

Fitted ALE scheme for Two-Phase Navier–Stokes Flow

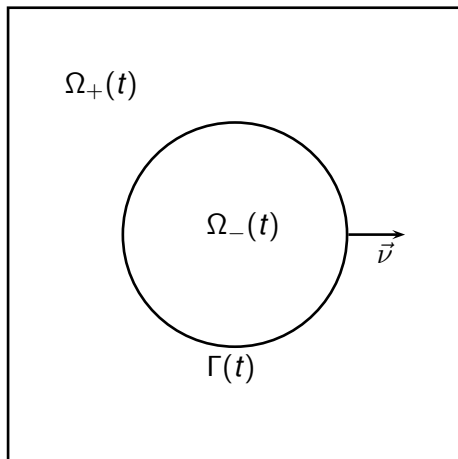
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Problem setting

Domain Ω in the 2-dimensional case.



Governing equations

- Bulk equations

$$\begin{aligned}\rho(\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}) - \nabla \cdot \underline{\underline{\sigma}} &= \vec{f} = \rho \vec{f}_1 + \vec{f}_2 && \text{in } \Omega_{\pm}(t), \\ \nabla \cdot \vec{u} &= 0 && \text{in } \Omega_{\pm}(t),\end{aligned}$$

where

$$\underline{\underline{\sigma}} = \mu(\nabla \vec{u} + (\nabla \vec{u})^T) - p \underline{\underline{\text{id}}} = 2\mu \underline{\underline{D}}(\vec{u}) - p \underline{\underline{\text{id}}}.$$

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► Interface equations

$$\begin{aligned}[\vec{u}]_{-}^{+} &= \vec{0} && \text{on } \Gamma(t), \\ [\underline{\underline{\sigma}} \vec{\nu}]_{-}^{+} &= -\gamma \kappa \vec{\nu} && \text{on } \Gamma(t), \\ \vec{\gamma} \cdot \vec{\nu} &= \vec{u} \cdot \vec{\nu} && \text{on } \Gamma(t).\end{aligned}$$

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- To close the system, we prescribe the initial data $\Gamma(0) = \Gamma_0$, the initial velocity \vec{u}_0 and some boundary condition for \vec{u} on $\partial\Omega$.

Interface treatment

- ▶ $\Gamma(t)$ is a sufficiently smooth evolving hypersurface without boundary that is parameterized by $\vec{x}(\cdot, t) : \Upsilon \rightarrow \mathbb{R}^d$, therefore

$$\Gamma(t) = \vec{x}(\Upsilon, t),$$

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- ▶ It holds that

$$\Delta_s \vec{\text{id}} = \varkappa \vec{\nu} \quad \text{on } \Gamma(t),$$

where $\Delta_s = \nabla_s \cdot \nabla_s$ is the Laplace-Beltrami operator on $\Gamma(t)$ with $\nabla_s \cdot$ and ∇_s denoting surface divergence and surface gradient on $\Gamma(t)$.

Navier–Stokes weak formulation

Using the function spaces

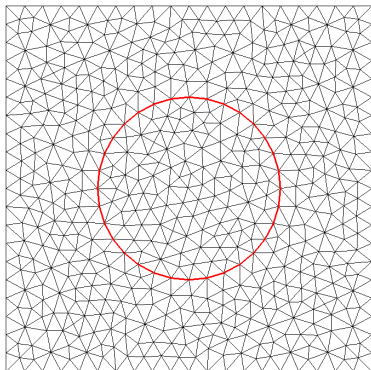
$$\mathbb{U} := [H_0^1(\Omega)]^d, \quad \mathbb{P} := L^2(\Omega) \quad \text{and}$$

$$\widehat{\mathbb{P}} := \{\eta \in \mathbb{P} : \int_{\Omega} \eta \, d\mathcal{L}^d = 0\},$$

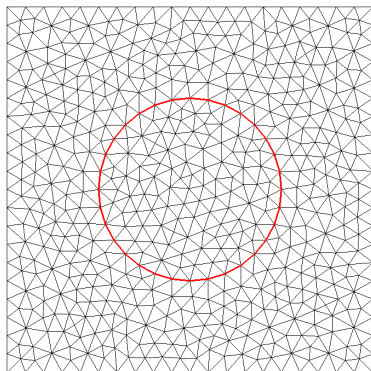
the Navier–Stokes weak formulation is

$$\begin{aligned} & \frac{1}{2} \left[\frac{d}{dt} (\rho \vec{u}, \vec{\xi}) + (\rho \vec{u}_t, \vec{\xi}) + (\rho, [(\vec{u} \cdot \nabla) \vec{u}] \cdot \vec{\xi} - [(\vec{u} \cdot \nabla) \vec{\xi}] \cdot \vec{u}) \right] \\ & + 2 \left(\mu \underline{\underline{D}}(\vec{u}), \underline{\underline{D}}(\vec{\xi}) \right) - \left(p, \nabla \cdot \vec{\xi} \right) - \gamma \left\langle \varkappa \vec{v}, \vec{\xi} \right\rangle_{\Gamma(t)} = \left(\vec{f}, \vec{\xi} \right) \quad \forall \vec{\xi} \in \mathbb{U}, \\ & (\nabla \cdot \vec{u}, \varphi) = 0 \quad \forall \varphi \in \widehat{\mathbb{P}}, \\ & \left\langle \vec{v} - \vec{u}, \chi \vec{v} \right\rangle_{\Gamma(t)} = 0 \quad \forall \chi \in H^1(\Gamma(t)), \\ & \left\langle \varkappa \vec{v}, \vec{\eta} \right\rangle_{\Gamma(t)} + \left\langle \nabla_s \text{id}, \nabla_s \vec{\eta} \right\rangle_{\Gamma(t)} = 0 \quad \forall \vec{\eta} \in [H^1(\Gamma(t))]^d. \end{aligned}$$

Fitted approach

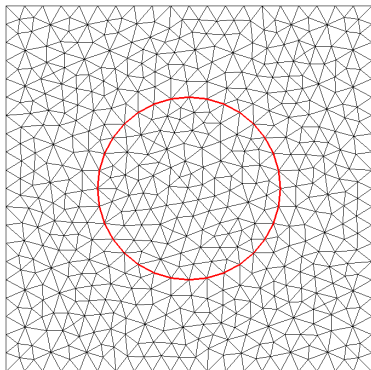


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- ▶ Pros: naturally captured discontinuity jumps in ρ, μ, p and no need to interpolate bulk quantities over interface.
- ▶ Cons: possible bulk mesh distortion and difficult bulk mesh adaptation.

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- ▶ For $m \geq 0$, let a discrete interface Γ^m be given. Here Γ^m is a polygonal curve in 2d, and a two-dimensional polyhedral surface in 3d.

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- ▶ Let $(\mathbf{U}^m, \mathbb{P}^m)$ be an LBB-stable pair of velocity/pressure finite element spaces in the bulk, e.g. P2-P0 (only in 2d) or P2-P1, with \mathbb{P}^m possibly extended by an additional basis function.

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- ▶ Then our finite element approximation is based directly on the discretization of the weak formulation.

FEM discretization

Find $(\vec{U}^{m+1}, \mathbf{P}^{m+1}, \kappa^{m+1}, \delta \vec{X}^{m+1})$ such that

$$\begin{aligned}
 & \left(\rho^m \frac{\vec{U}^{m+1} - \vec{l}_2^m \vec{U}^m}{\tau^m}, \vec{\xi} \right) \\
 & + \frac{1}{2} \left(\rho^m, [(\vec{l}_2^m \vec{U}^m \cdot \nabla) \vec{U}^{m+1}] \cdot \vec{\xi} - [(\vec{l}_2^m \vec{U}^m \cdot \nabla) \vec{\xi}] \cdot \vec{U}^{m+1} \right) \\
 & + 2 \left(\mu^m \underline{\underline{D}}(\vec{U}^{m+1}), \underline{\underline{D}}(\vec{\xi}) \right) - \left(\mathbf{P}^{m+1}, \nabla \cdot \vec{\xi} \right) - \gamma \left\langle \kappa^{m+1} \vec{\nu}^m, \vec{\xi} \right\rangle_{\Gamma^m} \\
 & = \left(\rho^m \vec{f}_1^{m+1} + \vec{f}_2^{m+1}, \vec{\xi} \right) \quad \forall \vec{\xi} \in \mathbb{U}^m, \\
 & \left(\nabla \cdot \vec{U}^{m+1}, \varphi \right) = 0 \quad \forall \varphi \in \hat{\mathbb{P}}^m, \\
 & \left\langle \frac{\vec{X}^{m+1} - \text{id}}{\tau}, \chi \vec{\nu}^m \right\rangle_{\Gamma^m}^h - \left\langle \vec{U}^{m+1}, \chi \vec{\nu}^m \right\rangle_{\Gamma^m} = 0 \quad \forall \chi \in W(\Gamma^m), \\
 & \left\langle \kappa^{m+1} \vec{\nu}^m, \vec{\eta} \right\rangle_{\Gamma^m}^h + \left\langle \nabla_s \vec{X}^{m+1}, \nabla_s \vec{\eta} \right\rangle_{\Gamma^m} = 0 \quad \forall \vec{\eta} \in \underline{V}(\Gamma^m)
 \end{aligned}$$

and set $\Gamma^{m+1} = \vec{X}^{m+1}(\Gamma^m)$.

Scheme properties

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- ▶ Pressure jumps at the interface are captured accurately for standard pressure finite element spaces without the need for XFEM extensions.
- ▶ The surface mesh quality is maintained and for the semidiscrete scheme an equidistribution property can be shown in 2d.

Solution method

Find $(\vec{U}^{m+1}, \mathbf{P}^{m+1}, \kappa^{m+1}, \delta \vec{X}^{m+1})$ such that

$$\begin{pmatrix} \vec{B}_\Omega & \vec{C}_\Omega & -\gamma \vec{N}_{\Gamma,\Omega} & 0 \\ \vec{C}_\Omega^T & 0 & 0 & 0 \\ \vec{N}_{\Gamma,\Omega}^T & 0 & 0 & -\frac{1}{\tau_m} \vec{N}_\Gamma^T \\ 0 & 0 & \vec{N}_\Gamma & \vec{A}_\Gamma \end{pmatrix} \begin{pmatrix} \vec{U}^{m+1} \\ \mathbf{P}^{m+1} \\ \kappa^{m+1} \\ \delta \vec{X}^{m+1} \end{pmatrix} = \begin{pmatrix} \vec{c} \\ 0 \\ 0 \\ -\vec{A}_\Gamma \vec{X}^m \end{pmatrix},$$

where $\vec{X}^{m+1} = \vec{X}^m + \delta \vec{X}^{m+1}$.

Schur complement approach

Let

$$\Xi_{\Gamma} := \begin{pmatrix} 0 & -\frac{1}{\tau_m} \vec{N}_{\Gamma}^T \\ \vec{N}_{\Gamma} & \vec{A}_{\Gamma} \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} \vec{B}_{\Omega} + \gamma (\vec{N}_{\Gamma, \Omega} \ 0) \Xi_{\Gamma}^{-1} \begin{pmatrix} \vec{N}_{\Gamma, \Omega}^T \\ 0 \end{pmatrix} & \vec{C}_{\Omega} \\ \vec{C}_{\Omega}^T & 0 \end{pmatrix} \begin{pmatrix} \vec{U}^{m+1} \\ \vec{P}^{m+1} \end{pmatrix} = \\ \begin{pmatrix} \vec{c} - \gamma (\vec{N}_{\Gamma, \Omega} \ 0) \Xi_{\Gamma}^{-1} \begin{pmatrix} 0 \\ \vec{A}_{\Gamma} \vec{X}^m \end{pmatrix} \\ 0 \end{pmatrix}$$

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$$\begin{pmatrix} \kappa^{m+1} \\ \delta \vec{X}^{m+1} \end{pmatrix} = \Xi_{\Gamma}^{-1} \begin{pmatrix} -\vec{N}_{\Gamma, \Omega}^T \vec{U}^{m+1} \\ -\vec{A}_{\Gamma} \vec{X}^m \end{pmatrix}.$$

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As preconditioner we use the matrix

$$\mathcal{P} = \begin{pmatrix} \vec{B}_{\Omega} & \vec{C}_{\Omega} \\ 0 & -M_{\Omega} \end{pmatrix}.$$

Mesh smoothing and remeshing

- Smoothing: find a displacement $\vec{\psi} \in [H^1(\Omega)]^d$ such that

$$\begin{aligned}\nabla \cdot \underline{\underline{S}} &= \vec{0} && \text{in } \Omega_{\pm}^m, \\ \vec{\psi} &= \delta \vec{X} && \text{on } \Gamma^m, \\ \vec{\psi} \cdot \vec{n} &= 0 && \text{on } \partial\Omega,\end{aligned}$$

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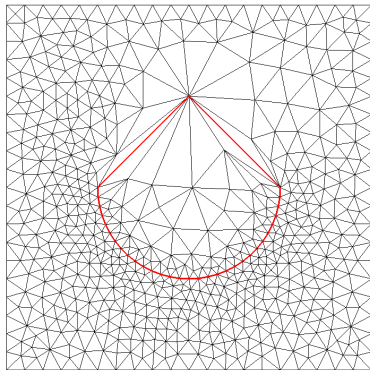
- Remeshing: perform remeshing when

$$\frac{\max_{o \in \mathcal{T}^{m+1}} (\mathcal{H}^d(o))}{\min_{o \in \mathcal{T}^{m+1}} (\mathcal{H}^d(o))} \geq C_r,$$

where $C_r \geq 1$ is a fixed constant.

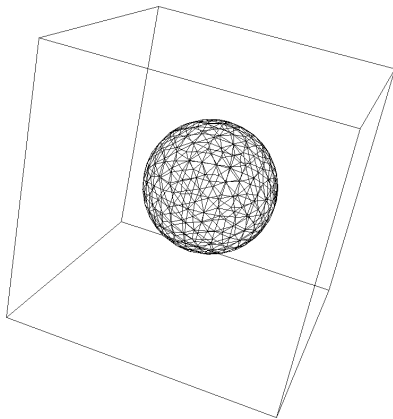
Equidistribution property experiment

$$\rho_{\pm} = 0, \quad \mu_{\pm} = 1, \quad \gamma = 1$$



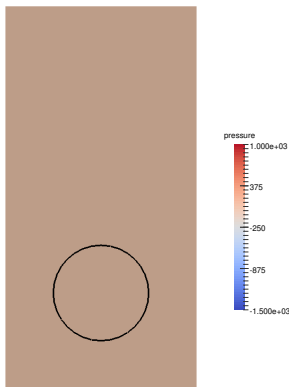
Shear flow experiment

$$\rho_{\pm} = 0, \quad \mu_{\pm} = 1, \quad \gamma = 3, \quad \vec{g}(\vec{x}) = x_3 \vec{e}_1 \quad \text{on } \partial\Omega$$



Rising bubble experiment

$$\rho_+ = 10^3, \rho_- = 10^2, \mu_+ = 10, \mu_- = 1, \gamma = 24.5, \vec{f} = -0.98\vec{e}_2$$



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- ▶ In order to reduce the computational cost of the interpolation, bulk elements are sorted and traversed following a continuous path; barycentric coordinates are used to locate the interpolation points in the previous bulk mesh.
- ▶ On the other hand, to improve the accuracy, the interpolation should be performed only after a remeshing.

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- ▶ This prescribed flow needs to be accounted for in the approximation of the momentum equation; in practice, it adds a convective-type term to the momentum balance.
- ▶ The movement of the bulk mesh is incorporated in the finite element approximation therefore it avoids the repeated interpolation onto the bulk mesh.

Outlook

- ▶ Test other solvers/preconditioners to solve the algebraic linear system more efficiently.
- ▶ Include surface active agents (surfactants) to the model.
- ▶ Use adaptive meshes to increase the accuracy of the scheme.
- ▶ Test higher order spaces to approximate the displacement of the interface.

References

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