Fitted ALE scheme for Two-Phase Navier–Stokes Flow

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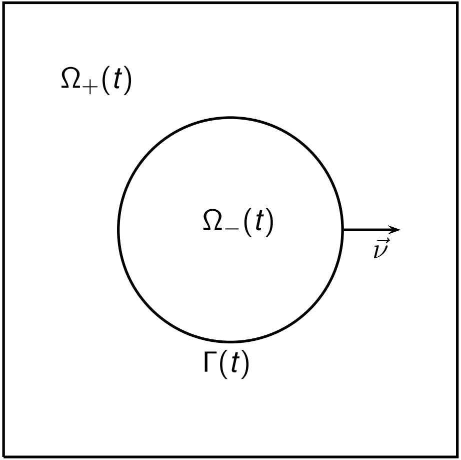
8th September 2016

Problem setting

Domain Ω in the 2-dimensional case.



Governing equations

- Bulk equations

$$\rho\,(\vec{u}_t + (\vec{u}\,.\,\nabla)\,\vec{u}) - \nabla\,.\,\underline{\underline{\sigma}} = \vec{f} = \rho\,\vec{f}_1 + \vec{f}_2 \qquad \text{in } \Omega_\pm(t)\,,$$
$$\nabla\,.\,\vec{u} = 0 \qquad \text{in } \Omega_\pm(t)\,,$$

where

$$\underline{\underline{\sigma}} = \mu\,(\nabla\,\vec{u} + (\nabla\,\vec{u})^T) - p\,\underline{\underline{\mathrm{id}}} = 2\,\mu\,\underline{\underline{D}}(\vec{u}) - p\,\underline{\underline{\mathrm{id}}}\,.$$

- Interface equations

$$[\vec{u}]_-^+ = \vec{0} \qquad \text{on } \Gamma(t)\,,$$
$$[\underline{\underline{\sigma}}\,\vec{\nu}]_-^+ = -\gamma\,\varkappa\,\vec{\nu} \qquad \text{on } \Gamma(t)\,,$$
$$\vec{\mathcal{V}}\,.\,\vec{\nu} = \vec{u}\,.\,\vec{\nu} \qquad \text{on } \Gamma(t)\,.$$

- To close the system, we prescribe the initial data $\Gamma(0) = \Gamma_0$, the initial velocity $\vec{u}_0$ and some boundary condition for $\vec{u}$ on $\partial\Omega$.

Interface treatment

- $\Gamma(t)$ is a sufficiently smooth evolving hypersurface without boundary that is parameterized by $\vec{x}(\cdot, t) : \Upsilon \to \mathbb{R}^d$, therefore
$$\Gamma(t) = \vec{x}(\Upsilon, t)\,,$$
where $\Upsilon \subset \mathbb{R}^d$ is a given reference manifold.
- It holds that
$$\Delta_s\, \vec{\mathrm{id}} = \varkappa\, \vec{\nu} \qquad \text{on } \Gamma(t)\,,$$
where $\Delta_s = \nabla_s\,.\,\nabla_s$ is the Laplace-Beltrami operator on $\Gamma(t)$ with $\nabla_s\,.$ and ∇_s denoting surface divergence and surface gradient on $\Gamma(t)$.

Arbitrary Lagrangian Eulerian approach

- In the ALE approach, a prescribed flow drives the movement of the bulk mesh vertices.
- Let $h : \Omega_{\pm}(t) \times [0, T] \to \mathbb{R}$ be a function defined on the Eulerian frame, the corresponding function on the ALE frame $\hat{h}$ is defined as

$$\hat{h} : \Upsilon_{\Omega_{\pm}} \times [0, T] \to \mathbb{R}, \qquad \hat{h}(\vec{q}, t) = h(\vec{x}(\vec{q}, t), t)$$

  and it holds

$$h_t = h_t|_{\Upsilon_{\Omega_{\pm}}} - \vec{\mathcal{W}} \cdot \nabla h.$$

- The movement of the bulk mesh is incorporated in the finite element approximation therefore it avoids the repeated interpolation of the velocity onto the bulk mesh.

Weak formulation

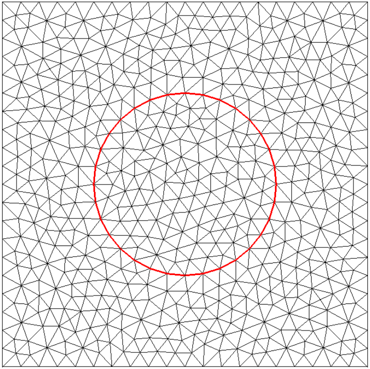
Using the function spaces

$$\mathbb{U} := [H^1_0(\Omega)]^d\,, \qquad \mathbb{P} := L^2(\Omega) \qquad \text{and}$$
$$\widehat{\mathbb{P}} := \{\eta \in \mathbb{P} : \int_\Omega \eta \,\mathrm{d}\mathcal{L}^d = 0\}\,,$$

the Navier–Stokes weak formulation is

$$\left(\rho\ \vec{u}_t|_{\Upsilon_{\Omega_\pm}}, \vec{\xi}\right) + \left(\rho\,(\vec{u} - \vec{\mathcal{W}})\,.\,\nabla\,\vec{u}\,,\,\vec{\xi}\right) + 2\left(\mu\,\underline{\underline{D}}(\vec{u}), \underline{\underline{D}}(\vec{\xi})\right)$$
$$-\left(p, \nabla\,.\,\vec{\xi}\right) - \gamma\left\langle \varkappa\,\vec{\nu}, \vec{\xi}\right\rangle_{\Gamma(t)} = \left(\vec{f}, \vec{\xi}\right) \quad \forall\,\vec{\xi} \in \mathbb{U}\,,$$
$$(\nabla\,.\,\vec{u}, \varphi) = 0 \quad \forall\,\varphi \in \widehat{\mathbb{P}}\,,$$
$$\left\langle \vec{\mathcal{V}} - \vec{u}, \chi\,\vec{\nu}\right\rangle_{\Gamma(t)} = 0 \quad \forall\,\chi \in H^1(\Gamma(t))\,,$$
$$\langle \varkappa\,\vec{\nu}, \vec{\eta}\rangle_{\Gamma(t)} + \left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\eta}\right\rangle_{\Gamma(t)} = 0 \quad \forall\,\vec{\eta} \in [H^1(\Gamma(t))]^d$$

Fitted approach



- Pros: naturally captured discontinuity jumps in ρ, μ, p and no need to interpolate bulk quantities over interface.
- Cons: possible bulk mesh distortion and difficult bulk mesh adaptation.

Scheme properties

- Simple stationary solutions are captured exactly, which means that no spurious velocities appear.
- The scheme conserves the volume of the two phases.
- Pressure jumps at the interface are captured accurately for standard pressure finite element spaces without the need for XFEM extensions.
- The surface mesh quality is maintained and for the semidiscrete scheme an equidistribution property can be shown in 2d.

Mesh smoothing and remeshing

- Smoothing: find a displacement $\vec{\psi} \in [H^1(\Omega)]^d$ such that

$$\begin{aligned} \nabla \,.\, \underline{\underline{S}} &= \vec{0} && \text{in } \Omega_\pm^m\,, \\ \vec{\psi} &= \delta\vec{X} && \text{on } \Gamma^m\,, \\ \vec{\psi}\,.\,\vec{\mathrm{n}} &= 0 && \text{on } \partial\Omega\,, \end{aligned}$$

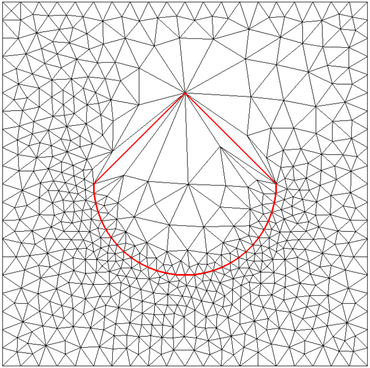
  where $\underline{\underline{S}} = 2\,\underline{\underline{D}}(\vec{\psi}) + (\nabla\,.\vec{\psi})\,\underline{\underline{\mathrm{id}}}$ is the stress tensor and where $\vec{\mathrm{n}}$ is the outer unit normal to Ω on $\partial\Omega$.
- Remeshing: perform remeshing when

$$\frac{\max_{o\in\mathcal{T}^{m+1}}(\mathcal{H}^d(o))}{\min_{o\in\mathcal{T}^{m+1}}(\mathcal{H}^d(o))} \geq C_r\,,$$

  where $C_r \geq 1$ is a fixed constant.

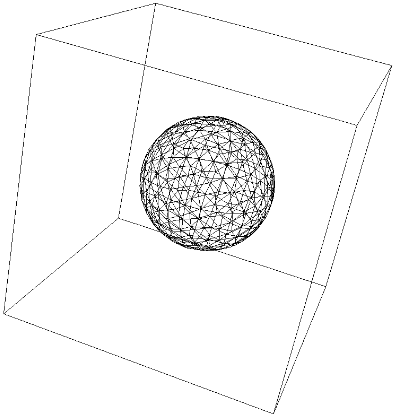
Equidistribution property experiment

$$\rho_{\pm} = 0\,, \quad \mu_{\pm} = 1\,, \quad \gamma = 1$$



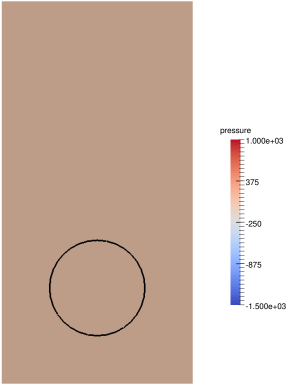
Shear flow experiment

$$\rho_\pm = 0\,, \quad \mu_\pm = 1\,, \quad \gamma = 3\,, \quad \vec{g}(\vec{x}) = x_3\,\vec{e}_1 \quad \text{on } \partial\Omega$$



Rising bubble experiment

$$\rho_+ = 10^3\,,\rho_- = 10^2\,,\mu_+ = 10\,,\mu_- = 1\,,\gamma = 24.5\,,\vec{f} = -0.98\vec{e}_2$$



Outlook

- Test other solvers/preconditioners to solve the algebraic linear system more efficiently.
- Include surface active agents (surfactants) to the model.
- Use adaptive meshes to increase the accuracy of the scheme.
- Test higher order spaces to approximate the displacement of the interface.

References

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