Fitted Finite element Discretization of Two-Phase Navier-Stokes Flow

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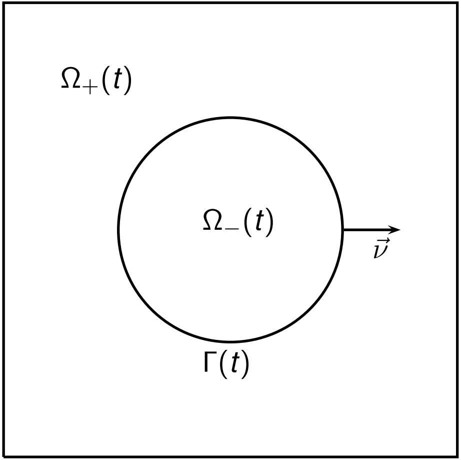
Problem setting

Domain Ω in the 2-dimensional case.



Governing equations

- Bulk equations

$$\begin{aligned} \rho\,(\vec{u}_t + (\vec{u}\,.\,\nabla)\,\vec{u}) - \nabla\,.\,\underline{\underline{\sigma}} = \vec{f} = \rho\,\vec{f}_1 + \vec{f}_2 \qquad & \text{in } \Omega_\pm(t)\,, \\ \nabla\,.\,\vec{u} = 0 \qquad & \text{in } \Omega_\pm(t)\,, \end{aligned}$$

where

$$\underline{\underline{\sigma}} = \mu\,(\nabla\,\vec{u} + (\nabla\,\vec{u})^T) - p\,\underline{\underline{\mathrm{id}}} \; = 2\,\mu\,\underline{\underline{D}}(\vec{u}) - p\,\underline{\underline{\mathrm{id}}}\,.$$

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- Interface equations

$$[\vec{u}]_-^+ = \vec{0} \qquad \text{on } \Gamma(t)\,,$$
$$[\underline{\underline{\sigma}}\,\vec{\nu}]_-^+ = -\gamma\,\varkappa\,\vec{\nu} \qquad \text{on } \Gamma(t)\,,$$
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- To close the system, we prescribe the initial data $\Gamma(0) = \Gamma_0$, the initial velocity $\vec{u}_0$ and some boundary condition for $\vec{u}$ on $\partial\Omega$.

Interface treatment

- $\Gamma(t)$ is a sufficiently smooth evolving hypersurface without boundary that is parameterized by $\vec{x}(\cdot, t) : \Upsilon \to \mathbb{R}^d$, therefore
$$\Gamma(t) = \vec{x}(\Upsilon, t)\,,$$
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- It holds that
$$\Delta_s\, \vec{\mathrm{id}} = \varkappa\, \vec{\nu} \qquad \text{on } \Gamma(t)\,,$$
where $\Delta_s = \nabla_s\,.\,\nabla_s$ is the Laplace-Beltrami operator on $\Gamma(t)$ with $\nabla_s\,.$ and ∇_s denoting surface divergence and surface gradient on $\Gamma(t)$.

Stokes weak formulation

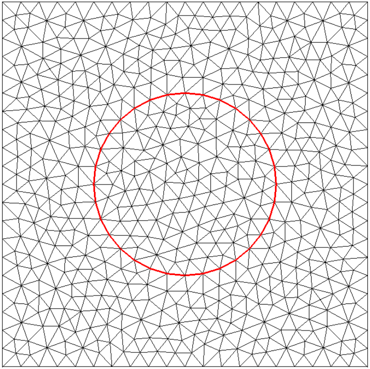
Using the function spaces

$$\mathbb{U} := [H_0^1(\Omega)]^d\,, \qquad \mathbb{P} := L^2(\Omega) \qquad \text{and}$$
$$\widehat{\mathbb{P}} := \{\eta \in \mathbb{P} : \int_\Omega \eta \,\mathrm{d}\mathcal{L}^d = 0\}\,,$$

the Stokes weak formulation is

$$2\left(\mu\,\underline{\underline{D}}(\vec{u}), \underline{\underline{D}}(\vec{\xi})\right) - \left(p, \nabla\,.\,\vec{\xi}\right) - \gamma\,\left\langle \varkappa\,\vec{\nu}, \vec{\xi}\right\rangle_{\Gamma(t)} = \left(\vec{f}, \vec{\xi}\right) \quad \forall\,\vec{\xi} \in \mathbb{U}\,,$$
$$(\nabla\,.\,\vec{u}, \varphi) = 0 \quad \forall\,\varphi \in \widehat{\mathbb{P}}\,,$$
$$\left\langle \vec{\mathcal{V}} - \vec{u}, \chi\,\vec{\nu}\right\rangle_{\Gamma(t)} = 0 \quad \forall\,\chi \in H^1(\Gamma(t))\,,$$
$$\langle \varkappa\,\vec{\nu}, \vec{\eta}\rangle_{\Gamma(t)} + \left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\eta}\right\rangle_{\Gamma(t)} = 0 \quad \forall\,\vec{\eta} \in [H^1(\Gamma(t))]^d\,.$$

Fitted approach



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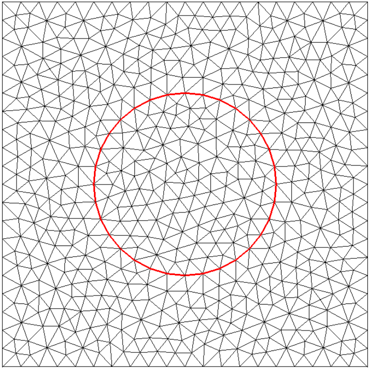

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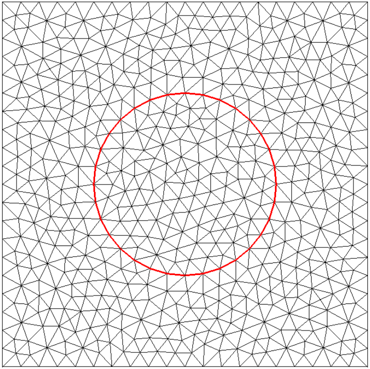

- Pros: naturally captured discontinuity jumps in ρ, μ, p and no need to interpolate bulk quantities over interface.
- Cons: possible bulk mesh distortion and difficult bulk mesh adaptation.

FEM discretization

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- Let $(\mathbb{U}^m, \mathbb{P}^m)$ be an LBB-stable pair of velocity/pressure finite element spaces in the bulk, e.g. P2-P0 (only in 2d) or P2-P1, with $\mathbb{P}^m$ possibly extended by an additional basis function.

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- Then our finite element approximation is based directly on the discretization of the weak formulation.

Scheme properties

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- Pressure jumps at the interface are captured accurately for standard pressure finite element spaces without the need for XFEM extensions.
- The surface mesh quality is maintained and for the semidiscrete scheme an equidistribution property can be shown in 2d.

Solution method

Find $(\vec{U}^{m+1}, P^{m+1}, \kappa^{m+1}, \delta\vec{X}^{m+1})$ such that

$$\begin{pmatrix} \vec{B}_\Omega & \vec{C}_\Omega & -\gamma\,\vec{N}_{\Gamma,\Omega} & 0 \\ \vec{C}_\Omega^T & 0 & 0 & 0 \\ \vec{N}_{\Gamma,\Omega}^T & 0 & 0 & -\frac{1}{\tau_m}\,\vec{N}_\Gamma^T \\ 0 & 0 & \vec{N}_\Gamma & \vec{A}_\Gamma \end{pmatrix} \begin{pmatrix} \vec{U}^{m+1} \\ P^{m+1} \\ \kappa^{m+1} \\ \delta\vec{X}^{m+1} \end{pmatrix} = \begin{pmatrix} \vec{c} \\ 0 \\ 0 \\ -\vec{A}_\Gamma\,\vec{X}^m \end{pmatrix},$$

where $\vec{X}^{m+1} = \vec{X}^m + \delta\vec{X}^{m+1}$.

Schur complement approach

Let

$$\Xi_\Gamma := \begin{pmatrix} 0 & -\frac{1}{\tau_m}\vec{N}_\Gamma^T \\ \vec{N}_\Gamma & \vec{A}_\Gamma \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} \vec{B}_\Omega + \gamma\,(\vec{N}_{\Gamma,\Omega}\ 0)\,\Xi_\Gamma^{-1}\,\binom{\vec{N}_{\Gamma,\Omega}^T}{0} & \vec{C}_\Omega \\ \vec{C}_\Omega^T & 0 \end{pmatrix} \begin{pmatrix} \vec{U}^{m+1} \\ P^{m+1} \end{pmatrix} =$$

$$\begin{pmatrix} \vec{c} - \gamma\,(\vec{N}_{\Gamma,\Omega}\ 0)\,\Xi_\Gamma^{-1}\,\binom{0}{\vec{A}_\Gamma\,\vec{X}^m} \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \kappa^{m+1} \\ \delta\vec{X}^{m+1} \end{pmatrix} = \Xi_\Gamma^{-1} \begin{pmatrix} -\vec{N}_{\Gamma,\Omega}^T\,\vec{U}^{m+1} \\ -\vec{A}_\Gamma\,\vec{X}^m \end{pmatrix}.$$

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As preconditioner we use the matrix

$$\mathcal{P} = \begin{pmatrix} \vec{B}_\Omega & \vec{C}_\Omega \\ 0 & -M_\Omega \end{pmatrix}.$$

Mesh smoothing and remeshing

- Smoothing: find a displacement $\vec{\psi} \in [H^1(\Omega)]^d$ such that

$$\begin{aligned} \nabla \,.\, \underline{\underline{S}} &= \vec{0} && \text{in } \Omega_\pm^m\,, \\ \vec{\psi} &= \delta \vec{X} && \text{on } \Gamma^m\,, \\ \vec{\psi} \,.\, \vec{\mathrm{n}} &= 0 && \text{on } \partial\Omega\,, \end{aligned}$$

where $\underline{\underline{S}} = 2\,\underline{\underline{D}}(\vec{\psi}) + (\nabla\,.\vec{\psi})\,\underline{\underline{\mathrm{id}}}$ is the stress tensor and where $\vec{\mathrm{n}}$ is the outer unit normal to Ω on $\partial\Omega$.

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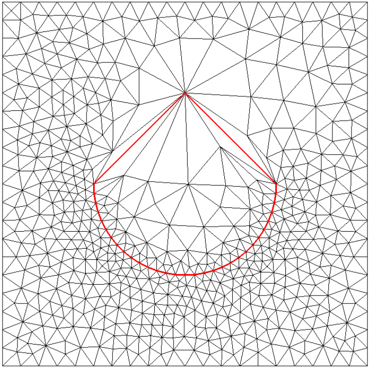
- Remeshing: perform remeshing when

$$\frac{\max_{o\in\mathcal{T}^{m+1}}(\mathcal{H}^d(o))}{\min_{o\in\mathcal{T}^{m+1}}(\mathcal{H}^d(o))} \geq C_r\,,$$

  where $C_r \geq 1$ is a fixed constant.

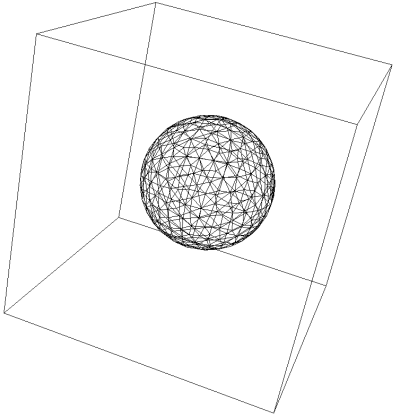
Equidistribution property experiment

$$\rho_\pm = 0\,, \quad \mu_\pm = 1\,, \quad \gamma = 1$$



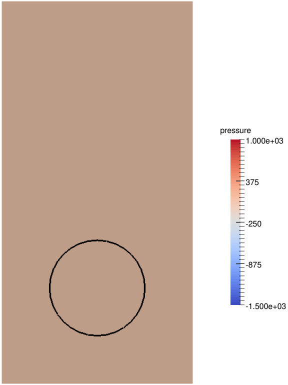
Shear flow experiment

$$\rho_\pm = 0\,, \quad \mu_\pm = 1\,, \quad \gamma = 3\,, \quad \vec{g}(\vec{x}) = x_3\,\vec{e}_1 \quad \text{on } \partial\Omega$$



Rising bubble experiment

$$\rho_+ = 10^3\,, \rho_- = 10^2\,, \mu_+ = 10\,, \mu_- = 1\,, \gamma = 24.5\,, \vec{f} = -0.98\vec{e}_2$$



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- In order to reduce the computational cost of the interpolation, bulk elements are sorted and traversed following a continuous path; barycentric coordinates are used to locate the interpolation points in the previous bulk mesh.
- On the other hand, to improve the accuracy, the interpolation should be performed only after a remeshing.

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- This prescribed flow needs to be accounted for in the approximation of the momentum equation; in practice, it adds a convective-type term to the momentum balance.
- The movement of the bulk mesh is incorporated in the finite element approximation therefore it avoids the repeated interpolation onto the bulk mesh.

Outlook

- Test other solvers/preconditioners to solve the algebraic linear system more efficiently.
- Include surface active agents (surfactants) to the model.
- Use adaptive meshes to increase the accuracy of the scheme.
- Test higher order spaces to approximate the displacement of the interface.

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