Aircraft Structural Analysis

Master Course in Aerospace Engineering

Session #02 — Parts 1 & 2

Goals for today

- Start building up the foundations for the analysis of thin-walled structures
- Stress and Strains (Part 1), Deflections (Part 2)
- Be comfortable with analytical approaches that can be easily programmed and tested against computational methodologies (Finite Element Method)

- References for follow-up study (and step-by-step solved exercises):
  - Book: Ugural, chapter 3 (**sections 3.1 - 3.6**), Part 1 of these slides
  - Book: Ugural, chapter 3 (**sections 3.7 - 3.17**), Part 2 of these slides

Pictures on these slides are digitalisations from the author's hard copies of the books of A. Ugural, and also J. Silva Gomes

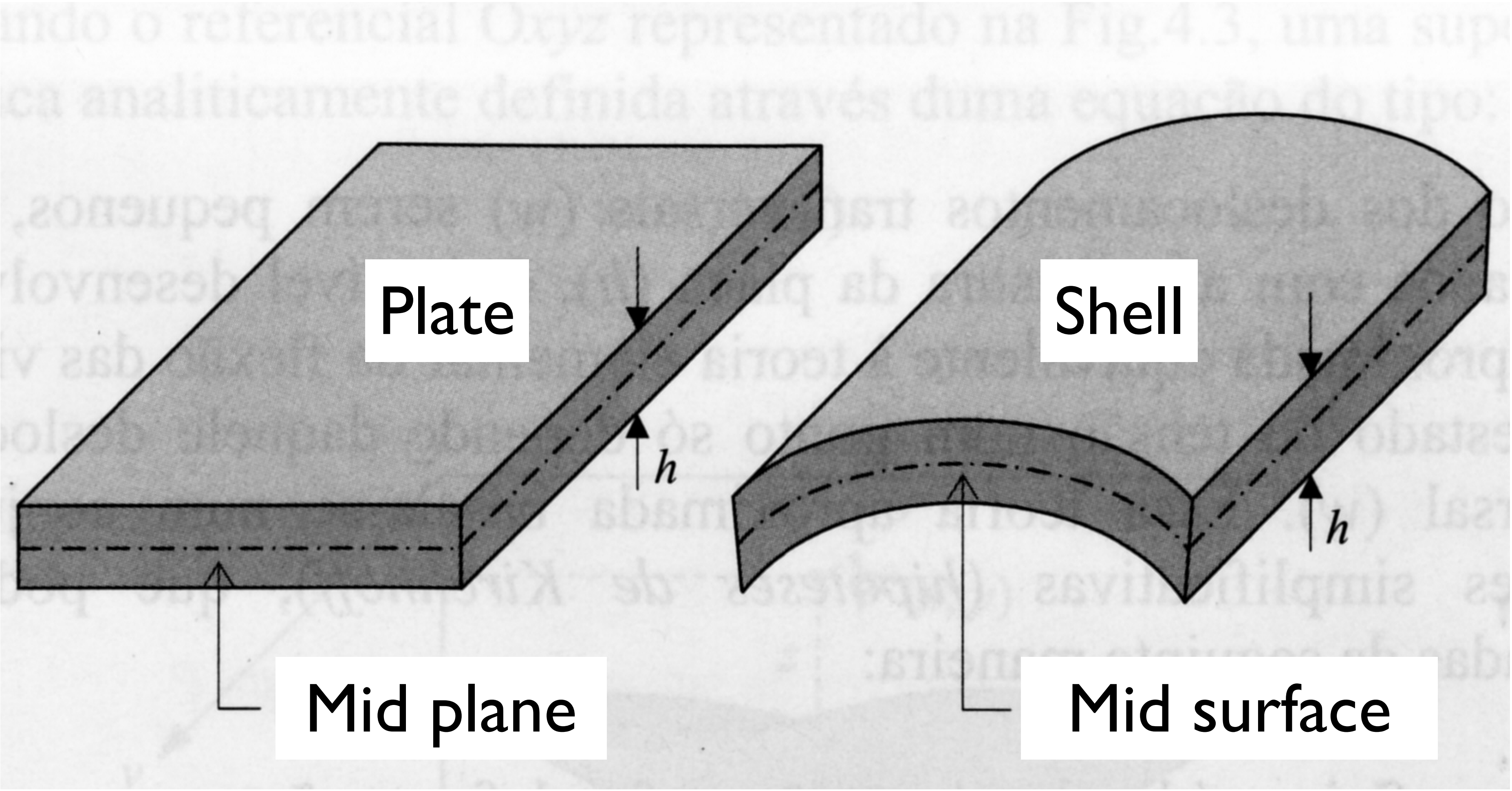
Introductory concepts

- **Plates**: structures initially flat (no curvature), where one of the dimensions (thickness) is smaller than the other dimensions
- **Shells**: the same, but with a geometry including one (or more) radius of curvature

- a wide number of structural applications can be seen as plates / shells
- they are typically subjected to bending effects, coming from loads perpendicular to their mid plane (mid surface)
- the theory involved can be seen as an extension of Solid Mechanics, Structural Mechanics, Strength of Materials, etc.

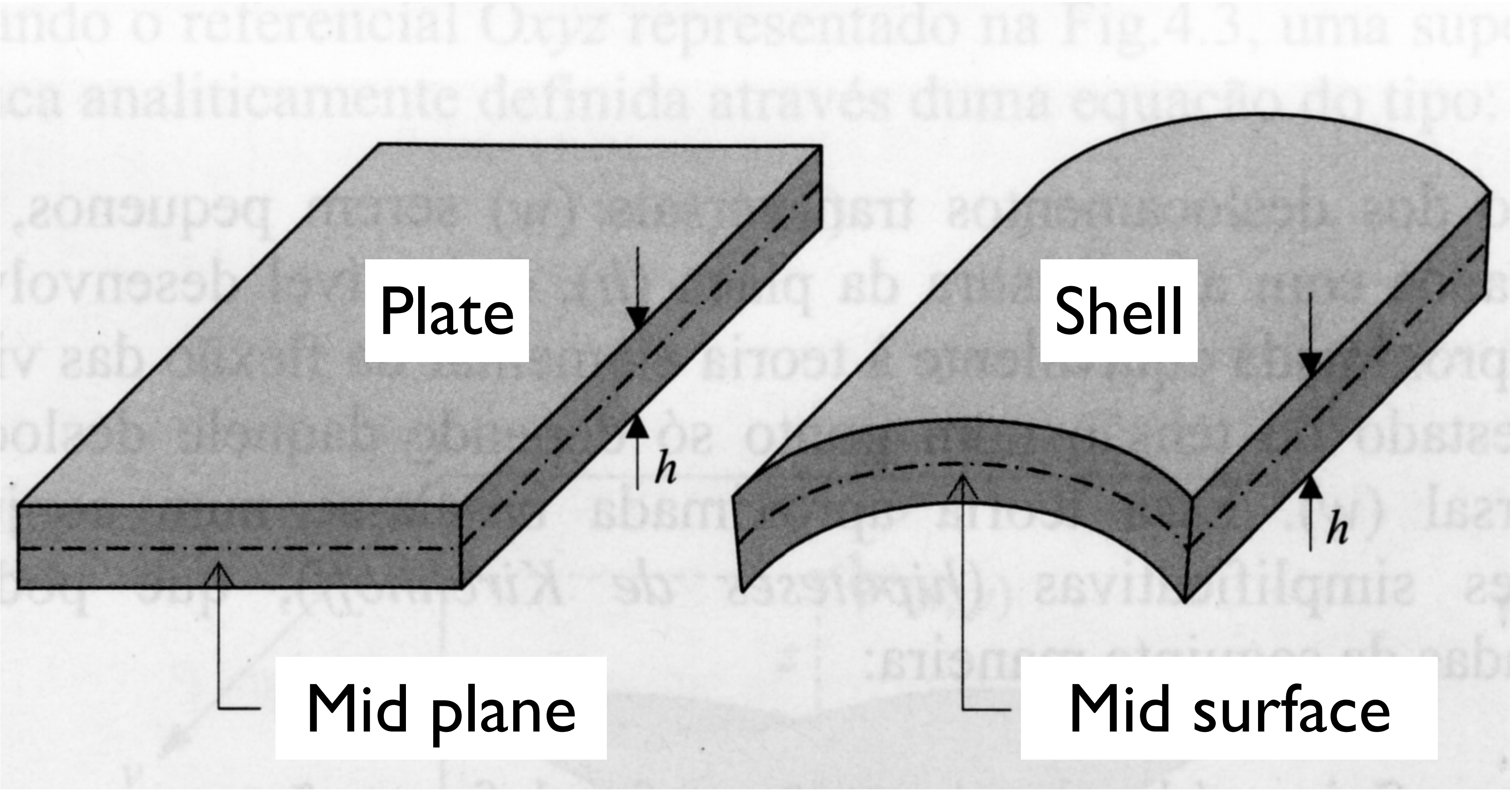
Part I

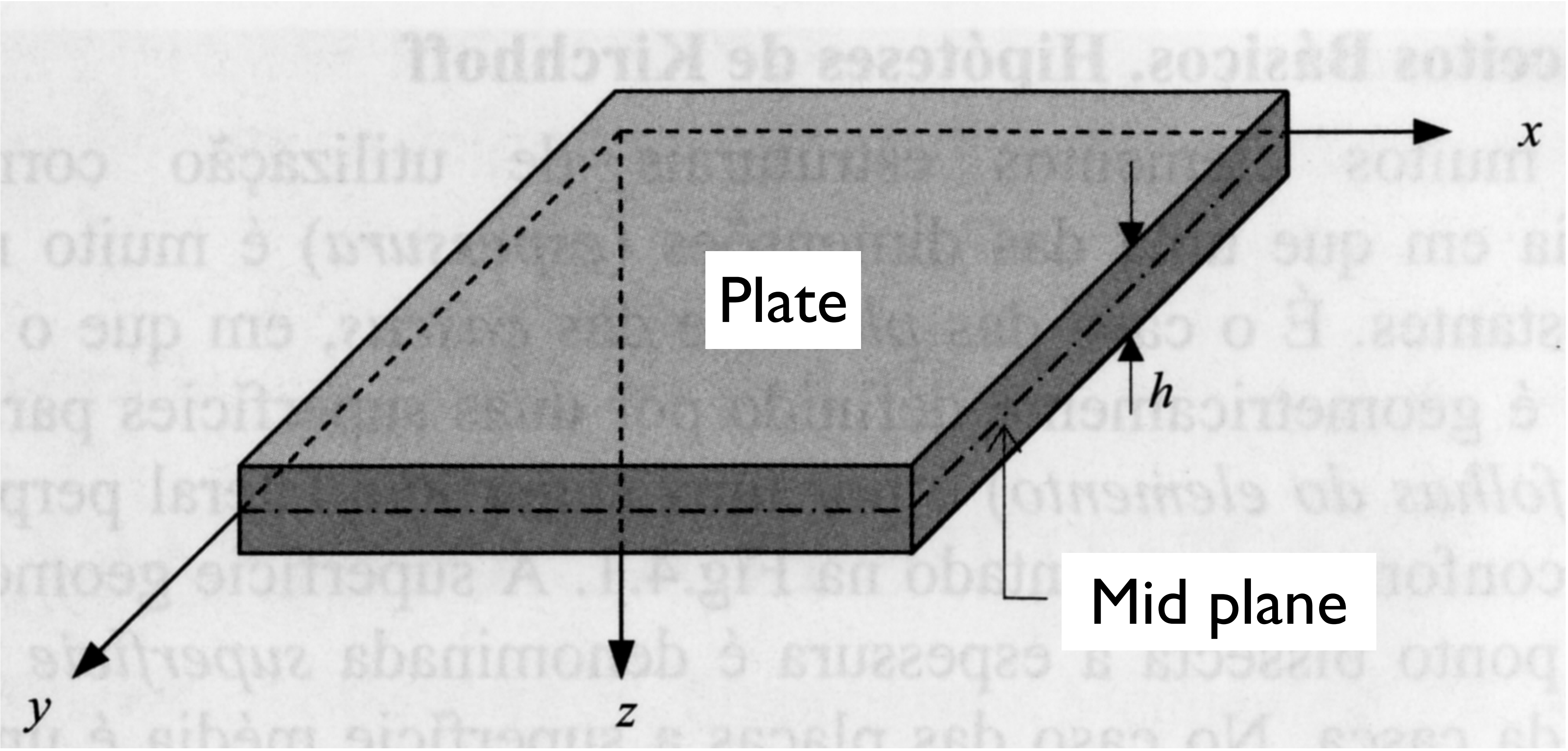
The geometry and notation



h (or t) = thickness value
mid plane = reference plane
mid surface = reference surface

The geometry and notation



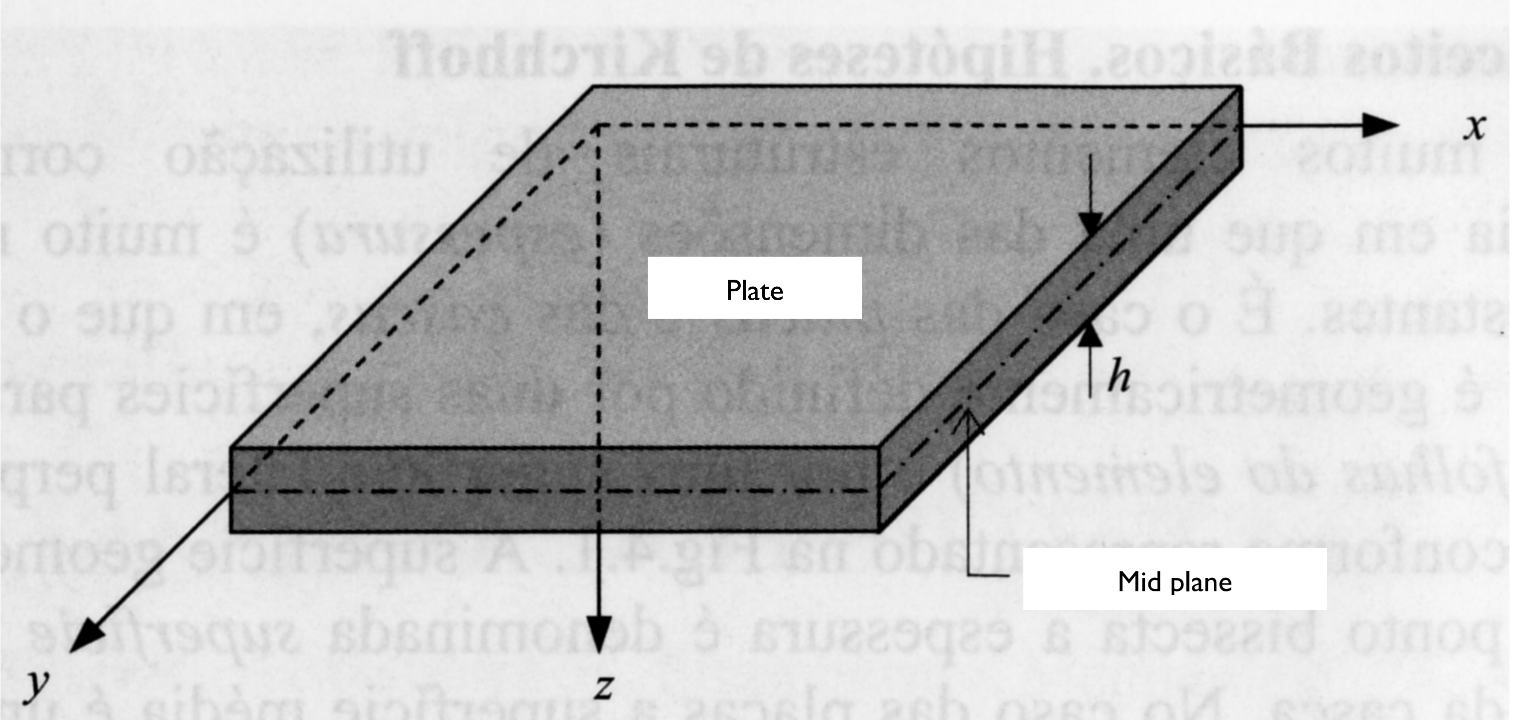


typically, x and y define the mid plane (reference plane)

What kind of models?

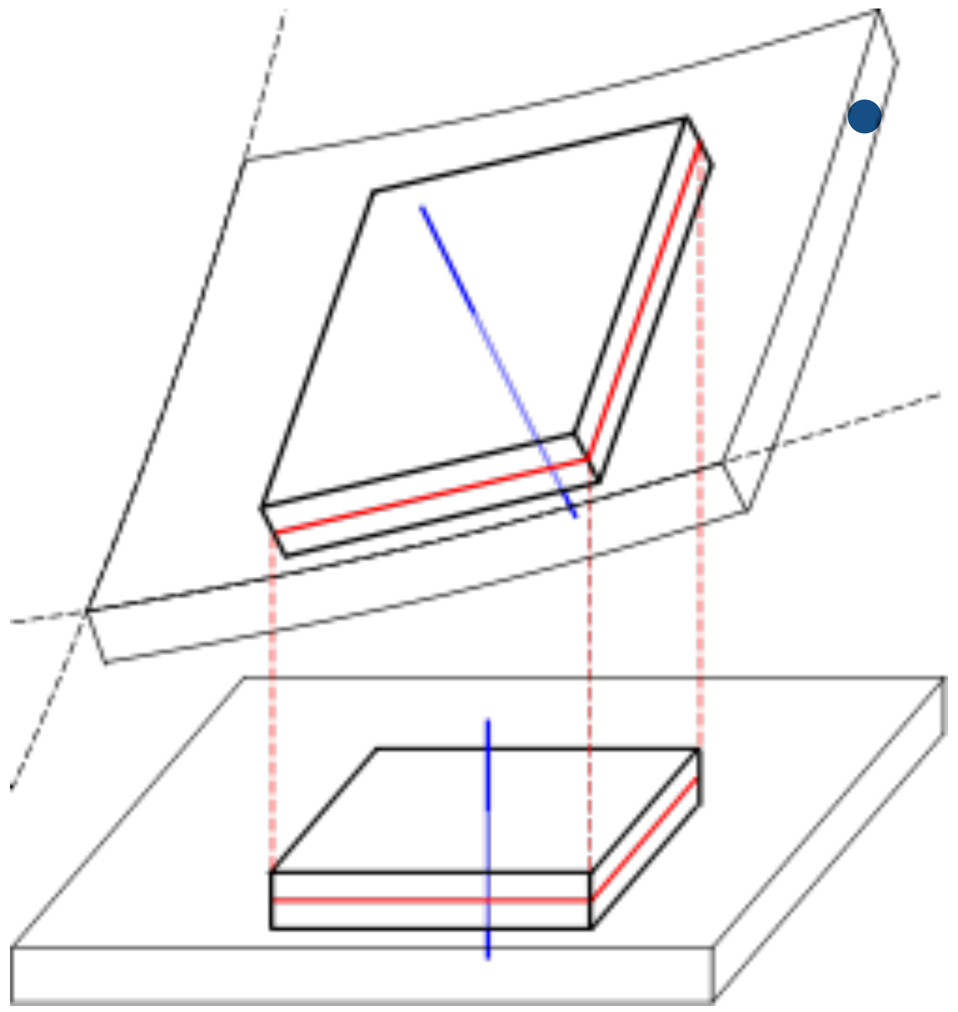
- **thin plates with small displacements** (exact theory available)
- thin plates with large displacements and deformations/strains
  (some limited theories, complemented by the Finite Element Method)

- what makes a plate to be thin? **thickness ratio equal ou lesser than 1/20, compared to the other dimensions**
- isotropic and homogeneous materials

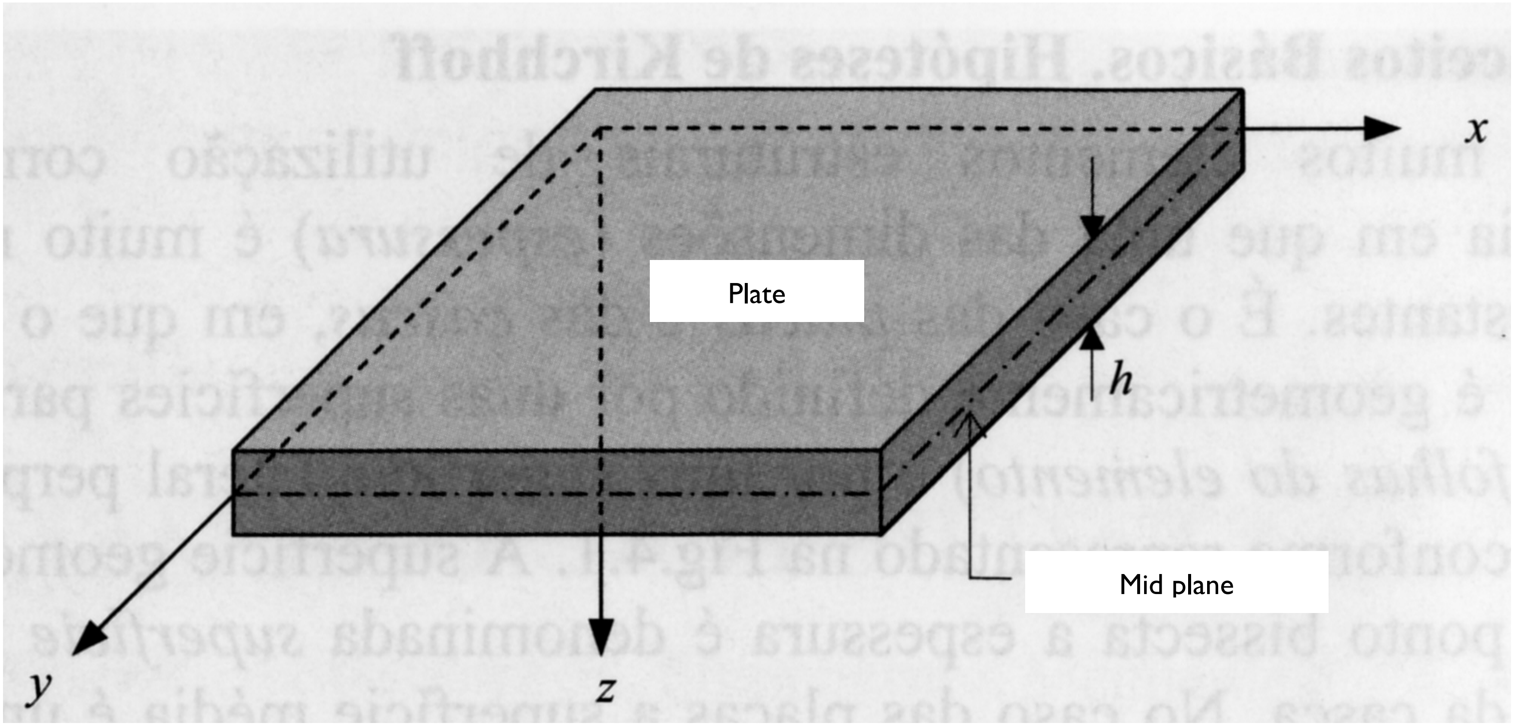


Kirchhoff theory for thin plates

- displacements along OZ direction are small (5 times, at least) than thickness
- stress state at a point does not depend on its position along thickness
- straight lines normal to the mid-surface remain straight after deformation
- straight lines normal to the mid-surface remain normal to the mid-surface after deformation
- the thickness of the plate does not change during a deformation



- **two-dimensional mathematical model** to determine the stresses and deformations in thin plates subjected to forces and moments



Introduction to the Kirchhoff theory for thin plates

Kirchhoff theory for thin plates (usual assumptions)

- straight lines normal to the mid-surface remain straight after deformation

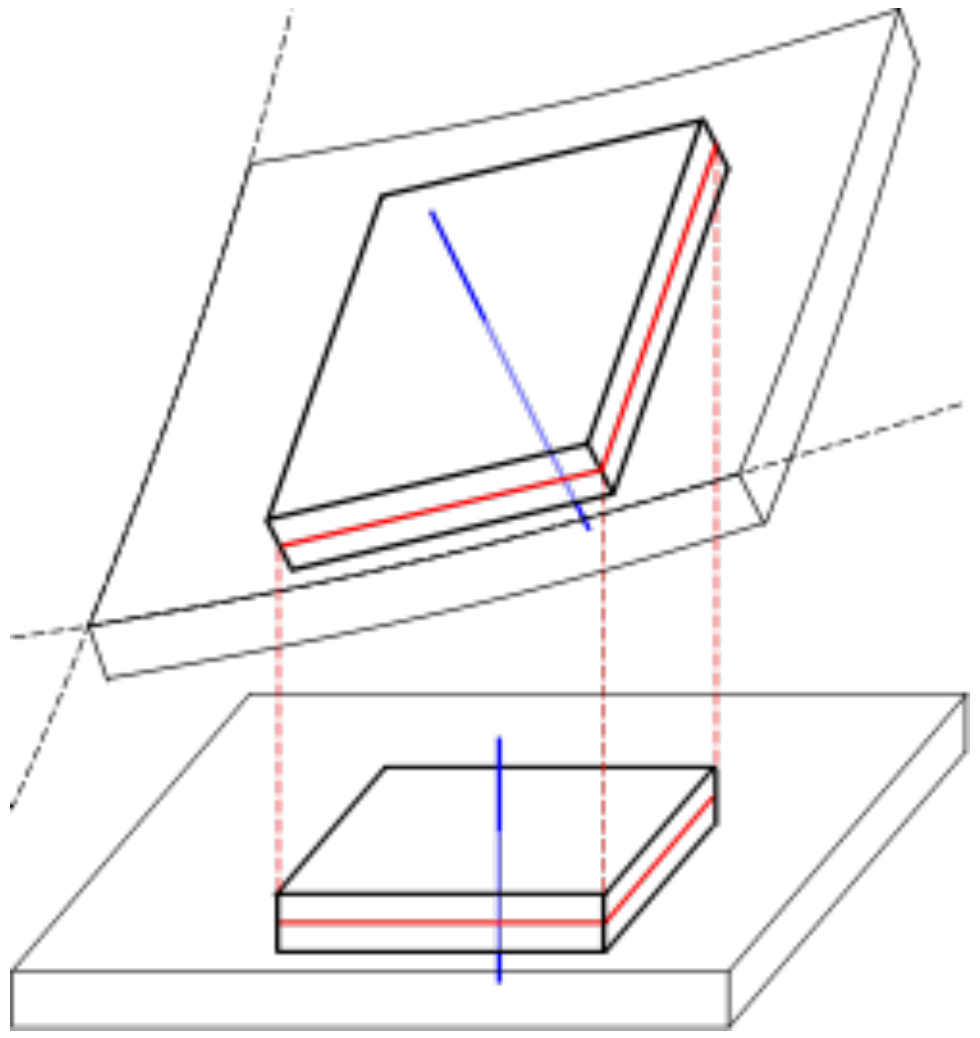
$$\text{for } z = 0 \longrightarrow \varepsilon_{xx} = \varepsilon_{yy} = \gamma_{xy} = 0$$

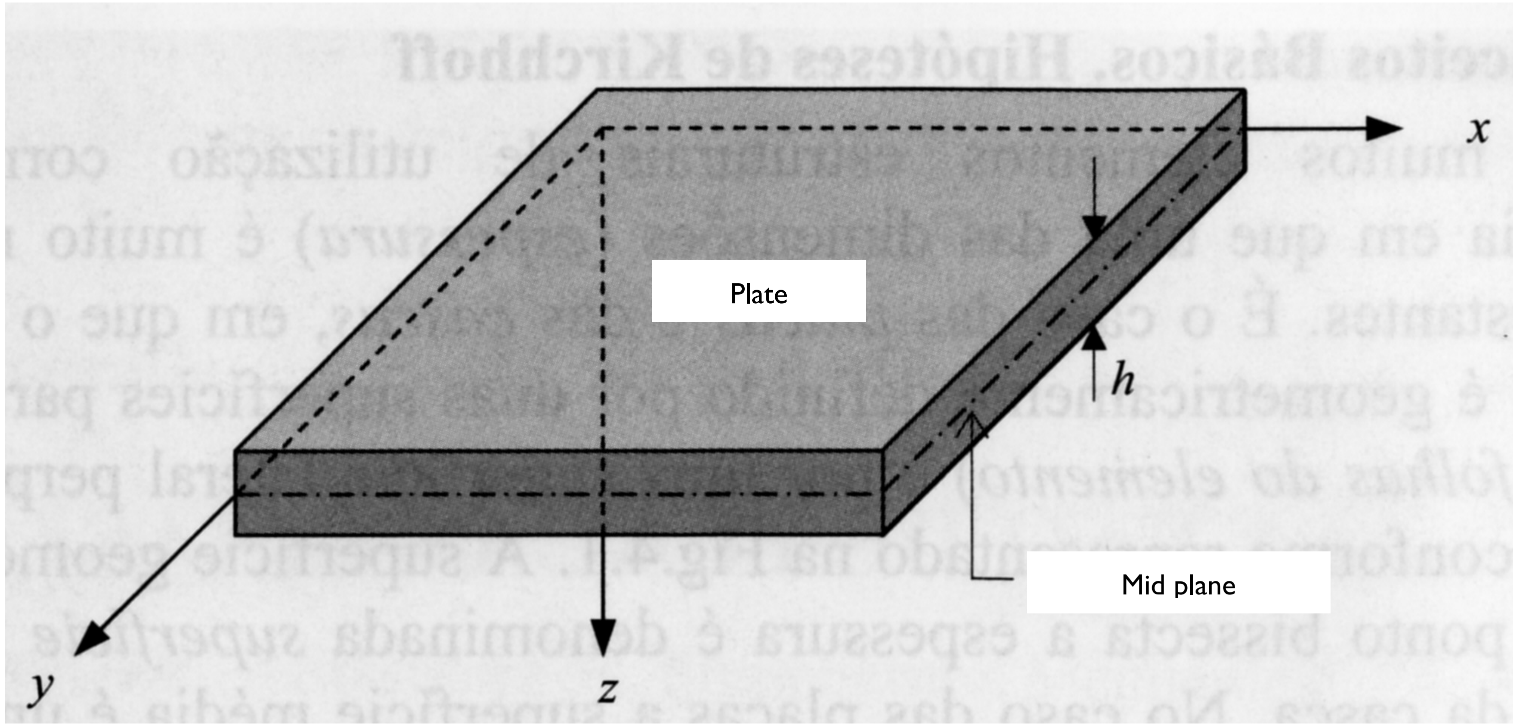
- straight lines normal to the mid-surface remain normal to the mid-surface after deformation

$$\gamma_{xz} = \gamma_{yz} = 0$$

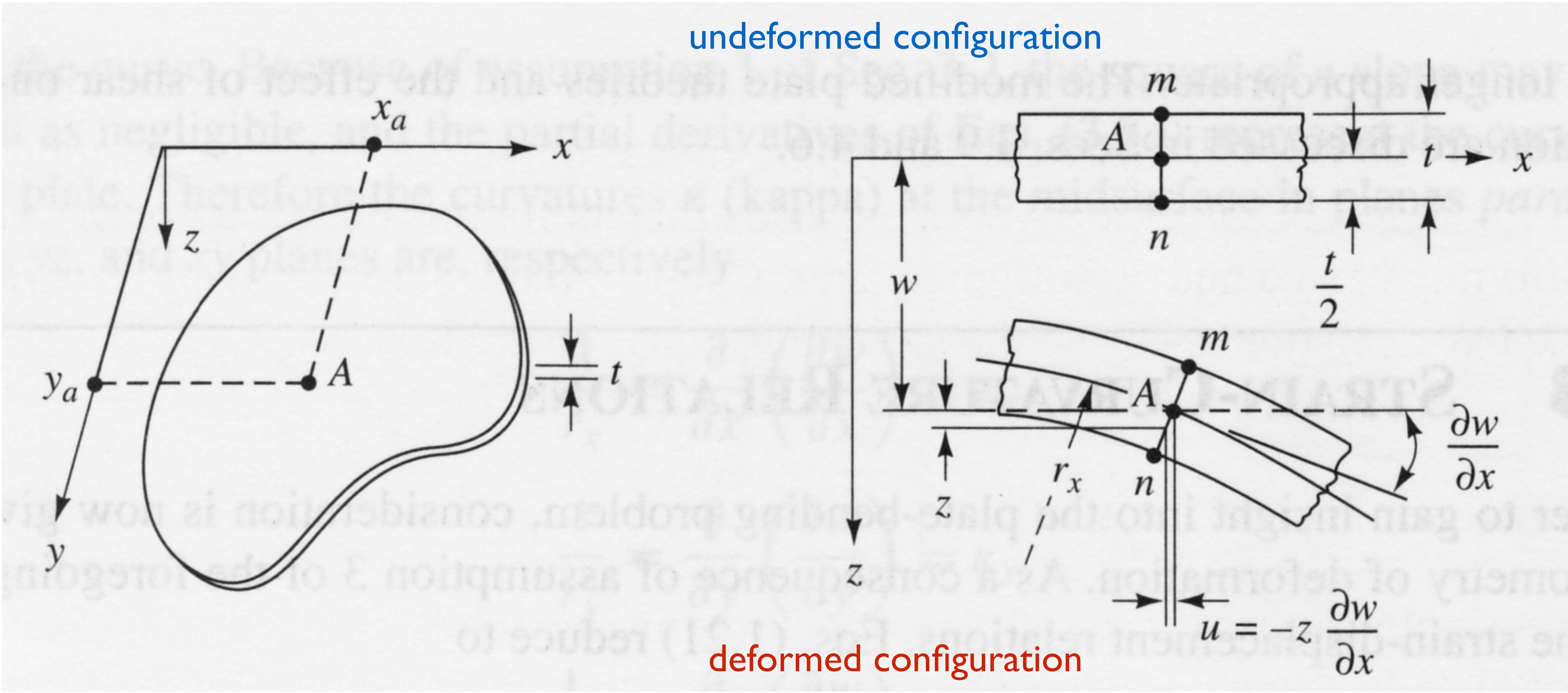
- the thickness of the plate does not change during a deformation

$$\sigma_{zz} \cong 0 \quad ; \quad \varepsilon_{zz} \cong 0$$



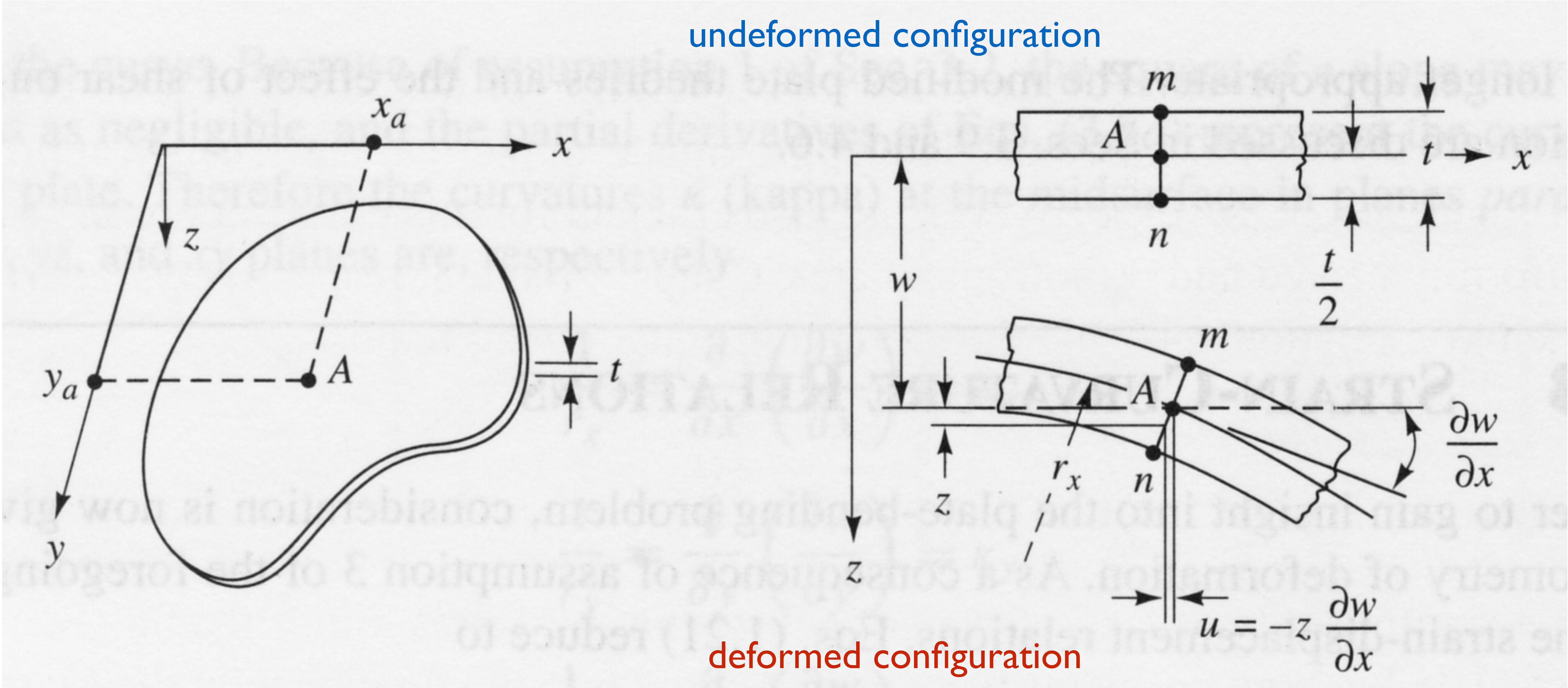


deformations at a given point



notations, new parameters, degrees of freedom, first mathematical relations…

deformations at a given point

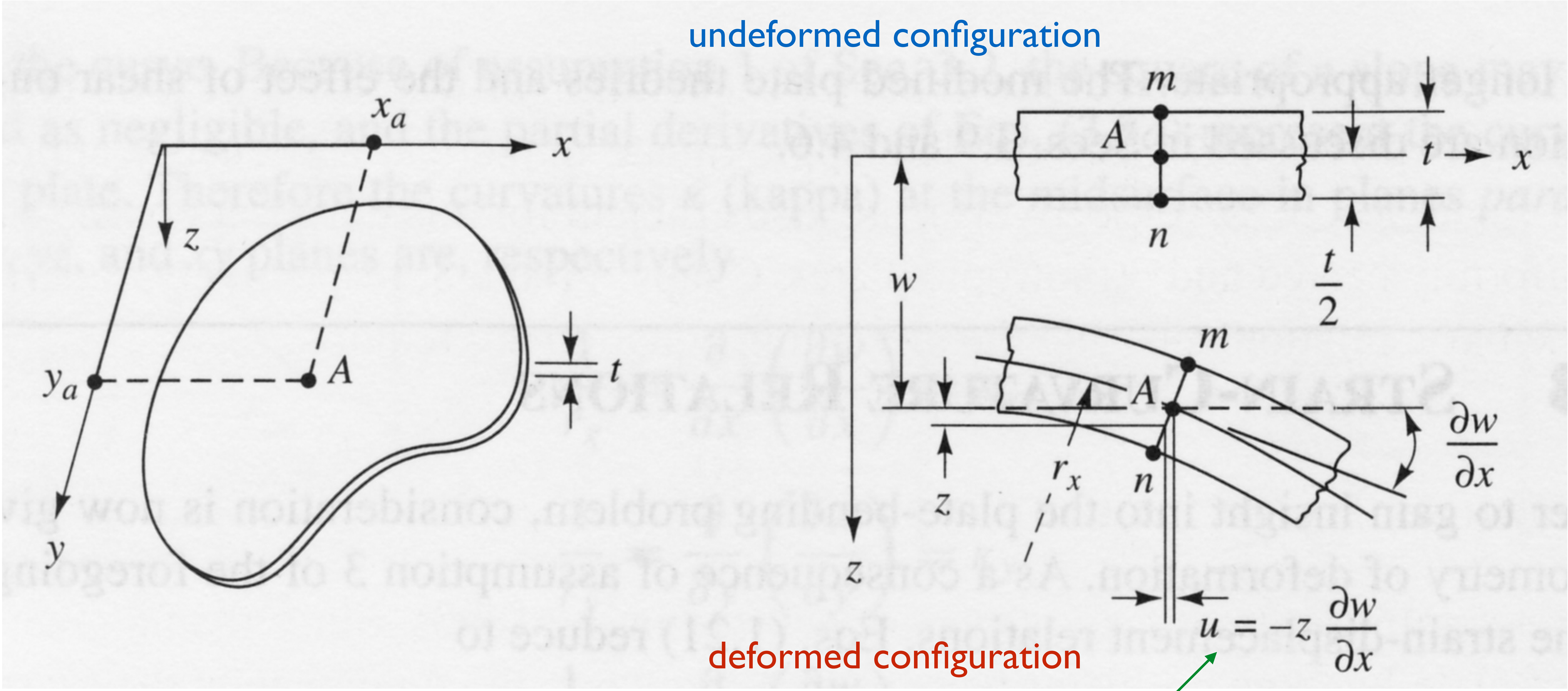


from Solid Mechanics (small strains)

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0$$

deformations at a given point

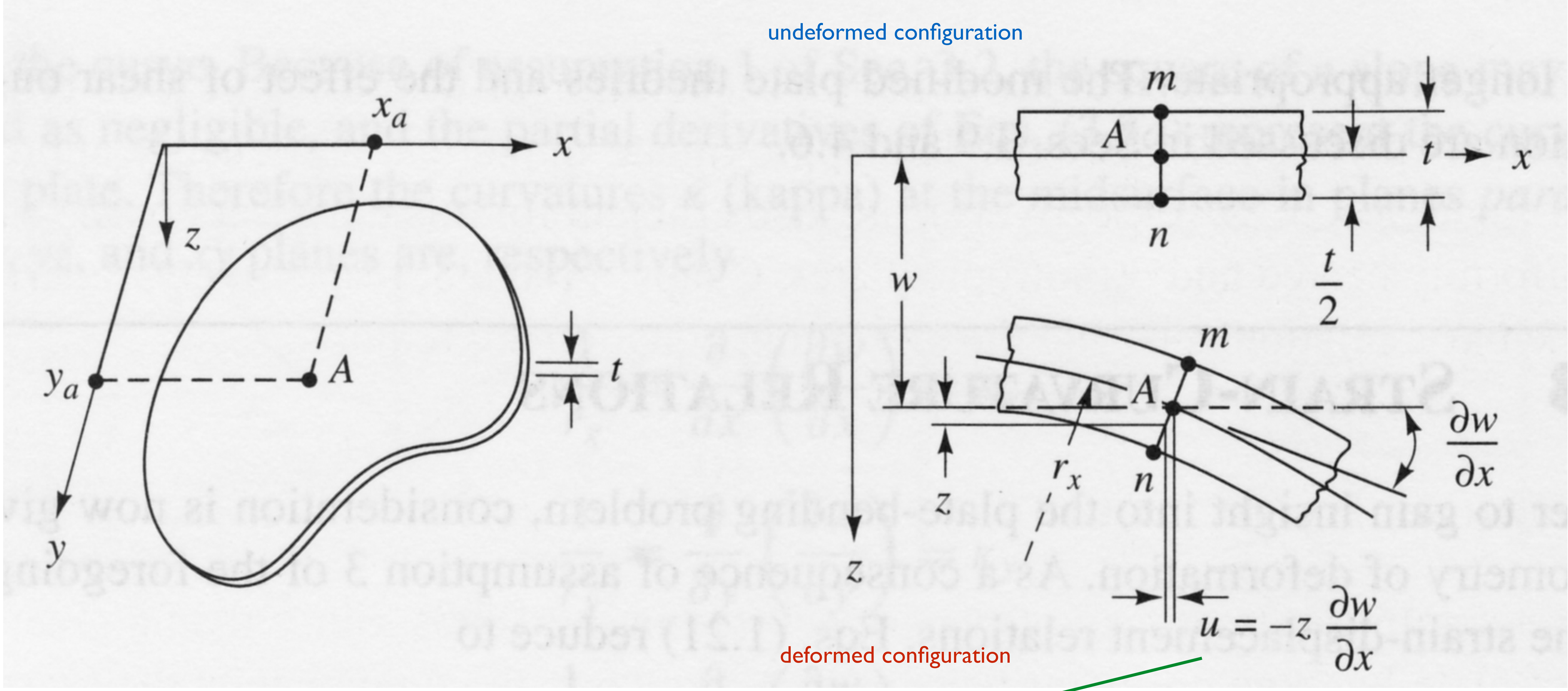


$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \quad \Rightarrow \quad w = w(x,y)$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 \quad \Rightarrow \quad u = -z\,\frac{\partial w}{\partial x} + \cancel{u_0(x,y)}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0 \quad \Rightarrow \quad v = -z\,\frac{\partial w}{\partial y} + \cancel{v_0(x,y)}$$

deformations at a given point

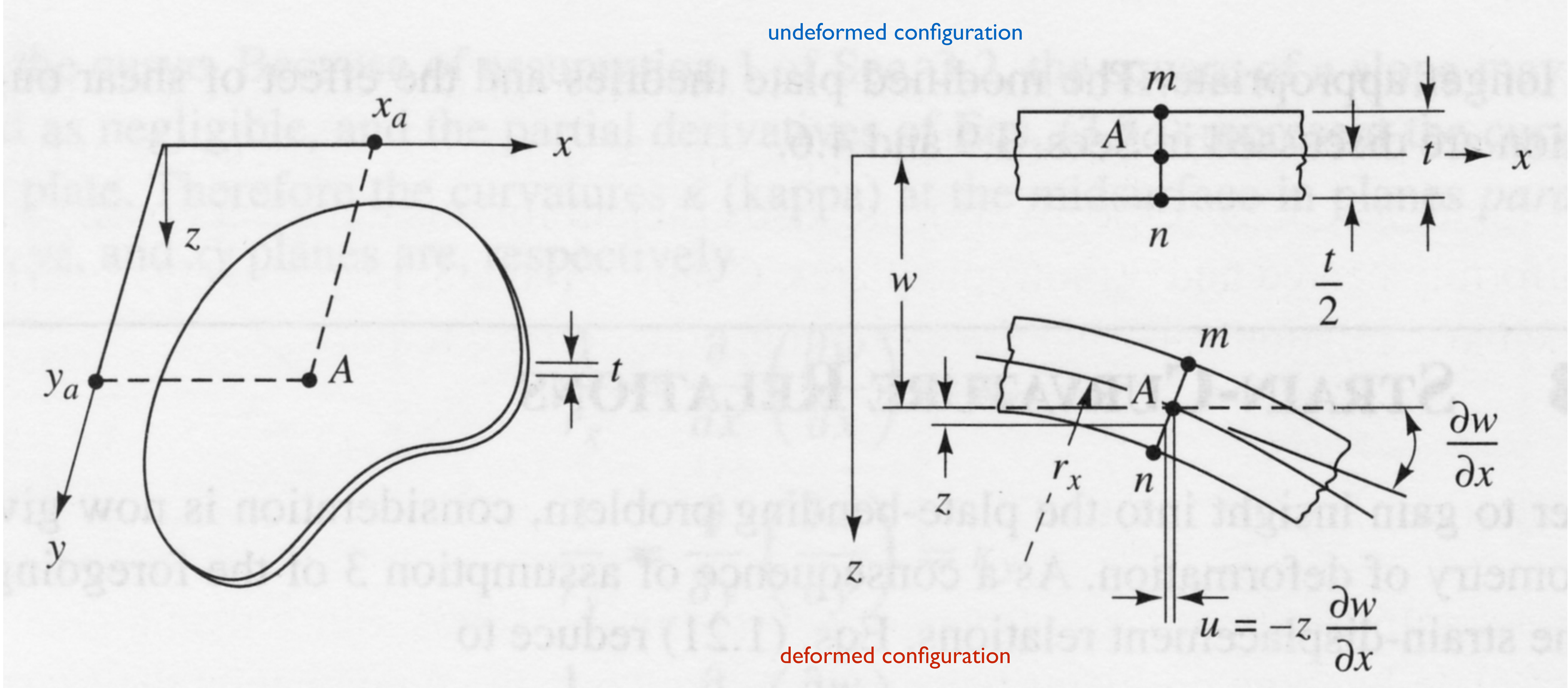


$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\,\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\,\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\,\frac{\partial^2 w}{\partial x \partial y}$$

deformations at a given point



$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\,\frac{\partial^2 w}{\partial x^2}$$

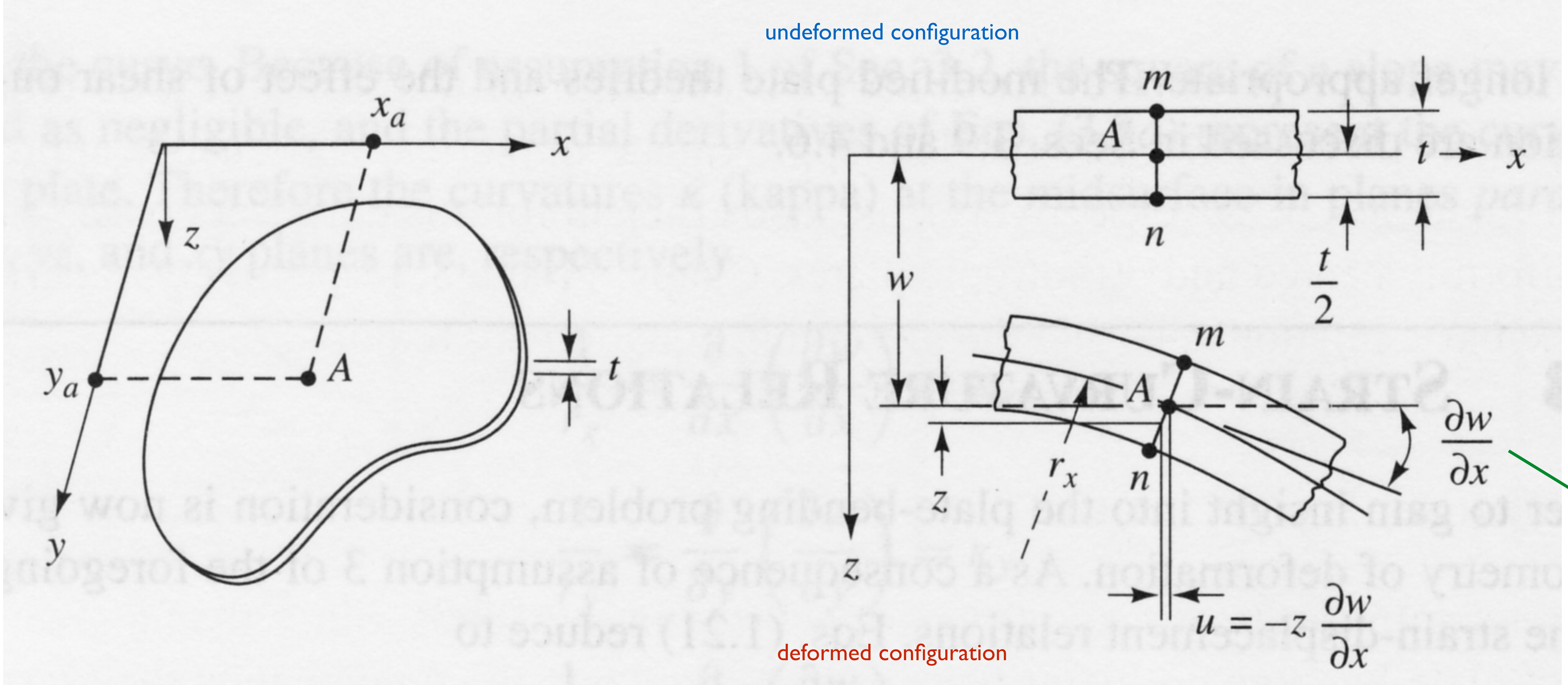
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\,\frac{\partial^2 w}{\partial y^2}$$

curvature

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\,\frac{\partial^2 w}{\partial x \partial y}$$

twist

deformations at a given point



slope angle

the curvature is the rate of change of the slope angles…

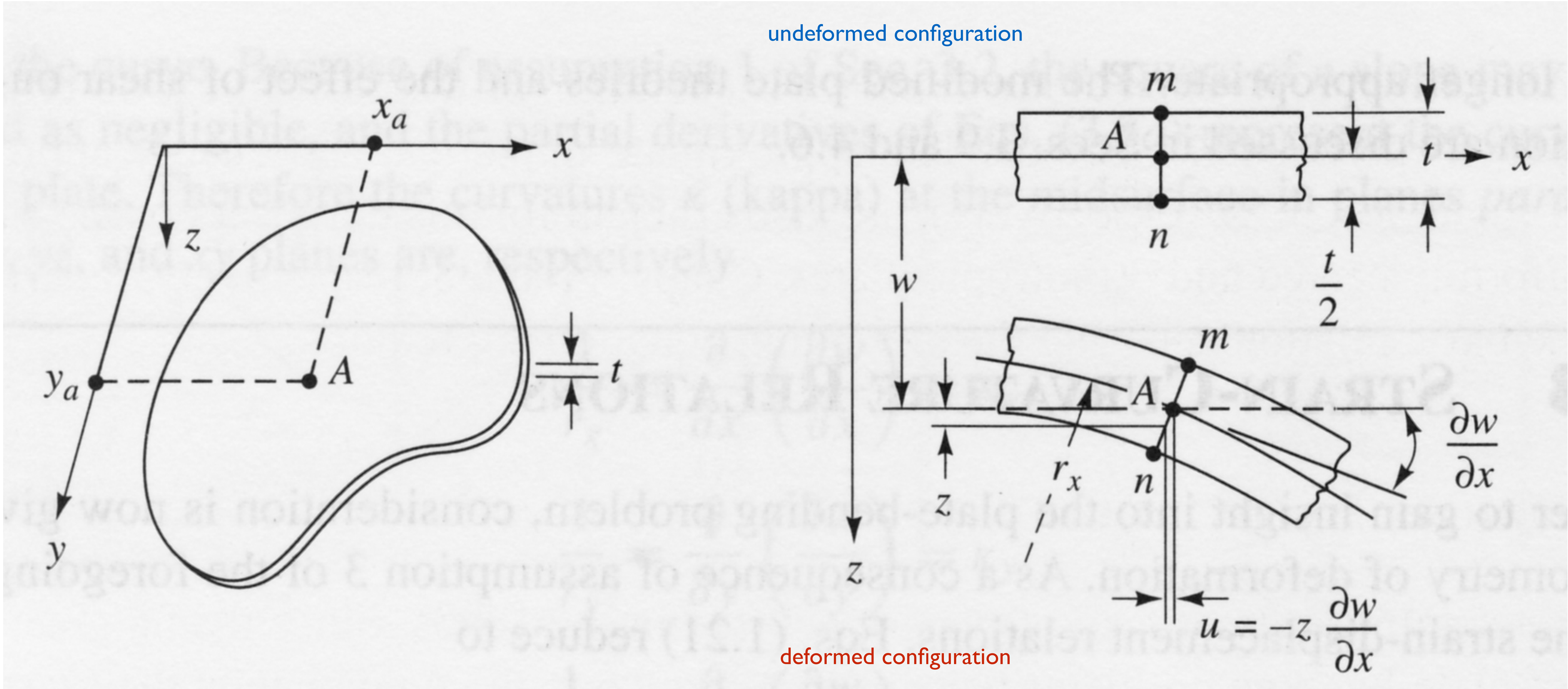
$$\kappa_x = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) \qquad \kappa_x = \frac{1}{r_x}$$

$$\kappa_y = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) \qquad \kappa_y = \frac{1}{r_y}$$

$$\kappa_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \kappa_{yx} \qquad \kappa_{xy} = \frac{1}{r_{xy}}$$

radius of curvature (m)

relations between strains and curvatures

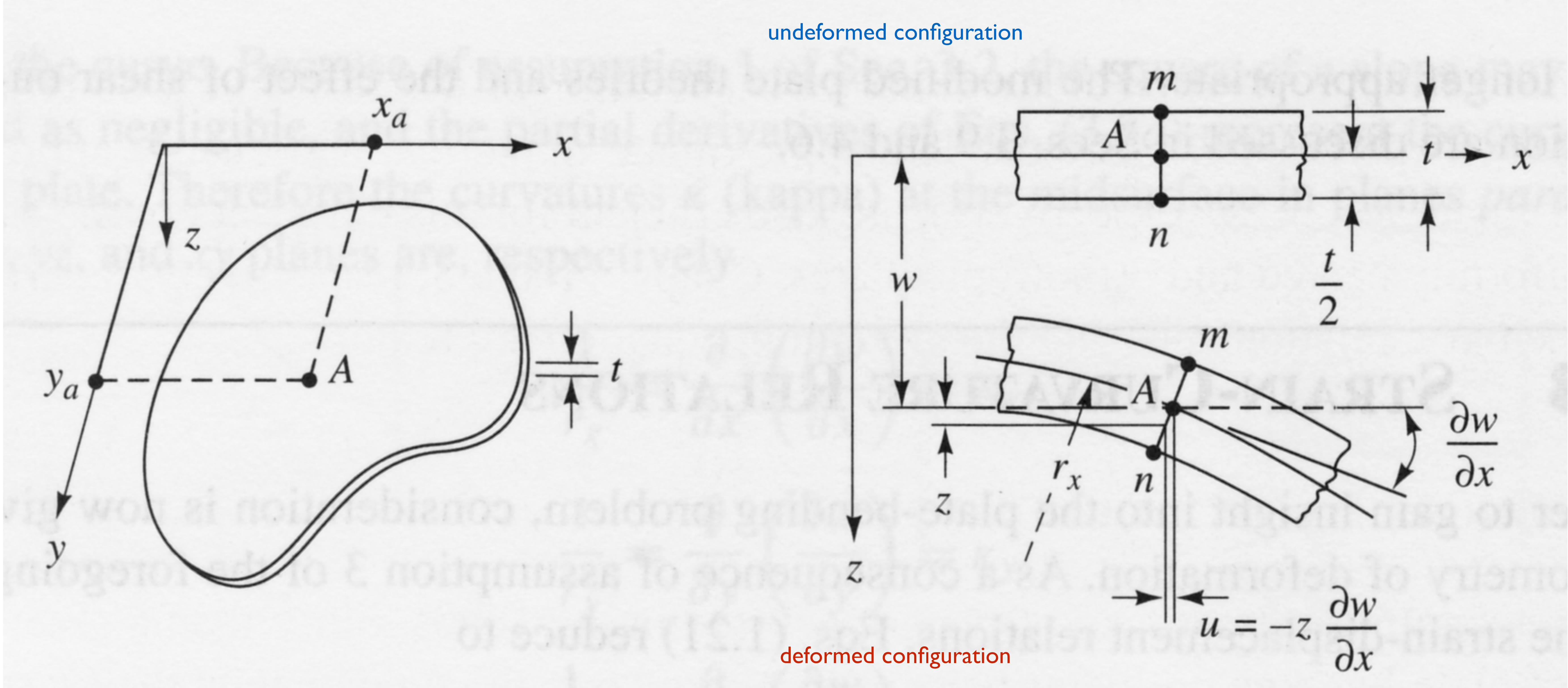


$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} = -z\kappa_x$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} = -z\kappa_y$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} = -2z\kappa_{xy}$$

relations between strains and curvatures



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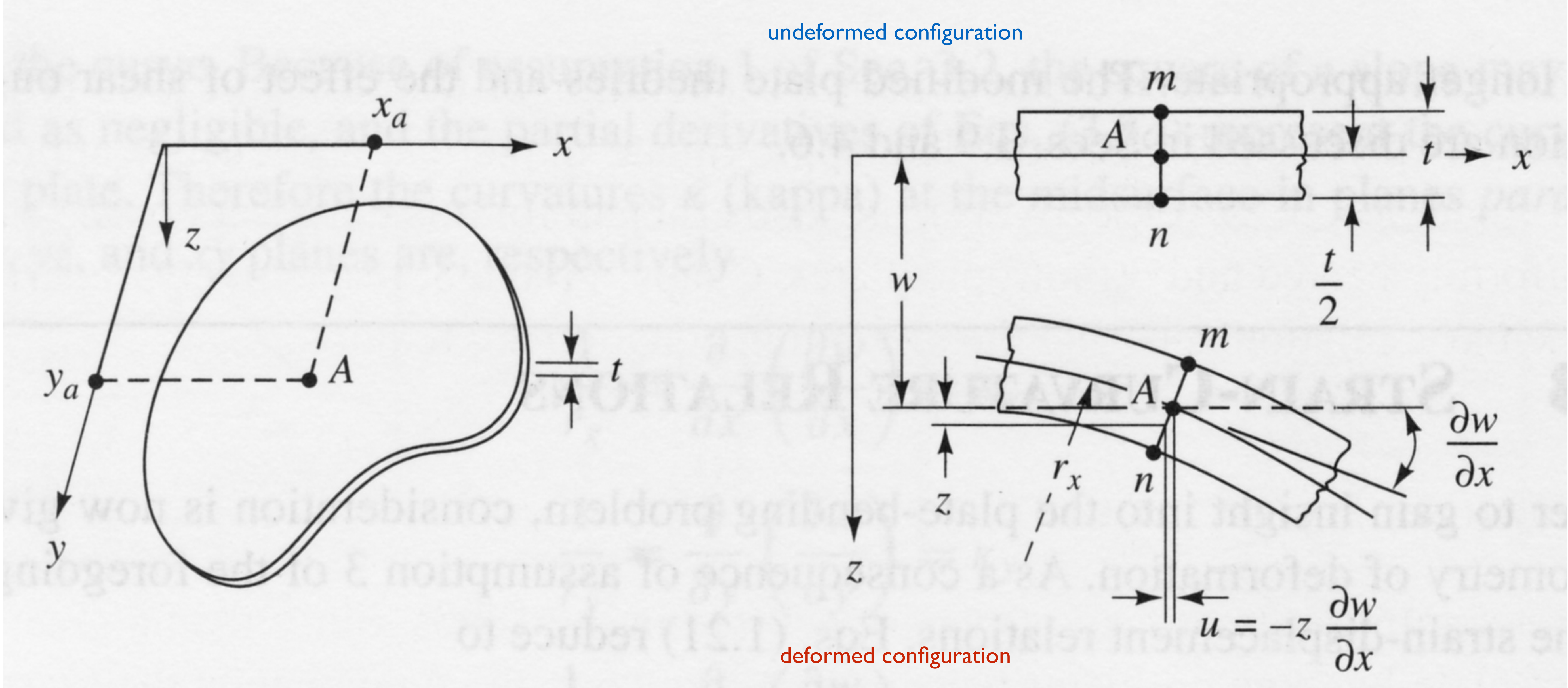
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\,\frac{\partial^2 w}{\partial x \partial y} = -2z\kappa_{xy}$$

“only” the geometry is needed (no loads)

“no” influence from the material…

linear increase of strains with position “z”

relations between strains and curvatures



$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} = -z\kappa_x$$

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$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} = -2z\kappa_{xy}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = -z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

Constitutive behaviour
(the influence coming from the material properties)

Hooke's Law for Elasticity

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \quad , \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] \quad , \quad \gamma_{xz} = \frac{\tau_{xz}}{G}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \quad , \quad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$G = \frac{E}{2\left(1 + \nu\right)}$$

Hooke's Law for Elasticity

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \quad , \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

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$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \quad , \quad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$G = \frac{E}{2(1+\nu)}$$

modulus of elasticity (Young)

shear modulus of elasticity

proportionality ratio (Poisson)

Hooke's Law for Elasticity

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \quad , \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] \quad , \quad \gamma_{xz} = \frac{\tau_{xz}}{G}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \quad , \quad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$G = \frac{E}{2\left(1 + \nu\right)}$$

and now for the particular case of Kirchhoff hypotheses?

$$\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$$

Hooke's Law + Kirchhoff = Plane stress state

$$\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0 \longrightarrow \begin{aligned} \sigma_{xx} &= \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right) \\ \sigma_{yy} &= \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right) \\ \tau_{xy} &= \frac{E}{2\left(1+\nu\right)}\gamma_{xy} \end{aligned}$$

Hooke's Law + Kirchhoff = Plane stress state

$$\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0 \longrightarrow$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right)$$
$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right)$$
$$\tau_{xy} = \frac{E}{2\left(1+\nu\right)}\gamma_{xy}$$

+

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} = -z\kappa_x$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} = -z\kappa_y$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} = -2z\kappa_{xy}$$

Hooke's Law + Kirchhoff = Plane stress state

$$\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0 \longrightarrow$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right)$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right)$$

$$\tau_{xy} = \frac{E}{2\left(1+\nu\right)}\gamma_{xy}$$

$$\boxed{\begin{aligned}
\sigma_{xx} &= -\frac{Ez}{1-\nu^2}\left(\kappa_x + \nu\kappa_y\right) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\\
\sigma_{yy} &= -\frac{Ez}{1-\nu^2}\left(\kappa_y + \nu\kappa_x\right) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)\\
\tau_{xy} &= -\frac{Ez}{1+\nu}\kappa_{xy} = -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x\partial y}
\end{aligned}}$$

Hooke's Law for the Kirchhoff theory

$$\boxed{\begin{aligned}
\sigma_{xx} &= -\frac{Ez}{1-\nu^2}\left(\kappa_x + \nu\kappa_y\right) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\\
\sigma_{yy} &= -\frac{Ez}{1-\nu^2}\left(\kappa_y + \nu\kappa_x\right) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)\\
\tau_{xy} &= -\frac{Ez}{1+\nu}\kappa_{xy} = -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x \partial y}
\end{aligned}}$$

or

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = -z \underbrace{\begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix}} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ 2\kappa_{xy} \end{Bmatrix}$$

stiffness matrix for the thin plate

Hooke's Law for the Kirchhoff theory

$$\sigma_{xx} = -\frac{Ez}{1-\nu^2}\left(\kappa_x + \nu\kappa_y\right) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_{yy} = -\frac{Ez}{1-\nu^2}\left(\kappa_y + \nu\kappa_x\right) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$\tau_{xy} = -\frac{Ez}{1+\nu}\kappa_{xy} = -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x \partial y}$$

zero stresses at the reference plane
($z = 0$)

maximum stresses at top and
bottom surfaces
($z = -t/2; +t/2$)

similar conclusions as those
from the bending of beams

or

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = -z \underbrace{\begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2\left(1+\nu\right)} \end{bmatrix}}_{\text{stiffness matrix for the thin plate}} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ 2\kappa_{xy} \end{Bmatrix}$$

An alternative point of view

(instead of using stresses)

An alternative point of view:

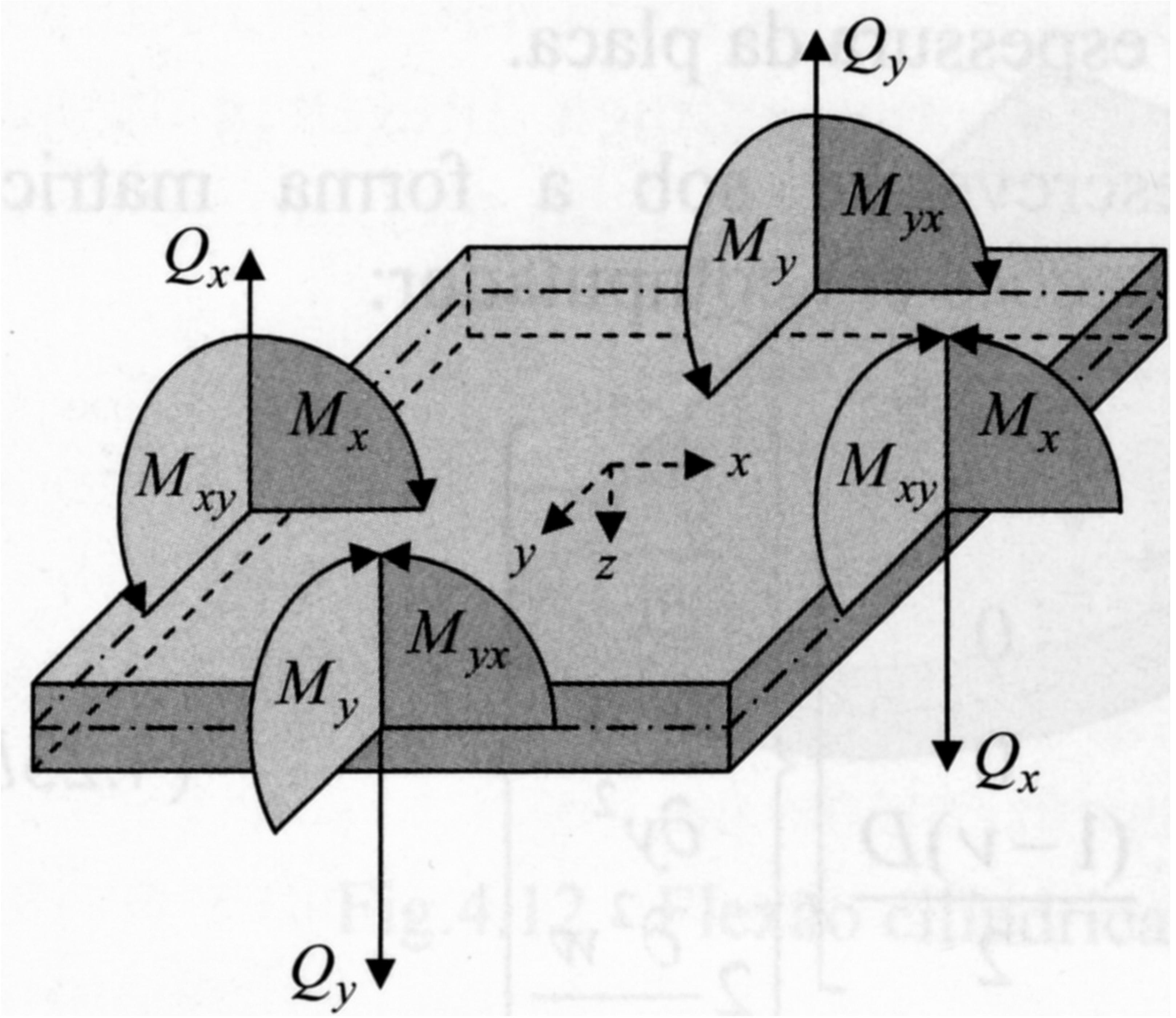
Stress Resultants

An alternative point of view: the concept of "stress resultants"

the stresses distributed over the thickness of the plate produce <u>bending moments</u>, <u>twisting moments</u>, and <u>vertical shear forces</u>

these moments and forces (per unit length) are also called <u>stress resultants</u>

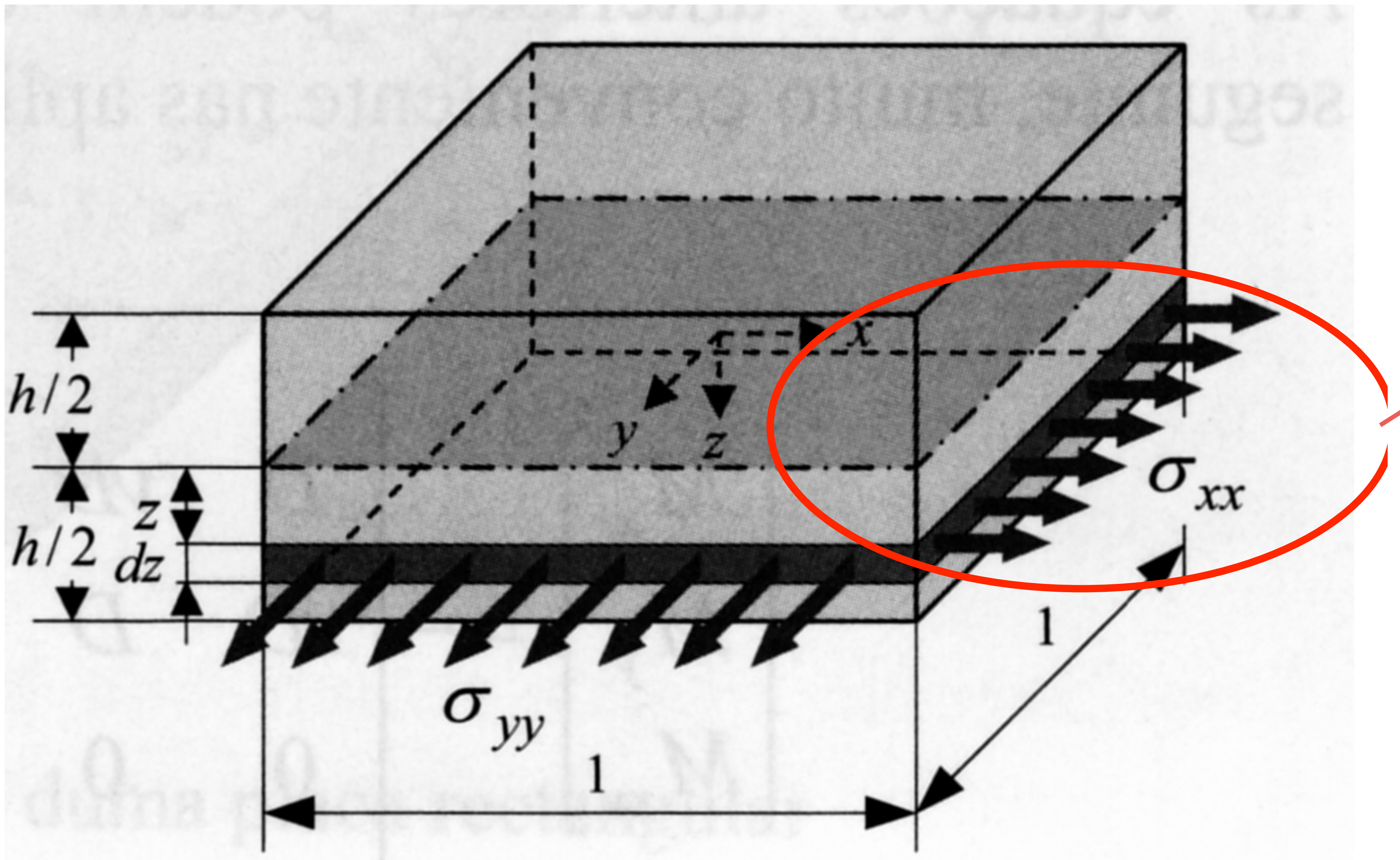
stress resultants: a visual representation…



(all signs are positive)

Mx , *My* : bending moments per unit of length
Mxy : torsion moments per unit of length
Qx , *Qy* : vertical shear forces per unit of length

relation between stress resultants and stresses: example for *Mx*

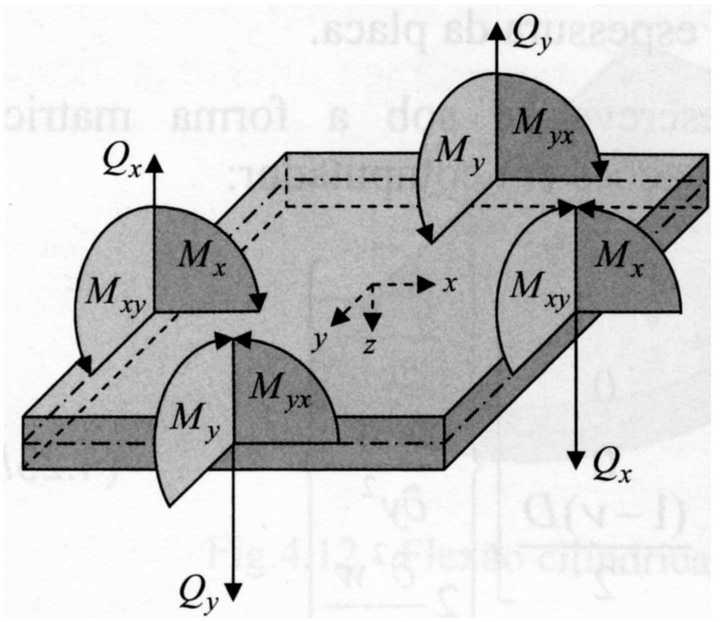


$$dA = 1 \times dz$$

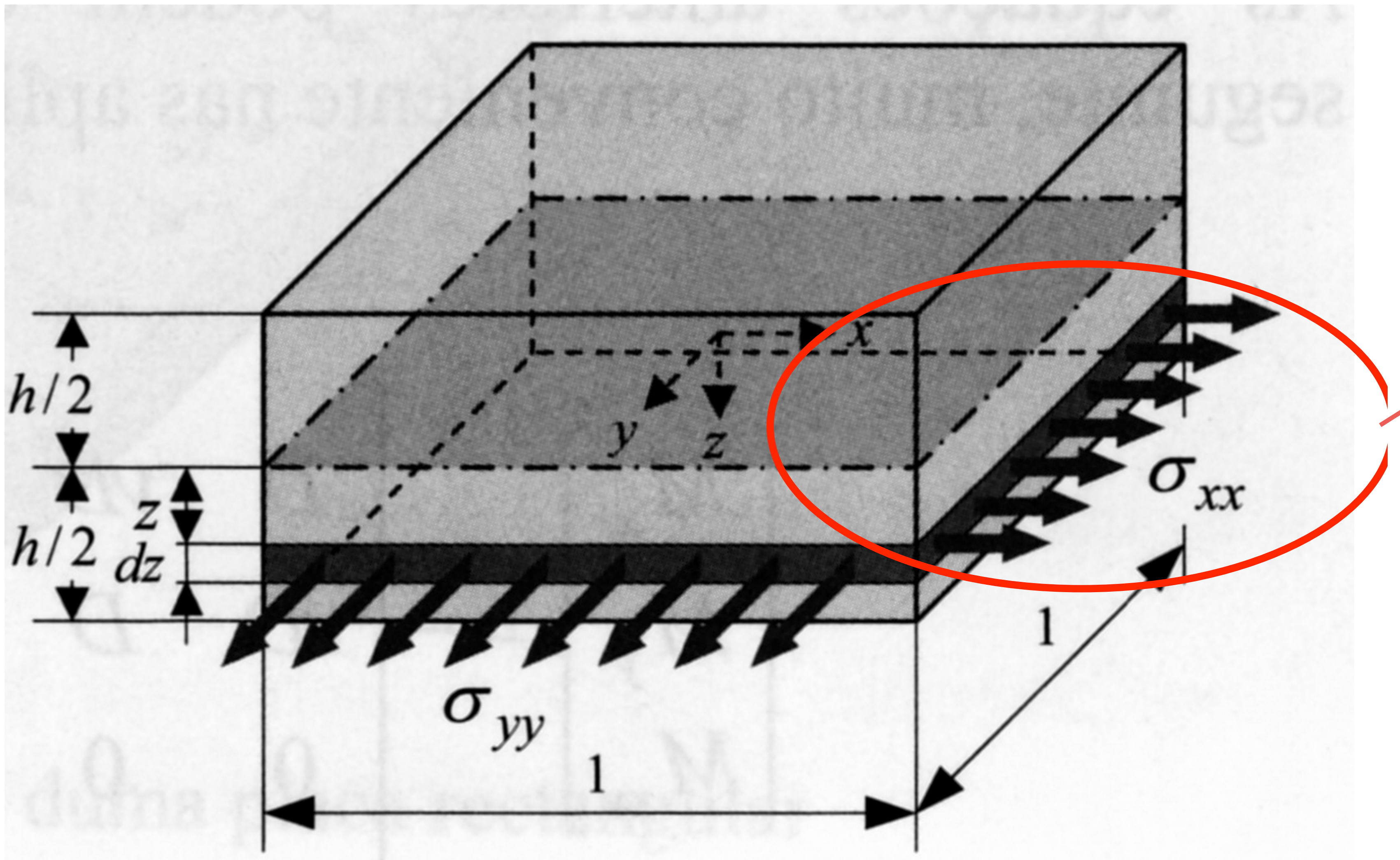
$$dF = \sigma_{xx} dA = \sigma_{xx} dz$$

$$dM = dF \times z = \left(\sigma_{xx} z\right) dz$$

$$M_x = \int_{-t/2}^{+t/2} \sigma_{xx} z dz$$



relation between stress resultants and stresses: example for *Mx*



$$dA = 1 \times dz$$

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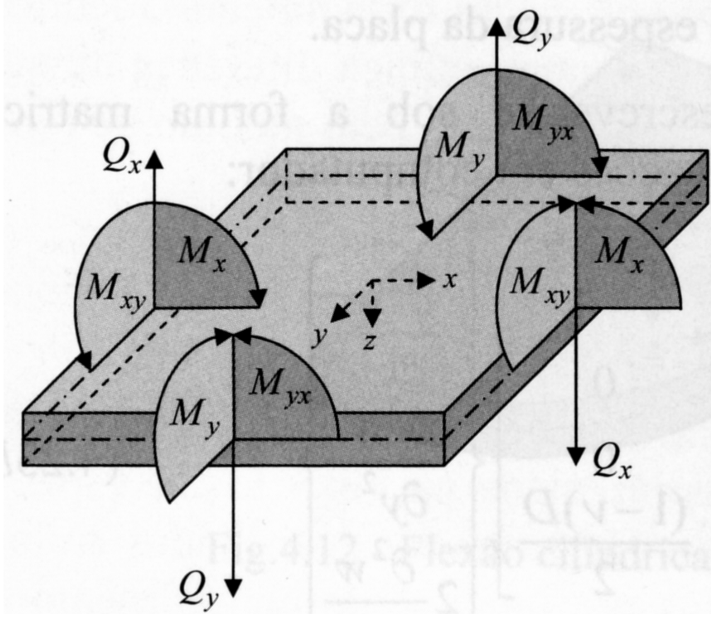
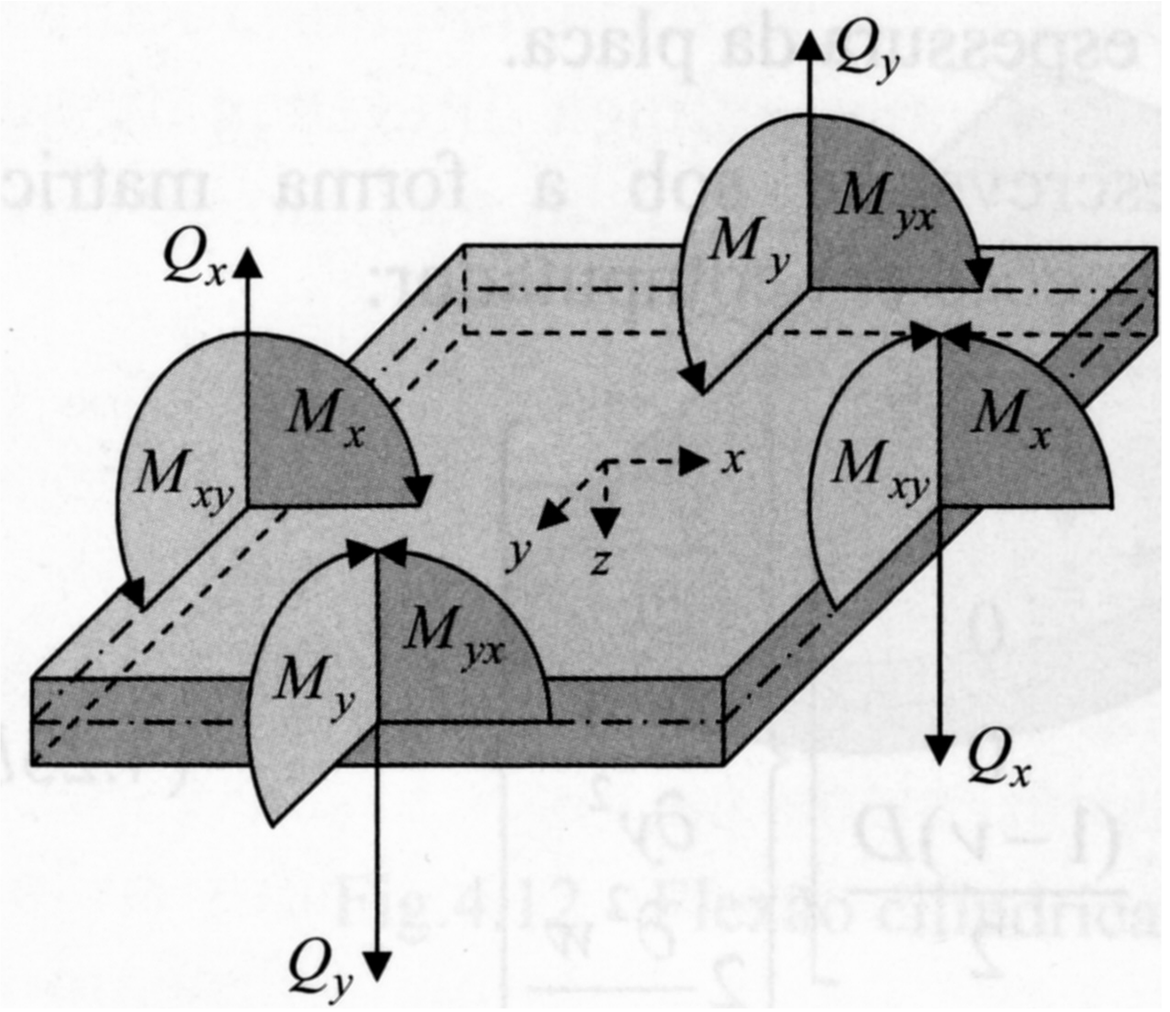


physically: *Mx* is the resultant moment (per unit length) acting along direction Oy and coming from the normal stress distributions through the thickness direction

relation between stress resultants and stresses

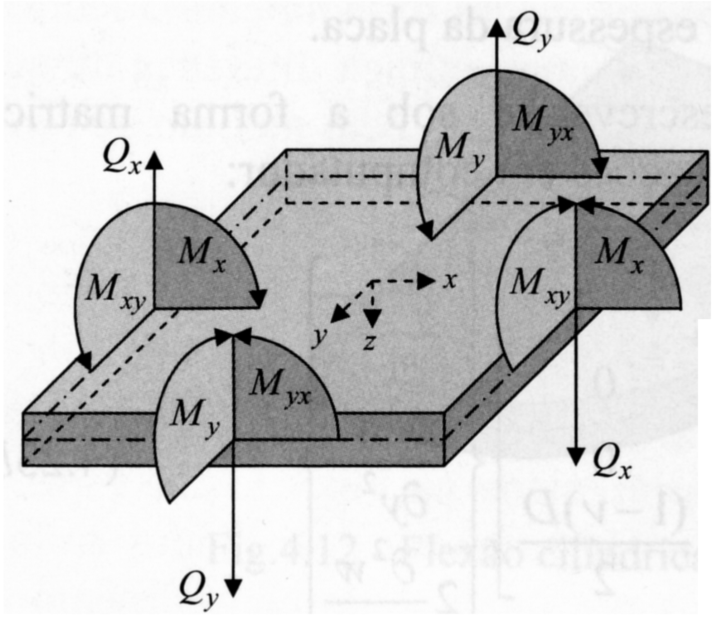


$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-\frac{t}{2}}^{+\frac{t}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} z\,dz$$

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \int_{-\frac{t}{2}}^{+\frac{t}{2}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} dz$$

Hooke's Law written in terms of stress resultants

combining the previous concepts…

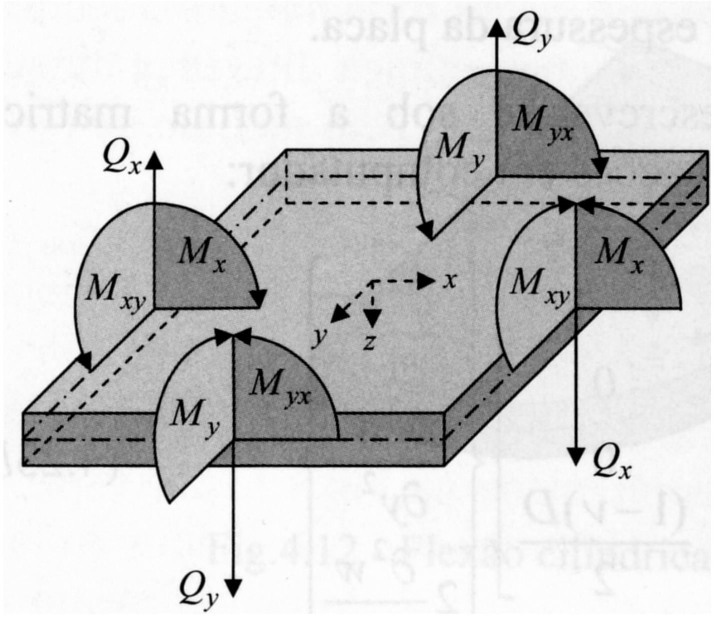


$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = -z \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ 2\kappa_{xy} \end{Bmatrix}$$

$$\Updownarrow$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-\frac{t}{2}}^{+\frac{t}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} z dz \qquad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \int_{-\frac{t}{2}}^{+\frac{t}{2}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} dz$$

… leads to Hooke's law for stress resultants



$$M_x = -D\left(\kappa_x + \nu\kappa_y\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\kappa_y + \nu\kappa_x\right) = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

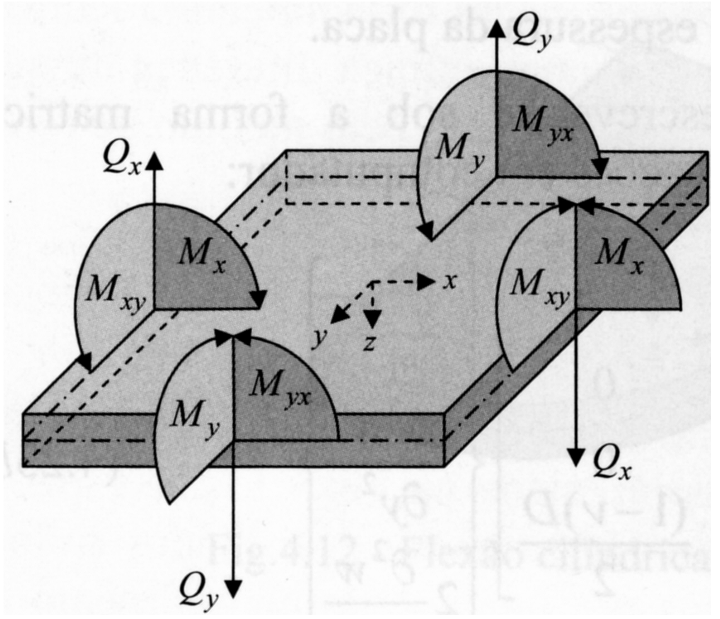
$$M_{xy} = -D\left(1-\nu\right)\kappa_{xy} = -D\left(1-\nu\right)\frac{\partial^2 w}{\partial x \partial y}$$

$$D = \frac{Et^3}{12\left(1-\nu^2\right)}$$

flexural rigidity of the plate
(geometry and material are now involved)

Hooke's law for stress resultants

(compact form)



$$M_x = -D\left(\kappa_x + \nu\kappa_y\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\kappa_y + \nu\kappa_x\right) = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = -D\left(1-\nu\right)\kappa_{xy} = -D\left(1-\nu\right)\frac{\partial^2 w}{\partial x \partial y}$$

$$D = \frac{Et^3}{12\left(1-\nu^2\right)}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = -\begin{bmatrix} D & \nu D & 0 \\ \nu D & D & 0 \\ 0 & 0 & \frac{\left(1-\nu\right)D}{2} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

$$\{\mathbf{M}\} = -[\mathbf{D}]\{\boldsymbol{\kappa}\}$$

stiffness matrix for plate bending

In summary: calculating stresses from resultants…

$$\sigma_{xx} = -\frac{Ez}{1-\nu^2}\left(\kappa_x + \nu\kappa_y\right) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_{yy} = -\frac{Ez}{1-\nu^2}\left(\kappa_y + \nu\kappa_x\right) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$\tau_{xy} = -\frac{Ez}{1+\nu}\kappa_{xy} = -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x \partial y}$$

$$M_x = -D\left(\kappa_x + \nu\kappa_y\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

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In summary: calculating stresses from resultants…

$$\sigma_{xx} = -\frac{Ez}{1-\nu^2}\left(\kappa_x + \nu\kappa_y\right) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

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$$M_x = -D\left(\kappa_x + \nu\kappa_y\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

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$$M_{xy} = -D\left(1-\nu\right)\kappa_{xy} = -D\left(1-\nu\right)\frac{\partial^2 w}{\partial x \partial y}$$

$$D = \frac{Et^3}{12\left(1-\nu^2\right)}$$

$$\sigma_x = \frac{12M_x z}{t^3}$$

$$\sigma_y = \frac{12M_y z}{t^3}$$

$$\tau_{xy} = \frac{12M_{xy} z}{t^3}$$

Part 2

Stresses coming from bending…

(as a function of the curvatures)

$$\sigma_{xx} = -\frac{E}{1-\nu^2} z\left(\kappa_x + \nu\kappa_y\right) = -\frac{E}{1-\nu^2} z\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_{yy} = -\frac{E}{1-\nu^2} z\left(\kappa_y + \nu\kappa_x\right) = -\frac{E}{1-\nu^2} z\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$\tau_{xy} = -\frac{E}{1-\nu^2} z\kappa_{xy} = -\frac{E}{1+\nu} z\frac{\partial^2 w}{\partial x \partial y}$$

important: linear evolution along the thickness direction

Stresses coming from bending…

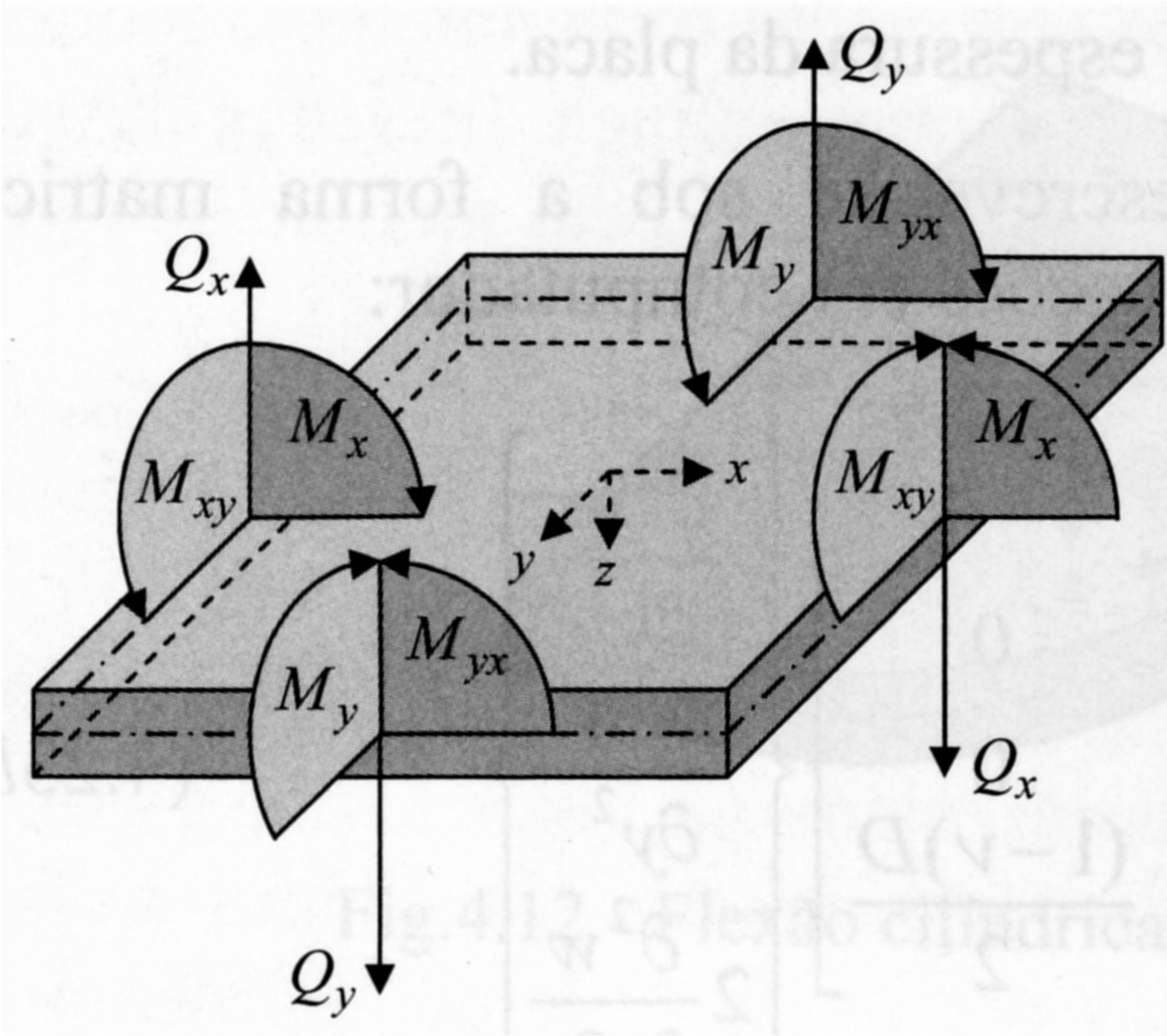
(matrix form)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = -z \begin{bmatrix} \dfrac{E}{1-\nu^2} & \dfrac{\nu E}{1-\nu^2} & 0 \\ \dfrac{\nu E}{1-\nu^2} & \dfrac{E}{1-\nu^2} & 0 \\ 0 & 0 & \dfrac{E}{2(1+\nu)} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

important: linear evolution along the thickness direction

Stress resultants

(per unit of length)



- Bending and Torsion moments

$$M_x = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

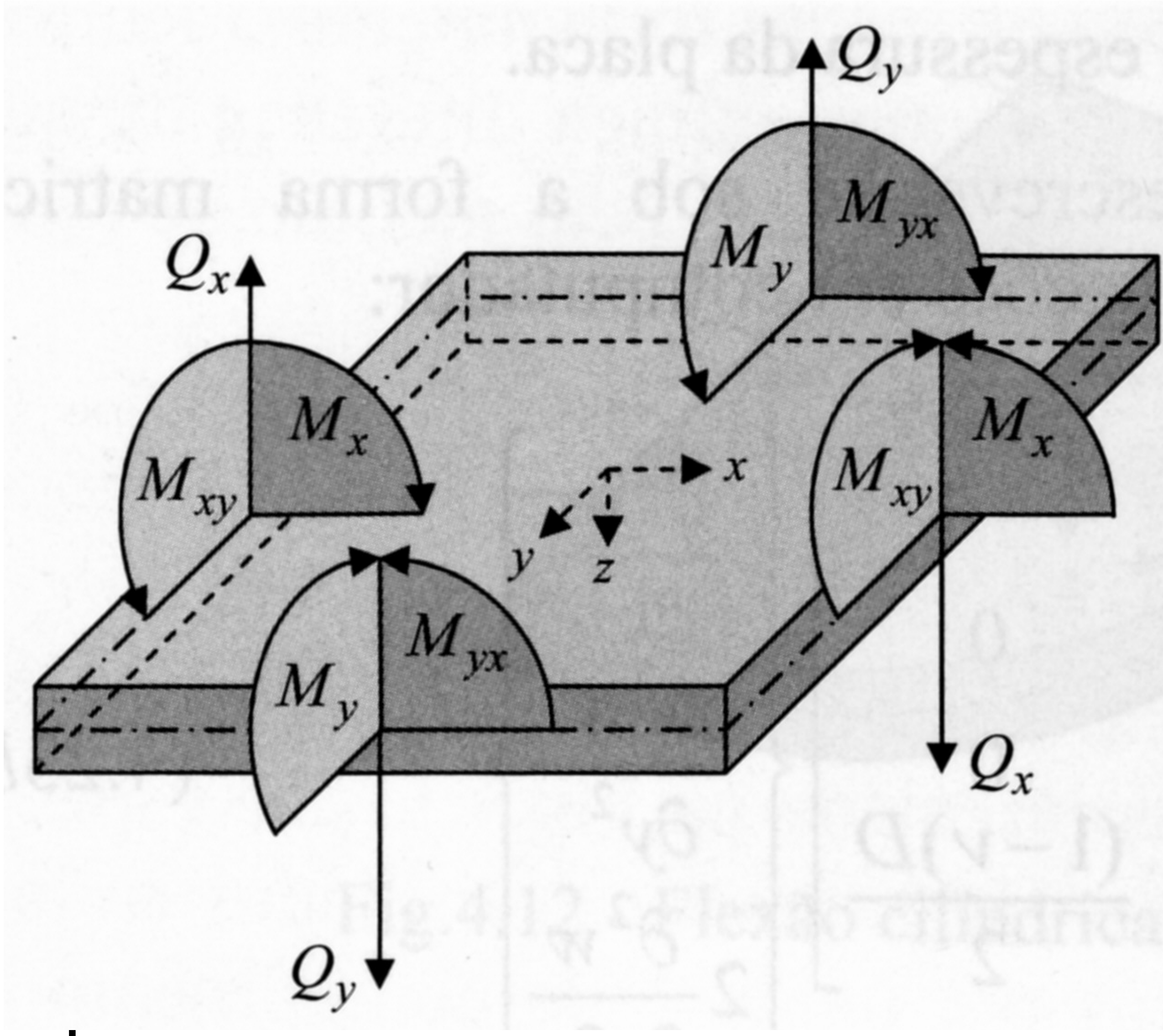
$$M_y = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = -\frac{Et^3}{12(1-\nu^2)}(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

as a function of the material and the curvature

Stress resultants

(per unit of length)



- Bending and Torsion moments (matrix form)

$$\left\{ \begin{array}{c} M_x \\ M_y \\ M_{xy} \end{array} \right\} = -\begin{bmatrix} D & \nu D & 0 \\ \nu D & D & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} D \end{bmatrix} \left\{ \begin{array}{c} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{array} \right\}$$

$$D = \frac{Et^3}{12(1-\nu^2)}$$

flexural rigidity of the plate
(geometry and material)

Stress Resultants vs. Stresses

- How to relate both of them?

$$\sigma_{xx} = \frac{12M_x}{t^3} z \qquad \Rightarrow \qquad \sigma_{xx}\Big|_{\max/\min} = \pm\frac{6M_x}{t^2}$$

$$\sigma_{yy} = \frac{12M_y}{t^3} z \qquad \Rightarrow \qquad \sigma_{yy}\Big|_{\max/\min} = \pm\frac{6M_y}{t^2}$$

$$\tau_{xy} = \frac{12M_{xy}}{t^3} z \qquad \Rightarrow \qquad \tau_{xy}\Big|_{\max/\min} = \pm\frac{6M_{xy}}{t^2}$$

comes from operating on the previous equations (using the curvatures)

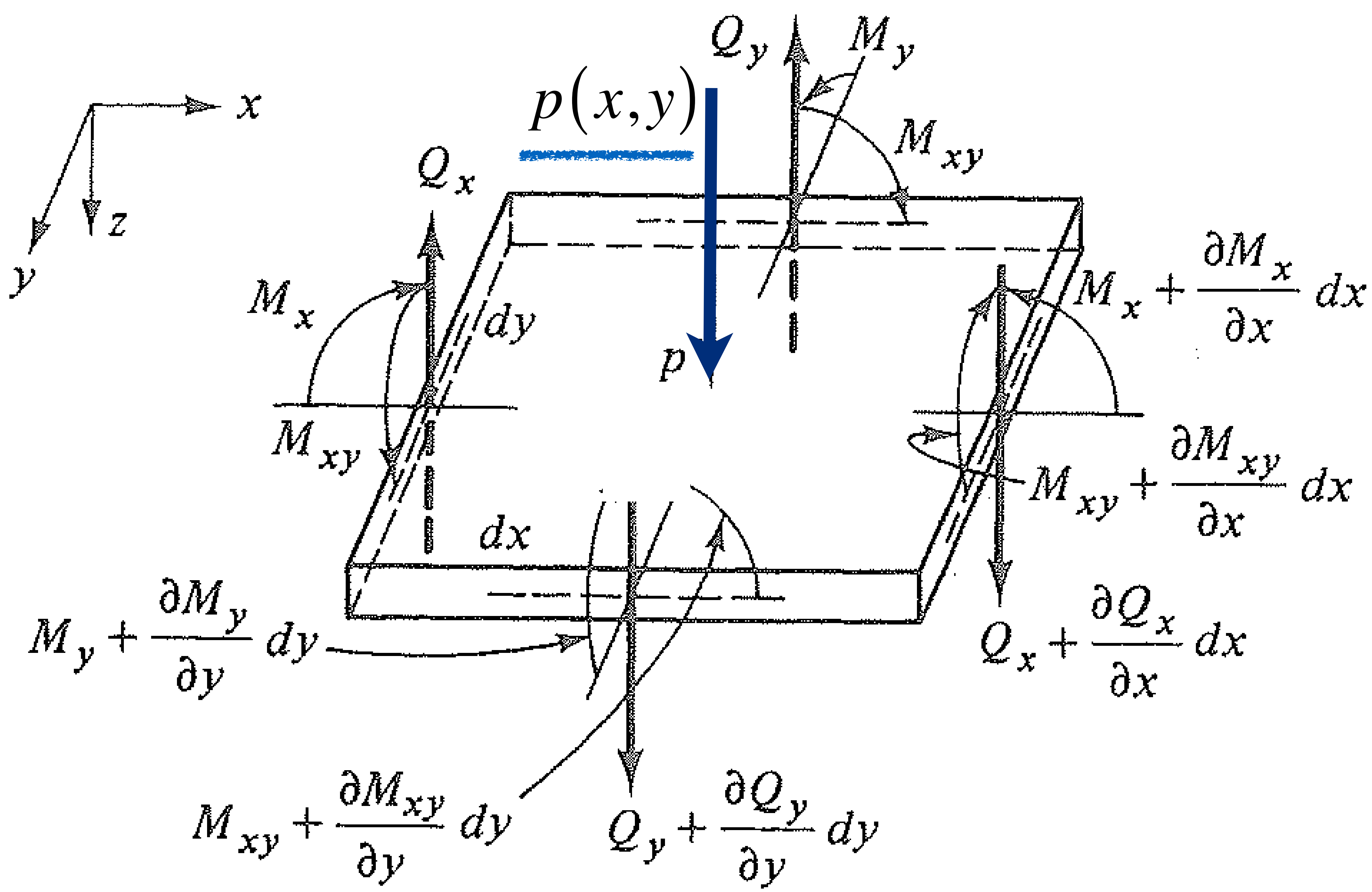
But…

for a given load, how can I calculate the plate **deflection**?

The governing **equation** for the **deflection** of plates

The governing equation for the deflection of plates

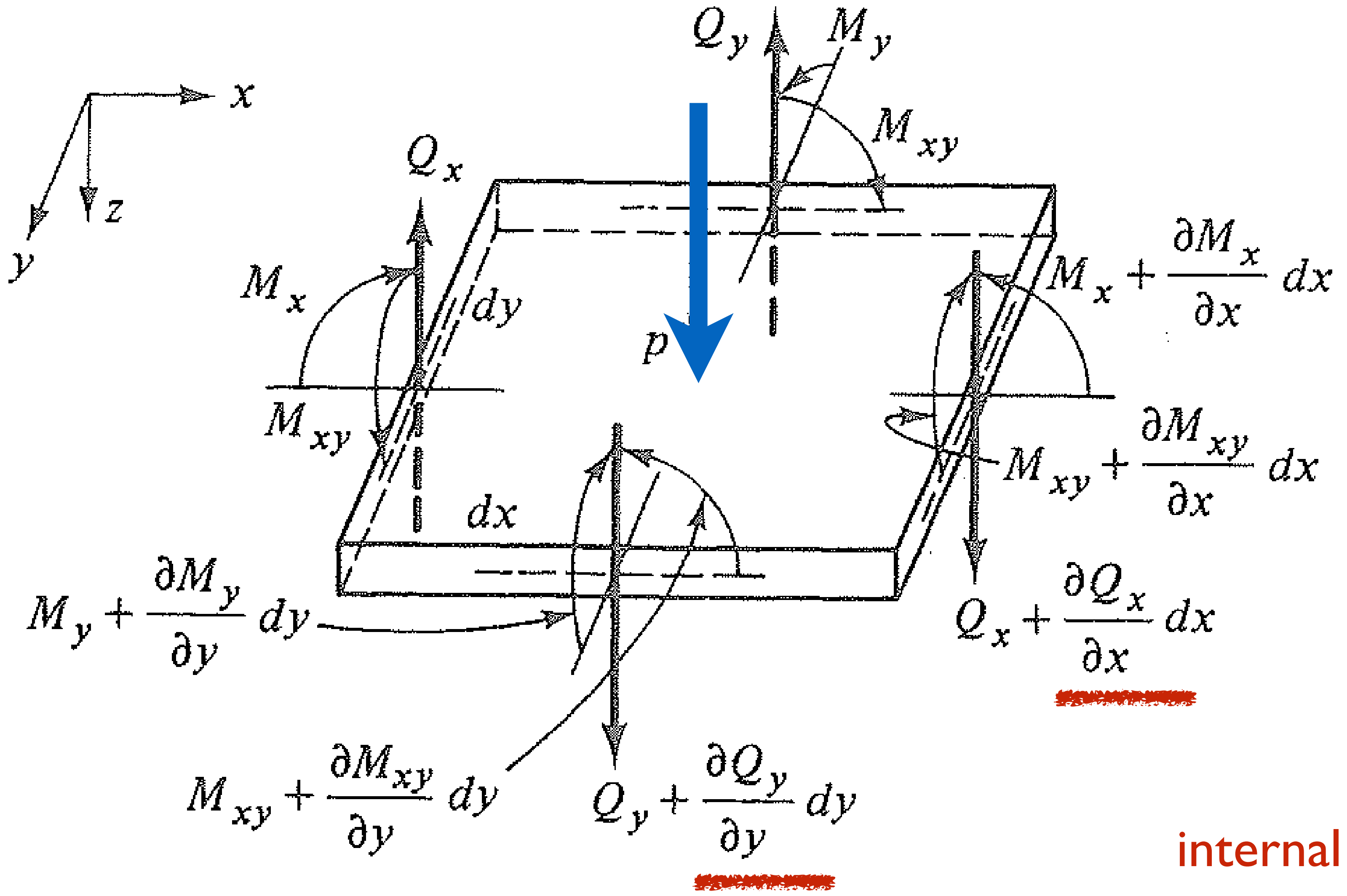
- external (*pressure*) and internal loads/moments (reactions) on an infinitesimal plate element



- non-uniform stress field over the surface of the plate…
- external pressure (p) is a function of (x,y), distributed on the reference surface (plane *oxy*)

The governing equation for the deflection of plates

- enforcing the static equilibrium of **forces** along *oz* direction…

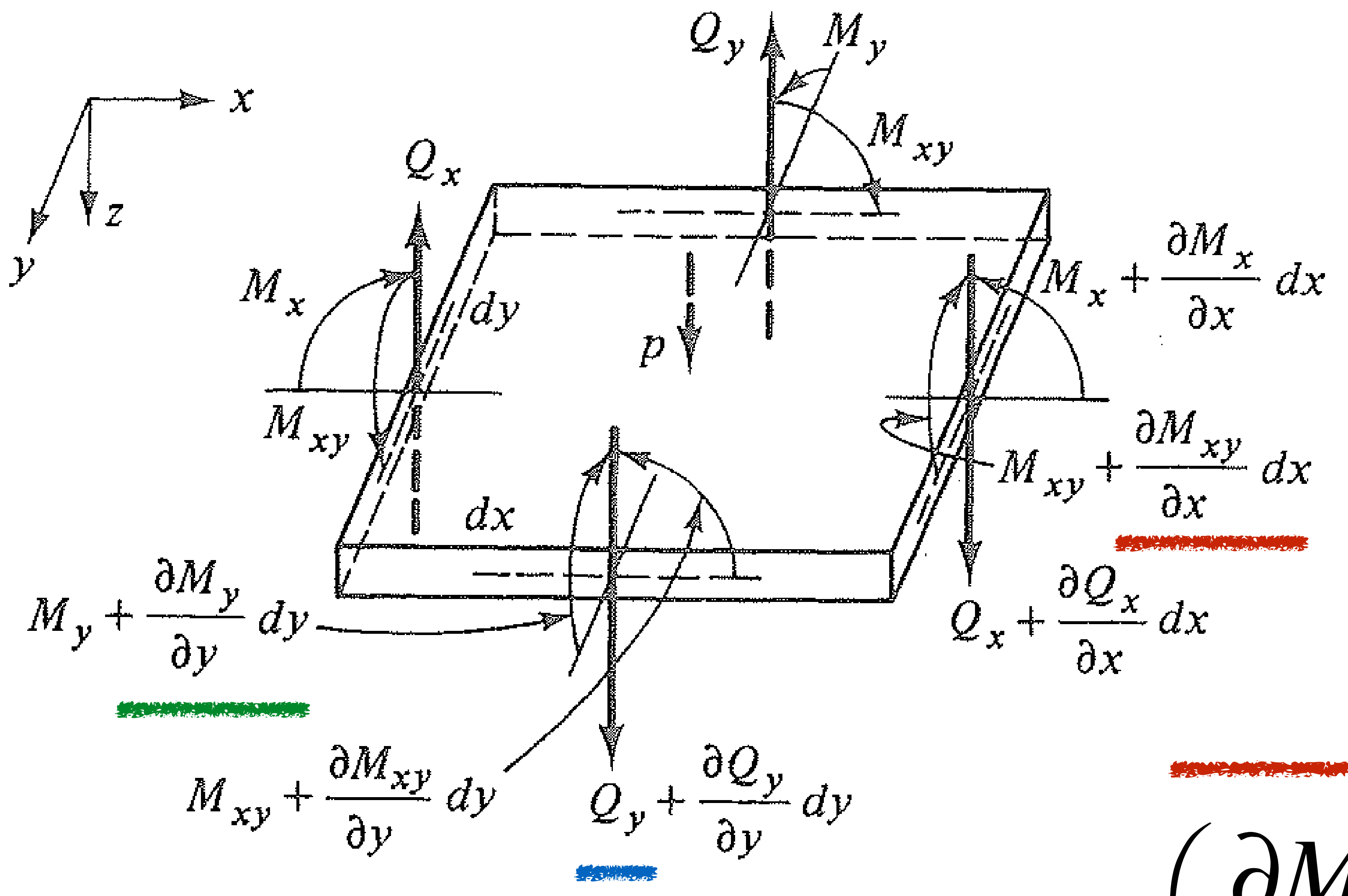


$$\underbrace{\left(\frac{\partial Q_x}{\partial x} dx\right) dy}_{\text{internal reaction}} + \underbrace{\left(\frac{\partial Q_y}{\partial y} dy\right) dx}_{\text{internal reaction}} + \underbrace{p\, dx dy}_{\text{external force}} = 0$$

$$\boxed{\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0}$$

The governing equation for the deflection of plates

- enforcing the static equilibrium of **moments** around/related to direction *ox*
- assuming the external pressure to be self-equilibrating
- ignoring higher order (infinitesimal) terms

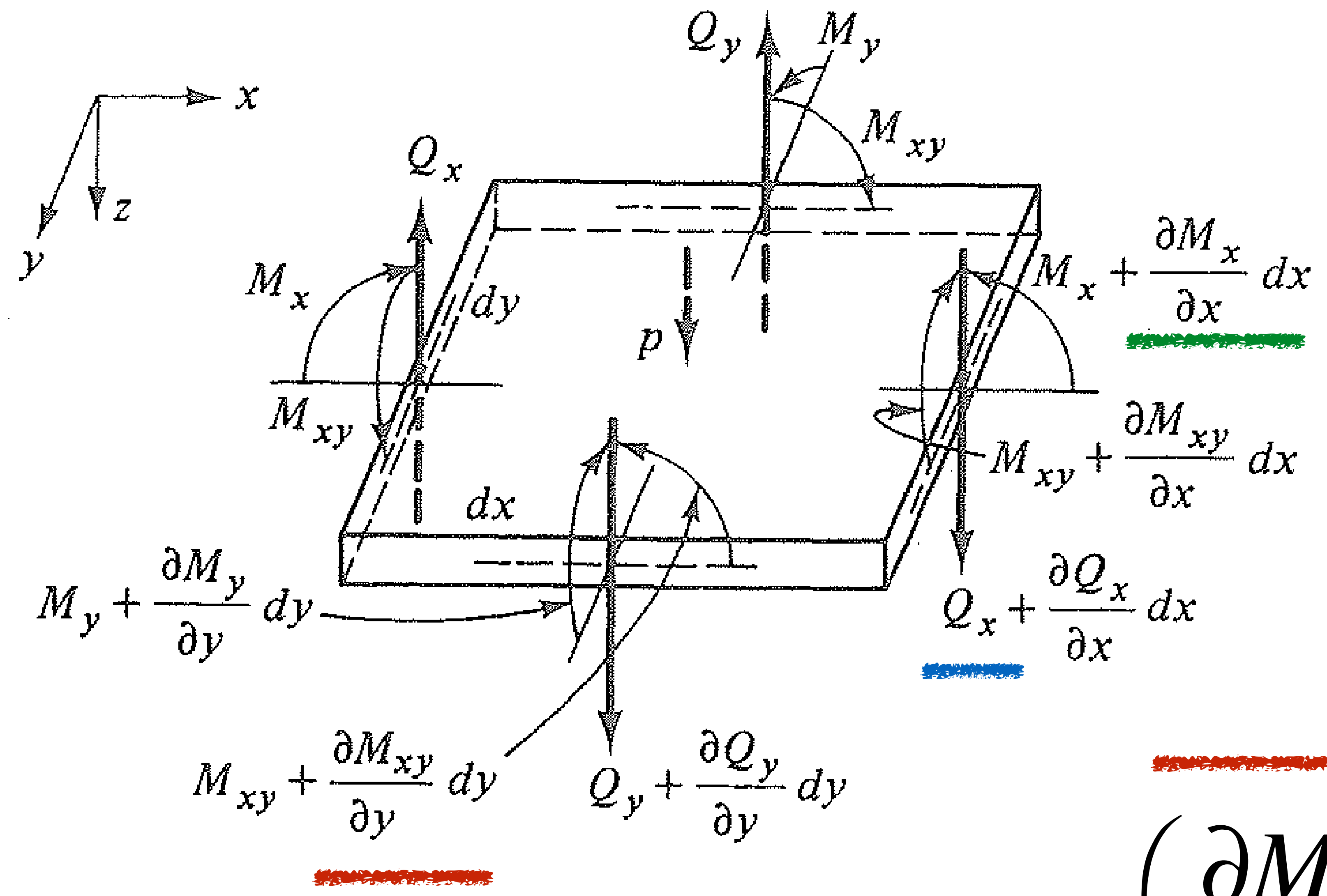


$$-\left(\frac{\partial M_{xy}}{\partial x} dx\right) dy - \left(\frac{\partial M_y}{\partial y} dy\right) dx + Q_y \, dxdy = 0$$

$$\boxed{\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0}$$

The governing equation for the deflection of plates

- enforcing the static equilibrium of **moments** around/related to direction *oy*
- assuming the external pressure to be self-equilibrating
- ignoring higher order (infinitesimal) terms

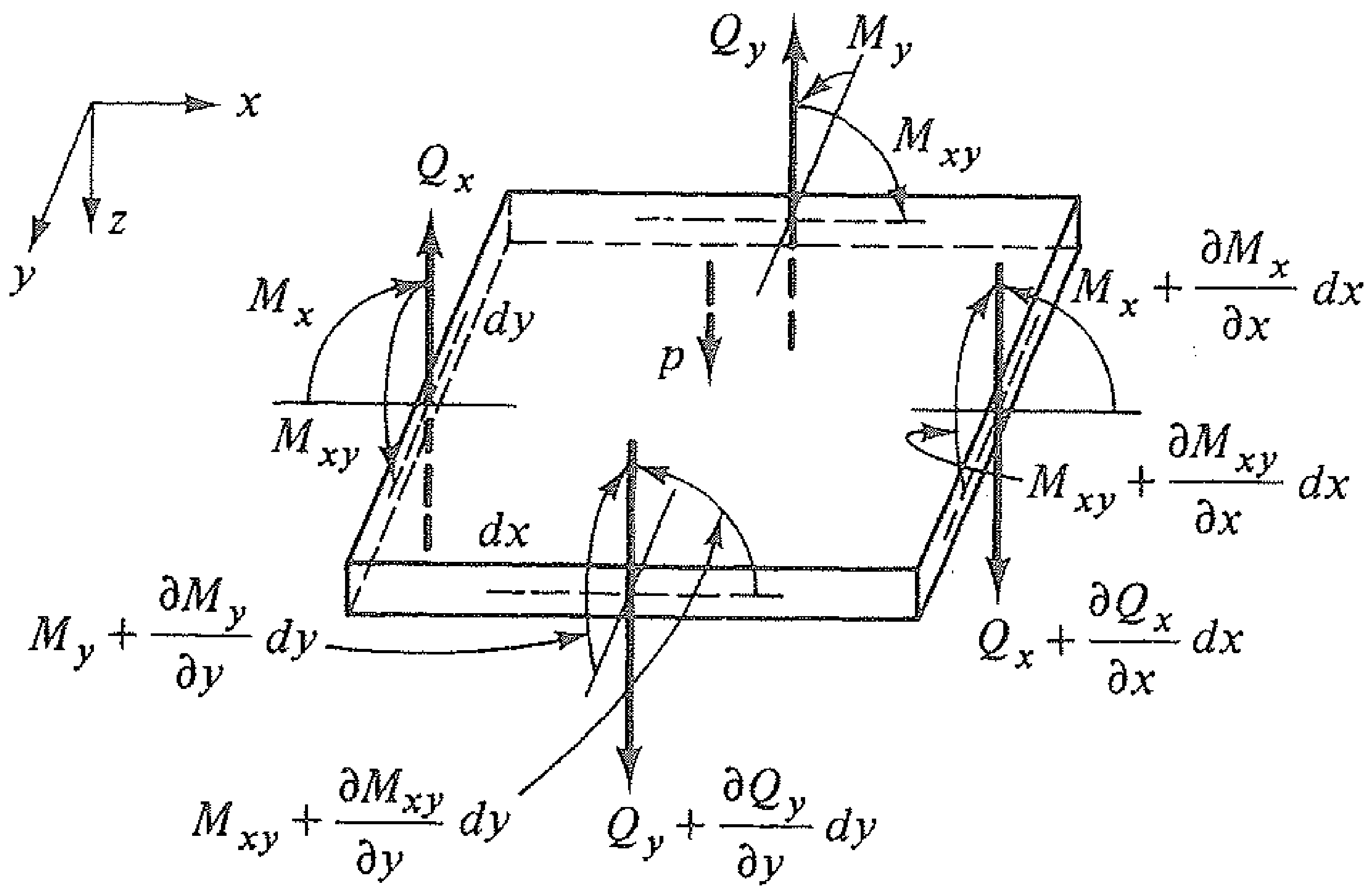


$$\left(\frac{\partial M_{xy}}{\partial y} dy\right) dx + \left(\frac{\partial M_x}{\partial x} dx\right) dy - Q_x \, dxdy = 0$$

$$\boxed{\frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0}$$

The governing equation for the deflection of plates

- putting together all the three equations seen before…

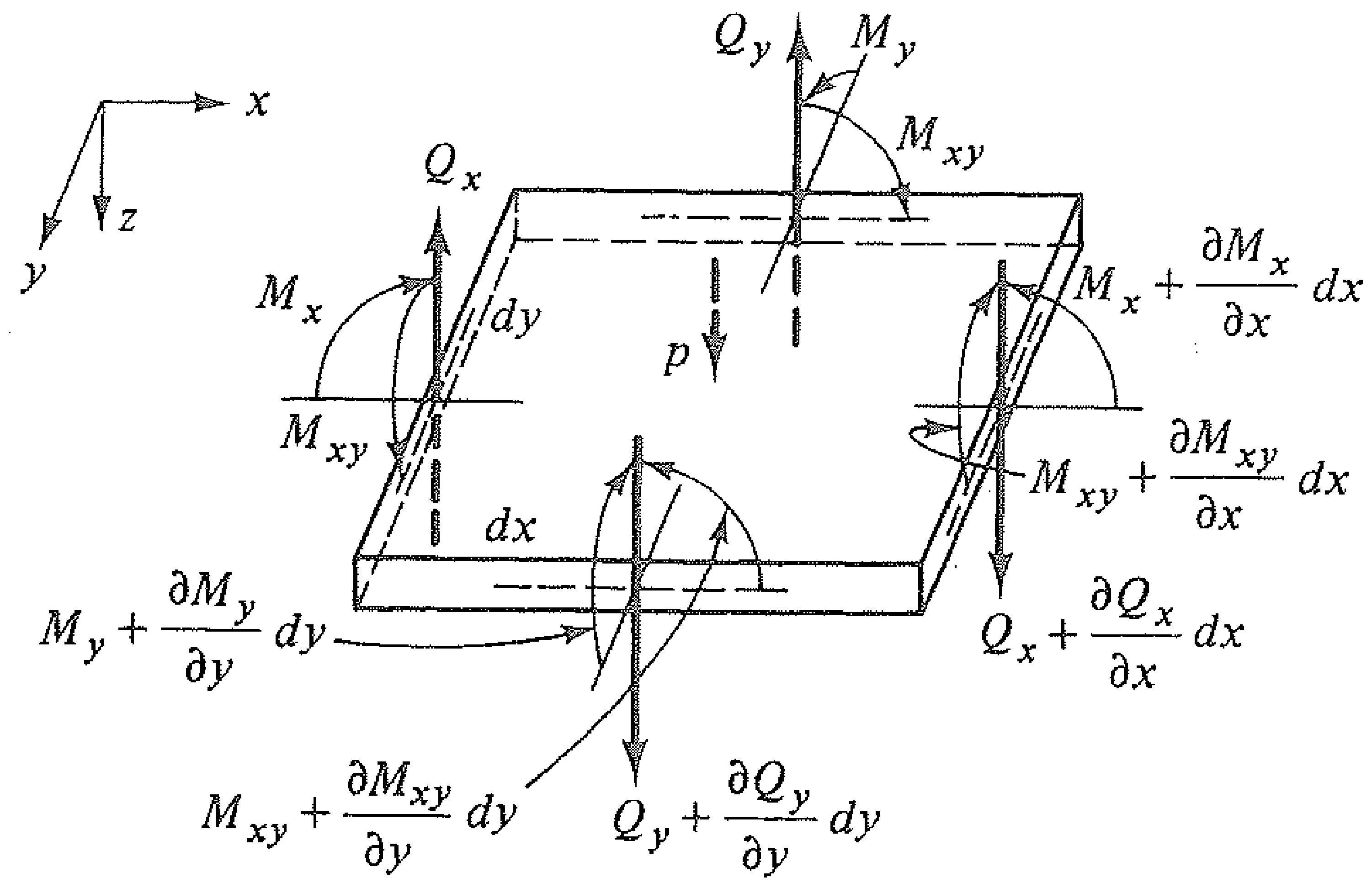


$$\left.\begin{aligned} \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \\ \frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0 \end{aligned}\right\} \quad \rightarrow \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0 \quad \Rightarrow \quad \boxed{\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p}$$

now we have a differential equilibrium equation, as a function of **bending** and **torsion** moments

The governing equation for the deflection of plates

- recalling some previous expressions for the stress resultants…



$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -D\frac{\partial}{\partial x}\left(\nabla^2 w\right)$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -D\frac{\partial}{\partial y}\left(\nabla^2 w\right)$$

$$D = \frac{Et^3}{12\left(1-\nu^2\right)}$$

The governing equation for the deflection of plates

$$\boxed{\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p}$$

$$M_x = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = -\frac{Et^3}{12(1-\nu^2)}(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -D\frac{\partial}{\partial x}\left(\nabla^2 w\right)$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -D\frac{\partial}{\partial y}\left(\nabla^2 w\right)$$

let's try to simplify this, writing the equation as a function of the deflection (*w*)…

The (**differential**) governing equation for the **deflection** of plates

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p \quad \Rightarrow \quad \boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}} \quad \Rightarrow \quad \boxed{\nabla^4 w = \frac{p}{D}}$$

$$M_x = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = -\frac{Et^3}{12(1-\nu^2)}(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -D\frac{\partial}{\partial x}\left(\nabla^2 w\right)$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -D\frac{\partial}{\partial y}\left(\nabla^2 w\right)$$

a guideline for solution

1°) integrate the differential equation for the deflection (*w*)
2°) impose a number of boundary conditions for the integration constants
3°) then you have a generic expression for the deflection at any point (x,y) on the plate surface
4°) calculate everything else (moments, strains, stresses, etc.)!

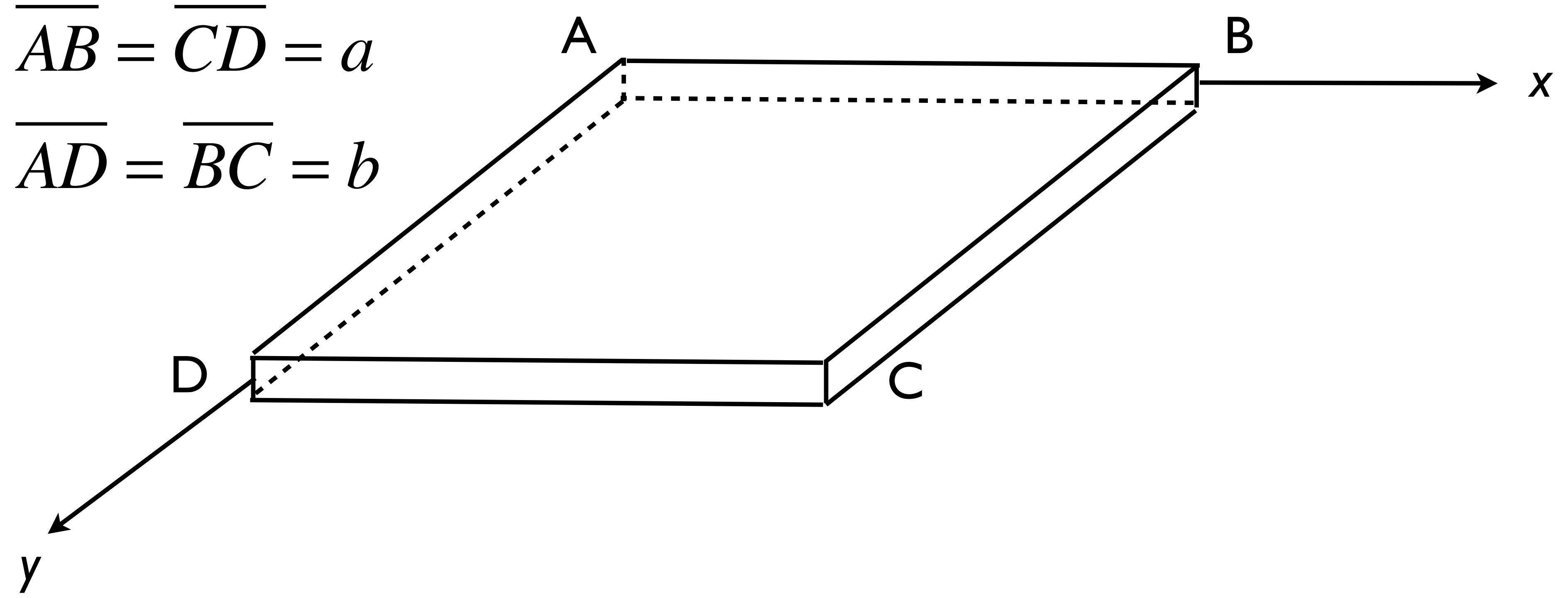
boundary conditions…

about the boundary conditions, some examples…

$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}} \quad \Rightarrow \quad \boxed{\nabla^4 w = \frac{p}{D}}$$

$\overline{AB} = \overline{CD} = a$

$\overline{AD} = \overline{BC} = b$



Clamped edges (no deflection, no rotation on these edges)

$$\overline{AB} / \overline{CD} \quad \rightarrow \quad y = 0 \quad y = b \quad \Rightarrow \quad \omega = 0 \quad \frac{\partial \omega}{\partial y} = 0$$

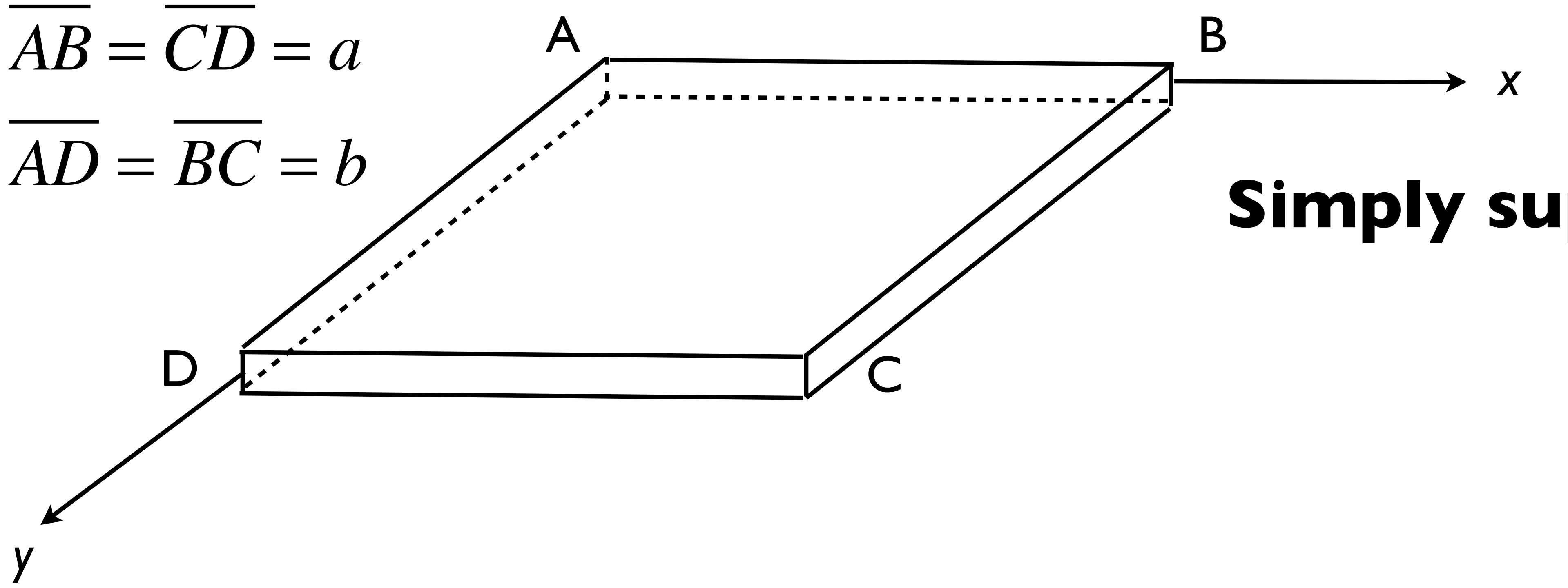
$$\overline{AD} / \overline{BC} \quad \rightarrow \quad x = 0 \quad x = a \quad \Rightarrow \quad \omega = 0 \quad \frac{\partial \omega}{\partial x} = 0$$

about the boundary conditions, some examples…

$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}} \quad \Rightarrow \quad \boxed{\nabla^4 w = \frac{p}{D}}$$

$$\overline{AB} = \overline{CD} = a$$

$$\overline{AD} = \overline{BC} = b$$



Simply supported edges (no deflection, no moments)

$$\overline{AB} / \overline{CD} \quad \rightarrow \quad y = 0 \quad y = b:$$

$$\omega = 0 \qquad \left(\frac{\partial^2 w}{\partial x^2} = 0 \right) \qquad M_y = 0$$

$$M_x = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = -\frac{Et^3}{12(1-\nu^2)}(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

$$\overline{AD} / \overline{BC} \quad \rightarrow \quad x = 0 \quad x = a:$$

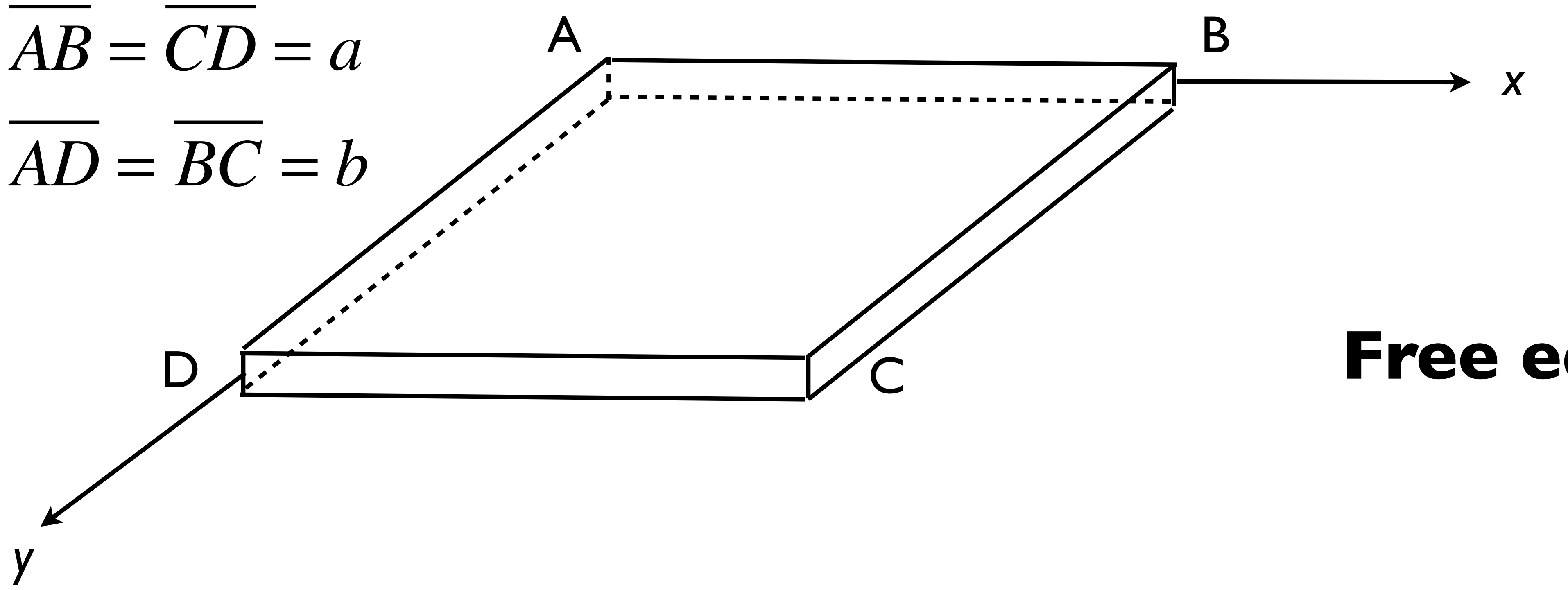
$$\omega = 0 \qquad \left(\frac{\partial^2 w}{\partial y^2} = 0 \right) \qquad M_x = 0$$

about the boundary conditions, some examples…

$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}} \quad \Rightarrow \quad \boxed{\nabla^4 w = \frac{p}{D}}$$

$\overline{AB} = \overline{CD} = a$

$\overline{AD} = \overline{BC} = b$



$$M_x = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = -\frac{Et^3}{12(1-\nu^2)}(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

Free edges (no moments and force reactions)

$$\overline{AB} / \overline{CD} \quad \rightarrow \quad y = 0 \quad y = b:$$

$$M_y = 0 \qquad M_{xy} = 0 \qquad Q_y = 0$$

$$\overline{AD} / \overline{BC} \quad \rightarrow \quad x = 0 \quad x = a:$$

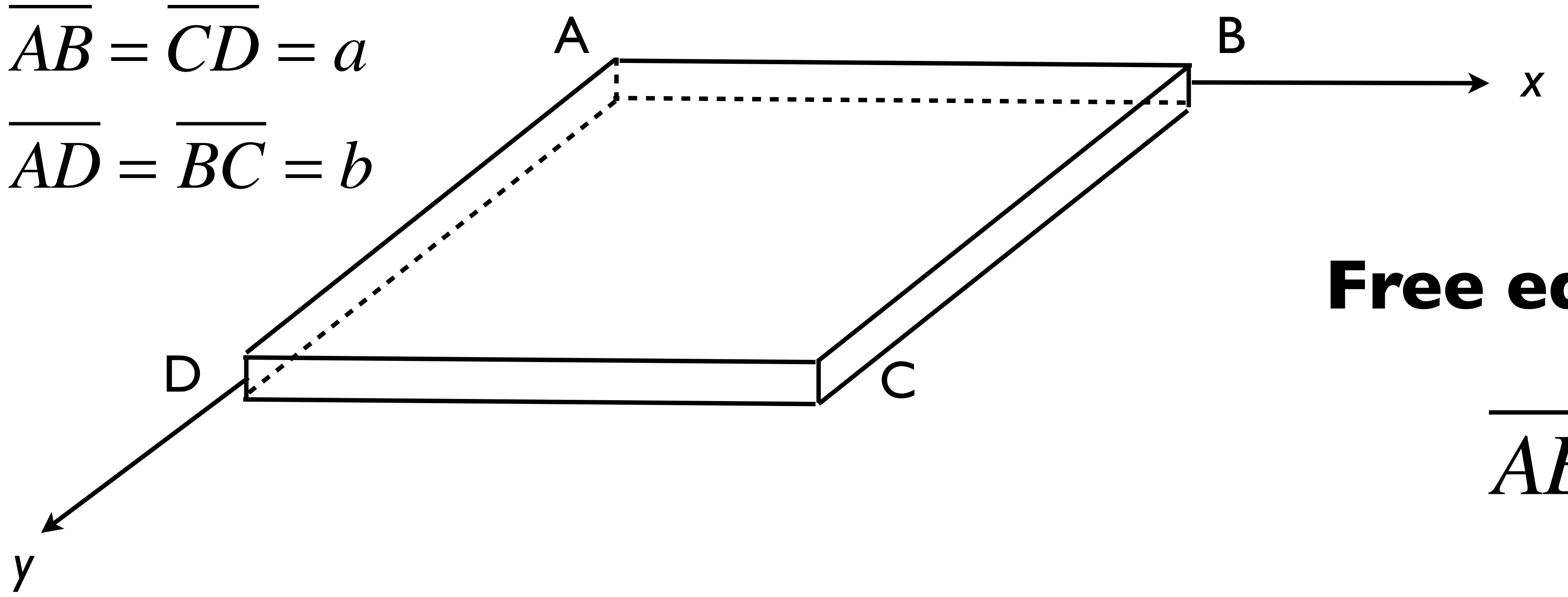
$$M_x = 0 \qquad M_{xy} = 0 \qquad Q_x = 0$$

about the boundary conditions, some examples…

$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}} \quad \Rightarrow \quad \boxed{\nabla^4 w = \frac{p}{D}}$$

$\overline{AB} = \overline{CD} = a$

$\overline{AD} = \overline{BC} = b$



Free edges, alternative way (no effective force)

$$\overline{AB} / \overline{CD} \quad \rightarrow \quad y = 0 \quad y = b:$$

$$M_y = 0 \qquad V_y = Q_y + \frac{\partial M_{xy}}{\partial x} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right] = 0$$

$$\overline{AD} / \overline{BC} \quad \rightarrow \quad x = 0 \quad x = a:$$

$$M_x = 0 \qquad V_x = Q_x + \frac{\partial M_{xy}}{\partial y} = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right] = 0$$

Summary: to solve a bending problem of a thin plate subjected to a pressure field (*p*) can be simplified to integrating a differential equation

$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}} \quad \Rightarrow \quad \boxed{\nabla^4 w = \frac{p}{D}}$$

- knowing the deflection over the reference surface *w(x,y)*, it is then possible to:
  - map (graphical representation) the deflections
  - calculate bending/torsion moments
  - calculate the stresses (including maximum and minimum values)
  - calculate the curvatures
  - calculate the strains
  - …

$$D = \frac{Et^3}{12\left(1-\nu^2\right)}$$

Summary chart

$$\boxed{\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{p}{D}} \Rightarrow \boxed{\nabla^4 w=\frac{p}{D}} \qquad D=\frac{Et^3}{12\left(1-\nu^2\right)}$$

$$M_x=-\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y=-\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy}=-\frac{Et^3}{12(1-\nu^2)}(1-\nu)\frac{\partial^2 w}{\partial x\partial y}=-D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}$$

$$\sigma_{xx}=-\frac{E}{1-\nu^2}z\left(\kappa_x+\nu\kappa_y\right)=-\frac{E}{1-\nu^2}z\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_{yy}=-\frac{E}{1-\nu^2}z\left(\kappa_y+\nu\kappa_x\right)=-\frac{E}{1-\nu^2}z\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$\tau_{xy}=-\frac{E}{1-\nu^2}z\kappa_{xy}=-\frac{E}{1+\nu}z\frac{\partial^2 w}{\partial x\partial y}$$

$$\sigma_{xx}=\frac{12M_x}{t^3}z \quad\Rightarrow\quad \sigma_{xx}\Big|_{\max/\min}=\pm\frac{6M_x}{t^2}$$

$$\sigma_{yy}=\frac{12M_y}{t^3}z \quad\Rightarrow\quad \sigma_{yy}\Big|_{\max/\min}=\pm\frac{6M_y}{t^2}$$

$$\tau_{xy}=\frac{12M_{xy}}{t^3}z \quad\Rightarrow\quad \tau_{xy}\Big|_{\max/\min}=\pm\frac{6M_{xy}}{t^2}$$

Aircraft Structural Analysis

Master Course in Aerospace Engineering

Session #02 — Parts 1 & 2