g_m(y)=\frac{1}{D\pi^4}\frac{a^4}{m^4}p_m(y)}$$

$$D=\frac{Et^3}{12\left(1-\nu^2\right)}$$

The complete solution for the deflection will finally come as

$$w(x,y) = w_h(x,y) + w_p(x,y)$$

$$w(x,y) = \sum_{m=1}^{\infty}\left[\frac{p_m a^4}{\pi^4 D m^4} + A_m \sinh\frac{m\pi y}{a} + B_m \cosh\frac{m\pi y}{a} + \frac{m\pi y}{a}\left(C_m \sinh\frac{m\pi y}{a} + D_m \cosh\frac{m\pi y}{a}\right)\right]\sin\frac{m\pi}{a}x$$

where the constants are calculated from the boundary conditions on y=+/- b/2

$$D = \frac{Et^3}{12\left(1-\nu^2\right)}$$

Summary chart (Levy method)

- represent the external load by a Fourier series: $$p(x,y)=\sum_{m=1}^{\infty} p_m(y)\sin\frac{m\pi}{a}x$$

- calculate the loading coefficients: $$p_m(y)=\frac{2}{a}\int_0^a p(x,y)\sin\frac{m\pi}{a}x\,\mathrm{d}x$$

- represent the homogeneous solution as a Fourier series: $$w_h(x,y)=\sum_{m=1}^{\infty} f_m(y)\sin\frac{m\pi}{a}x$$

- calculate the coefficients for the homogeneous part of the deflection:

$$f_m(y)=A_m\sinh\frac{m\pi y}{a}+B_m\cosh\frac{m\pi y}{a}+\frac{m\pi y}{a}\left(C_m\sinh\frac{m\pi y}{a}+D_m\cosh\frac{m\pi y}{a}\right)$$

- represent the particular solution as a Fourier series:

$$w_p(x,y)=\sum_{m=1}^{\infty} g_m(y)\sin\frac{m\pi}{a}x \qquad g_m(y)=\frac{1}{D\pi^4}\frac{a^4}{m^4}p_m(y)$$

- calculate the total deflection: $$w(x,y)=w_h(x,y)+w_p(x,y)$$

$$D=\frac{Et^3}{12\left(1-\nu^2\right)}$$

Summary chart (moments, stresses, general)

$$M_x = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_{xx} = -\frac{E}{1-\nu^2}z\left(\kappa_x + \nu\kappa_y\right) = -\frac{E}{1-\nu^2}z\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$\sigma_{yy} = -\frac{E}{1-\nu^2}z\left(\kappa_y + \nu\kappa_x\right) = -\frac{E}{1-\nu^2}z\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = -\frac{Et^3}{12(1-\nu^2)}(1-\nu)\frac{\partial^2 w}{\partial x\partial y} = -D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}$$

$$\tau_{xy} = -\frac{E}{1-\nu^2}z\kappa_{xy} = -\frac{E}{1+\nu}z\frac{\partial^2 w}{\partial x\partial y}$$

$$\sigma_{xx} = \frac{12M_x}{t^3}z \quad\Rightarrow\quad \sigma_{xx}\Big|_{\max/\min} = \pm\frac{6M_x}{t^2}$$

$$\sigma_{yy} = \frac{12M_y}{t^3}z \quad\Rightarrow\quad \sigma_{yy}\Big|_{\max/\min} = \pm\frac{6M_y}{t^2}$$

$$\tau_{xy} = \frac{12M_{xy}}{t^3}z \quad\Rightarrow\quad \tau_{xy}\Big|_{\max/\min} = \pm\frac{6M_{xy}}{t^2}$$

$$D = \frac{Et^3}{12\left(1-\nu^2\right)}$$

End!

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