Cap. 2 – cinemática (suplementar)

Sumário:

Velocidade e aceleração em coordenadas curvilíneas.

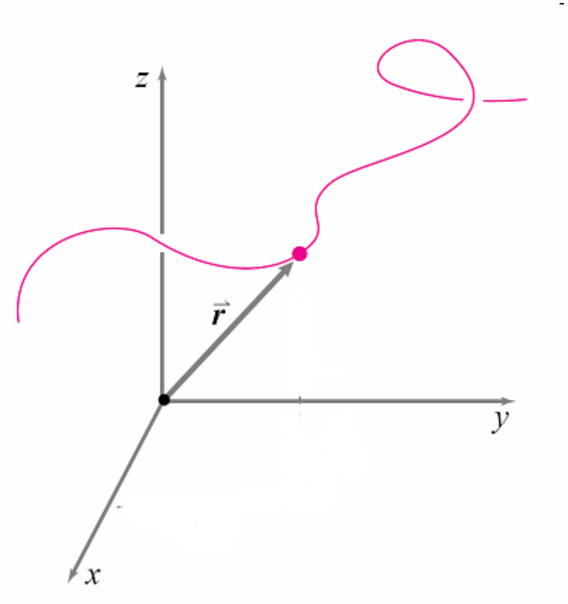
Movimento circular.

Vector posição

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$$\vec{r} = \rho\,\hat{e}_\rho + z\,\hat{e}_z \quad \text{(coord. cilínd.)}$$

$$\vec{r} = r\,\hat{e}_r \quad \text{(coord. esf.)}$$



veloc. e aceler. em coord. cartes.

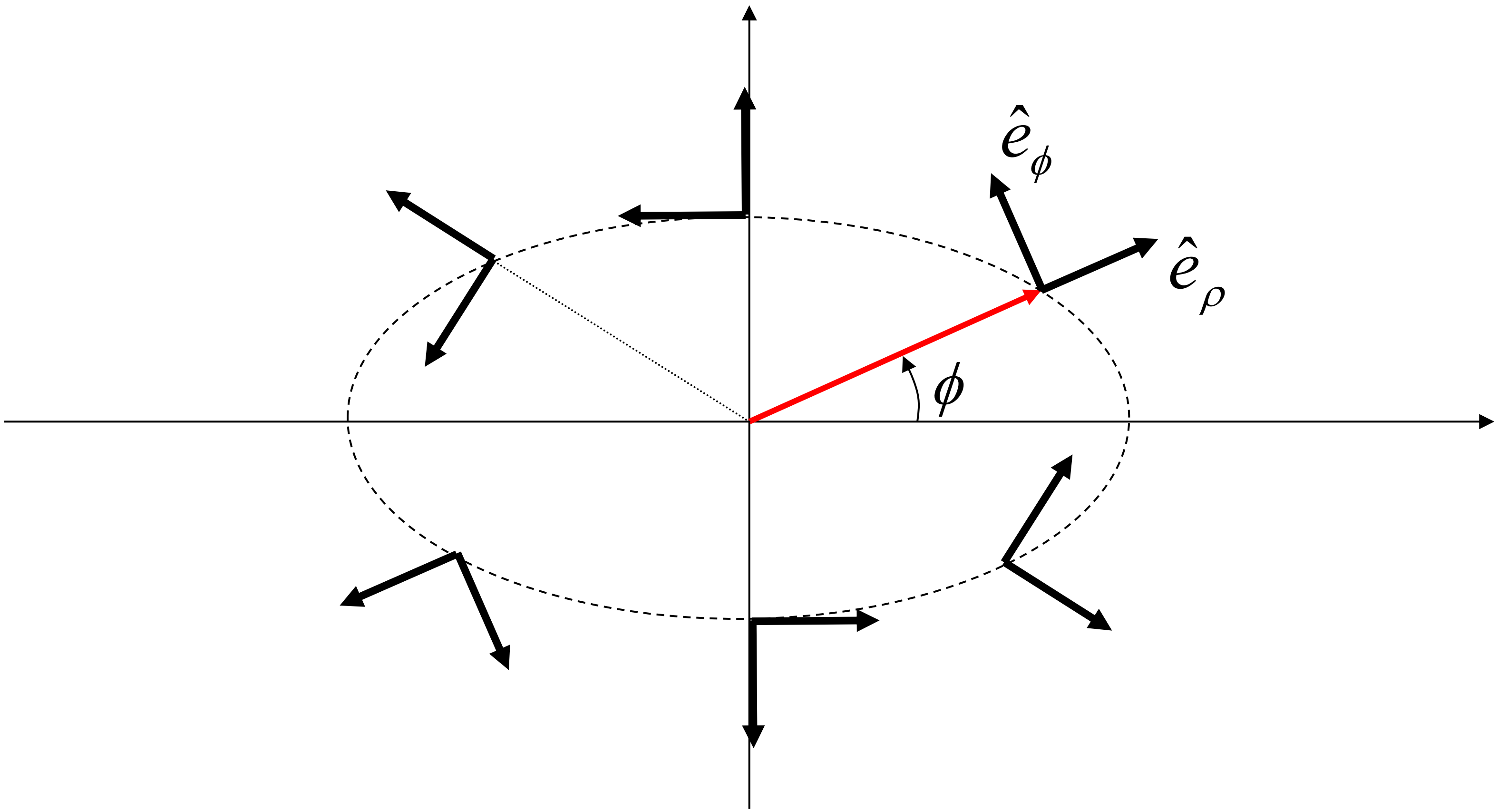
velocidade:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k}$$

aceleração:

$$\vec{a}(t) = \frac{d^2\vec{r}(t)}{dt^2} = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} + \ddot{z}\,\hat{k}$$

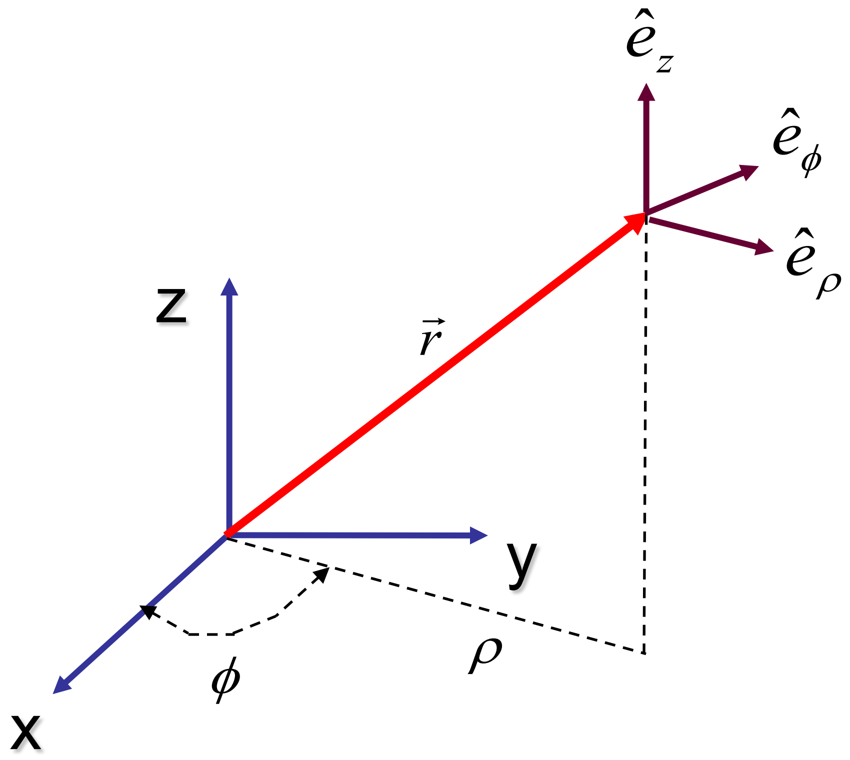
base das coordenadas polares $\left\{\hat{e}_\rho, \hat{e}_\phi\right\}$



coorden. cilíndricas

vector posição:

$$\vec{r}(t) = \rho(t)\,\hat{e}_\rho(t) + z(t)\,\hat{e}_z(t)$$



variação de $\hat{e}_\rho$ no tempo

$$\hat{e}_\rho = \left(\cos\phi, \sin\phi, 0\right)$$

$$\frac{d\hat{e}_\rho}{dt} = \underbrace{\frac{\partial \hat{e}_\rho}{\partial \rho}\cdot\frac{d\rho}{dt}}_{=0} + \frac{\partial \hat{e}_\rho}{\partial \phi}\cdot\frac{d\phi}{dt} + \underbrace{\frac{\partial \hat{e}_\rho}{\partial z}\cdot\frac{dz}{dt}}_{=0}$$

$$= \left(-\sin\phi, \cos\phi, 0\right)\dot{\phi}$$

$$= \dot{\phi}\,\hat{e}_\phi$$

variação de $\hat{e}_\phi$ e $\hat{e}_z$ no tempo

$$\hat{e}_\phi = \left(-\sin\phi, \cos\phi, 0\right)$$

$$\Rightarrow \frac{d\hat{e}_\phi}{dt} = \underbrace{\frac{\partial \hat{e}_\phi}{\partial \rho}\cdot\frac{d\rho}{dt}}_{=0} + \frac{\partial \hat{e}_\phi}{\partial \phi}\cdot\frac{d\phi}{dt} + \underbrace{\frac{\partial \hat{e}_\phi}{\partial z}\cdot\frac{dz}{dt}}_{=0}$$

$$= \left(-\cos\phi, -\sin\phi, 0\right)\dot{\phi}$$

$$= -\dot{\phi}\,\hat{e}_\rho$$

$$\hat{e}_z = \left(0,0,1\right) \Rightarrow \frac{d\hat{e}_z}{dt} = 0$$

variação de $\hat{e}_\rho$, $\hat{e}_\phi$ e $\hat{e}_z$ no tempo

$$\boxed{\begin{aligned} \dot{\hat{e}}_\rho &= \dot{\phi}\,\hat{e}_\phi \\ \dot{\hat{e}}_\phi &= -\dot{\phi}\,\hat{e}_\rho \\ \dot{\hat{e}}_z &= 0 \end{aligned}}$$

velocidade em coordenadas cilínd.

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\,\hat{e}_\rho + \rho\,\dot{\hat{e}}_\rho + \dot{z}\,\hat{e}_z$$

$$\boxed{\dot{\hat{e}}_\rho = \dot{\phi}\,\hat{e}_\phi}$$

$$= \underbrace{\dot{\rho}\,\hat{e}_\rho}_{v_\rho} + \underbrace{\rho\,\dot{\phi}\,\hat{e}_\phi}_{v_\phi} + \underbrace{\dot{z}\,\hat{e}_z}_{v_z}$$

aceleração em coordenadas cilínd.

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \frac{d}{dt}\left(\dot{\rho}\,\hat{e}_\rho + \rho\,\dot{\phi}\,\hat{e}_\phi + \dot{z}\,\hat{e}_z\right)$$

$$= \left(\ddot{\rho}\,\hat{e}_\rho + \dot{\rho}\,\dot{\phi}\,\hat{e}_\phi\right) + \left(\dot{\rho}\,\dot{\phi}\,\hat{e}_\phi + \rho\,\ddot{\phi}\,\hat{e}_\phi - \rho\,\dot{\phi}^2\,\hat{e}_\rho\right) + \ddot{z}\,\hat{e}_z$$

$$= \underbrace{\left(\ddot{\rho} - \rho\,\dot{\phi}^2\right)}_{a_\rho}\hat{e}_\rho + \underbrace{\left(\rho\,\ddot{\phi} + 2\dot{\rho}\,\dot{\phi}\right)}_{a_\phi}\hat{e}_\phi + \underbrace{\ddot{z}}_{a_z}\,\hat{e}_z$$

sumário (coord. cilínd.)

$$\vec{r} = \rho\,\hat{e}_\rho + z\,\hat{e}_z$$

$$\vec{v} = \dot{\rho}\,\hat{e}_\rho + \rho\,\dot{\phi}\,\hat{e}_\phi + \dot{z}\,\hat{e}_z$$

$$\vec{a} = \left(\ddot{\rho} - \rho\,\dot{\phi}^2\right)\hat{e}_\rho + \left(\rho\,\ddot{\phi} + 2\dot{\rho}\,\dot{\phi}\right)\hat{e}_\phi + \ddot{z}\,\hat{e}_z$$

exercício

Aceleração em coordenadas polares: Um insecto desloca-se ao longo da espiral de uma concha. A trajectória descrita pelo insecto é dada pela equação:

$$R = R_0 \cdot e^{a.\theta},$$

onde a = 0.182 e R_0 = 5 mm. A distância radial do insecto ao centro da espiral aumenta à razão constante de 2 mm/s. Determine as componentes a_x e a_y da aceleração do insecto quando $\theta = \pi$.

nota: coordenadas polares (R, θ)

solução

aceleração em coordenadas polares:

$$\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta .$$

derivando a equação da trajectória em ordem ao tempo:

$$\begin{aligned} R &= R_0 e^{a\theta}, \\ \dot{R} &= R_0 a e^{a\theta}\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{\dot{R}}{R_0 a\, e^{a\theta}}, \qquad 1 \\ \ddot{R} &= R_0 a e^{a\theta}\ddot{\theta} + R_0 a^2 e^{a\theta}\dot{\theta}^2 . \end{aligned}$$

solução (cont.)

Mas

$$\dot{R} = 2 \ \mathrm{mm}/s \Rightarrow \ddot{R} = 0$$

$$\Rightarrow R_0 a \mathrm{e}^{a\theta} (\ddot{\theta} + a\dot{\theta}^2) = 0$$

$$\Rightarrow \quad \ddot{\theta} = -a\dot{\theta}^2 \underset{1}{=} -\frac{a\dot{R}^2}{R_0^2 a^2 \mathrm{e}^{2a\theta}} = -\frac{\dot{R}^2}{R_0^2 a \mathrm{e}^{2a\theta}}.$$

solução (cont.)

substituindo na aceleração:

$$
\begin{aligned}
\vec{\boldsymbol{a}} &= (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta \\
&= \left(0 - R_0 \mathrm{e}^{a\theta} \cdot \frac{\dot{R}^2}{R_0^2 a^2 \mathrm{e}^{2a\theta}}\right)\hat{\boldsymbol{e}}_R + \left(\frac{2\dot{R}^2}{R_0 a\, \mathrm{e}^{a\theta}} + \overbrace{R_0 \mathrm{e}^{a\theta}}^{R} \cdot \overbrace{\frac{-\dot{R}^2}{R_0^2 a\, \mathrm{e}^{2a\theta}}}^{\ddot{\theta}}\right)\hat{\boldsymbol{e}}_\theta \\
&= \frac{\dot{R}^2}{R_0 a\, \mathrm{e}^{a\theta}}\left[-\frac{1}{a}\hat{\boldsymbol{e}}_R + (2-1)\hat{\boldsymbol{e}}_\theta\right].
\end{aligned}
$$

solução (cont.)

$$R_0 = 5\,\text{mm},$$

$$\dot{R} = 2\,\text{mm}/s$$

$$a = 0.182,$$

Substituindo $\theta = \pi$:

$$\vec{\boldsymbol{a}} = -13.63\,\text{mm}/\,\text{s}^2\hat{\boldsymbol{e}}_R + 2.48\,\text{mm}/\,\text{s}^2\hat{\boldsymbol{e}}_\theta\,.$$

solução (cont.)

Mas para $\theta = \pi$:

$$\hat{\boldsymbol{e}}_R = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\imath}}$$

$$\hat{\boldsymbol{e}}_\theta = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\jmath}}$$

logo

$$\vec{\boldsymbol{a}} \quad = \quad 13.63\,\mathrm{mm/\,s^2}\,\hat{\boldsymbol{\imath}} - 2.48\,\mathrm{mm/\,s^2}\,\hat{\boldsymbol{\jmath}}$$

$$\boxed{a_x = 13.63\,\mathrm{mm/\,s^2},\ a_y = -2.48\,\mathrm{mm/\,s^2}}$$

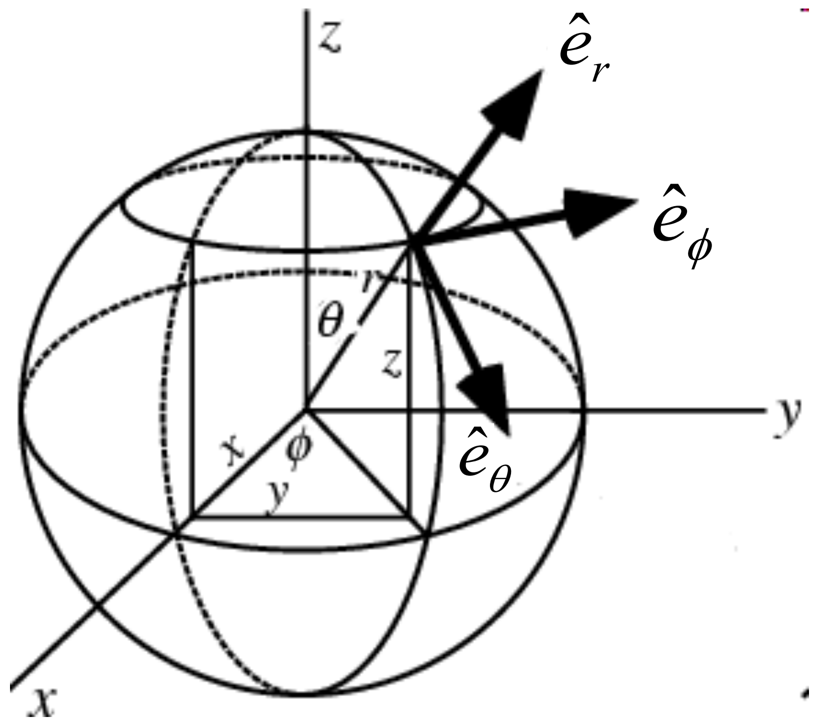
exercício

30. Uma partícula move-se com velocidade constante v ao longo da cardióide $r = k(1 + \cos\phi)$. Determine a aceleração, o módulo da aceleração e a velocidade angular da partícula (Greiner 10.2)

coordenadas esféricas.

vector posição:

$$\vec{r}(t) = r\,(t)\,\hat{e}_r(t)$$



variação de $\hat{e}_r$ no tempo

$$\hat{e}_r = \left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right)$$

$$\frac{d\hat{e}_r}{dt} = \underbrace{\frac{\partial\hat{e}_r}{\partial r}\cdot\frac{dr}{dt}}_{=0} + \frac{\partial\hat{e}_r}{\partial\phi}\cdot\frac{d\phi}{dt} + \frac{\partial\hat{e}_r}{\partial\theta}\cdot\frac{d\theta}{dt}$$

$$= \frac{\partial\hat{e}_r}{\partial\phi}\cdot\dot{\phi} + \frac{\partial\hat{e}_r}{\partial\theta}\cdot\dot{\theta}$$

$$= \left(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0\right)\dot{\phi}$$

$$+ \left(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta\right)\dot{\theta}$$

$$= \dot{\theta}\,\hat{e}_\theta + \sin\theta\,\dot{\phi}\,\hat{e}_\phi$$

variação de $\hat{e}_\theta$ no tempo

$$\hat{e}_\theta = \left(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta\right)$$

$$\frac{d\hat{e}_\theta}{dt} = \frac{\partial\hat{e}_\theta}{\partial\theta}\cdot\frac{d\theta}{dt} + \frac{\partial\hat{e}_\theta}{\partial\phi}\cdot\frac{d\phi}{dt}$$

$$= \left(-\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta\right)\dot{\theta}$$

$$+ \left(-\cos\theta\sin\phi, \cos\theta\cos\phi, 0\right)\dot{\phi}$$

$$= -\dot{\theta}\,\hat{e}_r + \cos\theta\,\dot{\phi}\,\hat{e}_\phi$$

variação de $\hat{e}_\phi$ no tempo

$$\hat{e}_\phi = \left(-\sin\phi, \cos\phi, 0\right)$$

$$\frac{d\hat{e}_\phi}{dt} = \left(-\cos\phi, -\sin\phi, 0\right)\dot{\phi}$$

$$= -\dot{\phi}\left[\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta\right]$$

posição, velocidade e aceleração

$$
\begin{aligned}
\mathbf{r} &= r\,\mathbf{e}_r\,,\\
\dot{\mathbf{r}} &= \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r\\
&= \dot{r}\,\mathbf{e}_r + r\,\dot{\vartheta}\,\mathbf{e}_\vartheta + r\sin\vartheta\;\dot{\varphi}\;\mathbf{e}_\varphi\,,\\
\ddot{\mathbf{r}} &= \ddot{r}\,\mathbf{e}_r + \dot{r}\,\dot{\mathbf{e}}_r + \dot{r}\,\dot{\vartheta}\,\mathbf{e}_\vartheta + r\,\ddot{\vartheta}\,\mathbf{e}_\vartheta + r\,\dot{\vartheta}\,\dot{\mathbf{e}}_\vartheta\\
&\qquad + \dot{r}\sin\vartheta\;\dot{\varphi}\;\mathbf{e}_\varphi + r\cos\vartheta\;\dot{\vartheta}\;\dot{\varphi}\;\mathbf{e}_\varphi +\\
&\qquad + r\sin\vartheta\;\ddot{\varphi}\;\mathbf{e}_\varphi + r\sin\vartheta\;\dot{\varphi}\;\dot{\mathbf{e}}_\varphi
\end{aligned}
$$

aceleração e velocidade em coordenadas esféricas

$$v_r = \dot{r},$$

$$v_\theta = r\dot{\theta},$$

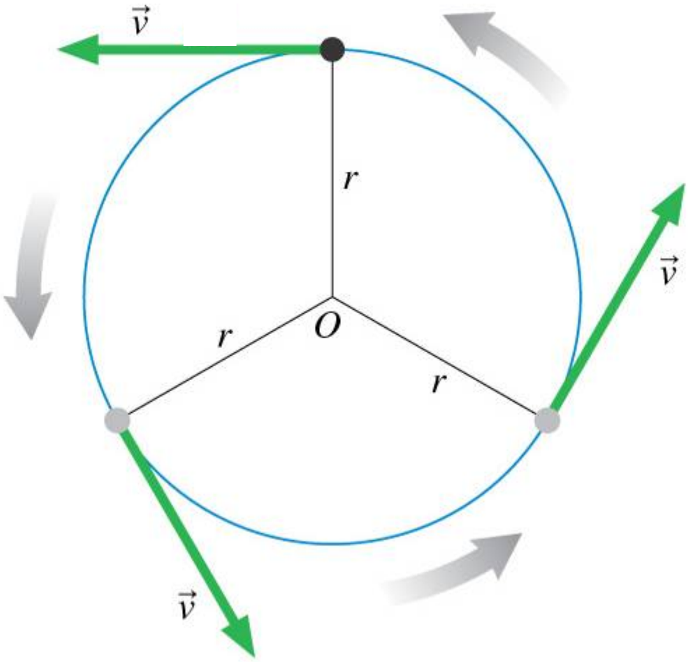
$$v_\varphi = r\sin\theta\dot{\varphi},$$

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\varphi}^2,$$

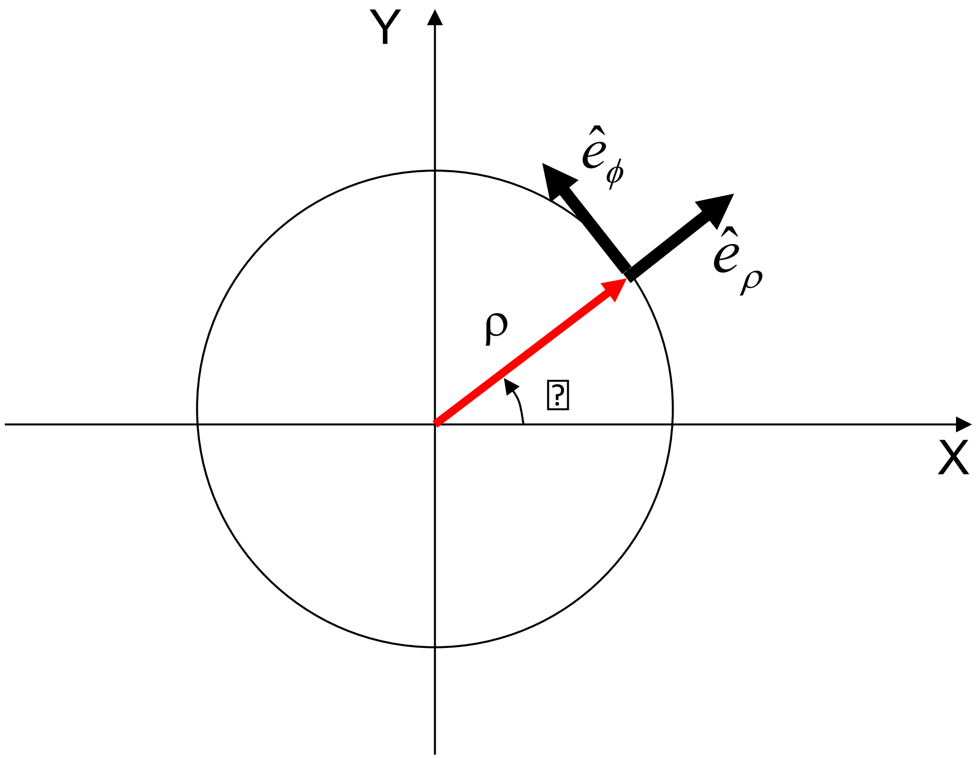
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\varphi}^2,$$

$$a_\varphi = r\sin\theta\ddot{\varphi} + 2\dot{r}\sin\theta\dot{\varphi} + 2r\cos\theta\dot{\theta}\dot{\varphi}.$$

movimento circular



movimento circular: coordenadas polares



movimento circular: coordenadas polares

$$\vec{r} = \rho\,\hat{e}_\rho$$

$$\vec{v} = \dot{\rho}\,\hat{e}_\rho + \rho\,\dot{\phi}\,\hat{e}_\phi$$

$$\vec{a} = \left(\ddot{\rho} - \rho\,\dot{\phi}^2\right)\hat{e}_\rho + \left(\rho\,\ddot{\phi} + 2\dot{\rho}\,\dot{\phi}\right)\hat{e}_\phi$$

$$\rho = const$$

$$\vec{r} = \rho\,\hat{e}_\rho$$

$$\vec{v} = \rho\,\dot{\phi}\,\hat{e}_\phi$$

$$\vec{a} = -\rho\,\dot{\phi}^2\,\hat{e}_\rho + \rho\,\ddot{\phi}\,\hat{e}_\phi$$

aceler. centripeta

aceler. tangencial

mov. circular uniforme

$$\vec{r} = \rho\,\hat{e}_\rho$$

$$\vec{v} = \rho\,\dot{\phi}\,\hat{e}_\phi$$

$$\vec{a} = -\rho\,\dot{\phi}^2\,\hat{e}_\rho$$

aceler.
centripeta

satélite