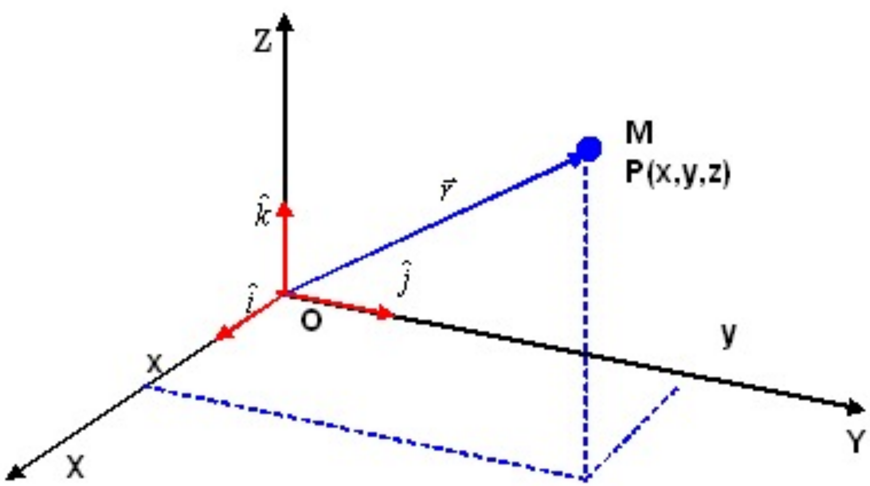
Cap. 1: material suplementar

Sumário:

Coordenadas generalizadas. Coordenadas cartesianas. Coordenadas curvilíneas: cilíndricas, esféricas, polares. Vectores de base.

coordenadas cartesianas

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$



$\hat{i}\ \hat{j}\ \hat{k}$ indep. de $\vec{r}$

$$\hat{i} \perp \hat{j} \perp \hat{k}$$

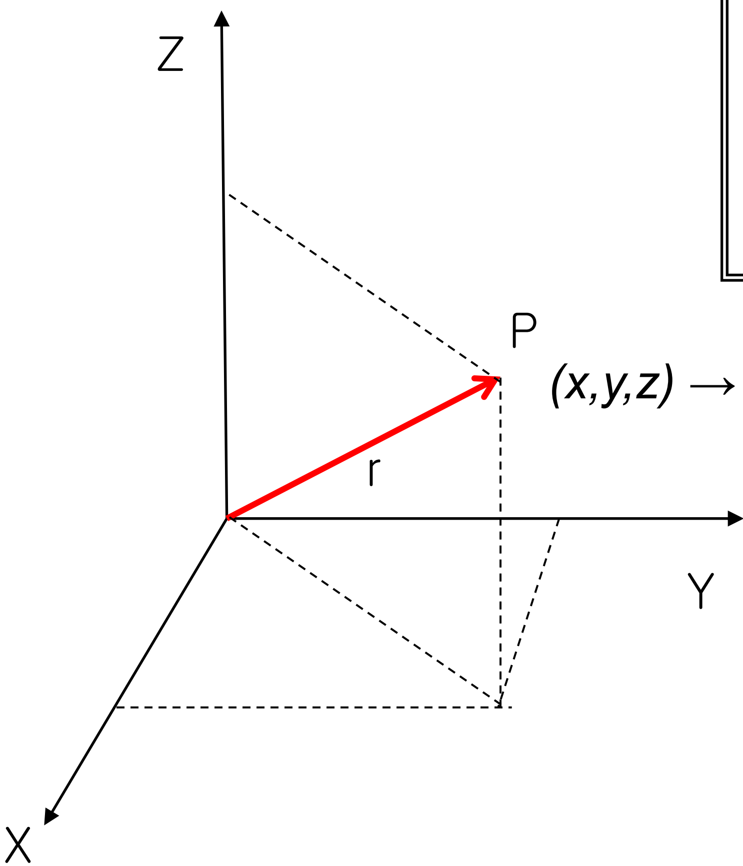
$$x = \vec{r} \cdot \hat{i}$$

$$y = \vec{r} \cdot \hat{j}$$

$$z = \vec{r} \cdot \hat{k}$$

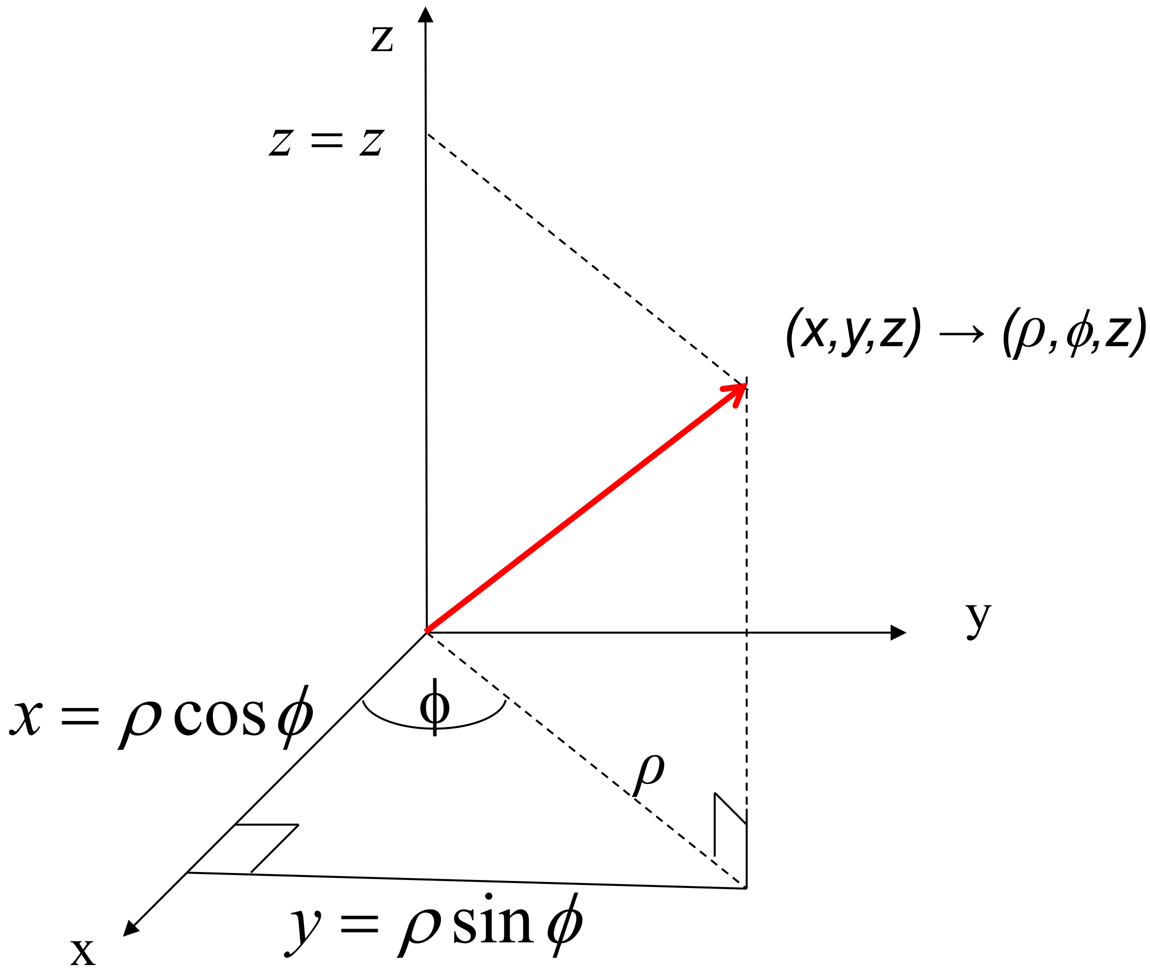
coordenadas curvilíneas

$$\vec{r}\,(q_1,q_2,q_3) = x(q_1,q_2,q_3)\,\hat{i} + y(q_1,q_2,q_3)\,\hat{j} + z(q_1,q_2,q_3)\,\hat{k}$$



$(x,y,z) \rightarrow (q_1,q_2,q_3)$

Exemplo: coordenadas cilíndricas



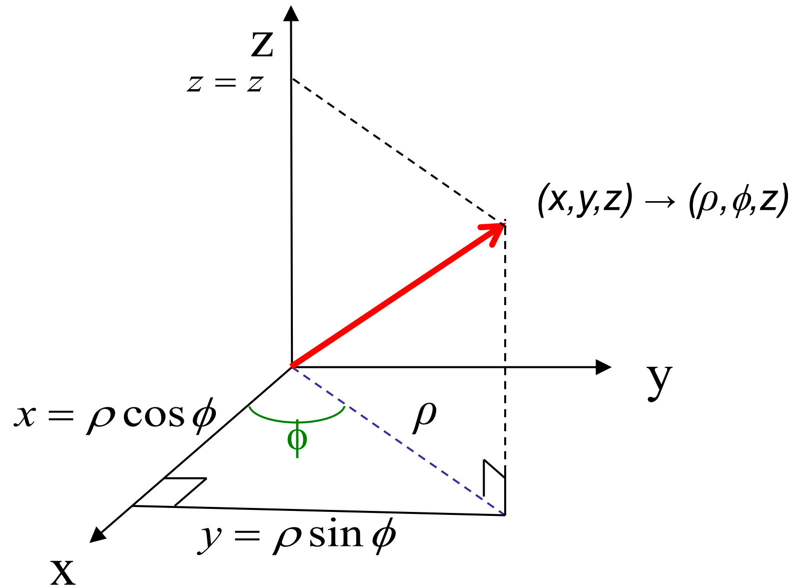
Exemplo: coordenadas cilíndricas

(ρ,ϕ,z):

ϕ: ângulo entre a projecção do vector posição no plano xy e o eixo do xx. ($0 \le \phi < 2\pi$)

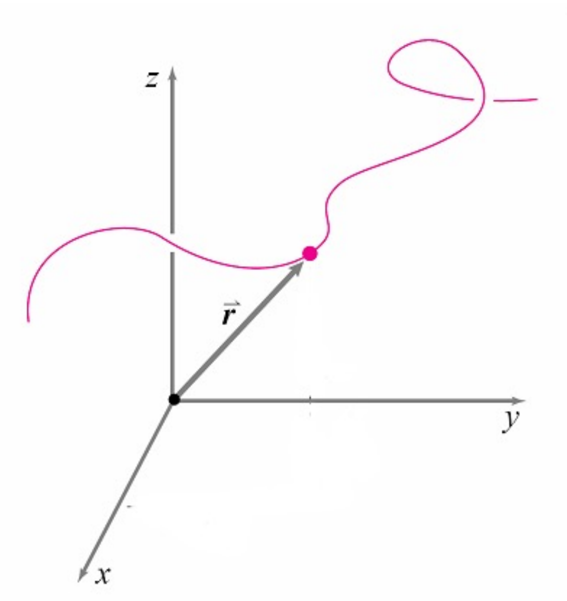
ρ : comprimento da projecção do vector posição no plano xy ($\rho \ge 0$)

z: componente do vector posição segundo o eixo dos zz



coordenadas curvilíneas

2 variáveis constantes:

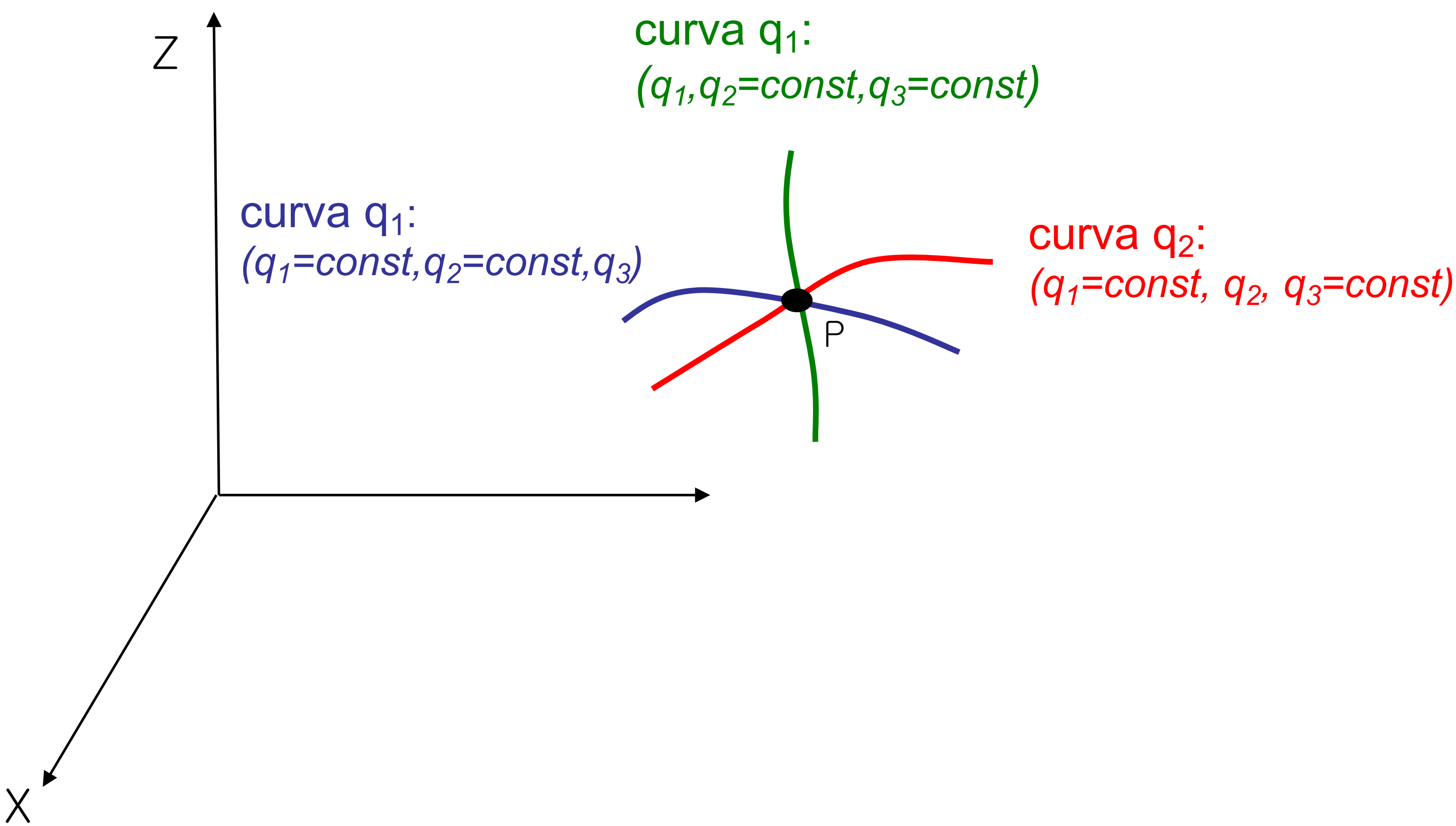


curva q_1: $(q_1, q_2\!=\!const, q_3\!=\!const)$

$$\frac{\partial \vec{r}}{\partial q_1} \text{ tangente a curva } q_1$$

coordenadas curvilíneas

<u>2 variáveis constantes:</u>



vectores da base

$\left\{ \hat{e}_{q_1}, \hat{e}_{q_2}, \hat{e}_{q_3} \right\}$ vectores unitários tangentes às curvas q_1, q_2 e q_3

$$\hat{e}_{q_1} = \frac{\left(\dfrac{\partial \vec{r}}{\partial q_1}\right)}{\left|\dfrac{\partial \vec{r}}{\partial q_1}\right|} \qquad \hat{e}_{q_2} = \frac{\left(\dfrac{\partial \vec{r}}{\partial q_2}\right)}{\left|\dfrac{\partial \vec{r}}{\partial q_2}\right|} \qquad \hat{e}_{q_3} = \frac{\left(\dfrac{\partial \vec{r}}{\partial q_3}\right)}{\left|\dfrac{\partial \vec{r}}{\partial q_3}\right|}$$

$h_i = \left|\dfrac{\partial \vec{r}}{\partial q_i}\right|$ factor de escala

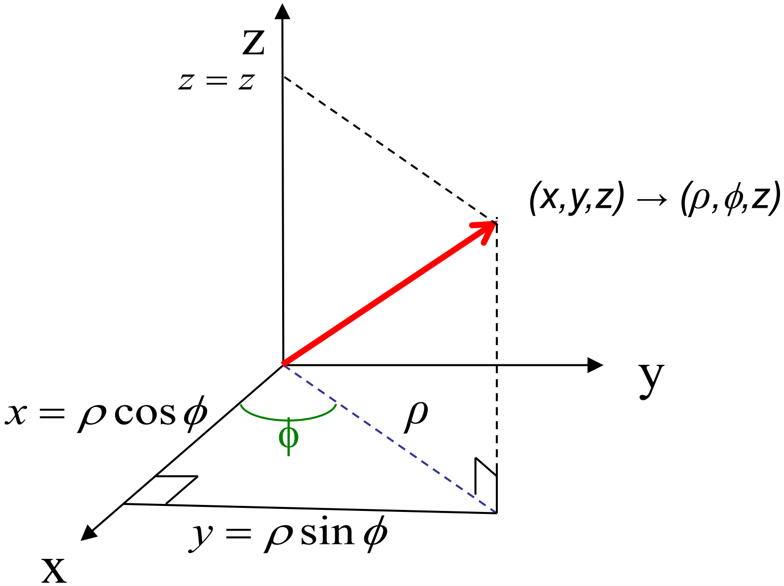
coordenadas cilíndricas

(ρ,ϕ,z):

ϕ: ângulo entre a projecção do vector posição no plano xy e o eixo do xx. ($0 \leq \phi < 2\pi$)

ρ : comprimento da projecção do vector posição no plano xy ($\rho \geq 0$)

z: componente do vector posição segundo o eixo dos zz



Coordenadas polares + coordenada z

equações de transformação

$$(x,y,z) \rightarrow (\rho,\phi,z)$$

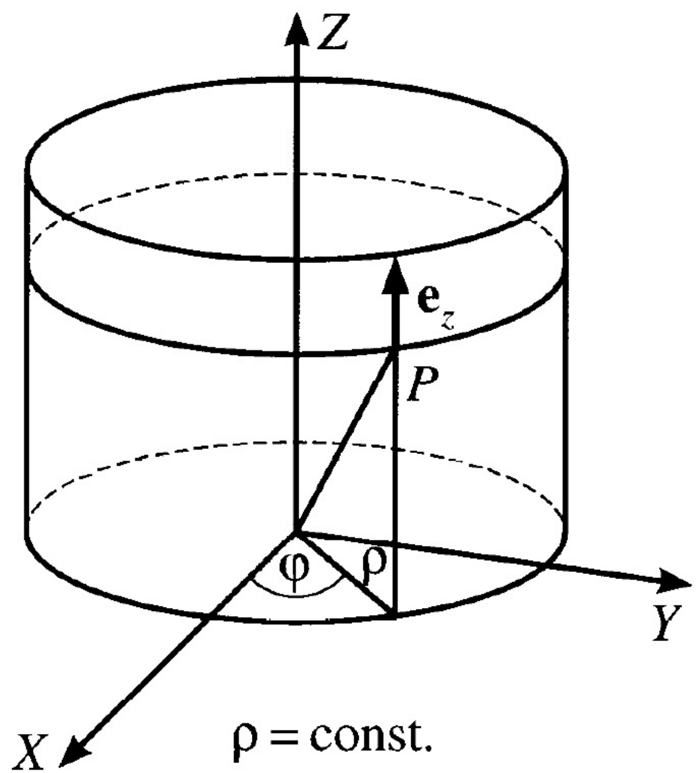
$$x = \rho\cos(\phi)$$
$$y = \rho\sin(\phi)$$
$$z = z$$

$$\Rightarrow$$

$$\rho^2 = x^2 + y^2$$
$$\tan(\phi) = \frac{y}{x}$$
$$z = z$$

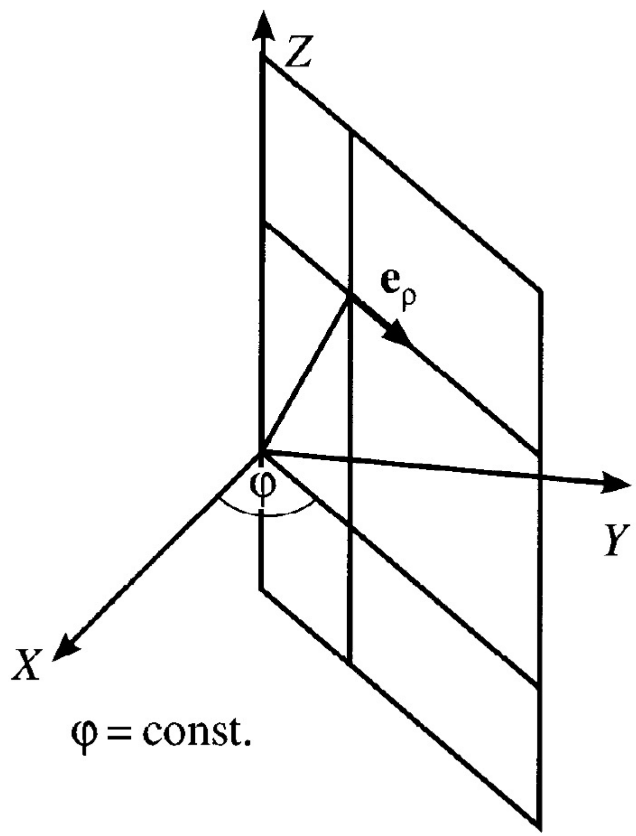
coordenadas cilíndricas

ρ =const → cilíndros com eixo nos zz



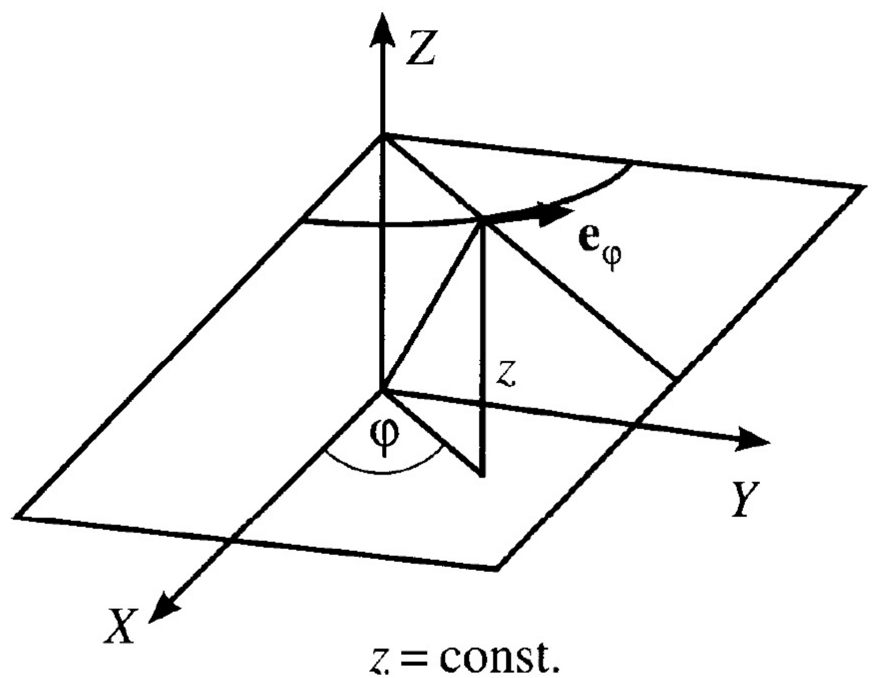
coordenadas cilíndricas

ϕ =const → semi-planos a partir do eixo dos zz

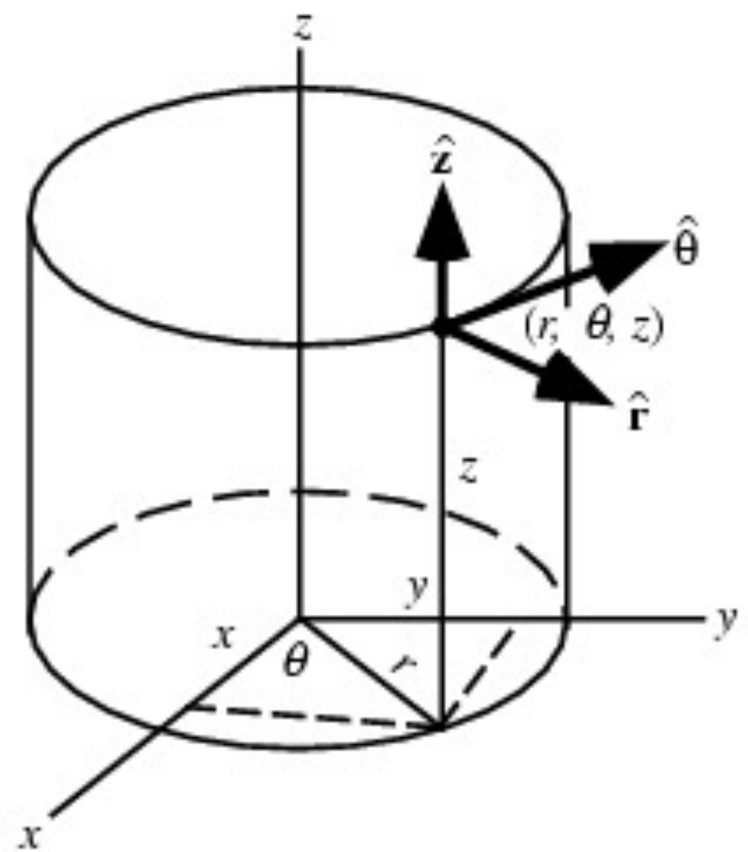


coordenadas cilíndricas

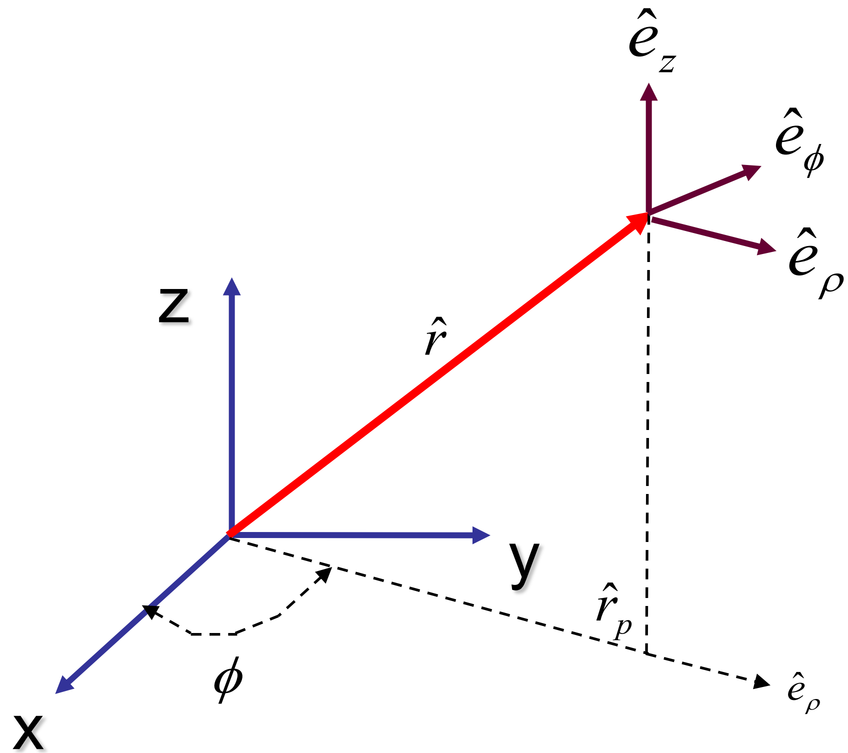
z = const $\rightarrow$ planos paralelos ao plano *xy*



vectores da base



coordenadas cilíndricas



Notas:

$\hat{e}_\rho$ é paralelo á projecção do vector posição.

$\hat{e}_\phi$ está no plano (x,y).

$$\hat{e}_z = \hat{k}$$

vectores da base (coord. cilínd.)

$$\vec{r} = (x, y, z) = (\rho\cos\phi, \rho\sin\phi, z)$$

$$\hat{e}_\rho = \frac{\left(\dfrac{\partial\vec{r}}{\partial\rho}\right)}{\left|\dfrac{\partial\vec{r}}{\partial\rho}\right|} = ? \qquad \hat{e}_\phi = \frac{\left(\dfrac{\partial\vec{r}}{\partial\phi}\right)}{\left|\dfrac{\partial\vec{r}}{\partial\phi}\right|} = ? \qquad \hat{e}_z = \frac{\left(\dfrac{\partial\vec{r}}{\partial z}\right)}{\left|\dfrac{\partial\vec{r}}{\partial z}\right|} = ?$$

vectores da base (coord. cilínd.)

$$\hat{e}_\rho = \frac{\left(\frac{\partial \vec{r}}{\partial \rho}\right)}{\left|\frac{\partial \vec{r}}{\partial \rho}\right|} = \frac{(\cos\phi, \sin\phi, 0)}{1} = (\cos\phi, \sin\phi, 0)$$

$$\hat{e}_\phi = \frac{\left(\frac{\partial \vec{r}}{\partial \phi}\right)}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|} = \frac{\rho(-\sin\phi, \cos\phi, 0)}{\rho} = (-\sin\phi, \cos\phi, 0)$$

$$\hat{e}_z = \frac{\left(\frac{\partial \vec{r}}{\partial z}\right)}{\left|\frac{\partial \vec{r}}{\partial z}\right|} = \frac{(0,0,1)}{1} = (0,0,1)$$

Nota

Note que $\hat{e}_\rho \cdot \hat{e}_\phi = \hat{e}_\phi \cdot \hat{e}_z = \hat{e}_\rho \cdot \hat{e}_z = 0$

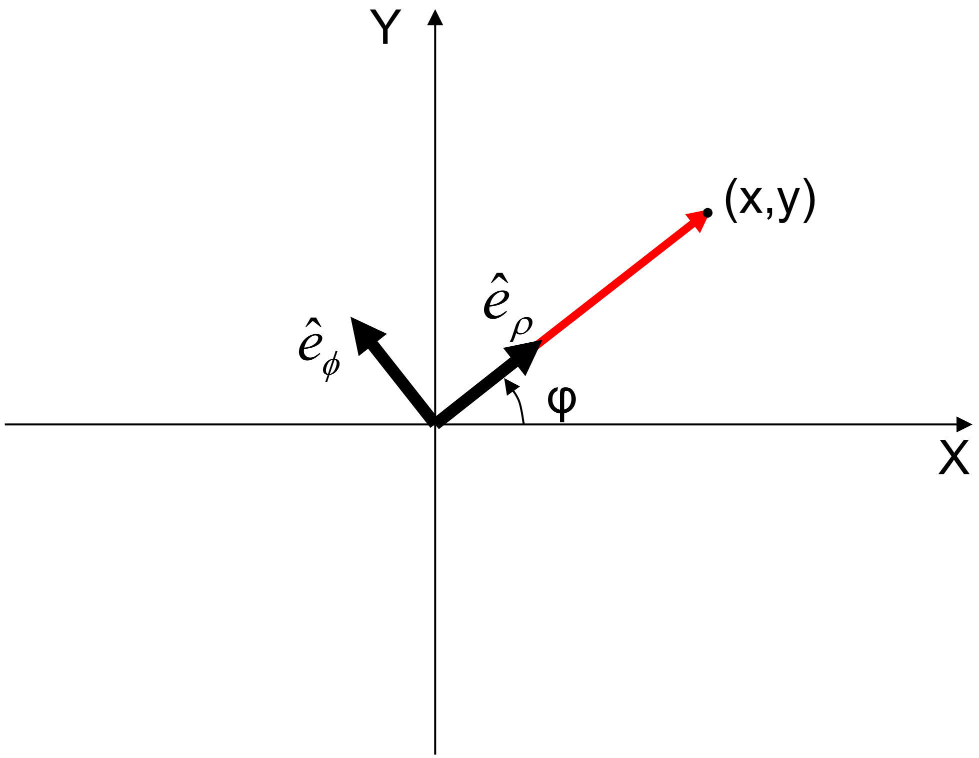
$\Rightarrow$ Sistema de coordenadas ortogonais

$$\boxed{\hat{e}_\rho \cdot \hat{e}_\phi \times \hat{e}_z = 1}$$

componentes na base de coord. cilínd.

$$\vec{v} = v_\rho \hat{e}_\rho + v_\phi \hat{e}_\phi + v_z \hat{e}_z$$

$\left\{\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z\right\}$ na base $\left\{\hat{i}, \hat{j}, \hat{k}\right\}$

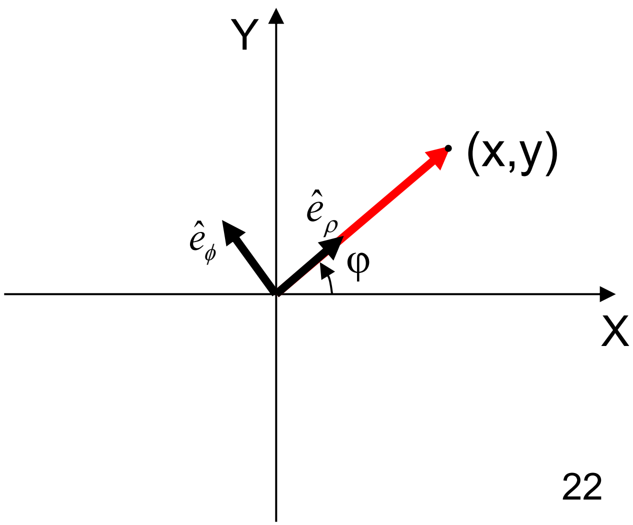


$\left\{\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z\right\}$ na base $\left\{\hat{i}, \hat{j}, \hat{k}\right\}$

$$\hat{e}_\rho = (\cos\phi, \sin\phi, 0) = \cos\phi\, \hat{i} + \sin\phi\, \hat{j} + 0\, \hat{k}$$
$$= \left(\hat{e}_\rho \cdot \hat{i}\right)\hat{i} + \left(\hat{e}_\rho \cdot \hat{j}\right)\hat{j} + \left(\hat{e}_\rho \cdot \hat{k}\right)\hat{k}$$

$$\hat{e}_\phi = (-\sin\phi, \cos\phi, 0) = -\sin\phi\, \hat{i} + \cos\phi\, \hat{j} + 0\, \hat{k}$$

$$\hat{e}_z = (0,0,1) = 0\, \hat{i} + 0\, \hat{j} + \hat{k}$$



$\left\{\hat{i},\hat{j},\hat{k}\right\}$ na base $\left\{\hat{e}_\rho,\hat{e}_\phi,\hat{e}_z\right\}$

$$\hat{i} = \cos\phi\ \hat{e}_\rho - \sin\phi\,\hat{e}_\phi + 0\,\hat{e}_z$$

$$= \left(\hat{e}_\rho\cdot\hat{i}\right)\hat{e}_\rho + \left(\hat{e}_\phi\cdot\hat{i}\right)\hat{e}_\phi + \left(\hat{e}_z\cdot\hat{i}\right)\hat{e}_z$$

$$\hat{j} = \sin\phi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\phi + 0\,\hat{e}_z$$

$$\hat{k} = 0\,\hat{e}_\rho + 0\,\hat{e}_\phi + 1\,\hat{e}_z$$

exercícios

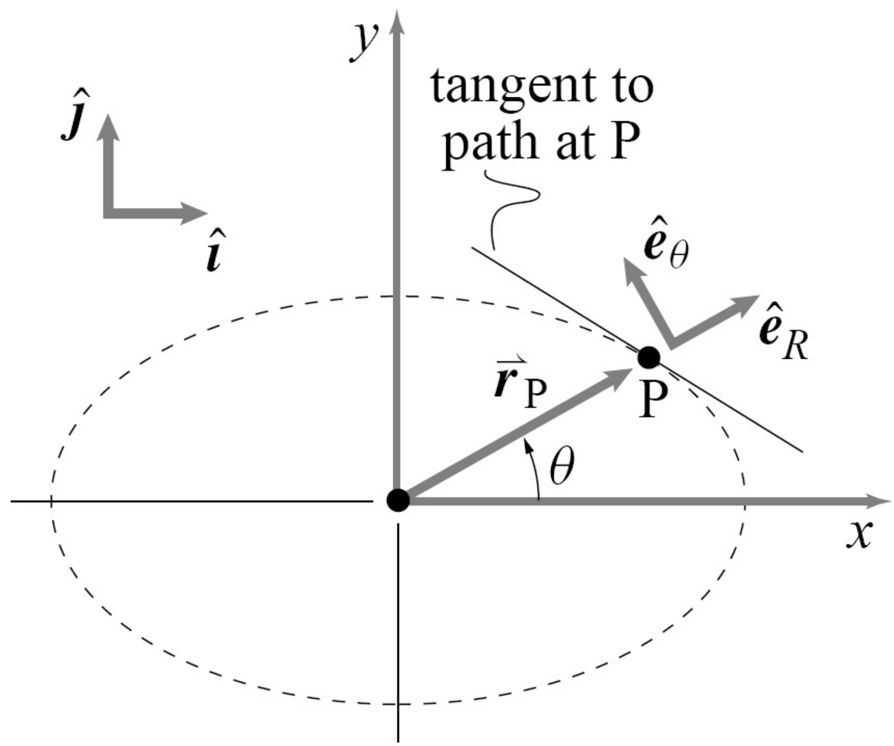
21. Determine as coordenadas cartesianas do ponto de coordenadas cilíndricas $(\rho, \theta, z) = (2, \pi/3, 1)$.

23. Determine as coordenadas cilíndricas do ponto de coordenadas cartesianas $(x, y, z) = (\sqrt{2}, -\sqrt{2}, 2)$.

exercícios

25. Escreva o vector $\mathbf{v} = z\mathbf{i} + 2x\mathbf{j} + y\,\mathbf{k}$ na base das coordenadas cilíndricas. (Greiner 10.2)

caso part.: coordenadas polares



equações de transformação

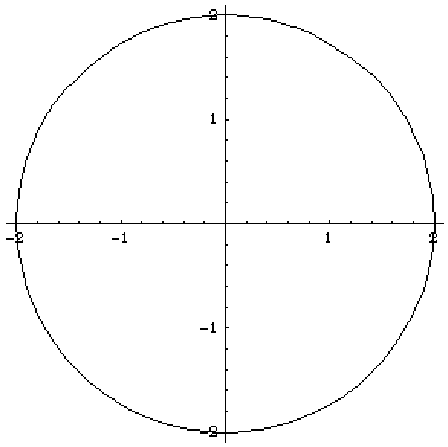
$(x,y) \rightarrow (r,\theta)$ nova notação!

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

$$r^2 = x^2 + y^2$$
$$\tan(\theta) = \frac{y}{x}$$

exemplos

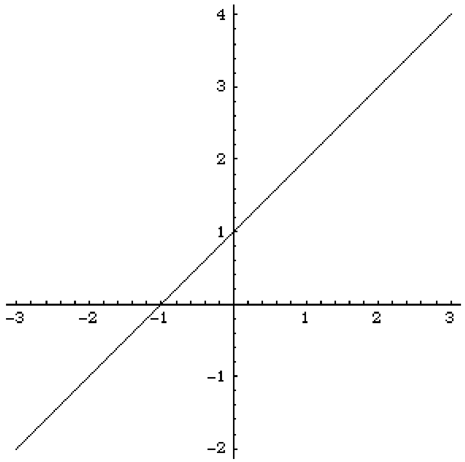
circunferência



coord. pol.: $r = 2$

coord. cart.: $x^2 + y^2 = 4$

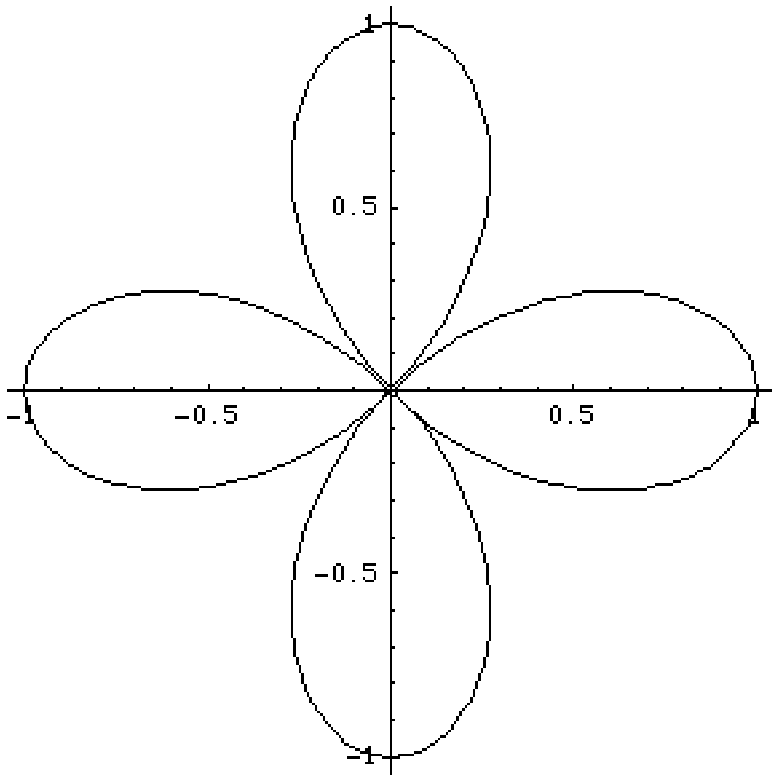
recta



coord. pol.: $r = \dfrac{1}{\sin\theta - \cos\theta}$

coord. cart.: $y = x + 1$

exemplos



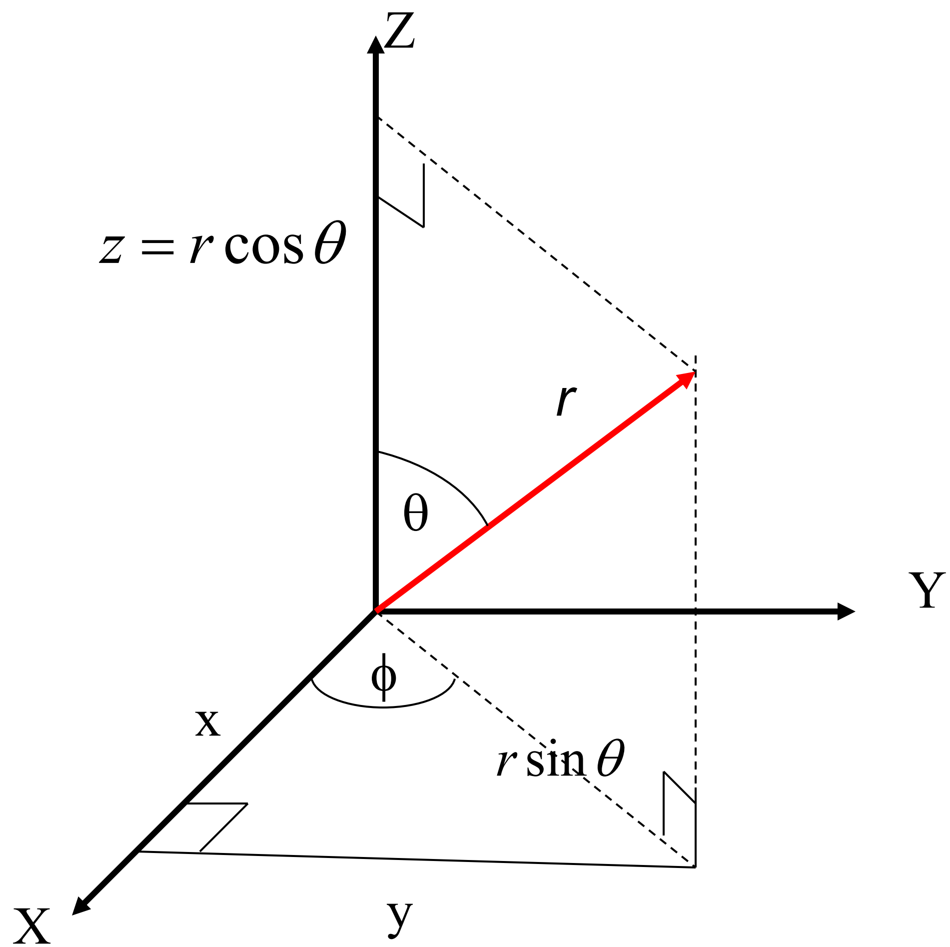
coord. polares:

$r = |\cos(2\theta)|$

coord. cart.:

$$(x^2 + y^2)^{\frac{3}{2}} = \pm(x^2 - y^2)$$

coordenadas esféricas



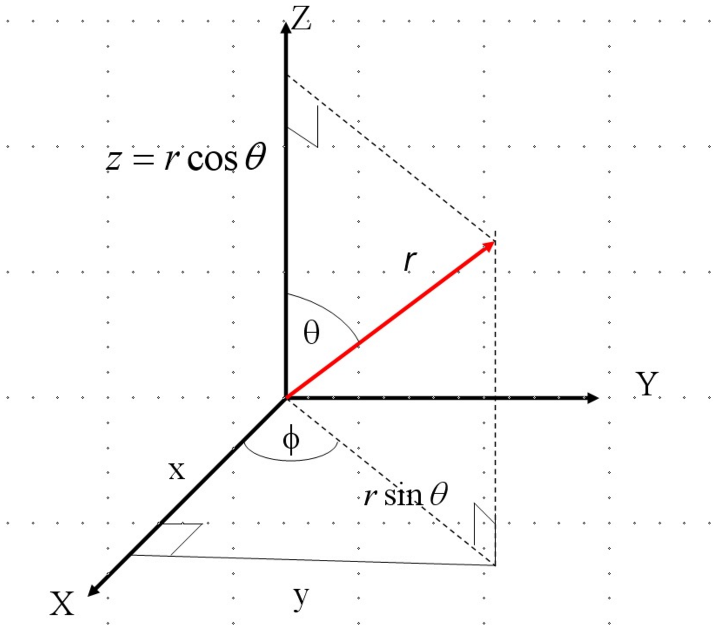
coordenadas esféricas

*(r,*θ*,ϕ):*

ϕ: ângulo entre a projecção do vector posição no plano xy e o eixo do xx. ($0 \leq \phi < 2\pi$)
(azimute)

r : comprimento do vector posição
($r \geq 0$)

θ: ângulo entre o vector posição e o eixo do zz. (0 ≤ ? ≤ π)
(zénite)



Raio, latitude and longitude?

equações de transformação

$$(x,y,z) \rightarrow (r,\theta,\phi)$$

$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\boldsymbol{\phi} = \tan^{-1}\frac{\boldsymbol{y}}{\boldsymbol{x}}$$
$$\boldsymbol{\theta} = \tan^{-1}\frac{\sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2}}{\boldsymbol{z}}$$

equações de transformação

$$(r,\theta,\phi) \rightarrow (\rho,\phi,z)$$

$$r = \sqrt{z^2 + \rho^2}$$

$$\theta = \tan^{-1}\frac{\rho}{z}$$

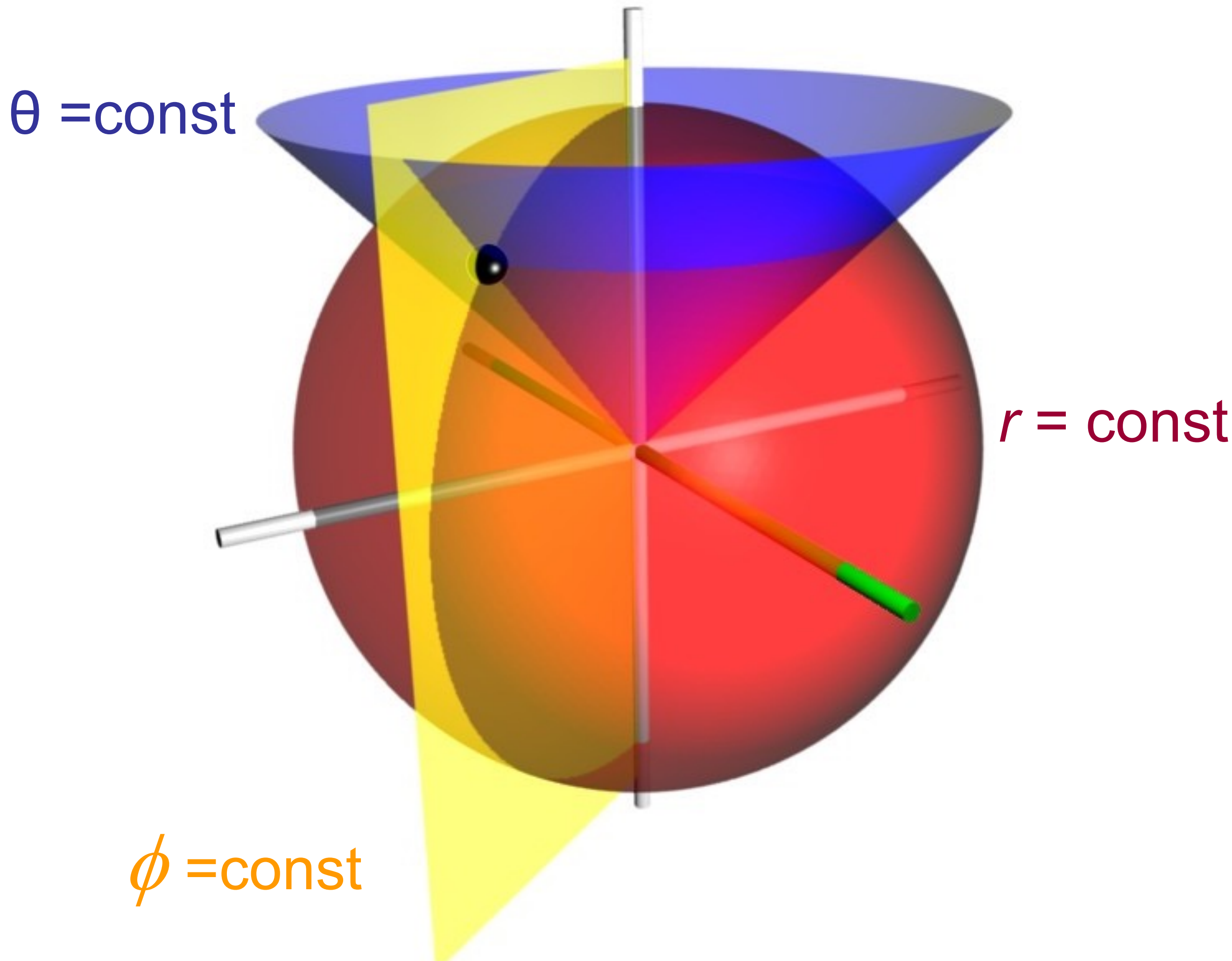
$$\phi = \phi$$

$$\rho = r\sin(\theta)$$

$$z = r\cos(\theta)$$

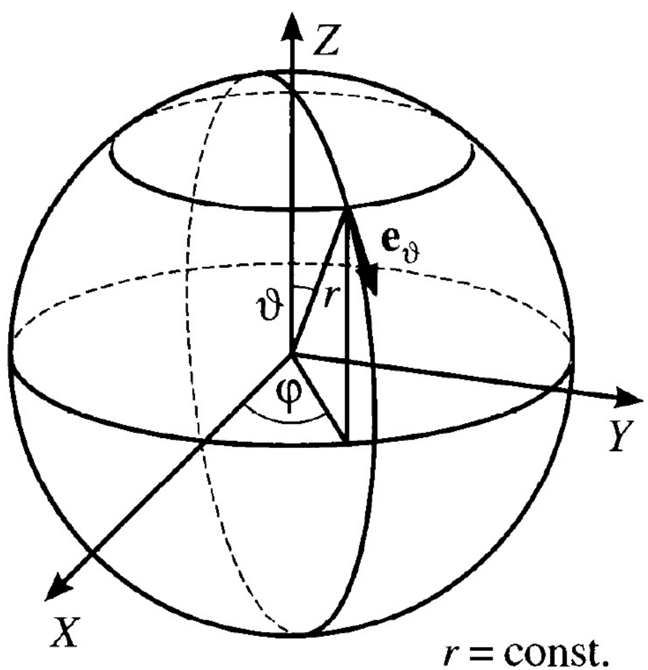
$$\phi = \phi$$

coordenadas esféricas



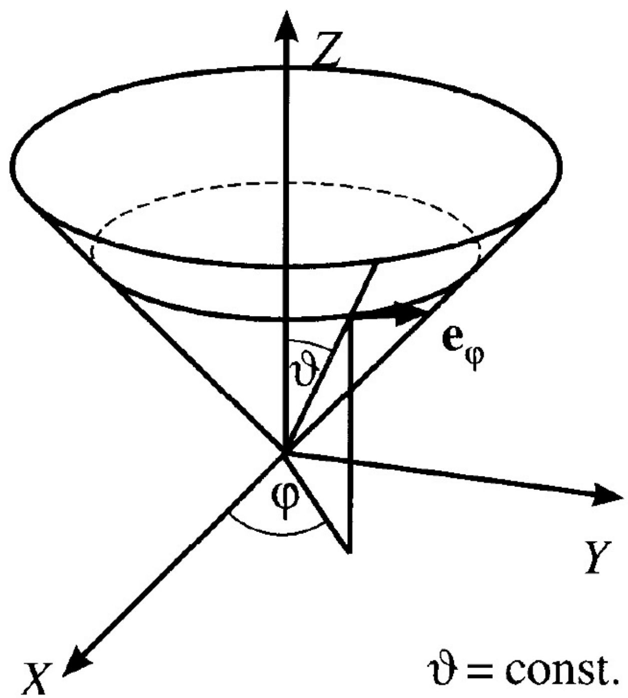
coordenadas esféricas

r = const → esferas com centro na origem



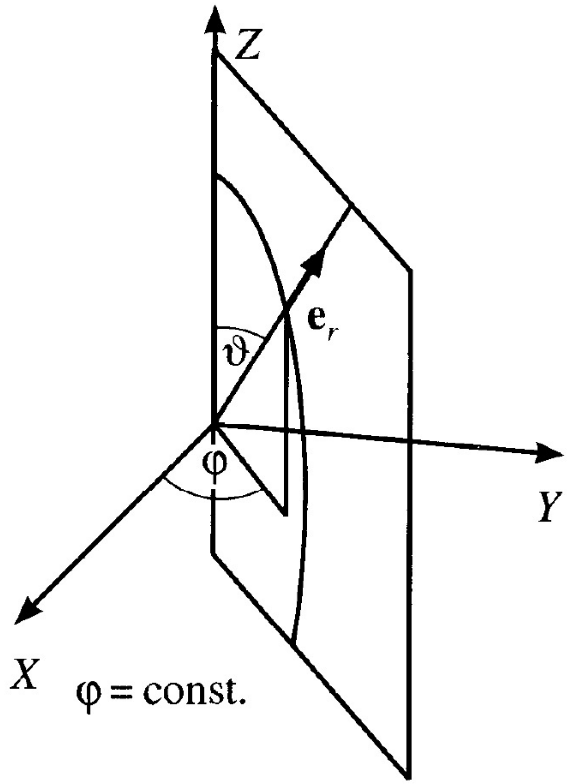
coordenadas esféricas

θ =const → cone

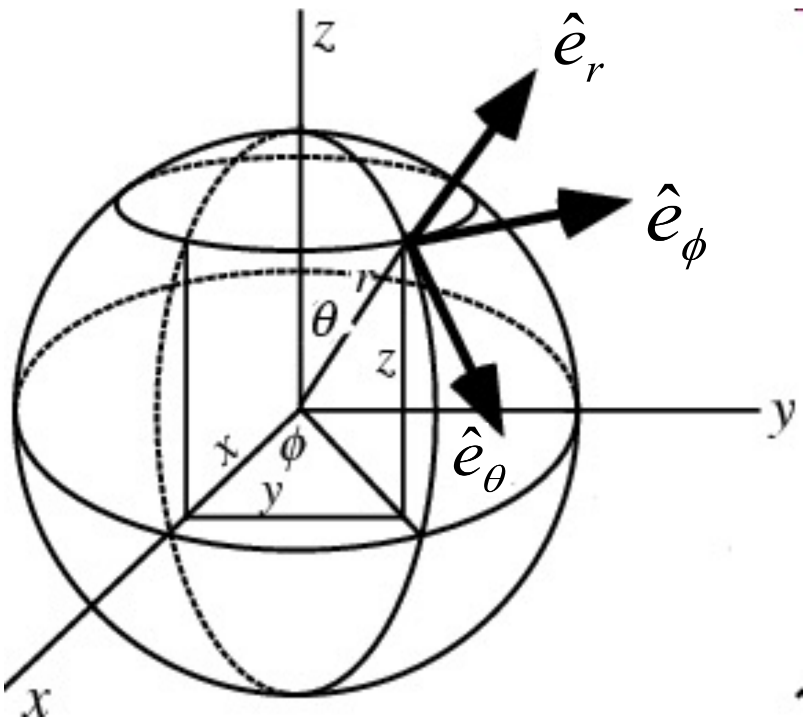


coordenadas esféricas

ϕ =const → semi-planos a partir do eixo dos zz



vectores da base



Notas:

$\hat{e}_r$ é paralelo ao vector posição.

$\hat{e}_\theta$ está no plano que contem o vector posição e o eixo dos *zz*

vectores da base (coord. esf.)

$$\vec{r} = \left(x, y, z\right) = \left(r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta\right)$$

$$\hat{e}_r = \frac{\left(\dfrac{\partial \vec{r}}{\partial r}\right)}{\left|\dfrac{\partial \vec{r}}{\partial r}\right|} = ? \qquad \hat{e}_\theta = \frac{\left(\dfrac{\partial \vec{r}}{\partial \theta}\right)}{\left|\dfrac{\partial \vec{r}}{\partial \theta}\right|} = ? \qquad \hat{e}_\phi = \frac{\left(\dfrac{\partial \vec{r}}{\partial \phi}\right)}{\left|\dfrac{\partial \vec{r}}{\partial \phi}\right|} = ?$$

vectores da base (coord. esf.)

$$\hat{e}_r = \frac{\left(\frac{\partial \vec{r}}{\partial r}\right)}{\left|\frac{\partial \vec{r}}{\partial r}\right|} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\hat{e}_\phi = \frac{\left(\frac{\partial \vec{r}}{\partial \phi}\right)}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|} = \frac{r(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)}{r\sin\theta} = (-\sin\phi, \cos\phi, 0)$$

$$\hat{e}_\theta = \frac{\left(\frac{\partial \vec{r}}{\partial \theta}\right)}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} = \frac{r(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)}{r} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

Nota

Note que $\hat{e}_r \cdot \hat{e}_\theta = \hat{e}_\theta \cdot \hat{e}_\phi = \hat{e}_\phi \cdot \hat{e}_r = 0$

⇨ Sistema de coordenadas ortogonais

$$\boxed{\hat{e}_r \cdot \hat{e}_\theta \times \hat{e}_\phi = 1}$$

componentes na base de coord. esf.

$$\vec{v} = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_\phi \hat{e}_\phi$$

exercício

Escreva os vectores da base de coordenadas cartesianas, $\left\{\hat{i}, \hat{j}, \hat{k}\right\}$, na base das coordenadas esféricas, $\left\{\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\right\}$

exercício

24. Determine as coordenadas esféricas do ponto de coordenadas cartesianas $(x, y, z) = (\sqrt{2}, 0, -\sqrt{2})$.

26. Escreva o vector $\mathbf{v} = (x+1)\mathbf{i} + \mathbf{j} + y\,\mathbf{k}$ na base das coordenadas cilíndricas e na base das coordenadas esféricas.