Aircraft Structural Analysis

Master Course in Aerospace Engineering

Extra Material I — Navier's approximate solution for rectangular plates (for comparison with FEM)

Reference material

Rectangular Plates - Approximate Solutions

Chapter 5: Navier's solution for simply supported rectangular plates

(sections 5.1, 5.2, 5.3)

of the reference book: Ansel C. Ugural, "Stresses in Plates and Shells", 2nd ed., McGraw-Hill

The main goal…

- to find a solution for the **out-of-plane displacements (deflections)**, but in an **approximate way**

- to write the approximate solution for deflections as a **double Fourier series**

- avoid using (directly) the fourth-order differential equation for displacements

- construct a method that is easy to be programmed (coded)

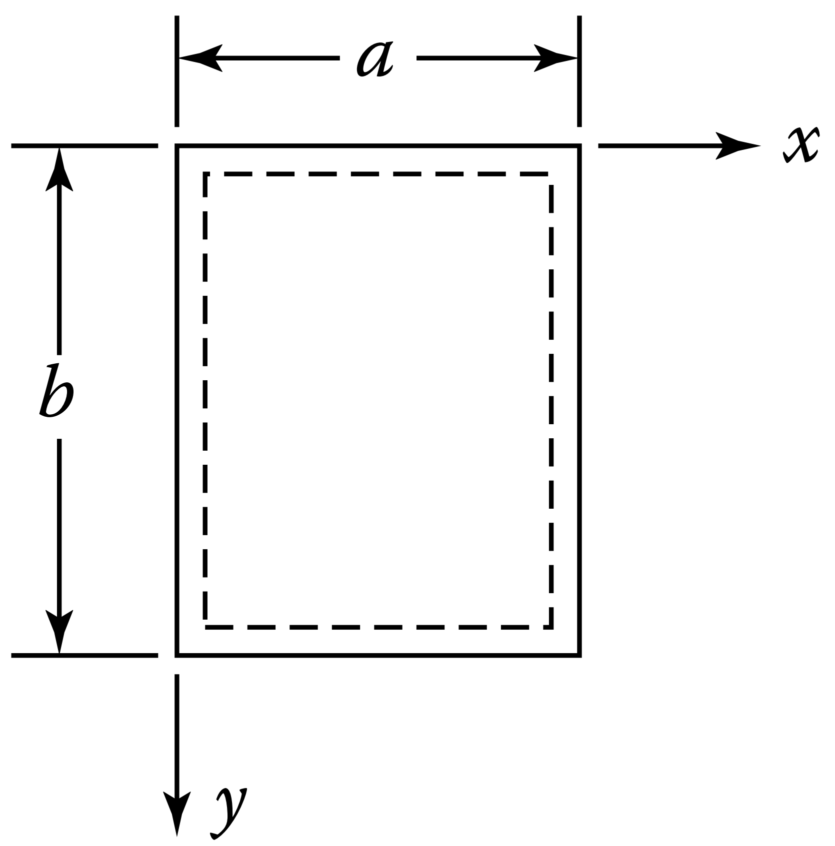
- the boundary conditions are pre-defined (simply supported)

- the referential (coordinate axes) are pre-defined

$$w(x,y) = ???$$

The (step by step) idea…

- write the applied loads (pressure) and out-of-plane displacements (deflection) in the form:



$$p(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}p_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$p_{mn} \rightarrow \textit{loading amplitude}$

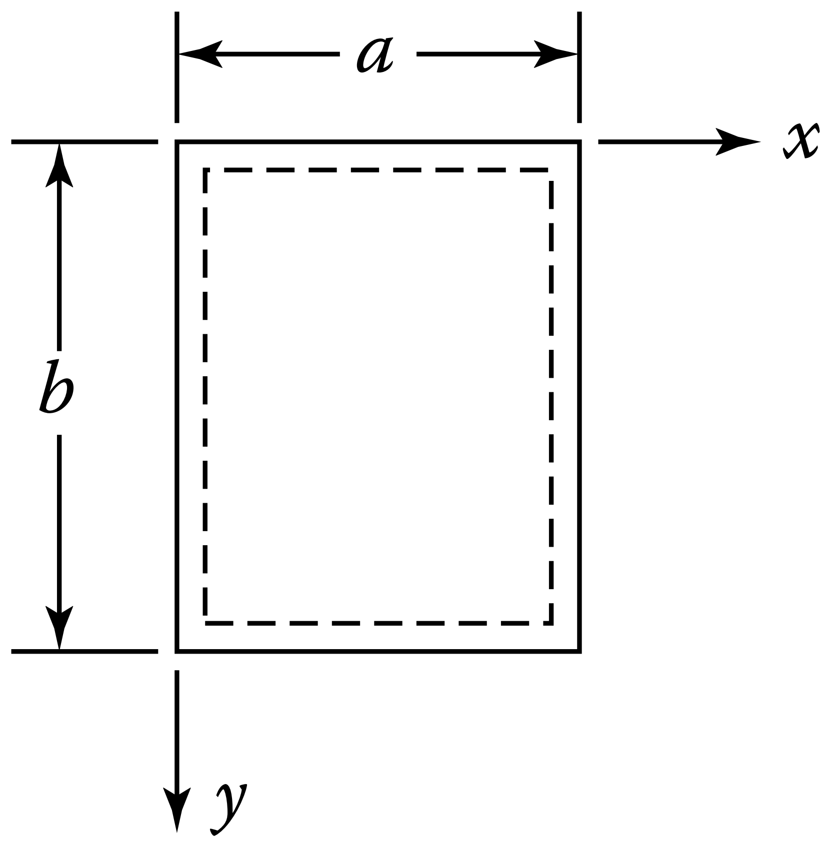
$a_{mn} \rightarrow \textit{displacement amplitude}$

$p_{mn} = ?$

$a_{mn} = ?$

Boundary conditions

- this approximate solution for the out-of-plane displacements (deflection) must respect the (simply supported) boundary conditions



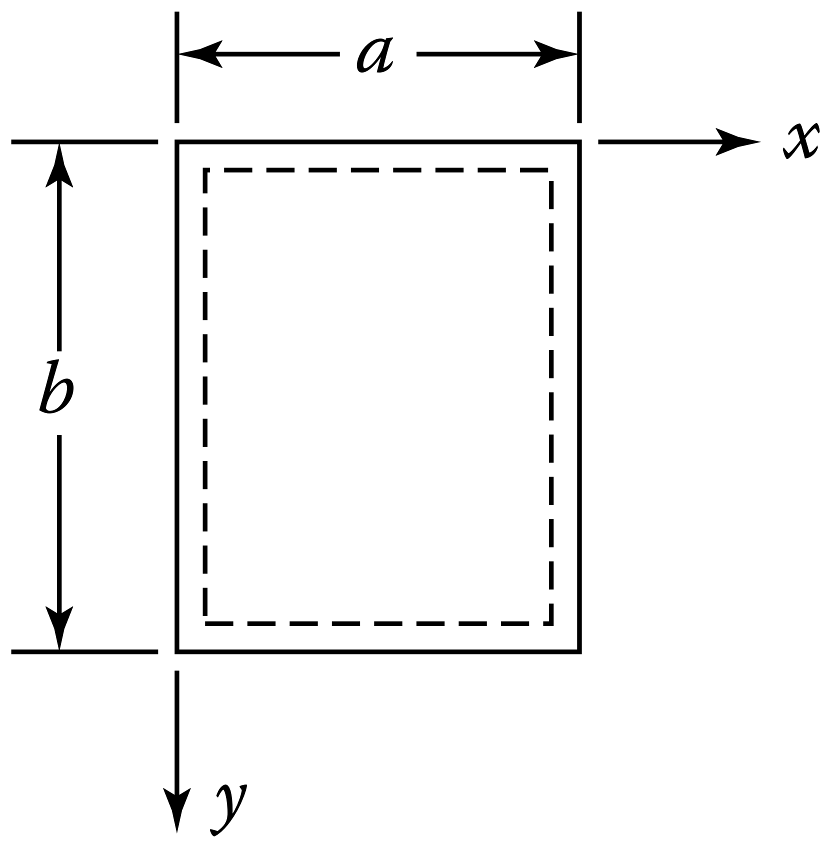
$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$w=0 \qquad \frac{\partial^2 w}{\partial x^2}=0 \qquad (x=0,\ \ x=a)$$

$$w=0 \qquad \frac{\partial^2 w}{\partial y^2}=0 \qquad (y=0,\ \ y=b).$$

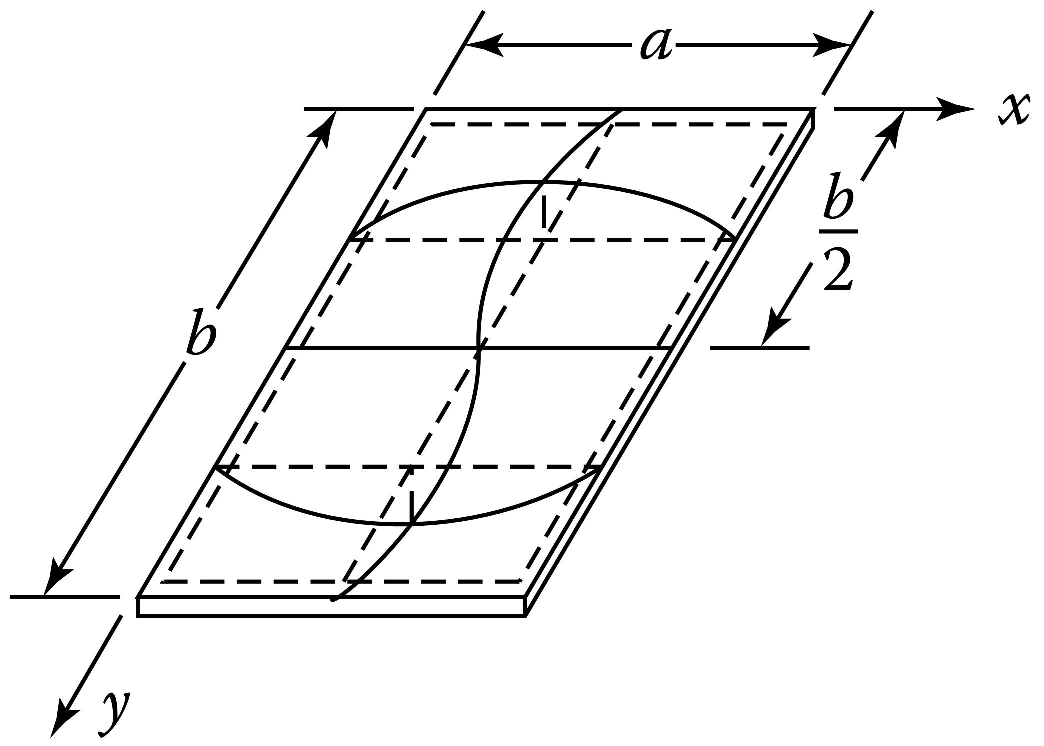
A visual interpretation for $w(x,y)$

- for the particular case of m=1 and n=2



$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

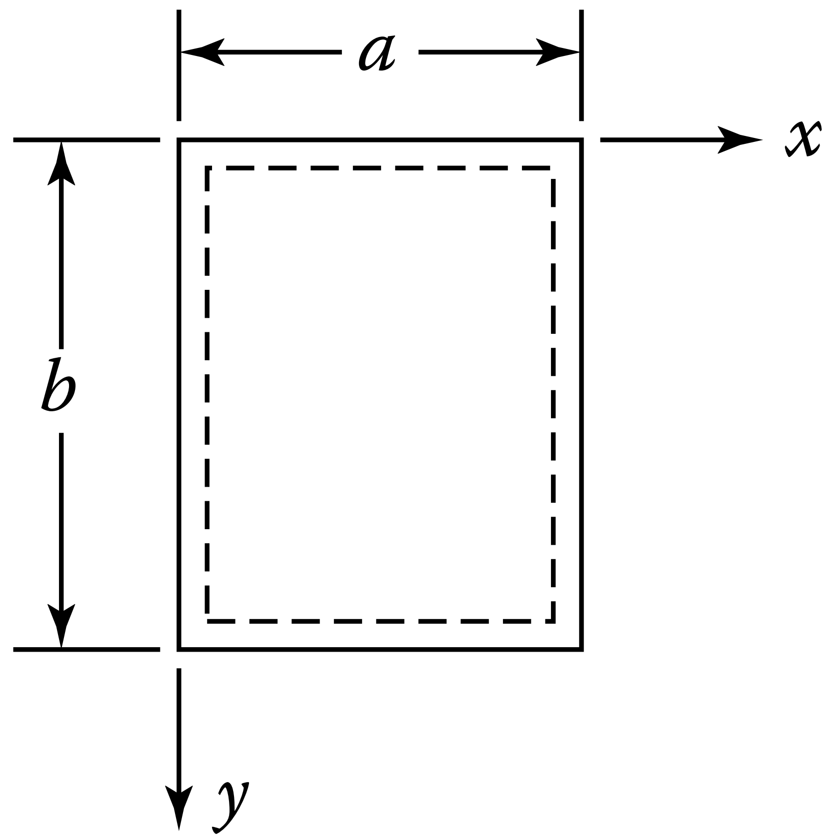
$$w(x,y)\Big|_{m=1,\,n=2}=a_{12}\sin\left(\frac{1\cdot\pi}{a}x\right)\sin\left(\frac{2\cdot\pi}{b}y\right)$$



Back to the differential equation…

$p_{mn} = ?$

$a_{mn} = ?$



$$\boxed{\nabla^4 w(x,y) = \frac{p(x,y)}{D}}$$

$$p(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

Back to the differential equation…

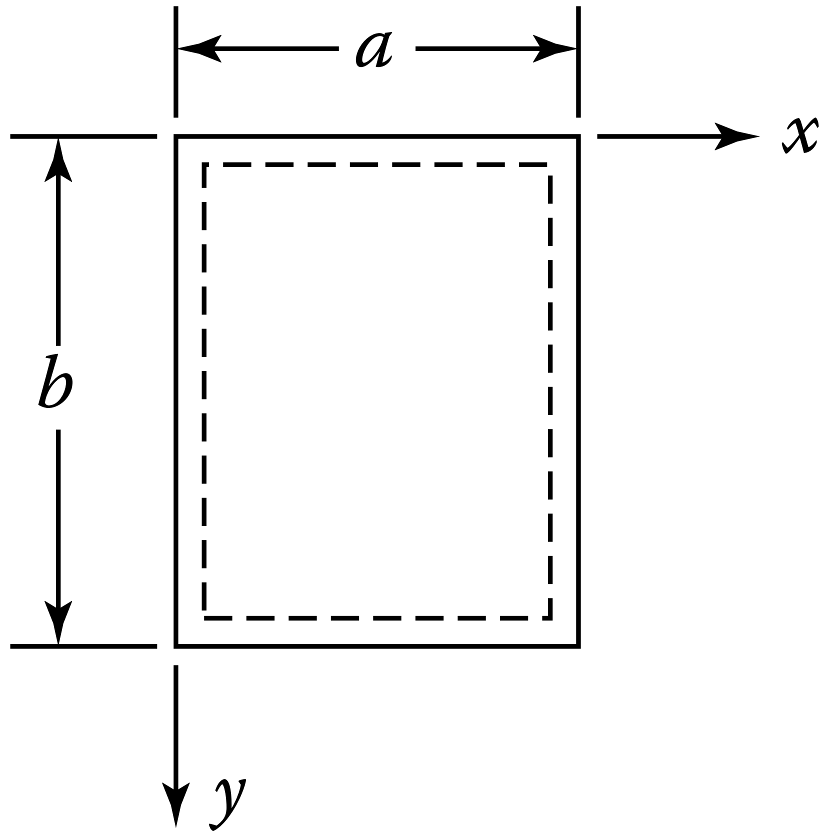
$p_{mn} = ?$

$a_{mn} = ?$

$$\boxed{\nabla^4 w(x,y) = \frac{p(x,y)}{D}}$$

$$p(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$



$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_{mn}}{D}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$D = \frac{Et^3}{12(1-\nu^2)}$$

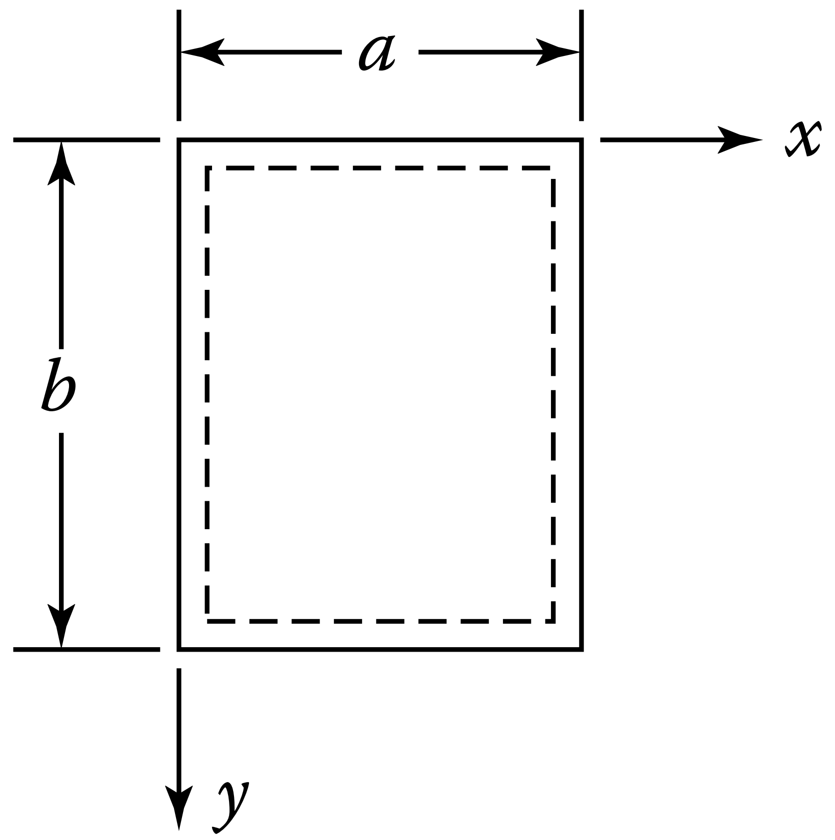
boundary conditions
$$\begin{cases} w = 0\big|_{x=0,x=a} \\ w = 0\big|_{y=0,y=b} \\ \left.\dfrac{\partial^2 w}{\partial x^2} = 0\right|_{x=0,x=a} \longleftarrow \left(\dfrac{\partial^2 w}{\partial y^2} = 0 \text{ and } M_x = 0\right) \\ \left.\dfrac{\partial^2 w}{\partial y^2} = 0\right|_{y=0,y=b} \end{cases}$$

Back to the differential equation…

$$\boxed{\nabla^4 w(x,y) = \frac{p(x,y)}{D}}$$

$$p(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$



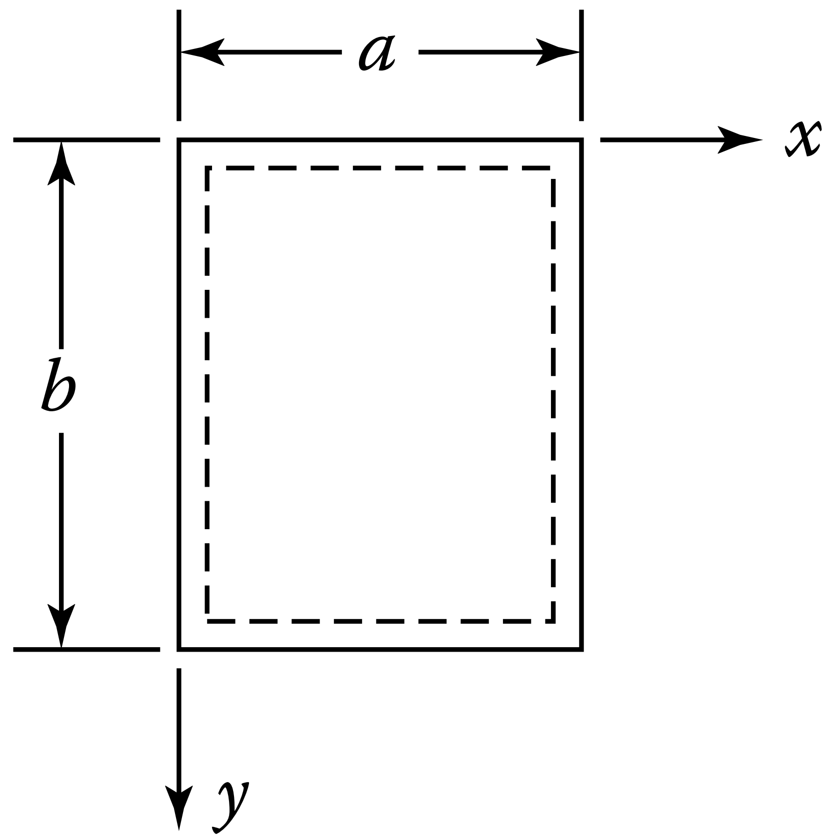
$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_{mn}}{D}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\boxed{a_{mn} = \frac{1}{\pi^4 D}\frac{p_{mn}}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}} \quad \checkmark$$

$$p_{mn} = ?$$

Now playing with the loading function…

$$p(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}p_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$



$$\int_0^a\int_0^b p(x,y)\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{k\pi y}{b}\right)\mathrm{d}x\mathrm{d}y=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}p_{mn}\int_0^a\int_0^b\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{k\pi y}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\mathrm{d}x\,\mathrm{d}y$$

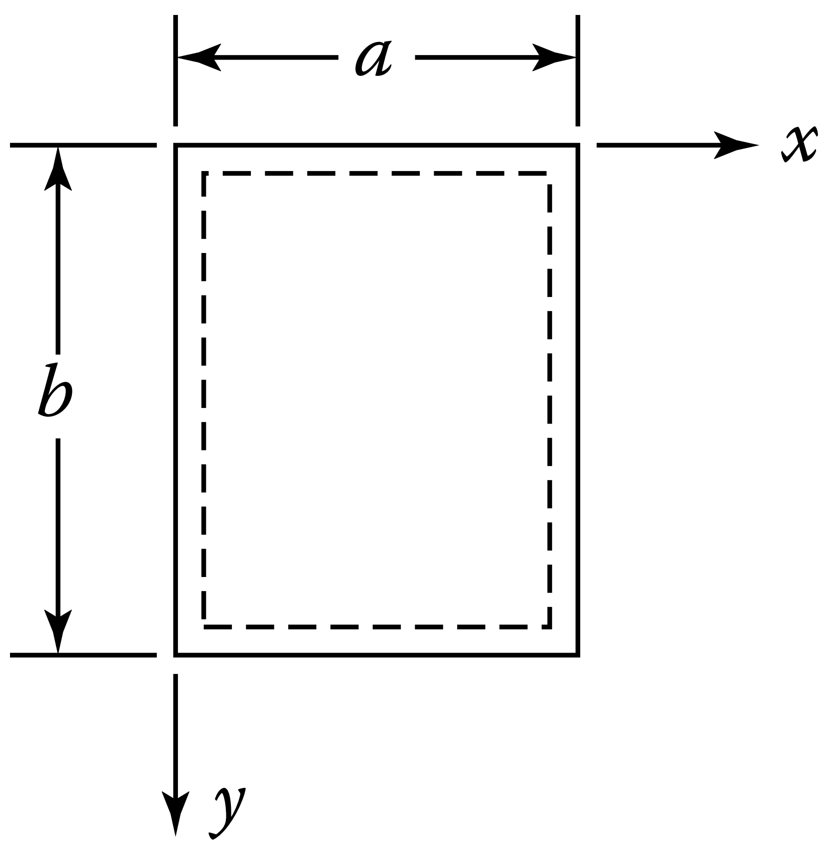
$$\int_0^a\sin\frac{m\pi x}{a}\sin\frac{l\pi x}{a}\mathrm{d}x=\begin{cases}0 & \forall\quad m\neq l\\ a/2 & \forall\quad m=l\end{cases}$$

$$\int_0^b\sin\frac{n\pi y}{b}\sin\frac{k\pi y}{b}\mathrm{d}x=\begin{cases}0 & \forall\quad n\neq k\\ b/2 & \forall\quad n=k\end{cases}$$

$$\boxed{p_{mn}=\frac{4}{ab}\int_0^a\int_0^b p(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\mathrm{d}x\mathrm{d}y}\quad\checkmark$$

$$\boxed{a_{mn}=\frac{1}{\pi^4 D}\frac{p_{mn}}{\left[\left(m/a\right)^2+\left(n/b\right)^2\right]^2}}\quad\checkmark$$

Flow chart: Navier's solution for thin plates



input:

geometry

(a, b, t)

load function

$p\,(x, y)$

material properties

(E, ν, D)

$$p_{mn} = \frac{4}{ab}\int_0^a\int_0^b p(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\mathrm{d}x\mathrm{d}y$$

$$a_{mn} = \frac{1}{\pi^4 D}\frac{p_{mn}}{\left[\left(m/a\right)^2 + \left(n/b\right)^2\right]^2}$$

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]$$

$$M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right]$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}$$

$$Q_x = -D\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]$$

$$Q_y = -D\frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]$$

$$\sigma_{xx} = \pm\frac{12M_x}{t^3}z$$

$$\sigma_{yy} = \pm\frac{12M_y}{t^3}z$$

$$\tau_{xy} = \frac{12M_{xy}}{t^3}z$$

Summary chart: moments, stresses (curvatures, strains, etc.)

$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}} \quad \Rightarrow \quad \boxed{\nabla^4 w = \frac{p}{D}} \qquad D = \frac{Et^3}{12\left(1-\nu^2\right)}$$

$$M_x = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = -\frac{Et^3}{12(1-\nu^2)}(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

$$\sigma_{xx} = -\frac{E}{1-\nu^2}z\left(\kappa_x + \nu\kappa_y\right) = -\frac{E}{1-\nu^2}z\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_{yy} = -\frac{E}{1-\nu^2}z\left(\kappa_y + \nu\kappa_x\right) = -\frac{E}{1-\nu^2}z\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$\tau_{xy} = -\frac{E}{1-\nu^2}z\kappa_{xy} = -\frac{E}{1+\nu}z\frac{\partial^2 w}{\partial x \partial y}$$

$$\sigma_{xx} = \frac{12M_x}{t^3}z \quad \Rightarrow \quad \sigma_{xx}\Big|_{\max/\min} = \pm\frac{6M_x}{t^2}$$

$$\sigma_{yy} = \frac{12M_y}{t^3}z \quad \Rightarrow \quad \sigma_{yy}\Big|_{\max/\min} = \pm\frac{6M_y}{t^2}$$

$$\tau_{xy} = \frac{12M_{xy}}{t^3}z \quad \Rightarrow \quad \tau_{xy}\Big|_{\max/\min} = \pm\frac{6M_{xy}}{t^2}$$

End!