As will become evident in the foregoing situations, a deformation pattern can be established from symmetry considerations alone.

EXAMPLE 3.1

Determine the deflection and stress in a very long and narrow rectangular plate if it is simply supported at edges $y = 0$ and $y = b$ (Figure 3.9). (a) The plate carries a nonuniform loading expressed by

$$p(y) = p_0 \sin\frac{\pi y}{b} \tag{a}$$

where the constant p_0 represents the load intensity along the line passing through $y = b/2$, parallel to the x axis. (b) The plate is under a uniform load p_0. Let $p_0 = 10$ kPa, $b = 0.4$ m, $t = 10$ mm, $\nu = 1/3$, and $E = 200$ GPa.

SOLUTION

Clearly, the loading described deforms the plate into a *cylindrical surface* possessing its generating line parallel to the x axis: The slope of the deflected plate along the x axis is zero. We thus have $\partial w / \partial x = 0$. It follows that $\partial^2 w / \partial x \partial y = 0$, and Equations 3.9 yield

$$M_x = -\nu D\frac{d^2 w}{dy^2} \qquad M_y = -D\frac{d^2 w}{dy^2}. \tag{3.33}$$

Equation 3.17 reduces to

$$\frac{d^4 w}{dy^4} = \frac{p}{D}. \tag{3.34}$$

This expression is of the same form as the beam equation. Hence, the solution proceeds as in the case of a beam.

Note that since the bent plate surface is of *developable* type and the edges are free to move horizontally, the formulas derived in this example also hold for large deflections $(w \geq t$ but $w < b)$.

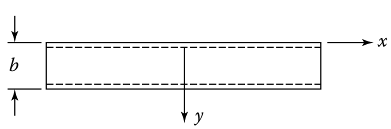


Figure 3.9 Simply supported plate strip.

(a) Substituting Equation a into Equation 3.34, integrating successively four times, and satisfying the boundary conditions ($w = 0$ and $d^2 w / dy^2 = 0$ at $y = 0$ and $y = b$), we have

$$w = \left(\frac{b}{\pi}\right)^4 \frac{p_0}{D} \sin\frac{\pi y}{b}. \tag{3.35}$$

The maximum stresses in the plate are obtained by substituting the above with $\nu = 1/3$ into Equations 3.11, 3.31, and 3.32:

$$\sigma_{x,\max} = 0.2 p_0\left(\frac{b}{t}\right)^2 \qquad \sigma_{y,\max} = 0.6\, p_0\left(\frac{b}{t}\right)^2 \qquad \left(z = \frac{t}{2},\, y = \frac{b}{2}\right).$$

$$\sigma_{z,\max} = -p_0 \qquad \left(z = -\frac{t}{2}\right).$$

$$\tau_{xy} = 0 \qquad \tau_{xz} = 0.$$

$$\tau_{yz,\max} = 0.5\, p_0\left(\frac{b}{t}\right) \qquad (z = 0,\ y = 0).$$

To gauge the magnitude of the deviation between the stress components, consider the ratios

$$\frac{\sigma_{z,\max}}{\sigma_{x,\max}} = 5\left(\frac{t}{b}\right)^2 \qquad \frac{\tau_{yz,\max}}{\sigma_{x,\max}} = 2.5\left(\frac{t}{b}\right).$$

If, for example, $b = 20t$, the above quotients are only $\frac{1}{80}$ and $\frac{1}{8}$, respectively. For a thin plate, $t / b < \frac{1}{20}$, and it is clear that stresses σ_z and τ_{yz} are very small compared with the normal stress components in the xy plane.

(b) Now Equation 3.34 with $p = p_0$, on successively integrating four times and satisfying $w = 0$ and $d^2 w / dy^2 = 0$ at $y = 0$ and $y = b$, yields

$$w = \frac{p_0 b^4}{24D}\left(\frac{y^4}{b^4} - 2\frac{y^3}{b^3} + \frac{y}{b}\right). \tag{3.36}$$

This represents the deflection of a uniformly loaded and simply supported plate strip parallel to the y axis. The maximum deflection of the plate is found by substituting $y = b/2$ in Equation 3.36, yielding $w_{\max} = 5 p_0 b^4 / 384D$. The largest moment and stress also occur at $y = b/2$, in the direction of the shorter span b. These are readily calculated (by means of Equations 3.36, 3.9, and 3.11) as $p_0 b_2/8$ and $3p_0 b_2/4t^2$ respectively. It is observed that for very long and narrow plates, the supports along the short sides have little effect on the action in the plate, and hence the *plate behaves as would a simple beam* of span b.

Introducing the given numerical values into Equation 3.10,

$$D = \frac{200(10^9)(0.01)^3}{12\left(1-\frac{1}{9}\right)} = 18{,}750.$$

Similarly,

$$w_{\max} = \frac{5p_0b^4}{384D} = \frac{5(10\times10^3)(0.4)^4}{384(18{,}750)} = 0.18 \text{ mm}$$

$$\sigma_{y,\max} = \frac{3p_0b^2}{4t^2} = \frac{3(10\times10^3)}{4}\left(\frac{400}{10}\right)^2 = 12 \text{ MPa}$$

$$\sigma_{x,\max} = \nu\sigma_{y,\max} = 4 \text{ MPa}.$$

Interestingly, Hooke's law, with $\sigma_z = 0$, gives

$$\varepsilon_{y,\max} = \frac{1}{200(10^9)}\left(12-\frac{4}{3}\right)10^6 = 53\mu$$

and, from Equation 3.3b, the radius of curvature is

$$r_y = -\frac{0.01}{2(53\times10^{-6})} = -94.3 \text{ m}.$$

Thus, $w_{\max}/t = 0.018$ and $r_y/b = 236$. These results show that the deflection curve is *extremely flat*, as is usually the case for small deformations.

EXAMPLE 3.2

A circular plate, clamped at the edge, is under a uniform pressure of intensity p_0 (Figure 3.10). Derive an expression for the surface deflection.

SOLUTION

The equation of the boundary of the plate is

$$1-\frac{x^2}{a^2}-\frac{y^2}{a^2} = 0. \tag{b}$$

The conditions for the edge

$$w = 0 \qquad \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0 \qquad (r = a) \tag{c}$$

are satisfied by taking the deflection w in the form

$$w = k\left(1-\frac{x^2}{a^2}-\frac{y^2}{a^2}\right)^2. \tag{d}$$

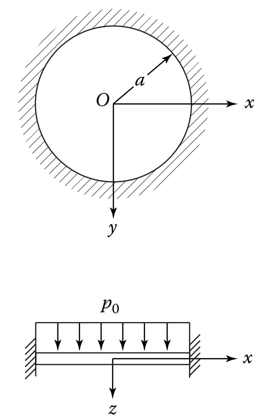


Figure 3.10 Circular plate under uniform pressure.

Here k represents an unknown constant. Note that Equation d and its first derivatives with respect to x and y become zero at the boundary by virtue of Equation b. The fourth derivatives of w are

$$\frac{\partial^4 w}{\partial x^4} = 24\frac{k}{a^4} \qquad \frac{\partial^4 w}{\partial x^2\partial y^2} = 8\frac{k}{a^4} \qquad \frac{\partial^4 w}{\partial y^4} = 24\frac{k}{a^4}.$$

Substitution of these into Equation 3.17 leads to

$$\frac{k}{a^4}(24+16+24) = \frac{p_0}{D}$$

or

$$k = \frac{p_0a^4}{64D}. \tag{e}$$

The deflection is thus

$$w = \frac{p_0a^4}{64D}\left(1-\frac{x^2}{a^2}-\frac{y^2}{a^2}\right)^2. \tag{3.37}$$

The moments and the stresses corresponding to Equation 3.37 are derived from Equations 3.9 and 3.7, respectively.

Similarly, the deflection of a uniformly loaded elliptical plate with clamped edge is readily derived (see Section 6.4). However, the circular plate problems are usually treated by employing polar coordinates, as will be illustrated in Chapter 4.

EXAMPLE 3.3

Determine the displacement of a rectangular plate with free edges of lengths *a* and *b* and subjected to transverse corner forces of magnitude *P*, as shown in Figure 3.11.

SOLUTION

The plate is free from surface loading, and hence $p = 0$. The boundary conditions are represented by

$$\begin{aligned} M_x &= 0 \qquad V_x = 0 \qquad \left(x = \pm\frac{a}{2}, y\right) \\ M_y &= 0 \qquad V_y = 0 \qquad \left(x, y = \pm\frac{b}{2}\right). \end{aligned} \tag{f}$$

Inasmuch as the center of the plate is free of displacement, we may assume the deflection w in the form

$$w = c_1 xy. \tag{g}$$

where c_1 is a constant. It is readily shown that Equations 3.18 and f are fulfilled by this expression. On the basis of Equations 3.24, 3.9, and g, the corner conditions, $F_c = -2M_{xy}$, lead to $P = 2D(1-\nu)c_1$ or $c_1 = P/2D(1-\nu)$.

Substituting this value of c_1 into Equation g, the surface deflection is found to be

$$w = \frac{P}{2b(1-\nu)}xy. \tag{3.38}$$

The corner displacements are then

$$w_B = w_D = -w_A = -w_C = \frac{Pab}{8D(1-\nu)}. \tag{3.39}$$

Here the negative sign indicates an upward direction.

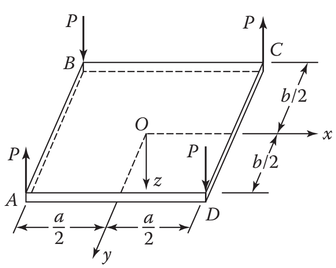


Figure 3.11 Rectangular plate subjected to corner loads.

EXAMPLE 3.4

A rectangular bulkhead of an elevator shaft is subjected to uniformly distributed bending moments $M_x = M_b$ and $M_y = M_a$, applied along its edges (Figure 3.12a). Derive the equation governing the surface deflection for two cases: (a) $M_a \neq M_b$, (b) $M_a = -M_b$.

SOLUTION

(a) We have positive edge moments applied to the plate (Figure 3.12a). Substituting $M_b = M_x$ and $M_a = M_y$ into Equations 3.9 gives

$$\frac{\partial^2 w}{\partial x^2} = -\frac{M_b - \nu M_a}{D(1-\nu^2)} \qquad \frac{\partial^2 w}{\partial y^2} = -\frac{M_a - \nu M_b}{D(1-\nu^2)} \qquad \frac{\partial^2 w}{\partial x \partial y} = 0. \tag{h}$$

Integrating the above leads to

$$w = -\frac{M_b - \nu M_a}{2D(1-\nu^2)}x^2 - \frac{M_a - \nu M_b}{2D(1-\nu^2)}y^2 + c_1 x + c_2 y + c_3.$$

If the origin of xyz is located at the center and midsurface of the *deformed plate*, the constants of integration vanish, and we have

$$w = -\frac{M_b - \nu M_a}{2D(1-\nu^2)}x^2 = \frac{M_a - \nu M_b}{2D(1-\nu^2)}y^2. \tag{3.40}$$

(b) Now y-edges are subjected to negative moments $-M_a$. Thus, by letting $M_a = -M_b$ in Equations h, the result is

$$\kappa_x = -\kappa_y = \frac{\partial^2 w}{\partial x^2} = -\frac{M_b}{D(1-\nu)}. \tag{i}$$

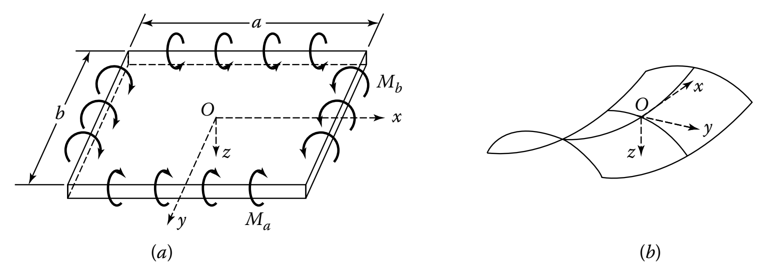


Figure 3.12 A large rectangular plate in pure bending.

This reveals that there is a saddle point 0 at the center of the plate (Figure 3.12b). Integrating and locating the origin *xyz* as before, Equation i leads to

$$w = -\frac{M_b}{2D(1-\nu)}(x^2 - y^2). \tag{3.41}$$

It is clear that the above expression represents an *anticlastic* or saddle-shaped surface with a negative Gaussian curvature. We note that in the particular case where $M_a = M_b$, Equation 3.40 yields a *paraboloid of revolution* (see Problem 3.18).

3.12 Strain Energy of Plates

The strain energy stored in an elastic body, for a general state of stress, is expressed by substituting Equation 2.42 into Equation 2.45. In so doing, we obtain

$$U = \frac{1}{2}\iiint_V (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})\, dx\, dy\, dz. \tag{3.42}$$

Integration extends over the entire body volume. According to the assumptions of Section 3.2, for thin plates, σ_z, γ_{xz}, and γ_{yz} can be omitted. Thus, by Hooke's law and Equations 3.6, the above expression reduces to the following form involving only stresses and elastic constants:

$$U = \iiint_V \left[\frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y\right) + \frac{1}{2G}\tau_{xy}^2\right] dx\, dy\, dz. \tag{3.43}$$

For a plate of *uniform thickness*, Equation 3.43 may be written in terms of deflection *w* by use of Equations 3.7 and 3.10 as follows:

$$U = \frac{1}{2}\iint_A D\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dx\, dy$$

or, alternately,

$$U = \frac{1}{2}\iint_A D\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dx\, dy. \tag{3.44}$$

where *A* is the area of the plate surface.

An alternate form of Equation 12.6a may be obtained from Equation 12.1b and Equation b. The membrane forces are then

$$\begin{aligned} N_s &= -\frac{F}{2\pi r_0 \sin\phi} \\ N_\theta &= -\frac{p_z r_0}{\sin\phi}. \end{aligned} \tag{12.8}$$

It is observed that given an external load distribution, hoop and meridional stresses can be computed *independently*.

12.6.3 Circular Cylindrical Shell

To obtain the stress resultants in a circular cylindrical shell (Figure 12.5b), we can begin with the cone equations, setting $\phi = \pi/2$, $p_z = p_r$ and mean radius $a = r_0 =$ constant. Hence, Equations 12.8 become

$$N_s = N_x = -\frac{F}{2\pi a} \qquad N_\theta = -p_r a \tag{12.9}$$

in which *x* is measured in the axial direction.

For a closed-end cylindrical vessel under *constant internal pressure*, $p = -p_r$ and $F = -\pi a^2 p$. Equations 12.9 then yield the following axial and hoop stresses:

$$\sigma_x = \frac{pa}{2t} \qquad \sigma_\theta = \frac{pa}{t}. \tag{12.10}$$

From Hooke's law, the extension of the radius of the cylinder under the action of the stresses given above is

$$\delta_c = \frac{a}{E}(\sigma_\theta - \nu\sigma_x) = \frac{pa^2}{2Et}(2-\nu). \tag{12.11}$$

Solutions of various other cases of practical significance may be obtained by employing a procedure similar to that described in the foregoing paragraphs, as demonstrated in Examples 12.1 through 12.6.

EXAMPLE 12.1

Consider a simply supported *covered market dome* of radius *a* and thickness *t*, carrying only its own weight *p* per unit area. (a) Determine the stresses for a dome of half-spherical geometry (Figure 12.6a). (b) Assume that the hemispherical dome is constructed of 70-mm-thick concrete of unit weight 23 kN/m³ and span 2*a* = 56 m. Apply the maximum principal stress theory to evaluate the shell's ability to resist failure by fracture. The ultimate compressive strength or crushing strength

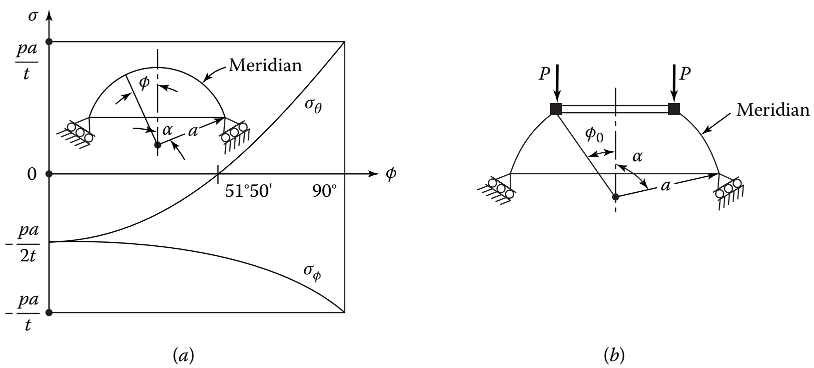


Figure 12.6 (a) Hemispherical dome; (b) truncated hemispherical dome.

of concrete σ_{uc} = 21 MPa, and E = 20 GPa. Also check the possibility of local buckling. (c) Determine the stresses in a done which is a truncated half-sphere (Figure 12.6b).

SOLUTION

The *components of the dome weight* are

$$p_x = 0 \qquad p_y = p\sin\phi \qquad p_z = p\cos\phi. \tag{12.12}$$

(a) Referring to Figure 12.6a, the weight of that part of the done subtended by ϕ is

$$F = \int_0^{\phi} p\cdot 2\pi a\sin\phi\cdot a\,d\phi = 2\pi a^2 p(1-\cos\phi). \tag{c}$$

Introduction into Equations 12.3 of p_z and F given by Equations 12.12 and c and division of the results by t yields the membrane stresses:

$$\begin{aligned}\sigma_{\phi} &= -\frac{ap(1-\cos\phi)}{t\sin^2\phi} = -\frac{ap}{t(1+\cos\phi)}\\ \sigma_{\theta} &= -\frac{ap}{t}\left(\cos\phi-\frac{1}{1+\cos\phi}\right).\end{aligned} \tag{12.13}$$

These stresses are plotted for a *hemisphere* in Figure 12.6a. Clearly, σ_ϕ is always compressive; its value increases with ϕ from $-pa/2t$ at the crown to pa/t at the edge. The *sign* of σ_θ, on the other hand, *changes* with the value of ϕ. The second of the above equations yields $\sigma_\theta = 0$ for $\phi = 51°50'$. When

ϕ is smaller than this value, σ_θ is compressive. For $\phi > 51°50'$, σ_θ is tensile, as shown in the figure.

(b) The maximum compressive stress in the dome is $\sigma_\phi = pa/t = (0.023t)a/t = 0.023\times 28 = 0.644$ MPa. Note that no failure occurs, as $|\sigma_\phi| < |\sigma_{uc}|$. Clearly, even for large domes, *the stress level due to self-weight is far from the limit stress of the material, at least in compression.* Note also that concrete is weak in tension and that a different conclusion may emerge from consideration of failure due to direct tensile forces. If the tensile strength of the material is smaller than 0.644 MPa, an assessment of tensile reinforcement will be required to ensure satisfactory design.

On application of Equation f of Section 12.3, the stress level at which local buckling occurs in the dome is found to be

$$\sigma_{cr} = 0.25(20\times 10^9)\left(\frac{7}{2800}\right) = 12.5\,\text{MPa}.$$

It is observed that there is no possibility of local buckling, as $\sigma_\phi < \sigma_{cr}$.

(c) Most domes are not closed at the upper portion and have a lantern, a small tower for lighting, and ventilation. In this case, a reinforcing ring is used to support the upper structure as shown in Figure 12.6b. Let $2\phi_0$ be the angle corresponding to the opening and P be the vertical load per unit length acting on the reinforcement ring. The resultant of the total load on that portion of the dome subtended by the angle ϕ, Equation c, is then

$$\begin{aligned}F &= 2\pi\int_{\phi_0}^{\phi} a^2 p\sin\phi\,d\phi + P\cdot 2\pi a\sin\phi_0\\ &= 2\pi a^2 p(\cos\phi_0-\cos\phi) + 2\pi Pa\sin\phi_0.\end{aligned}$$

Using Equations 12.3, we obtain

$$\begin{aligned}\sigma_\phi &= -\frac{ap}{t}\frac{\cos\phi_0-\cos\phi}{\sin^2\phi} - \frac{P}{t}\frac{\sin\phi_0}{\sin^2\phi}\\ \sigma_\theta &= \frac{ap}{t}\left(\frac{\cos\phi_0-\cos\phi}{\sin^2\phi}-\cos\phi\right) + \frac{P}{t}\frac{\sin\phi_0}{\sin^2\phi}\end{aligned} \tag{12.14}$$

for the hoop and the meridional stresses.

Note that the circumferential strain in the dome under the action of membrane stresses may be computed from Hooke's law:

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_\phi).$$

This strain contributes to a change $\varepsilon_\theta\cdot a\sin\alpha$ in edge radius. The simple support shown in the figure, free to move as the shell deforms under loading, ensures that no bending is produced in the neighborhood of the edge.

EXAMPLE 12.2

Consider a planetarium dome that may be approximated as an edge-supported *truncated cone.* Derive expressions for the hoop and meridional forces for two

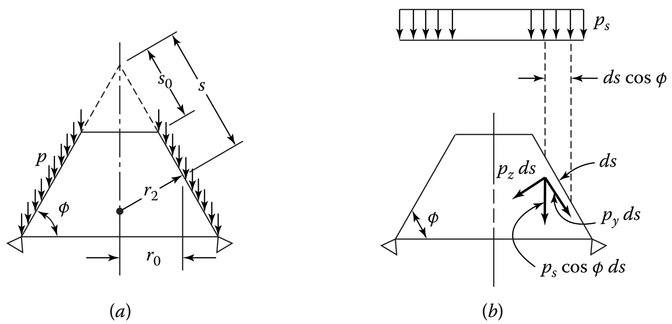


Figure 12.7 Truncated cone.

conditions of loading: (a) the shell carries its own weight p per unit area (Figure 12.7a); (b) the shell carries a snow load assumed to be uniformly distributed over the plan (Figure 12.7b).

SOLUTION

(a) Referring to Figure 12.7a,

$$r_2 = s\cot\phi \qquad r_0 = s\cos\phi \qquad p_z = p\cos\phi. \tag{d}$$

The weight of that part of the cone defined by $s - s_0$ is determined from

$$F = \int_{s0}^{s} p\cdot 2\pi r_2 \sin\phi \cdot ds = 2\pi p\int_{s0}^{s} s\cot\phi\sin\phi\, ds$$

or

$$F = \pi p\cos\phi(s^2 - s_0^2) + c.$$

As no force acts at the top edge, $c = 0$. Now the s- and θ-directed stress resultants can readily be obtained. Substituting F and Equations d into Equations 12.8 gives

$$\begin{aligned} N_s &= -\frac{p}{2s}\frac{s^2 - s_0^2}{\sin\theta} \\ N_\theta &= -ps\frac{\cos^2\phi}{\sin\phi}. \end{aligned} \tag{12.15}$$

(b) To analyze the components of the *snow load* p_s, we use a sketch of the forces acting on a midsurface element ds (Figure 12.7b). Referring to this figure, we have

$$p_x = 0 \qquad p_y = p_s\sin\phi\cos\phi \qquad p_z = p_s\cos^2\phi. \tag{12.16}$$

From Equations 12.8, d, and 12.16, we have

$$N_\theta = -p_s s\frac{\cos^3\phi}{\sin\phi}. \tag{12.17a}$$

Similarly, Equations 12.7, d, and 12.16 yield

$$\begin{aligned} N_s &= -\frac{1}{s}\int(p_s\sin\phi\cos\phi + p_s\cos^2\phi\cot\phi)s\,ds + \frac{c}{s} \\ &= \frac{p_s s}{2}\cot\phi + \frac{c}{s}. \end{aligned}$$

The condition that $N_s = 0$ at $s = s_0$ leads to $c = \frac{1}{2}p_s s_0^2\cot\phi$. Hence,

$$N_s = -\frac{p_s}{2s}(s^2 - s_0^2)\cot\phi. \tag{12.17b}$$

On dividing Equations 12.17 by t, the membrane stresses are obtained.

It is interesting to note that the *three typical loads* (weight per unit surface area, snow load per plan area, and wind load per surface area) are of the *same order*. For ordinary structures these might approximate 1500 to 2000 Pa.

EXAMPLE 12.3

A shell in the shape of a *torus* or *doughnut* of circular cross section is subjected to internal pressure p (Figure 12.8). Determine membrane stresses σ_ϕ and σ_θ.

SOLUTION

Consider the portion of the shell defined by ϕ. The vertical equilibrium of forces leads to

$$2\pi r_0\cdot N_\phi\sin\phi = \pi p(r_0^2 - b^2)$$

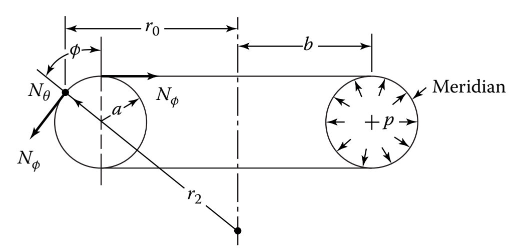


Figure 12.8 Toroidal shell.

or

$$N_\phi = \frac{p(r_0^2 - b^2)}{2r_0 \sin\phi} = \frac{pa(r_0 + b)}{2r_0}.$$

Introducing N_ϕ into Equation 12.1a, setting $p_z = -p$ and $r_1 = a$, we obtain

$$N_\theta = \frac{pr_2(r_0 - b)}{2r_0} = \frac{pa}{2}.$$

The stresses are then

$$\sigma_\phi = \frac{pa(r_0 + b)}{2r_0 t} \qquad \sigma_\theta = \frac{pa}{2t}. \tag{12.18}$$

It is noted that σ_θ is constant throughout the shell from the condition of symmetry.

EXAMPLE 12.4

Figure 12.9 represents the end enclosure of a cylindrical vessel in the form of a half *ellipsoid* of semiaxes a and b. Determine the membrane stresses resulting from an internal steam pressure p.

SOLUTION

Expressions for the principal radii of curvature r_1 and r_2 will be required. The equation of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ leads $y = \pm b\sqrt{a^2 - x^2}/a$. The magnitude of the derivatives of this expression are [18]

$$y' = \frac{bx}{a\sqrt{a^2 - x^2}} = \frac{b^2x}{a^2y} \qquad y'' = \frac{b^4}{a^2y^3}. \tag{e}$$

Referring to the figure,

$$\tan\phi = y' = \frac{x}{h} \qquad r_2 = \sqrt{h^2 + x^2}. \tag{f}$$

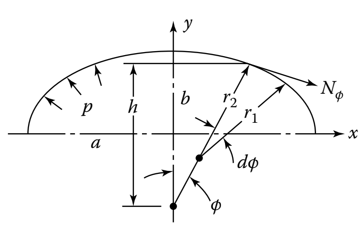


Figure 12.9 Ellipsoidal shell.

From the first of Equations e and f, $h = a\sqrt{a^2 - x^2}/b$, which, when substituted into the second of Equations f, yields r_2. Introduction of Equations e into the familiar expression for the curvature, $[1 + (y')^2]^{3/2}/y''$, gives the radius r_1. Thus,

$$r_1 = \frac{(a^4y^2 + b^4x^2)^{3/2}}{a^4b^4} \qquad r_2 = \frac{(a^4y^2 + b^4x^2)^{1/2}}{b^2}. \tag{12.19}$$

The load resultant is represented by $F = \pi p r_2^2 \sin^2\phi$. The membrane forces can then be determined from Equations 12.1 in terms of the principal curvatures. It follows that

$$\sigma_\phi = \frac{pr_2}{2t} \qquad \sigma_\theta = \frac{p}{t}\left(r_2 - \frac{r_2^2}{2r_1}\right). \tag{12.20}$$

At the *crown* (top of the shell), $r_1 = r_2 = a^2/b$, and Equations 12.20 reduce to

$$\sigma_\phi = \sigma_\theta = \frac{pa^2}{2bt}.$$

At the *equator* (base of the shell), $r_1 = b^2/a$ and $r_2 = a$, and Equations 12.20 appear as

$$\sigma_\phi = \frac{pa}{2t} \qquad \sigma_\theta = \frac{pa}{t}\left(1 - \frac{a^2}{2b^2}\right).$$

It is observed that the hoop stress σ_θ becomes compressive for $a^2 > 2b^2$. Clearly, the meridian stresses σ_ϕ are always tensile. A ratio $a/b = 1$, the case of a sphere, yields the lowest stress.

EXAMPLE 12.5

Analyze the membrane stresses in a thin metal *container* of *conical shape*, supported from the top. Consider two specific cases: (a) the shell is subjected to an internal pressure p; (b) the shell is filled with a liquid of specific weight γ (Figure 12.10).

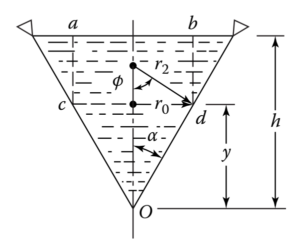


Figure 12.10 Conical tank.

SOLUTION

For such a tank, we have

$$\phi = \frac{\pi}{2} - \alpha \qquad r_0 = y\tan\alpha. \tag{g}$$

(a) Equations 12.8 with $F = -p\pi r_0^2$, after dividing by the thickness t, then become

$$\sigma_s = \frac{pr_0}{2t\cos\alpha} \qquad \sigma_\theta = \frac{pr_0}{t\cos\alpha}. \tag{12.21}$$

(b) According to the familiar laws of hydrostatics, the pressure at any point in the shell equals *the weight of a column of unit cross-sectional area of the liquid* at that point. At any arbitrary level *y*, the *pressure* is therefore

$$p = -p_z = \gamma(h-y). \tag{h}$$

Substituting Equations g and h into the second of Equations 12.8, after division by *t*, we have

$$\sigma_\theta = \frac{\gamma(h-y)y}{t}\frac{\tan\alpha}{\cos\alpha}. \tag{12.22a}$$

Differentiating with respect to *y* and equating to zero reveals that the maximum value of the above stress occurs at $y = h/2$ and is given by

$$\sigma_{\theta,\max} = \frac{\gamma h^2}{4t}\frac{\tan\alpha}{\cos\alpha}.$$

The load is equal to the weight of the liquid of volume *acOdb*. That is,

$$F = -\pi\gamma y^2\left(h - y + \frac{1}{3}y\right)\tan^2\alpha.$$

Introducing this value into the first of Equations 12.8 and dividing the resulting expression by *t* leads to

$$\sigma_s = \frac{\gamma(h-2y/3)y}{2t}\frac{\tan\alpha}{\cos\alpha}. \tag{12.22b}$$

The maximum value of this stress, $\sigma_{s,\max} = 3h^2\gamma\tan\alpha/16t\cos\alpha$, occurs at $y = 3h/4$.

EXAMPLE 12.6

Determine the membrane forces in a *spherical strong tank* filled with liquid of specific weight γ and supported on a cylindrical pipe (Figure 12.11a).

SOLUTION

The loading is expressed as

$$p = -p_z = \gamma a(1-\cos\phi).$$

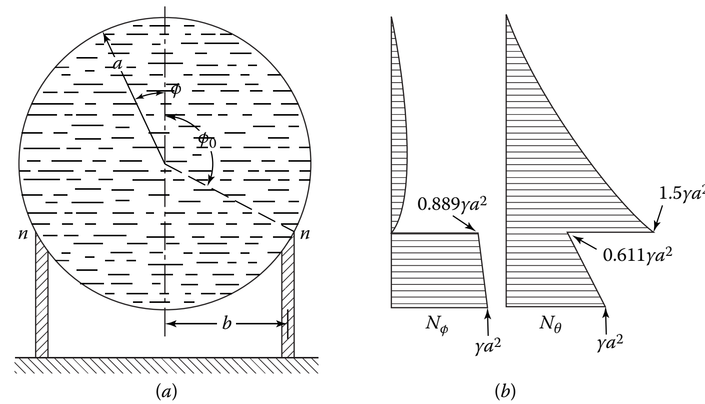


Figure 12.11 Spherical tank supported on a circular pipe.

Because of this pressure, the resultant force *F* for the portion intercepted by ϕ is

$$\begin{aligned} F &= -2\pi a^2\int_0^\phi \gamma a(1-\cos\phi)\sin\phi\cos\phi\, d\phi \\ &= -2\pi a^3\gamma\left[\frac{1}{6} - \frac{1}{2}\cos^2\phi\left(1 - \frac{2}{3}\cos\phi\right)\right]. \end{aligned}$$

Inserting the above into Equations 10.3,

$$\begin{aligned} N_\phi &= \frac{\gamma a^2}{6\sin^2\phi}[1-\cos^2\phi(3-2\cos\phi)] = \frac{\gamma a^2}{6}\left(1 - \frac{2\cos^2\phi}{1+\cos\phi}\right) \\ N_\theta &= \frac{\gamma a^2}{6}\left(5 - 6\cos\phi + \frac{2\cos^2\phi}{1+\cos\phi}\right). \end{aligned} \tag{12.23}$$

Equations 12.23 are valid for $\phi < \phi_0$.

In determining *F* for $\phi > \phi_0$, the sum of the vertical support reactions, which is equal to the total weight of the liquid ($4\gamma\pi a^3/3$), must also be taken into account in addition to the internal pressure loading. That is,

$$F = -\frac{4}{3}\pi a^3\gamma - 2\pi a^3\gamma\left[\frac{1}{6} - \frac{1}{2}\cos^2\phi\left(1 - \frac{2}{3}\cos\phi\right)\right].$$

Equations 12.3 now yield

$$\begin{aligned} N_\phi &= \frac{\gamma a^2}{6}\left(5 + \frac{2\cos^2\phi}{1-\cos\phi}\right) \\ N_\theta &= \frac{\gamma a^2}{6}\left(1 - 6\cos\phi - \frac{2\cos^2\phi}{1-\cos\phi}\right). \end{aligned} \tag{12.24}$$

From Equations 12.23 and 12.24, it is observed that both forces N_ϕ and N_θ change values abruptly at the support ($\phi = \phi_0$). This is illustrated for a support position at $\phi = 120°$ in Figure 12.11b. A discontinuity in N_θ means a discontinuity of the deformation of the parallel circles on the immediate sides of *nn*. Thus, the deformation associated with the membrane solution is *not compatible* with the continuity of the structure at support *nn*.

12.7 Axially Symmetric Deformation

We now discus the displacements in *symmetrically loaded* shells of revolution by considering an element *AB* of length $r_1\, d\phi$ of the meridian in an unstrained shell. Let the displacements in the direction of the tangent to the meridian and in the direction normal to the midsurface be denoted by v and w, respectively (Figure 12.12). After straining, *AB* is displaced to position A′B′. In the analysis that follows, the small deformation approximation is employed, and higher-order infinitesimal terms are neglected. The deformation experienced by an element of infinitesimal length $r_1\, d\phi$ may be regarded as composed of an increase in length $(dv/d\phi)\, d\phi$ due to the tangential displacements and a decrease in length $w\, d\phi$ produced by the radial displacement w. The meridional strain ε_ϕ, the total deformation per unit length of the element *AB*, is thus

$$\varepsilon_\phi = \frac{1}{r_1}\frac{dv}{d\phi} - \frac{w}{r_1}. \tag{12.25a}$$

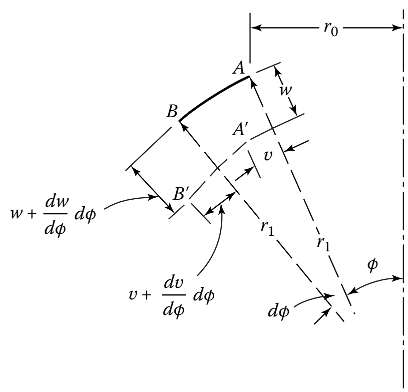
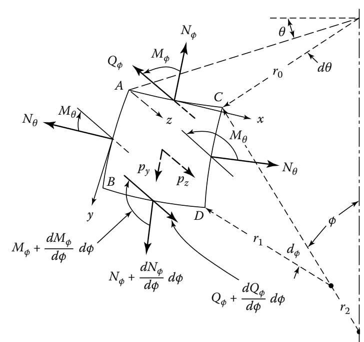


Figure 12.12 Displacements of midsurface.



$$\sum F_\theta = 0 \Rightarrow \frac{\partial}{\partial\phi}\left(r_0 N_{\phi\theta}\right) + r_1\frac{\partial N_\theta}{\partial\theta} + r_1 N_{\theta\phi}\cos\phi - r_1 Q_\theta \sin\phi + r_0 r_1 p_\theta = 0$$

$$\sum F_\phi = 0 \Rightarrow \frac{\partial}{\partial\phi}\left(r_0 N_\phi\right) + r_1\frac{\partial N_{\theta\phi}}{\partial\theta} - r_1 N_\theta\cos\phi - r_0 Q_\phi + r_0 r_1 p_\phi = 0$$

$$\sum F_z = 0 \Rightarrow r_0 N_\phi + r_1 N_\theta \sin\phi + r_1\frac{\partial Q_\theta}{\partial\theta} + \frac{\partial}{\partial\phi}\left(r_0 Q_\phi\right) + r_0 r_1 p_z = 0$$

$$\sum M_\theta = 0 \Rightarrow \frac{\partial}{\partial\phi}\left(r_0 M_\phi\right) + r_1\frac{\partial M_{\theta\phi}}{\partial\theta} - r_1 M_\theta\cos\phi - r_0 r_1 Q_\phi = 0$$

$$\sum M_\phi = 0 \Rightarrow \frac{\partial}{\partial\phi}\left(r_0 M_{\phi\theta}\right) + r_1\frac{\partial M_\theta}{\partial\theta} + r_1 M_{\theta\phi}\cos\phi - r_0 r_1 Q_\theta = 0$$

$$\sum M_z = 0 \Rightarrow r_0 r_1 N_{\theta\phi} + r_1 M_{\theta\phi}\sin\phi - r_0 r_1 N_{\phi\theta} - r_0 M_{\phi\theta} = 0$$