Aircraft Structural Analysis

Master Course in Aerospace Engineering

Membrane stresses in shells of revolution

Reference material

Membrane stresses in shells of revolution
(chapter 12)

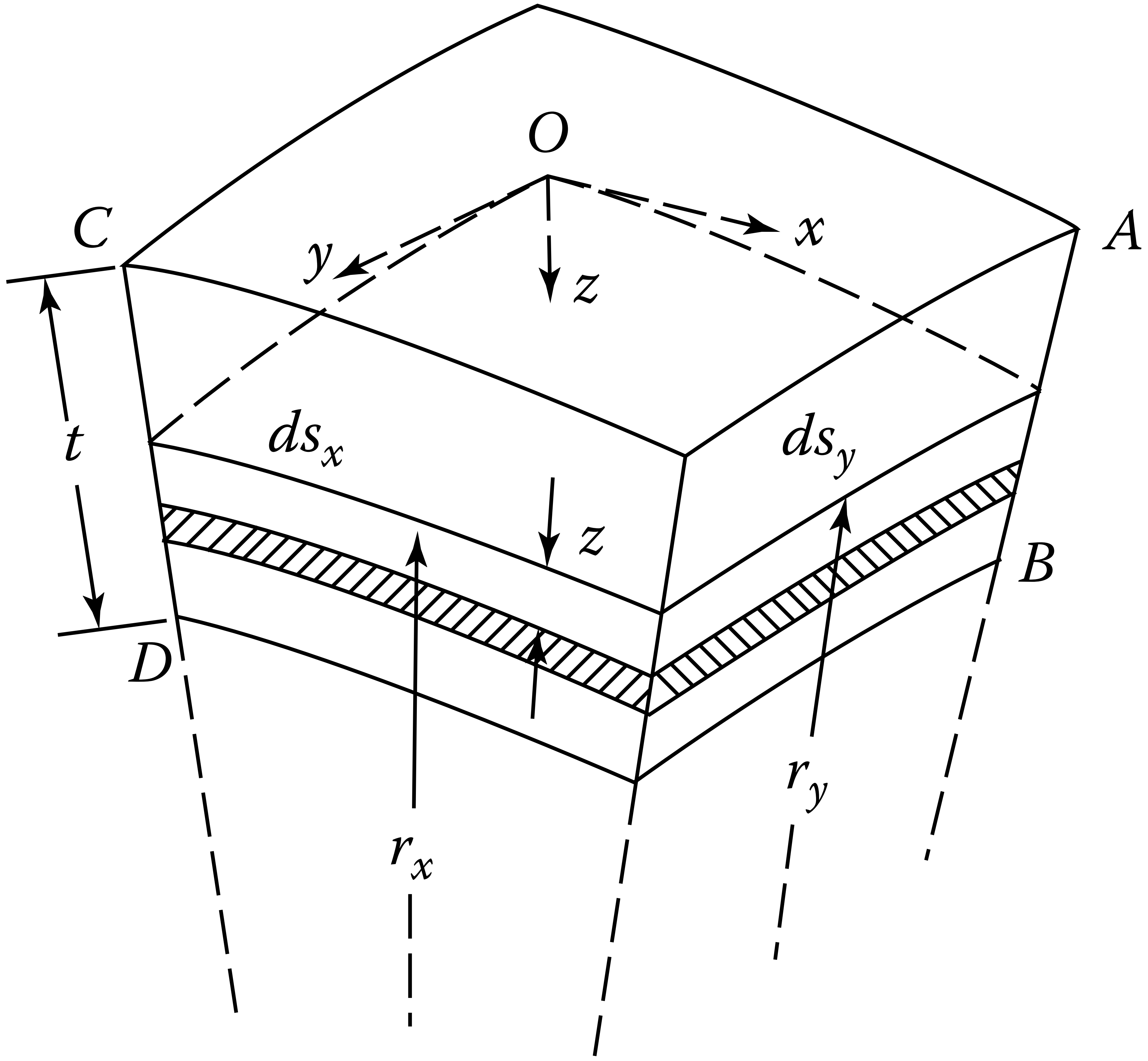
of the reference book: Ansel C. Ugural, "Stresses in Plates and Shells", 2nd ed., McGraw-Hill

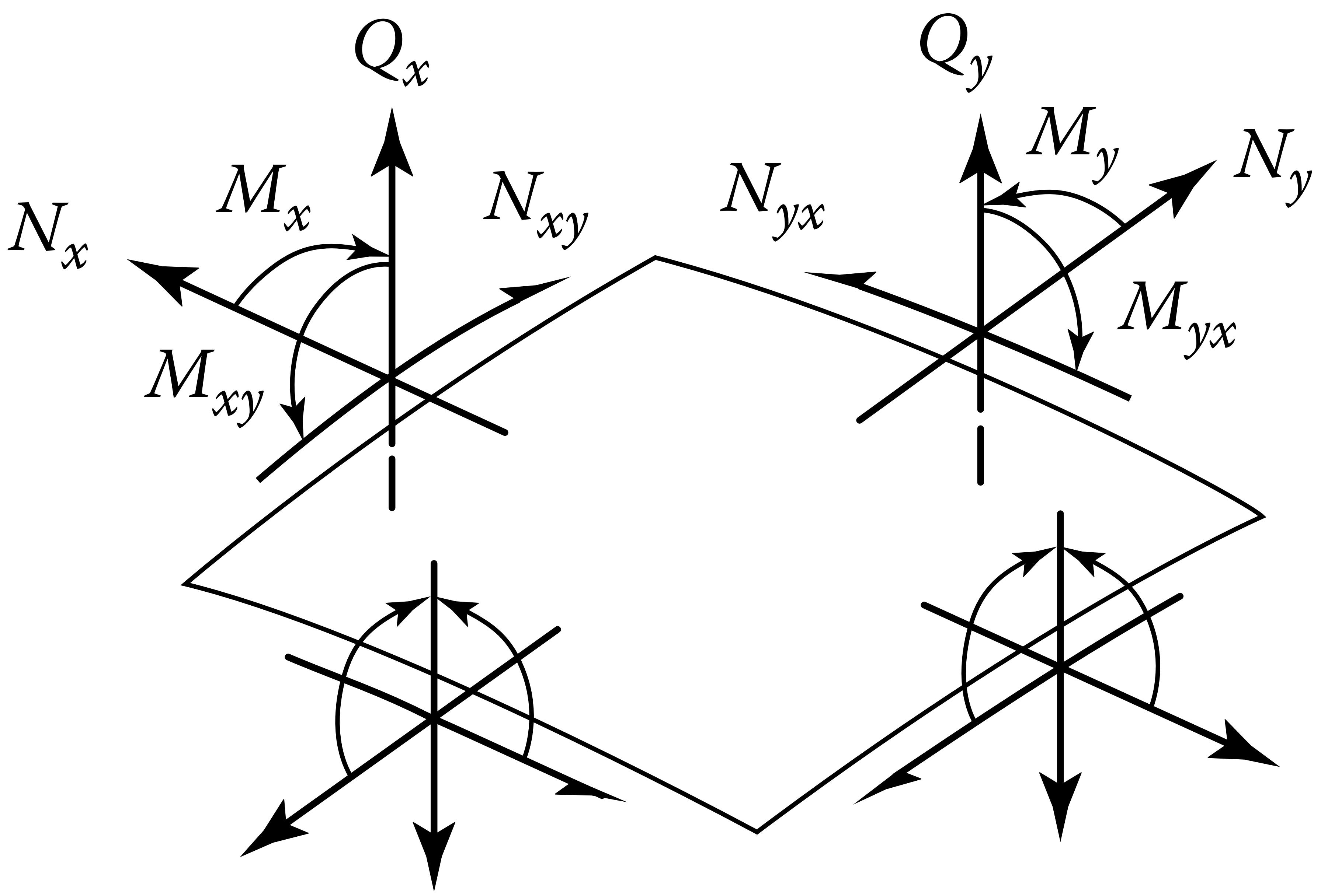
basic assumptions

- material: elastic, homogeneous, isotropic
- ratio of shell thickness to radius of curvature of the mid surface of the shell is small (compared with unity)
- deflections are small compared with shell thickness
- plane sections through a shell, taken normal to the mid surface, remain plane and normal to the deformed mid surface (negligible transverse strains)
- stresses and strains along thickness direction are negligible

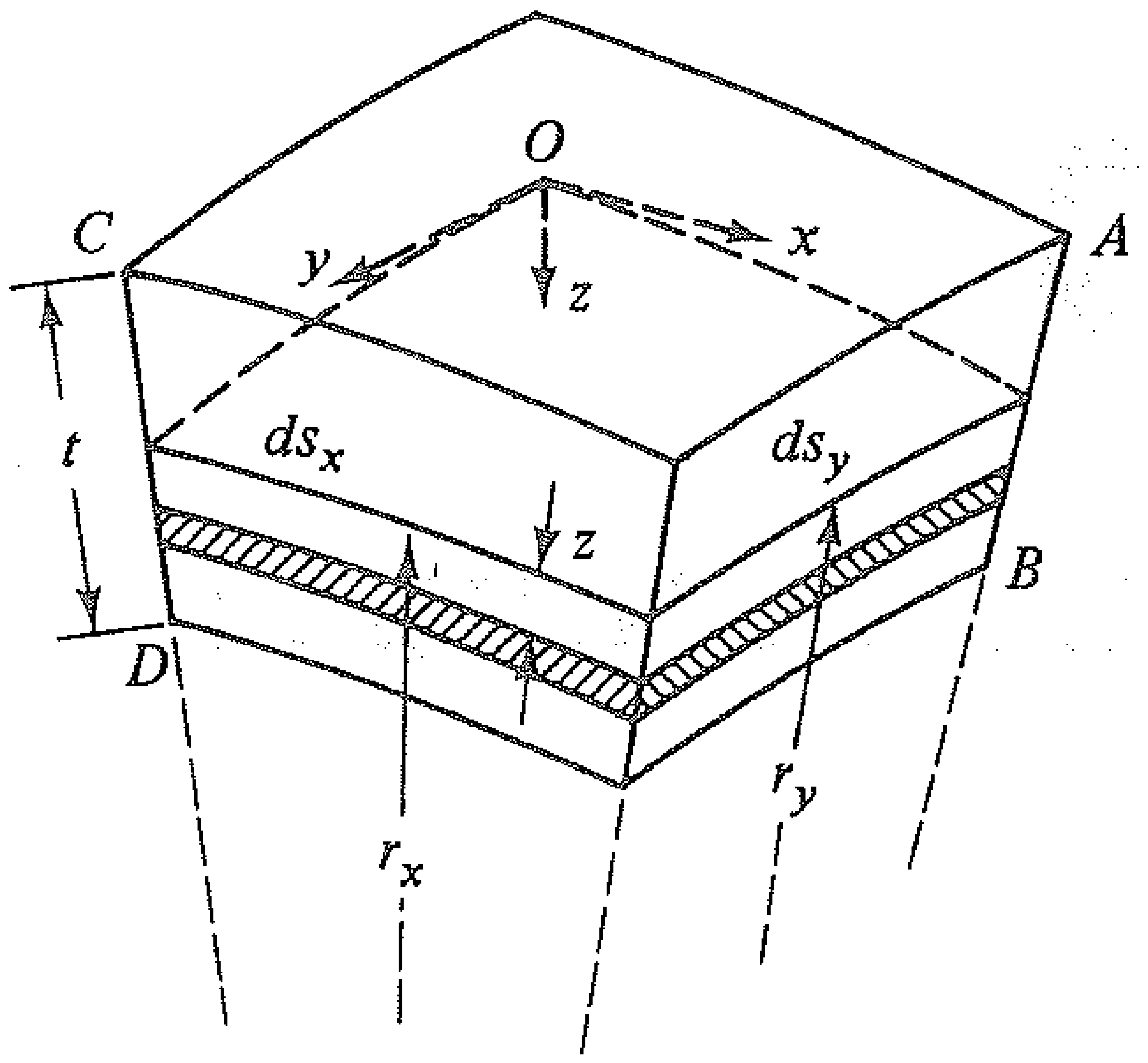
- bending stresses will be considered negligibly small (i.e., membrane theory)

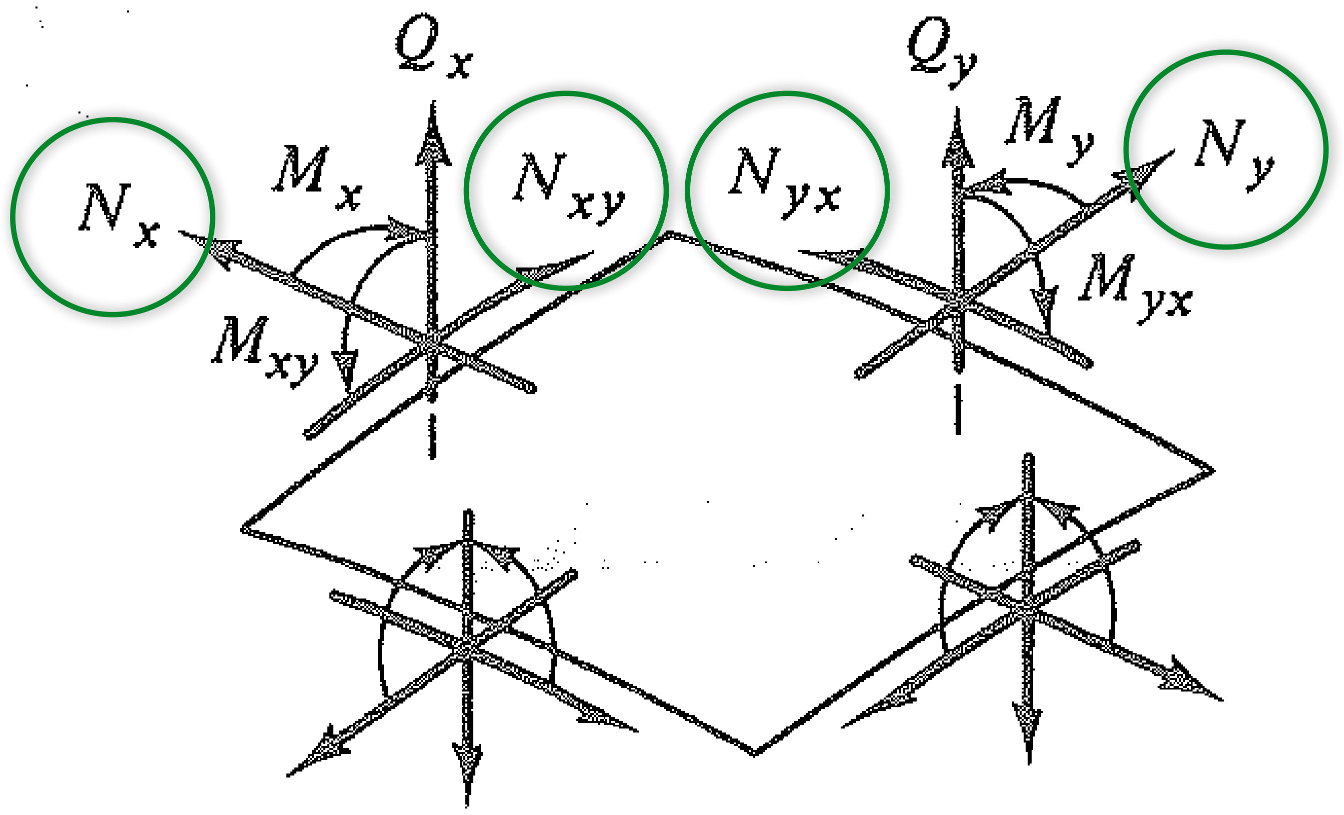
geometry and (generic) stress resultants





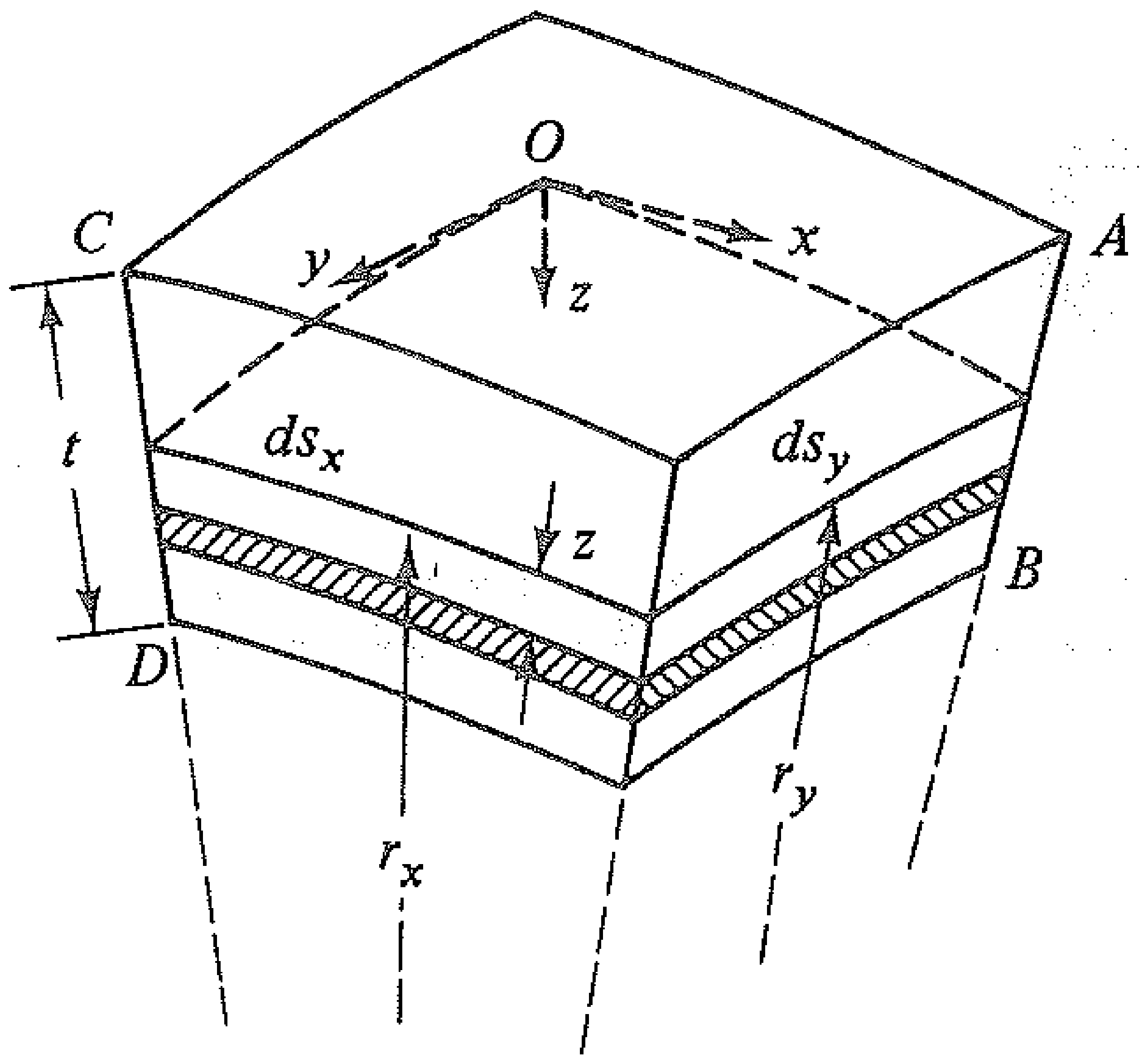
stress resultants

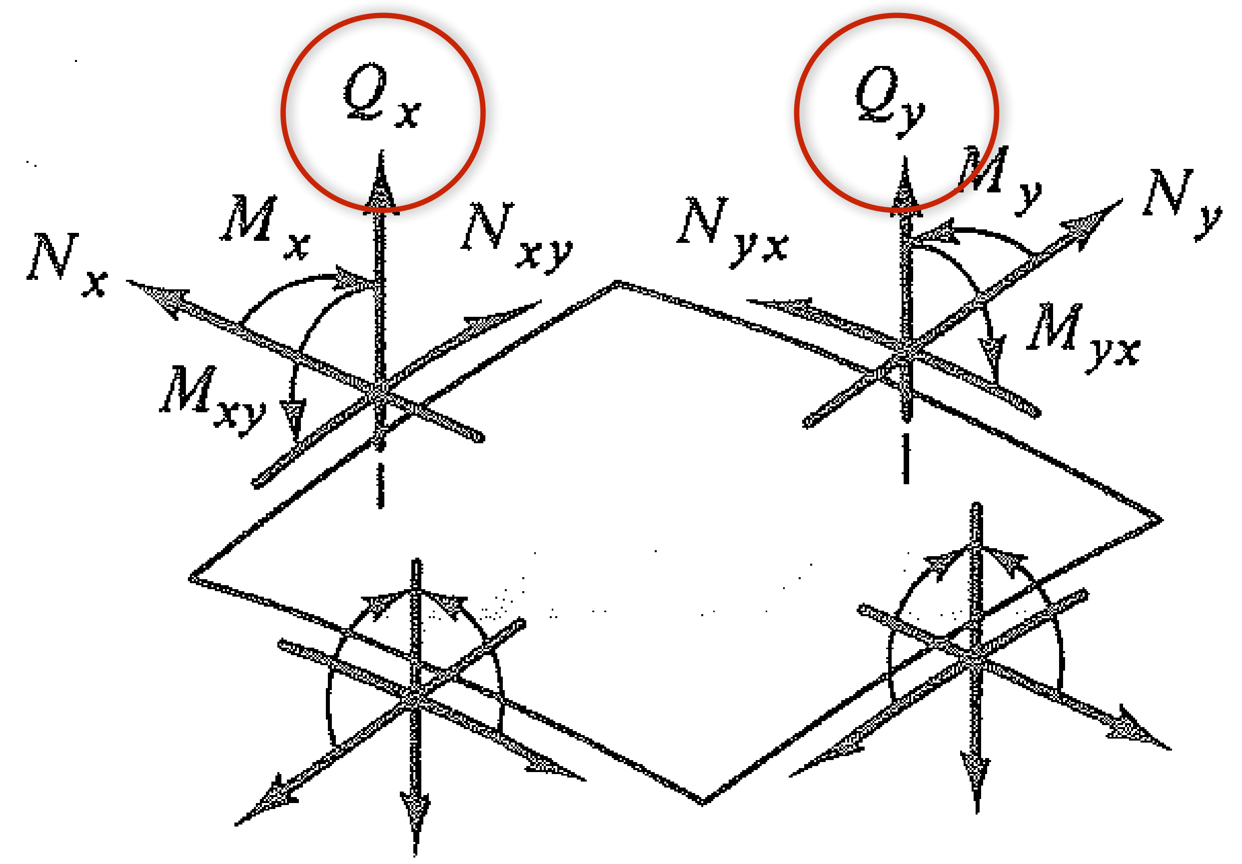




- membrane normal and shear resultants (N)

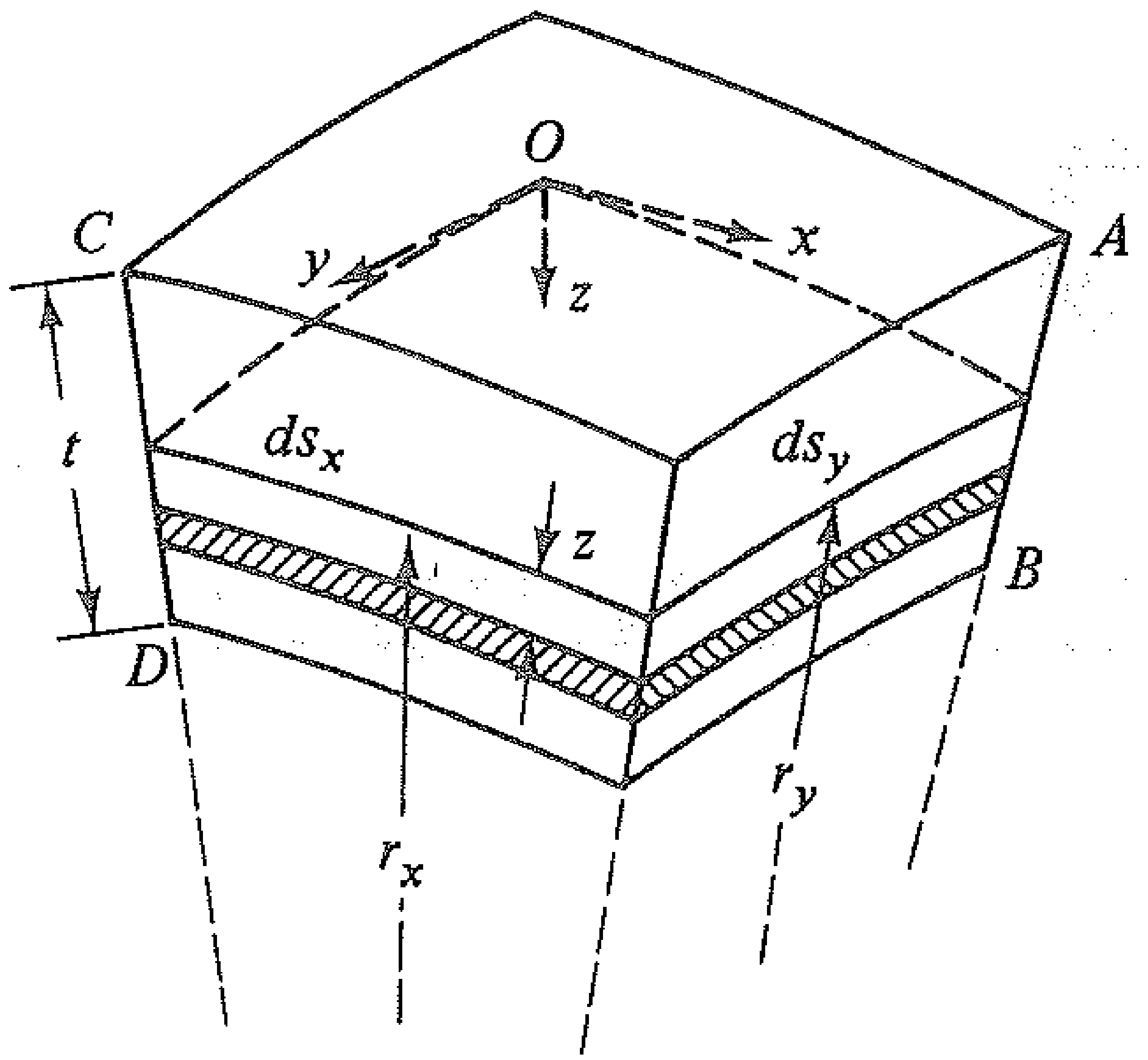
stress resultants

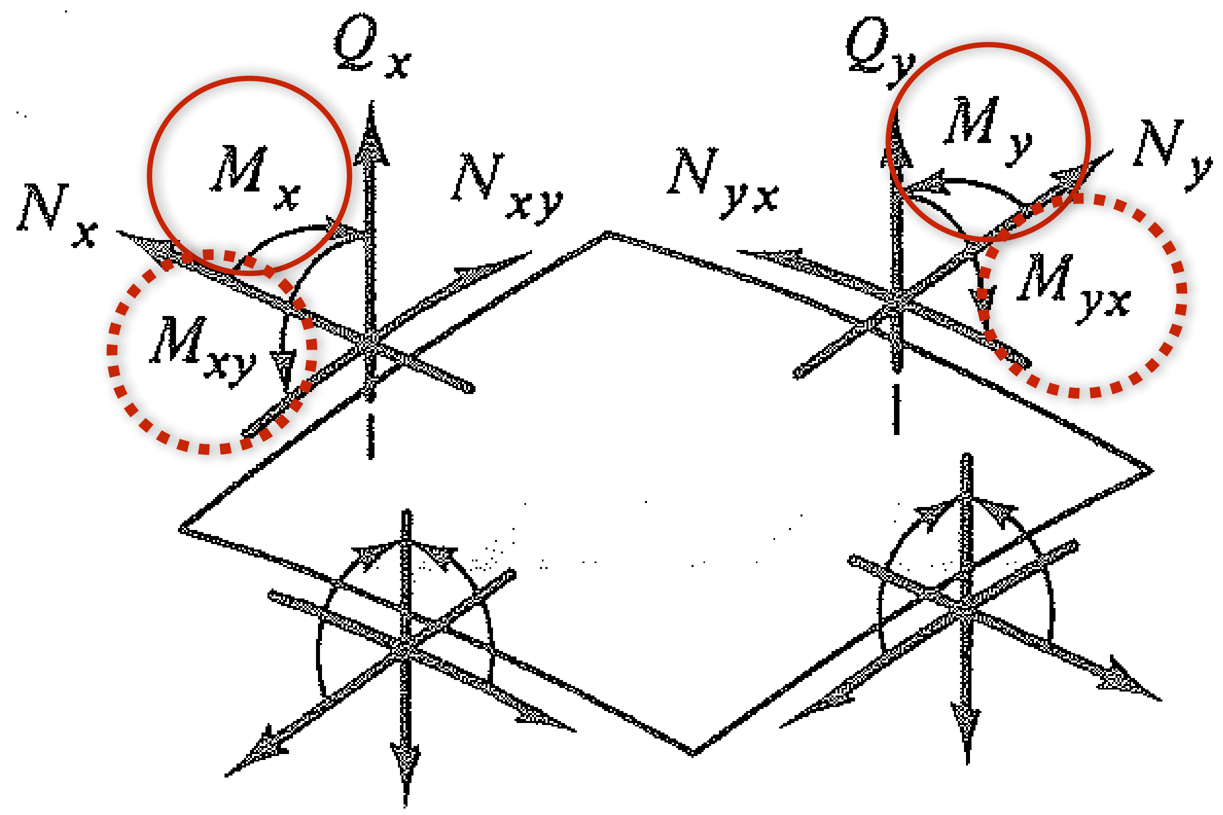




- transverse stress resultants (Q)

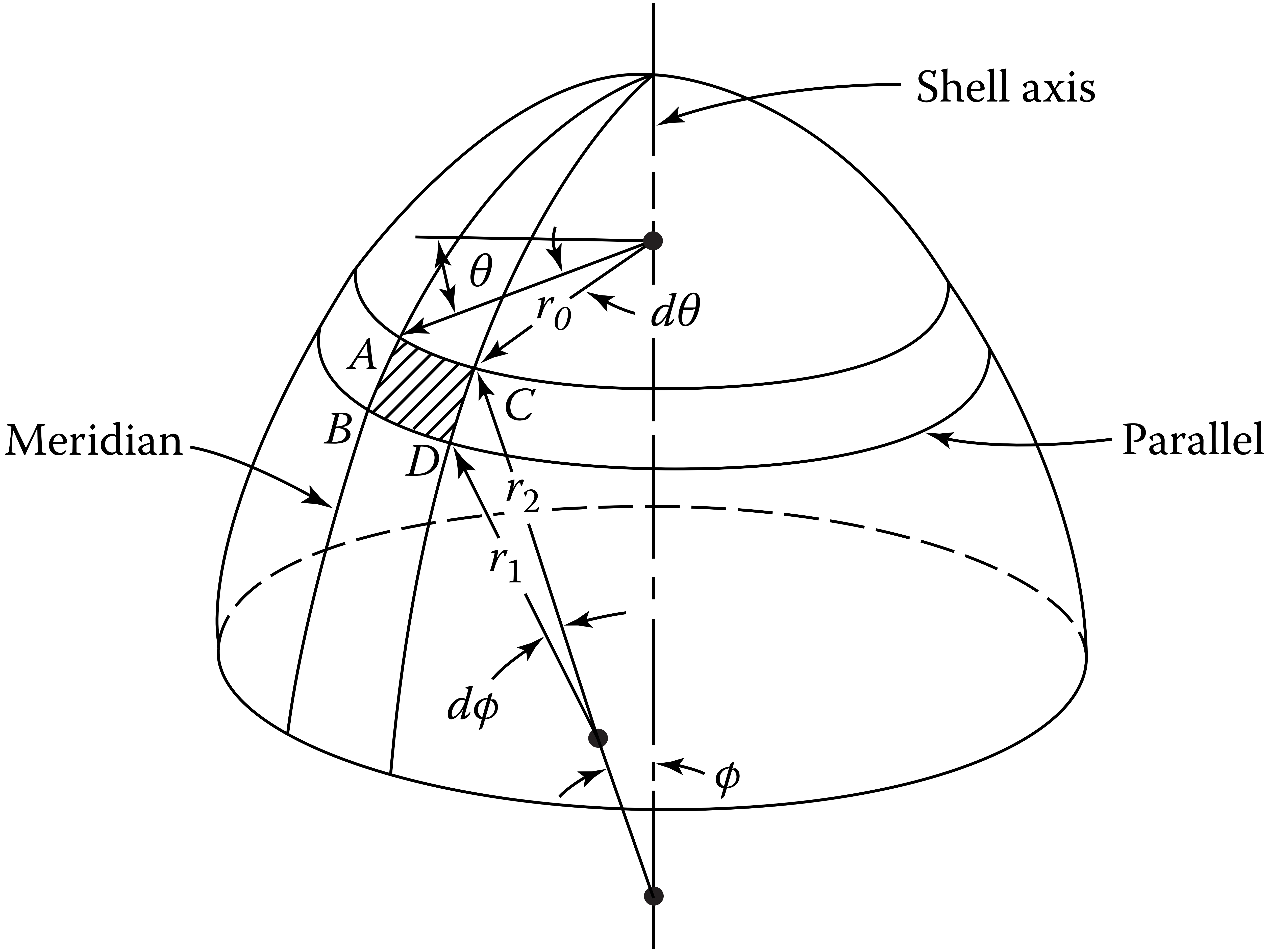
stress resultants





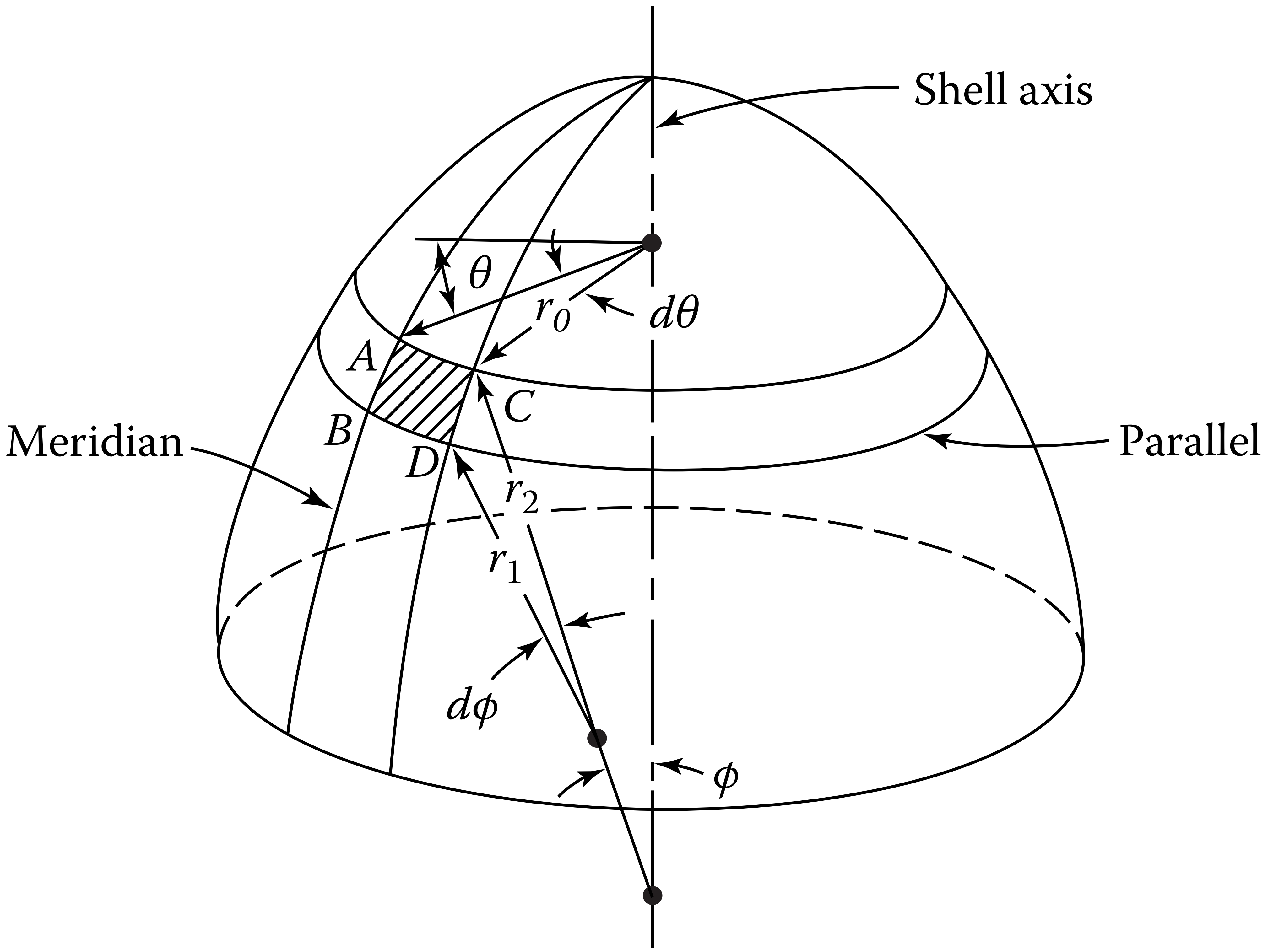
- bending and torsion moments (M)

geometry of shells of revolution (12.4)



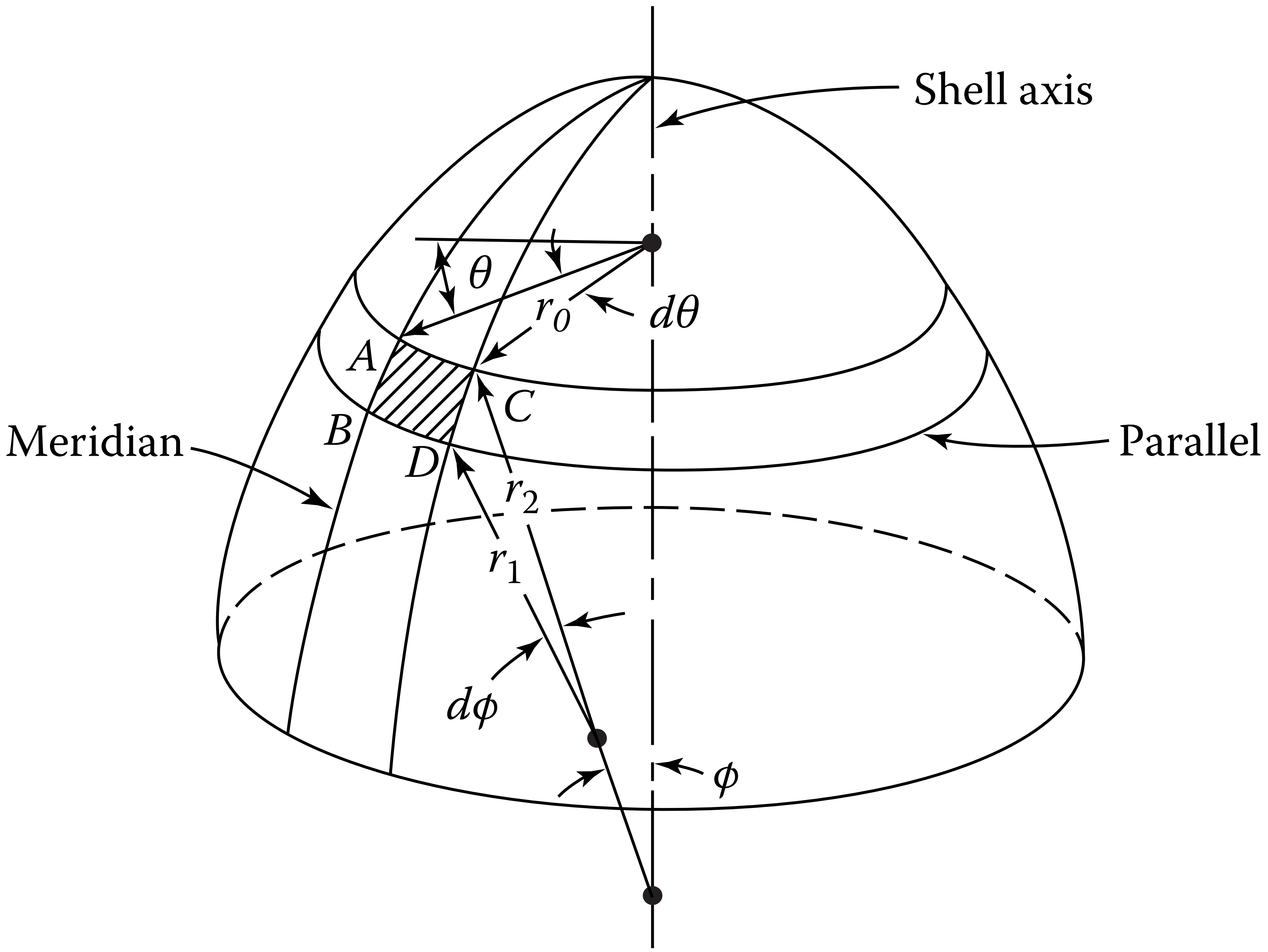
any point over a shell surface can be given by the coordinates (r_0, θ, ϕ)

geometry of shells of revolution



any surface element of a shell (ABCD) can be defined by
2 meridian and 2 parallel lines

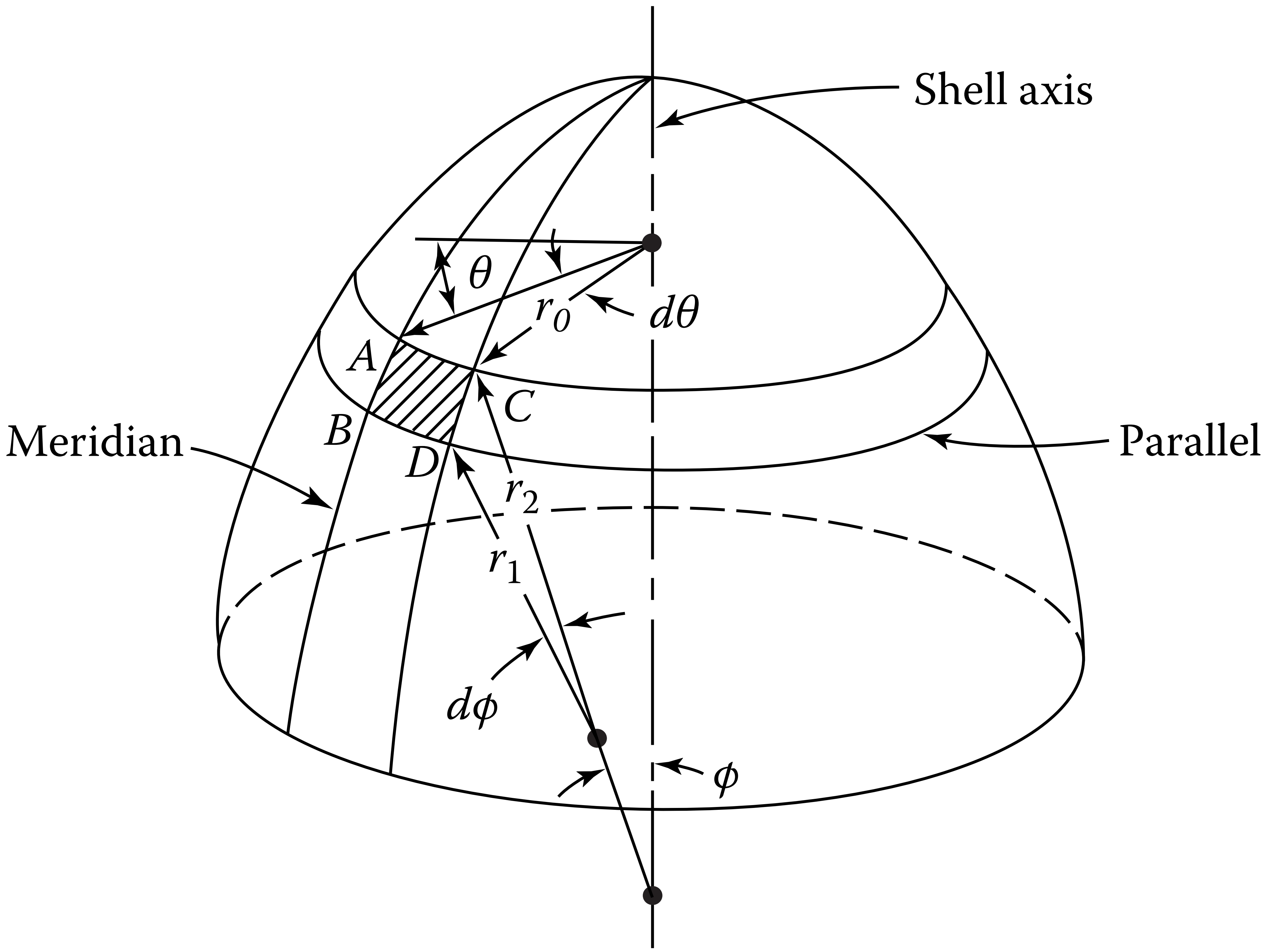
geometry of shells of revolution



principal radiuses of curvature

- projected to the parallel lines (r_0)
- as measured over the meridian lines (r_1)
- tangential radius of curvature (r_2)

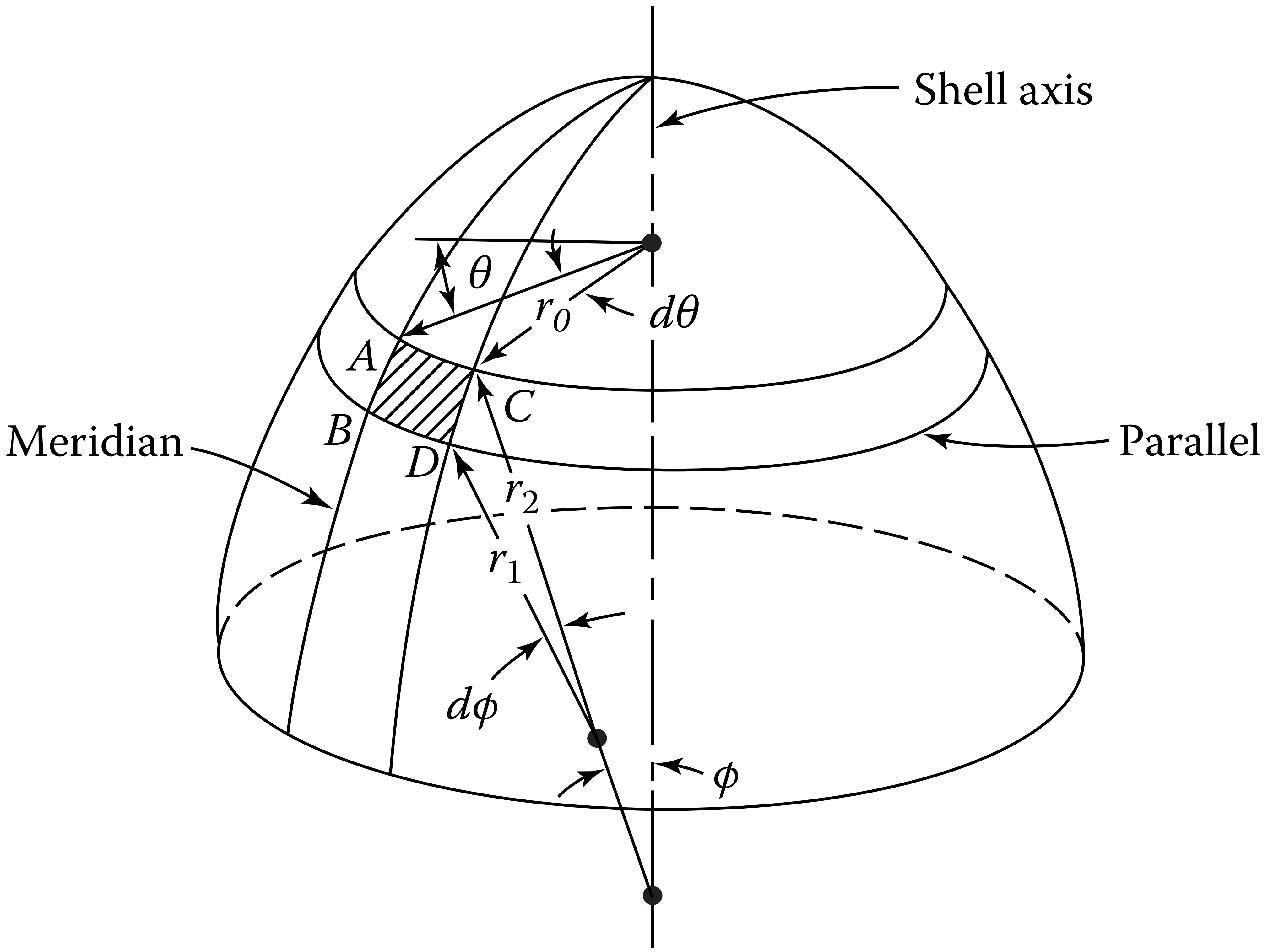
geometry of shells of revolution

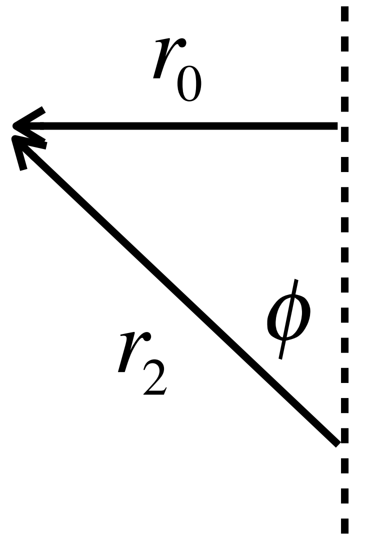


principal radiuses of curvature

- as measured over the meridian lines: CD (r_1)
- tangential radius of curvature: AC (r_2)

geometry of shells of revolution

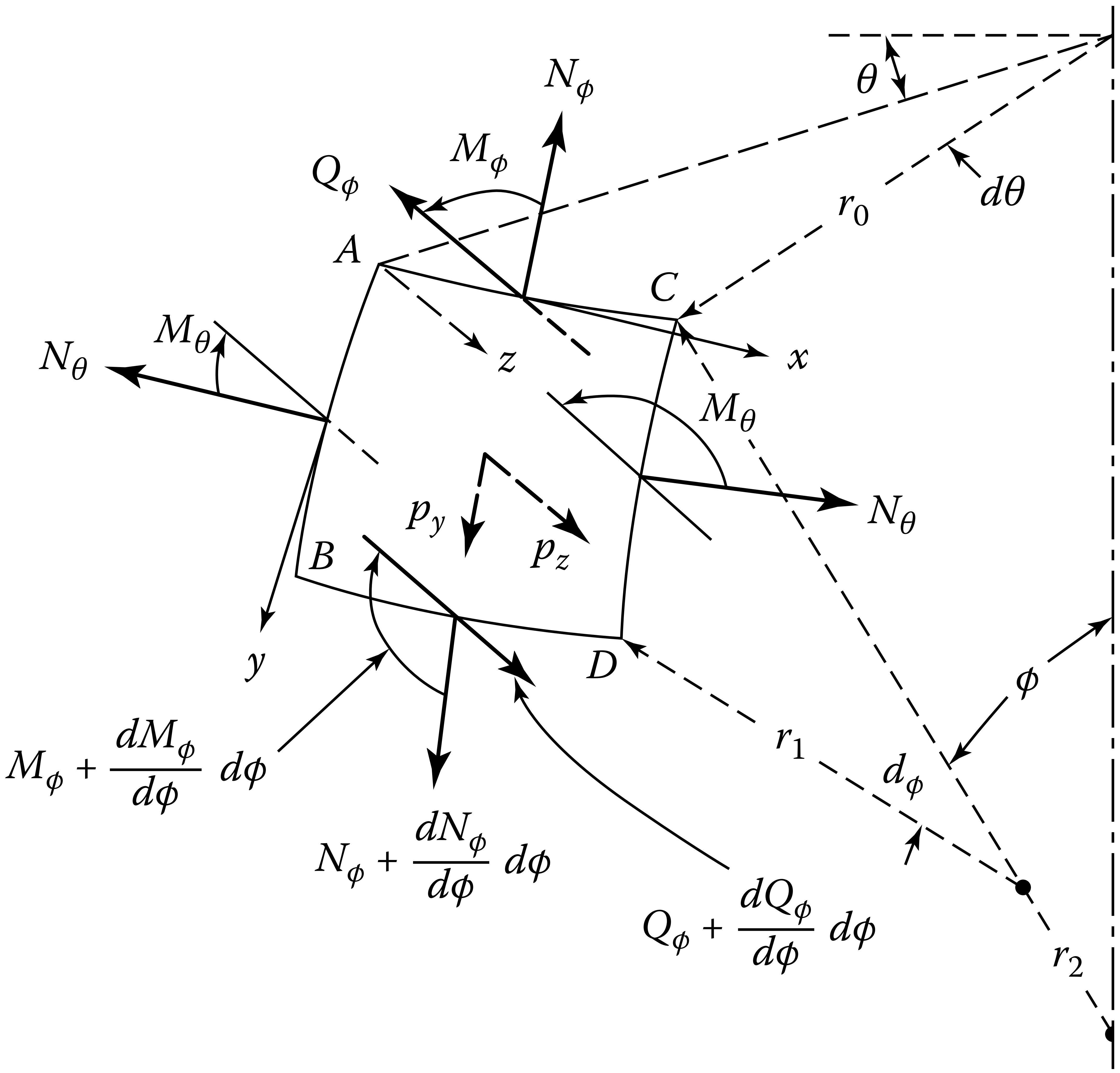




$$\overline{CD} = r_1 \mathrm{d}\phi$$

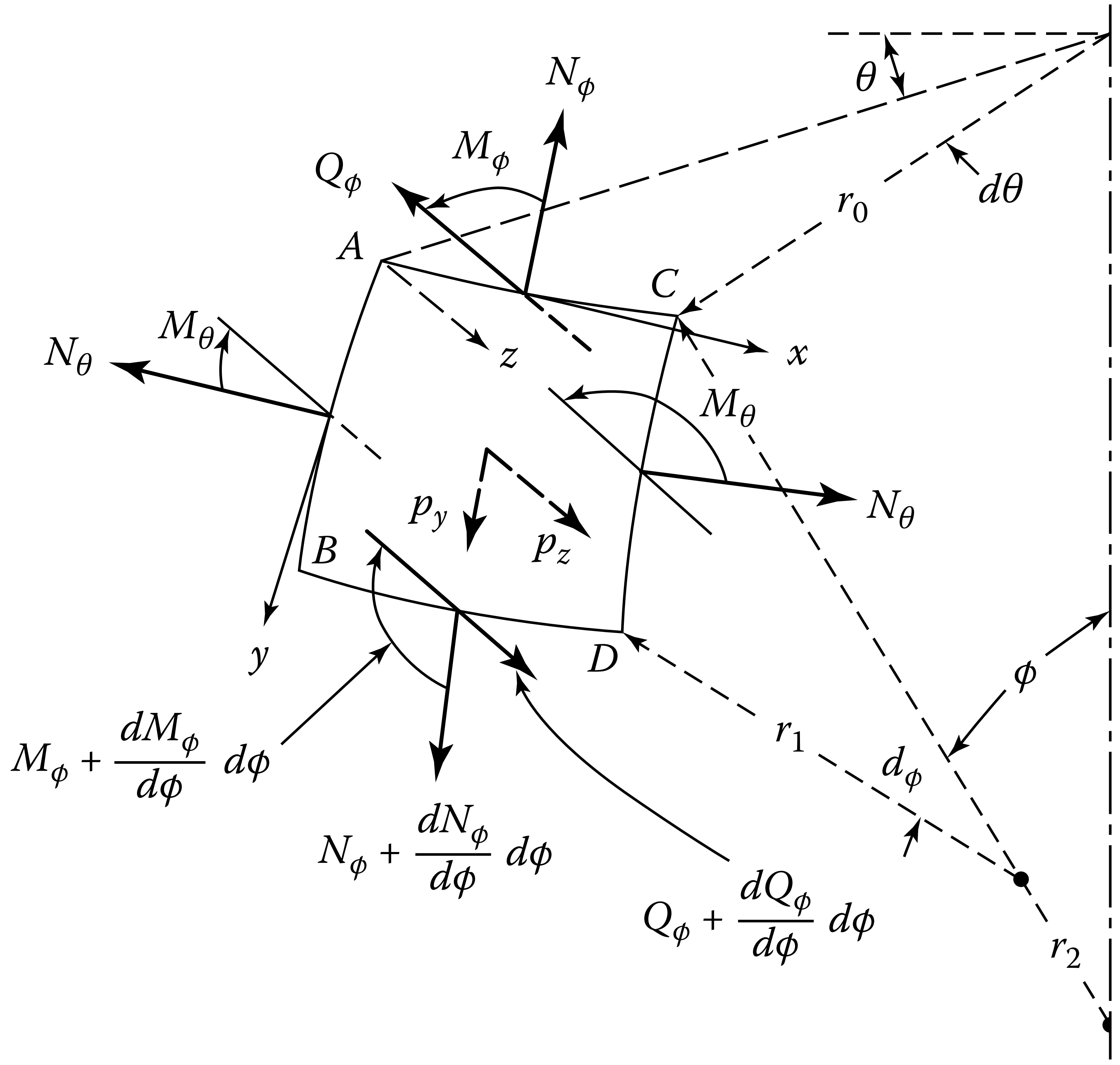
$$\overline{AC} = r_0 \mathrm{d}\theta = \left(r_2 \sin\phi\right)\mathrm{d}\theta$$

generic stress resultants



generic stress resultants

(membrane shear stresses and torsion moments are not represented)



<u>membrane stress resultants</u>

meridional stresses

$$N_\phi$$

tangential (longitudinal) stresses

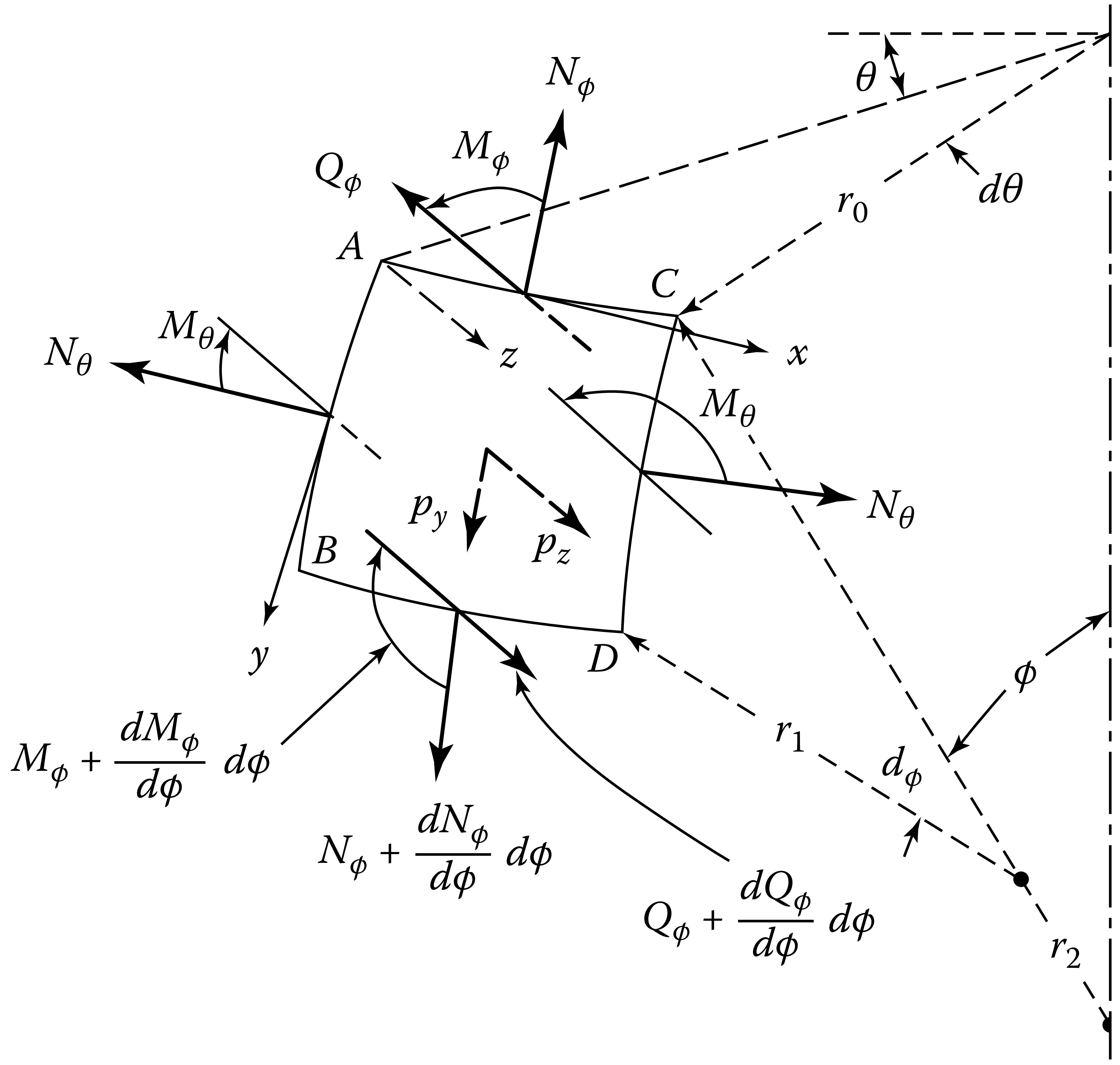
$$N_\theta$$

shear stresses

$$N_{\theta\phi} = N_{\phi\theta}$$

generic stress resultants

(membrane shear stresses and torsion moments are not represented)



<u>transverse stress resultants</u>

meridional stresses

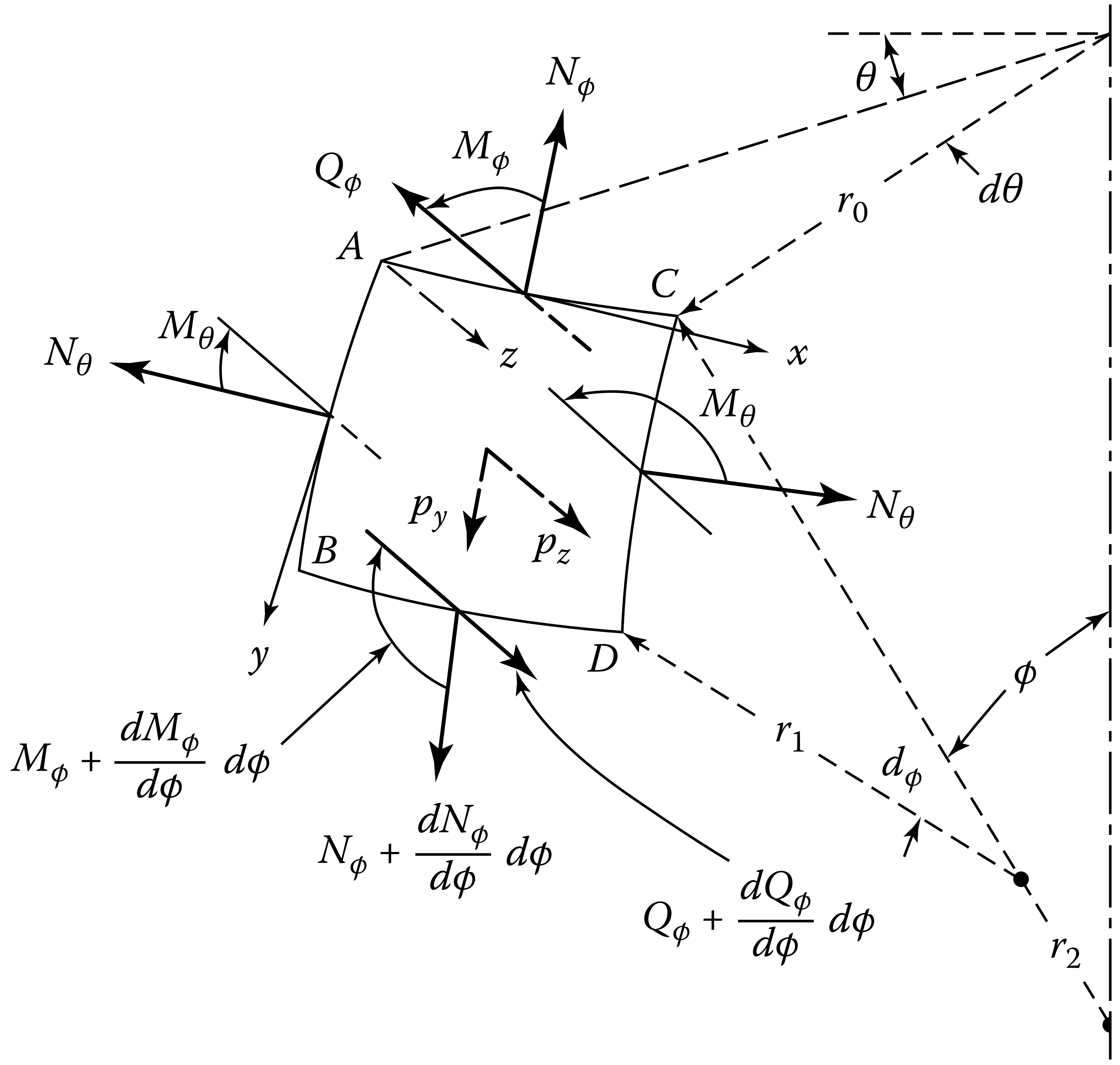
$$Q_\phi$$

tangential (longitudinal) stresses

$$Q_\theta$$

generic stress resultants

(membrane shear stresses and torsion moments are not represented)



<u>bending and torsion moments</u>

meridional moment

$$M_\phi$$

tangential (longitudinal) moment

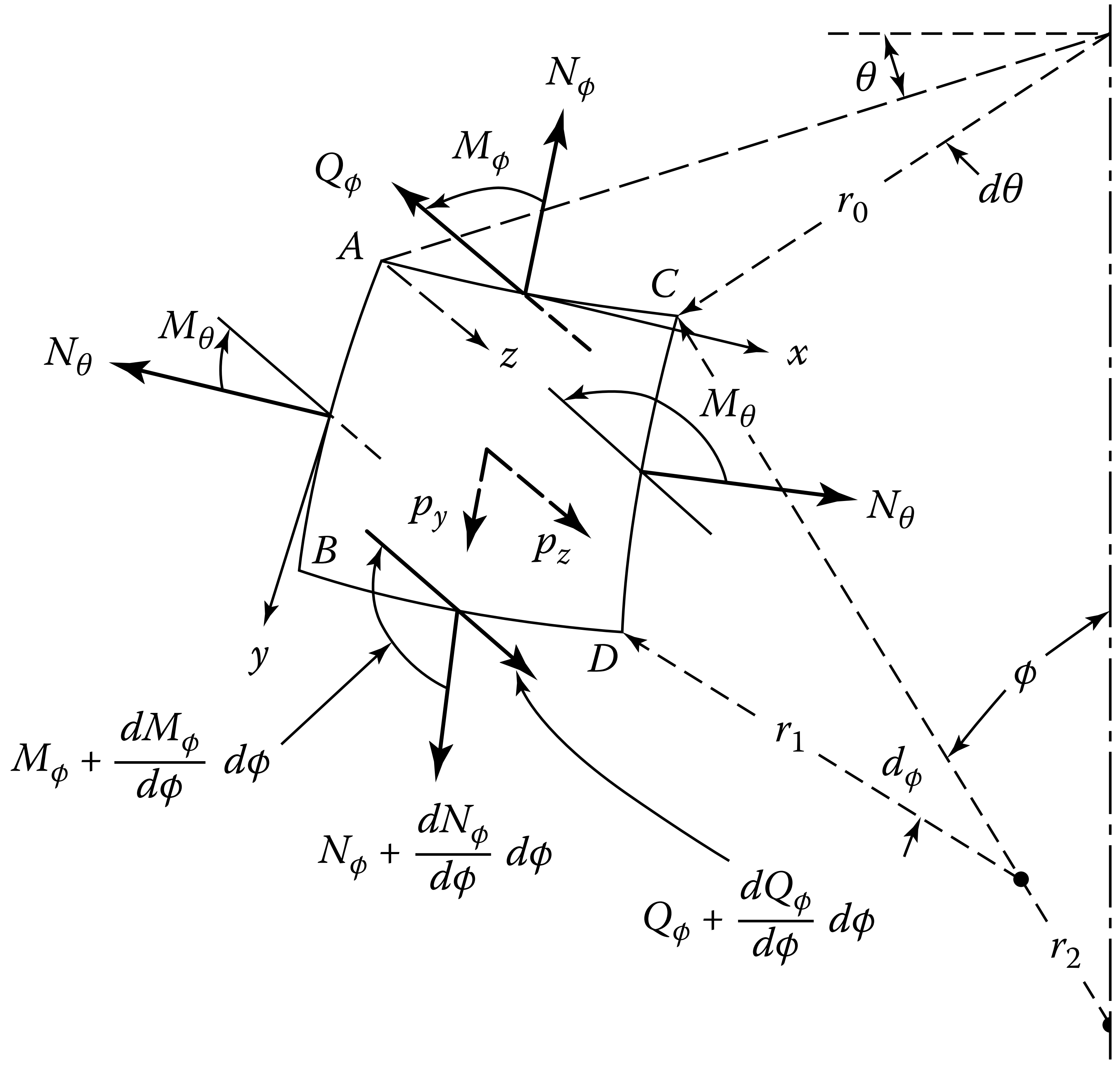
$$M_\theta$$

torsion moments

$$M_{\phi\theta} = M_{\theta\phi}$$

generic stress resultants

(membrane shear stresses and torsion moments are not represented)



<u>external loads (stresses)</u>

meridional pressure

$$p_y = p_\phi$$

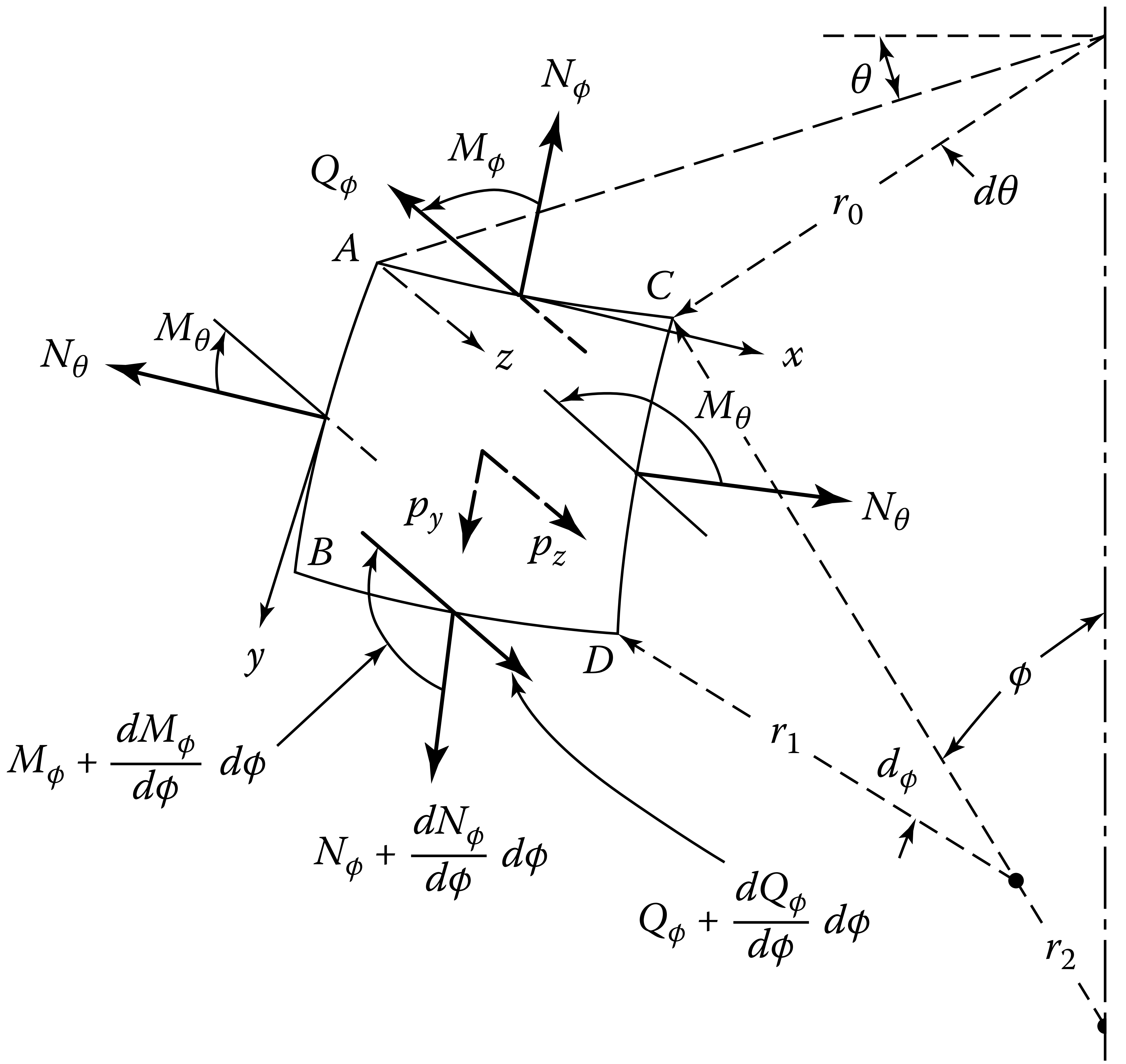
circumferential pressure

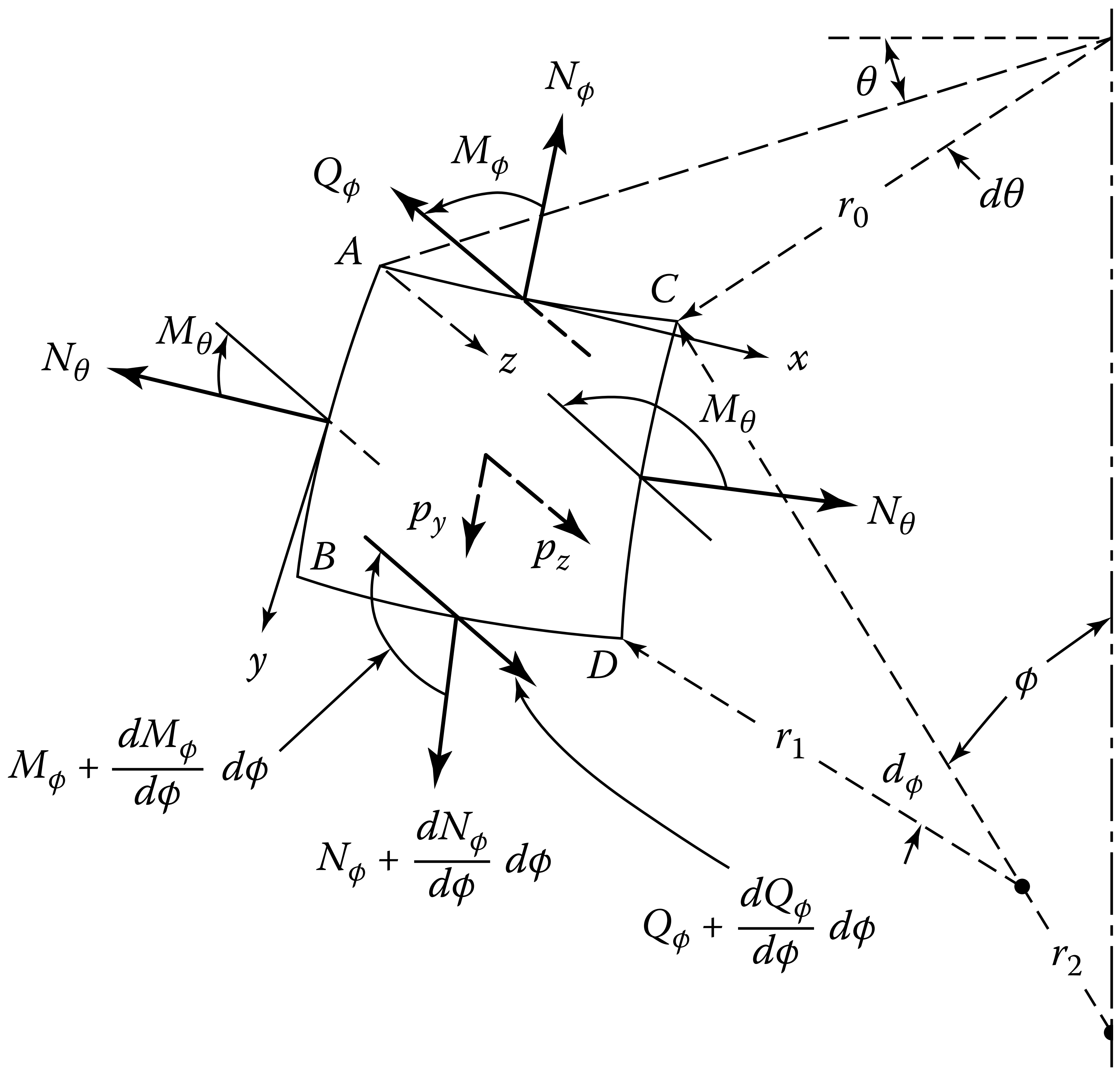
$$p_x = p_\theta$$

normal pressure

$$p_z$$

general equations: static equilibrium among all (internal/external) stresses, in all directions





$$\sum F_\theta = 0 \Rightarrow \frac{\partial}{\partial\phi}\left(r_0 N_{\phi\theta}\right) + r_1 \frac{\partial N_\theta}{\partial\theta} + r_1 N_{\theta\phi}\cos\phi - r_1 Q_\theta \sin\phi + r_0 r_1 p_\theta = 0$$

$$\sum F_\phi = 0 \Rightarrow \frac{\partial}{\partial\phi}\left(r_0 N_\phi\right) + r_1 \frac{\partial N_{\theta\phi}}{\partial\theta} - r_1 N_\theta \cos\phi - r_0 Q_\phi + r_0 r_1 p_\phi = 0$$

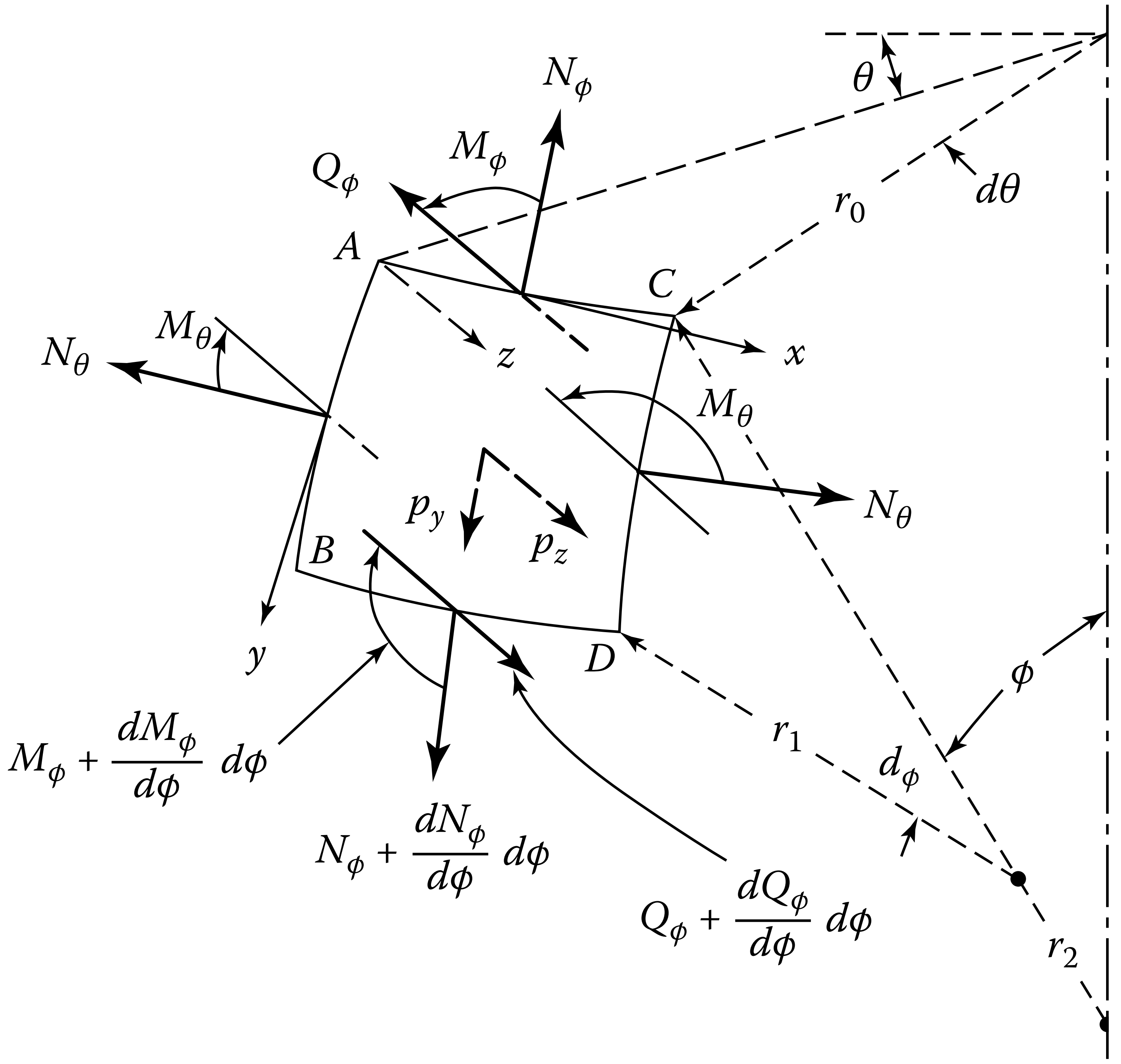
$$\sum F_z = 0 \Rightarrow r_0 N_\phi + r_1 N_\theta \sin\phi + r_1 \frac{\partial Q_\theta}{\partial\theta} + \frac{\partial}{\partial\phi}\left(r_0 Q_\phi\right) + r_0 r_1 p_z = 0$$

$$\sum M_\theta = 0 \Rightarrow \frac{\partial}{\partial\phi}\left(r_0 M_\phi\right) + r_1 \frac{\partial M_{\theta\phi}}{\partial\theta} - r_1 M_\theta \cos\phi - r_0 r_1 Q_\phi = 0$$

$$\sum M_\phi = 0 \Rightarrow \frac{\partial}{\partial\phi}\left(r_0 M_{\phi\theta}\right) + r_1 \frac{\partial M_\theta}{\partial\theta} + r_1 M_{\theta\phi}\cos\phi - r_0 r_1 Q_\theta = 0$$

$$\sum M_z = 0 \Rightarrow r_0 r_1 N_{\theta\phi} + r_1 M_{\theta\phi}\sin\phi - r_0 r_1 N_{\phi\theta} - r_0 M_{\phi\theta} = 0$$

some useful relations…



$$N_{\phi\theta} = N_{\theta\phi}$$

$$M_{\theta\phi} \cong M_{\phi\theta}$$

$$M_{\theta\phi} = M_{\phi\theta}\left(r_2 = r_1\right)$$

$$\sum M_z = 0 \Rightarrow r_0 r_1 N_{\theta\phi} + r_1 M_{\theta\phi}\sin\phi - r_0 r_1 N_{\phi\theta} - r_0 M_{\phi\theta} = 0$$

$$\Downarrow$$

$$r_1 M_{\theta\phi}\sin\phi - r_0 M_{\phi\theta} = 0$$

from the general equations, comes a useful simplification: the "**membrane solution**" for revolution shells

- small values of thickness (compared to the radiuses involved)
- only distributed loads are accepted over the surface of the shell
- concentrated loads are accepted, only if tangential to the shell surface
- strength of the shell only depends on the membrane stresses
- boundary conditions only induce tangential effects over the surface

"membrane solution" for revolution shells

1st simplification : ignore the bending effects from the general equilibrium equations

$$M_\theta = M_\phi = M_{\theta\phi} = Q_\theta = Q_\phi = 0$$

$$\sum F_\theta = 0 \Rightarrow \frac{\partial}{\partial \phi}\left(r_0 N_{\phi\theta}\right) + r_1 \frac{\partial N_\theta}{\partial \theta} + r_1 N_{\theta\phi} \cos\phi + r_0 r_1 p_\theta = 0$$

$$\sum F_\phi = 0 \Rightarrow \frac{\partial}{\partial \phi}\left(r_0 N_{\phi}\right) + r_1 \frac{\partial N_{\theta\phi}}{\partial \theta} - r_1 N_{\theta} \cos\phi + r_0 r_1 p_\phi = 0$$

$$\sum F_z = 0 \Rightarrow r_0 N_\phi + r_1 N_\theta \sin\phi + r_0 r_1 p_z = 0$$

“membrane solution” for revolution shells

2nd simplification : external loads are also axisymmetric

$$\boxed{\sum F_\phi = 0 \Rightarrow \frac{\partial}{\partial\phi}\left(r_0 N_\phi\right) + r_1 \cancel{\frac{\partial N_{\theta\phi}}{\partial\theta}} - r_1 N_\theta \cos\phi - r_0 \cancel{Q_\phi} + r_0 r_1 p_\phi = 0}$$

$$\boxed{\sum F_z = 0 \Rightarrow r_0 N_\phi + r_1 N_\theta \sin\phi + r_1 \cancel{\frac{\partial Q_\theta}{\partial\theta}} + \frac{\partial}{\partial\phi}\left(r_0 \cancel{Q_\phi}\right) + r_0 r_1 p_z = 0}$$

“membrane solution” for revolution shells

only two equilibrium equations (perpendicular and meridional equilibrium)

$$\frac{\partial}{\partial\phi}\left(r_0 N_\phi\right) - r_1 N_\theta \cos\phi + r_0 r_1 p_\phi = 0 \quad (1)$$

$$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} = -p_z \quad (2)$$

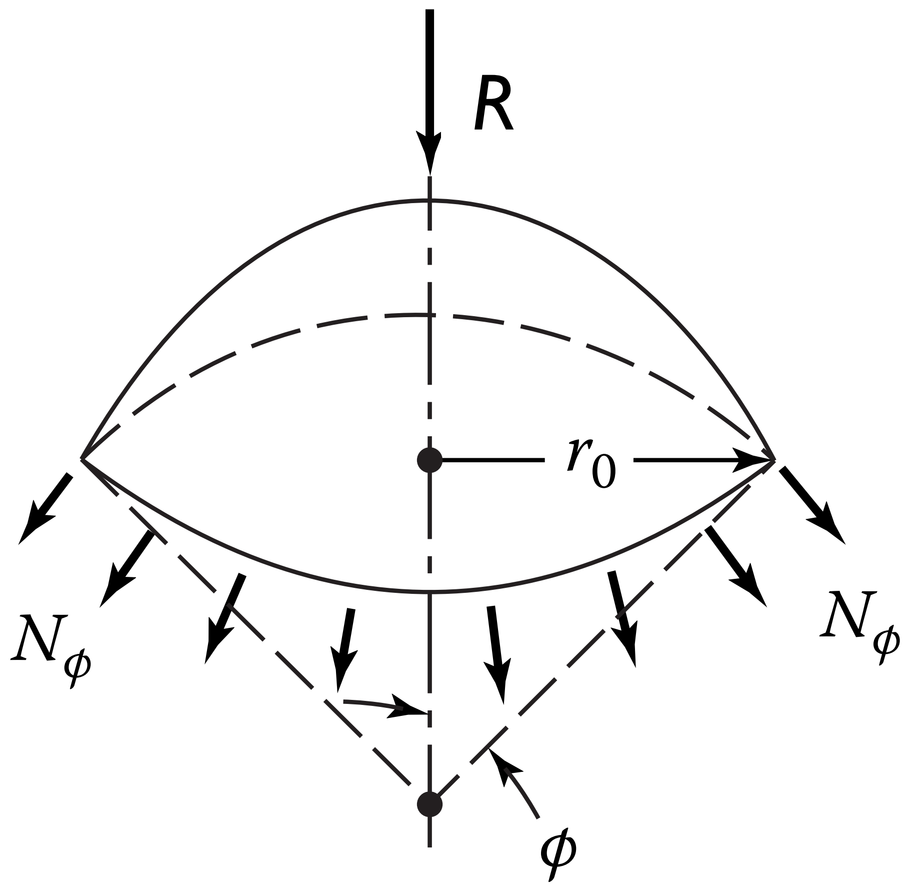
“membrane solution” for revolution shells

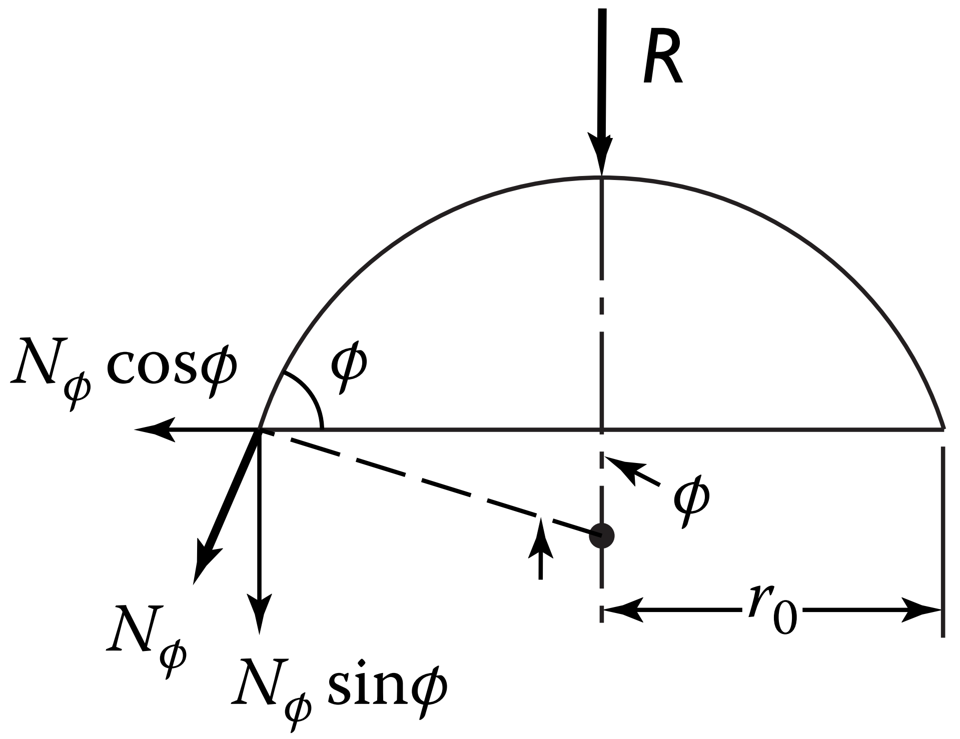
only two equilibrium equations (perpendicular and meridional equilibrium): alternative form

$$\cancel{\frac{\partial}{\partial \phi}\left(r_0 N_\phi\right) - r_1 N_\theta \cos\phi + r_0 r_1 p_\phi = 0 \quad (1)}$$

$$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} = -p_z \quad (2)$$

$$2\pi r_0 N_\phi \sin\phi + R = 0 \quad (3)$$





where R is the resultant (positive downwards) of all external forces acting over the parallel circle of radius (r_0)

summary: "membrane solution" for revolution shells

$$2\pi r_0 N_\phi \sin\phi = -R \qquad \frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} = -p_z$$

$$N_\phi = -\frac{R}{2\pi r_0 \sin\phi} \qquad N_\theta = \frac{R}{2\pi r_1 \sin^2\phi} - \frac{r_0 p_z}{\sin\phi}$$

$$\sigma_{\phi\phi} = \frac{N_\phi}{t} \qquad \sigma_{\theta\theta} = \frac{N_\theta}{t}$$

$$\varepsilon_{\phi\phi} = \frac{1}{Et}\left(N_\phi - \nu N_\theta\right) \qquad \varepsilon_{\theta\theta} = \frac{1}{Et}\left(N_\theta - \nu N_\phi\right)$$

End!