Fair Allocation with Special Externalities

Shaily Mishra, Manisha Padala, and Sujit Gujar

Machine Learning Lab, International Institute of Information Technology, Hyderabad
{shaily.mishra,manisha.padala}@research.iiit.ac.in,
sujit.gujar@iiit.ac.in

Abstract. Most of the existing algorithms for fair division do not consider externalities. Under externalities, the utility of an agent depends not only on its allocation but also on other agents' allocation. An agent has a positive (negative) value for the assigned goods (chores). This work studies a special case of externality which we refer to as 2-D. In 2-D, an agent receives a positive or negative value for unassigned items independent of who receives them. We propose a simple valuation transformation and show that we can adapt existing algorithms using it to retain some of the fairness and efficiency notions in 2-D. However, proportionality doesn't extend in 2-D. We redefine PROP and its relaxation and show that we can adapt existing algorithms. Further, we prove that maximin share (MMS) may not have any multiplicative approximation in this setting. Studying this domain is a stepping stone towards full externalities where ensuring fairness is much more challenging.

Keywords: Resource Allocation · Fairness · Externalities

1 Introduction

We consider the problem of allocating m *indivisible* items fairly among n agents who report their valuations for the items. These scenarios often arise in the division of inheritance among family members, divorce settlements and distribution of tasks among workers [12, 33, 38–40]. Economists have proposed many fairness and efficiency notions widely applicable in such real-world settings. Researchers also explore the computational aspects of some widely accepted fairness notions [15, 9, 18, 21, 36]. Such endeavours have led to web-based applications like Spliddit, The Fair Proposals System, Coursematch, etc. However, most approaches do not consider agents with *externalities*, which we believe is restrictive.

In the absence of externality, the utility corresponding to an unallocated item is zero. Externality implies that the agent's utility depends not only on their bundle but also on the bundles allocated to other agents. Such a scenario is relatively common, mainly in allocating necessary commodities. For example, the COVID-19 pandemic resulted in a sudden and steep requirement for life-supporting resources like hospital beds, ventilators, and vaccines. There has been a heavy disparity in handling resources across the globe. Even though there was a decrease in GDP worldwide, low-income countries suffered more than high-income countries. We can categorize externality into positive and negative; i.e., if it affects the agent positively, we refer to it as a positive externality and vice versa. Getting a vaccination affects an agent positively. The agent values it positively, possibly less, even if others get vaccinated instead of it. However, not receiving a ventilator results in negative utility for the patient and family. While there has been an increase in demand for pharmaceuticals, we see a steep decrease in travel. Such a complex valuation structure is modeled via externalities.

Generally with externalities, the utility of not receiving an item depends on which other agent receives it. That is, each agent's valuation for an item is an n-dimensional vector. The j^{th} component corresponds to the value an agent obtains if the item is allocated to agent j. In this work, we consider a special case of externalities in which the agents incur a cost/benefit for not receiving an item. Yet, the cost/benefit is *independent* of which other agent receives the item. This setting is referred to as 2-D, i.e., value v for receiving an item and v' otherwise. When there are only two agents, the 2-D domain is equivalent to the domain with general externalities. We refer to the agent valuations without externalities as 1-D. For the 2-D domain, we consider both goods/chores with positive/negative externality for the following fairness notions.

Fairness Notions. Envy-freeness (EF) is the most common fairness notion. It ensures that no agent has higher utility for other agent's allocation [20]. Consider 1-D setting with two agents - $\{1, 2\}$ and two goods - $\{g_1, g_2\}$; agent 1 values g_1 at 6 and g_2 at 5, while agent 2 values g_1 at 5 and g_2 at 6. Allocating g_1 and g_2 to agent 1 and 2, respectively, is EF. However, if agent 1 receives a utility of -1 and -100 for not receiving g_1 and g_2. And agent 2 receives a utility of -100 and -1 for not receiving g_1 and g_2; this allocation is no longer EF.

Externalities introduce complexity, so much that the definition of proportionality cannot be adapted to the 2-D domain. Proportionality (PROP) ensures that every agent receives at least $1/n$ of its complete bundle value [39]. In the above example, each agent should receive goods worth at least $11/2$. Guaranteeing this amount is impossible in 2-D, as it does not consider the dis-utility of not receiving goods. Moreover, it is known that EF implies PROP in the presence of additive valuations. However, in the case of 2-D, it need not be true, i.e., assigning g_2 to agent 1 and g_1 to agent 2 is EF but not PROP.

We consider a relaxation of PROP, the maximin share (MMS) allocation. Imagine asking an agent to divide the items into n bundles and take the minimum valued bundle. The agent would divide the bundles to maximize the minimum utility, i.e., the MMS share of the agent. An MMS allocation guarantees every agent its MMS share. Even for 1-D valuations, MMS allocation may not exist; hence researchers find multiplicative approximation α-MMS. An α-MMS allocation guarantees at least α fraction of MMS to every agent. [25] provides an algorithm that guarantees $3/4 + 1/12n$-MMS for goods and authors in [27] guarantees $11/9$-MMS for chores. In contrast, we prove that for 2-D valuation, it is impossible to guarantee multiplicative approximation to MMS. Thus, in order to guarantee existence results, we propose relaxed multiplicative approximation and also explore additive approximations of MMS guarantees.

In general, it is challenging to ensure fairness in the settings with full externality, hence the special case of 2-D proves promising. Moreover, in real-world applications, the 2-D valuations helps model various situations (e.g., COVID-19 resource allocation mentioned above). Studying 2-D domain is especially significant for α-MMS. We prove that there cannot exist any multiplicative approximation for MMS in 2-D. Therefore, we define Shifted α-MMS that always exists in 2-D. In summary our approach and contributions are as follows,

Our Approach. There is extensive literature available for fair allocations, and we primarily focus on leveraging existing algorithms to 2-D. We demonstrate in Section 3 that existing algorithms cannot directly be applied to 2-D. Towards guaranteeing fairness notion in 2-D, we propose a property preserving transformation $\mathfrak{T}$ that converts 2-D valuations to 1-D; i.e., an allocation that satisfies a property in 2-D also satisfies it in transformed 1-D and vice-versa.

Contributions.

1. We demonstrate in Section 3 that studying fair allocation with externalities is non-trivial and propose $\mathfrak{T}$ to retain fairness notions such as EF, MMS, and its additive relaxations and efficiency notions such as MUW and PO (Theorem 1). Thus, we can adapt the existing algorithms for the same.
2. We introduce PROP-E for general valuations for full externalities (Section 2) and derive relation with existing PROP extensions (Section 4).
3. We prove that α-MMS may not exist in 2-D (Theorem 2). We propose Shifted α-MMS, a novel way of approximating MMS in 2-D (Section 5.3).

Related Work

While fair resource division has an extremely rich literature, externalities is less explored. Velez [41] extended EF in externalities. [13] generalized PROP and EF for divisible goods with positive externalities. Seddighin et al. [37] proposed average-share, an extension of PROP, and studied MMS for goods with positive externalities. Authors in [6] explored EF1/EFX for the specific setting of two and three agents and provided PROP extension. For two agents, their setting is equivalent to 2-D, hence existing algorithms [15, 35, 4] suffice. Beyond two agents, the setting is more general and they proved the non-existence of EFX for three agents. In contrast, EFX always exists for three agents in our setting.

Envy-freeness up to one item (EF1) [14, 32] and Envy-freeness up to any item(EFX) [15] are prominent relaxation of EF. EF may not exist for indivisible items. We consider two prominent relaxations of EF, Envy-freeness up to one item (EF1) [14, 32] and Envy-freeness up to any item(EFX) [15]. We have poly-time algorithms to find EF1 in general monotone valuations for goods [32] and chores [11]. For additive valuations, EF1 can be found using Round Robin [15] in goods or chores, and Double Round Robin [2] in combination. [35] present an algorithm to find EFX allocation under identical general valuations for goods. [16] proved that an EFX allocation exists for three agents. Researchers have also studied fair division in presence of strategic agents, i.e., designing truthful mechanisms [8, 10, 34]. A great deal of research has been done on mechanism design [23, 22]. PROP1 and PROPX are popular relaxation of PROP. For additive valuations, EF1 implies PROP1, and EFX implies PROPX. Unfortunately, in paper [4], the authors showed the PROPX for goods may not always exists. [31] explored (weighted) PROPX and showed it exists in polynomial time. MMS do not always exist [29, 36]. The papers [36, 1, 7, 24] showed that 2/3-MMS for goods always exists. Paper [26, 25] showed that 3/4-MMS for goods always exists. Authors in [25] provides an algorithm that guarantees $3/4 + 1/12n$-MMS for goods. Authors in [5] presented a polynomial-time algorithm for 2-MMS for chores. . The algorithm presented in [7] gives 4/3-MMS for chores. Authors in [27] showed that 11/9-MMS for chores always exists. [28] explored $\alpha-$MMS for a combination of goods and chores. In [15] showed that MNW is EF1 and PO for indivisible goods and [9] gave a pseudo-polynomial time algorithm. [2] presented algorithm to find EF1 and PO for two agents. [4] presented an algorithm to find PROP1 and fPO for combination. [3] proposed a pseudo-polynomial time algorithm for finding utilitarian maximizing among EF1 or PROP1 in goods.

2 Preliminaries

We consider a resource allocation problem $(N, M, \mathcal{V})$ for determining an allocation A of $M = [m]$ indivisible items among $N = [n]$ interested agents, $m, n \in \mathbb{N}$. We only allow complete allocation and no two agents can receive the same item. That is, $A = (A_1, \ldots, A_n)$, s.t., $\forall i, j \in N$, $i \neq j; A_i \cap A_j = \emptyset$ and $\bigcup_i A_i = M$. A_{-i} denotes the set $M \setminus A_i$.

2-D Valuations. The valuation function is denoted by $\mathcal{V} = \{V_1, V_2, \ldots, V_n\}$; $\forall i \in N$, $V_i : 2^M \to \mathbb{R}^2$, $\forall S \subseteq M$, $V_i(S) = (v_i(S), v'_i(S))$, where $v_i(S)$ denotes the value for receiving bundle S and $v'_i(S)$ for not receiving S. The value of an agent i for item k in 2-D is (v_{ik}, v'_{ik}). If k is a good (chore), then $v_{ik} \geq 0$ ($v_{ik} \leq 0$). For positive (negative) externality $v'_{ik} \geq 0$ ($v'_{ik} \leq 0$).

The utility an agent $i \in N$ obtains for a bundle $S \subseteq M$ is, $u_i(S) = v_i(S) + v'_i(M \setminus S)$ Also, $u_i(\emptyset) = 0 + v'_i(M)$ and utilities in 2-D are not normalized[1]. When agents have additive valuations, $u_i(S) = \sum_{k \in S} v_{ik} + \sum_{k \notin S} v'_{ik}$. We assume monotonicity of utility for goods, i.e., $\forall S \subseteq T \subseteq M$, $u_i(S) \leq u_i(T)$ and anti-monotonicity of utility for chores, i.e, $u_i(S) \geq u_i(T)$. We use the term *full externalities* to represent complete externalities, i.e., each agent has n-dimensional vector for its valuation for an item. We next define fairness and efficiency notions.

[1] Utility is normalized when $u_i(\emptyset) = 0, \forall i$

Definition 1 (Envy-free (EF) and relaxations [2, 14, 15, 20, 41]). *For the items (chores or goods) an allocation A that satisfies $\forall i, j \in N$,*[2]

$$u_i(A_i) \geq u_i(A_j) \text{ is EF}$$

$$\left.\begin{array}{l} v_{ik} < 0, u_i(A_i \setminus \{k\}) \geq u_i(A_j); \forall k \in A_i \\ v_{ik} > 0, u_i(A_i) \geq u_i(A_j \setminus \{k\}); \forall k \in A_j \end{array}\right\} \text{ is EFX}$$

$$u_i(A_i \setminus \{k\}) \geq u_i(A_j \setminus \{k\}); \exists k \in \{A_i \cup A_j\} \text{ is EF1}$$

Definition 2 (Proportionality (PROP) [39]). *An allocation A is said to be proportional, if $\forall i \in N$, $u_i(A_i) \geq \frac{1}{n} \cdot u_i(M)$.*

For 2-D, achieving PROP is impossible as discussed in Section 1. To capture proportional under externalities, we introduce *Proportionality with externality* (PROP-E). Informally, while PROP guarantees $1/n$ share of the entire bundle, PROP-E guarantees $1/n$ share of the sum of utilities for all bundles. Note that, PROP-E is not limited to 2-D and applies to a full externalities. Formally,

Definition 3 (Proportionality with externality (PROP-E)). *An allocation A satisfies PROP-E if, $\forall i \in N$,$u_i(A_i) \geq \frac{1}{n} \cdot \sum_{j \in N} u_i(A_j)$*

Definition 4 (PROP-E relaxations). *An allocation A $\forall i, \forall j \in N$, satisfies PROPX-E if it is PROP-E up to any item, i.e.,*

$$\left.\begin{array}{l} v_{ik} > 0, u_i(A_i \cup \{k\}) \geq \frac{1}{n} \sum_{j \in N} u_i(A_j); \forall\ k \in \{M \setminus A_i\} \\ v_{ik} < 0, u_i(A_i \setminus \{k\}) \geq \frac{1}{n} \sum_{j \in N} u_i(A_j); \forall\ k \in A_i \end{array}\right\}$$

Next, A satisfies PROP1-E if it is PROP-E up to an item, i.e.,

$$\left.\begin{array}{l} u_i(A_i \cup \{k\}) \geq \frac{1}{n} \sum_{j \in N} u_i(A_j); \exists\ k \in \{M \setminus A_i\} \text{ or,} \\ u_i(A_i \setminus \{k\}) \geq \frac{1}{n} \sum_{j \in N} u_i(A_j); \exists\ k \in A_i \end{array}\right\}$$

Finally, we state the definition of MMS and its multiplicative approximation.

Definition 5 (Maxmin Share MMS [14]). *An allocation A is said to be MMS if $\forall i \in N, u_i(A_i) \geq \mu_i$, where*

$$\mu_i = \max_{(A_1, A_2, \ldots, A_n) \in \prod_n(M)} \min_{j \in N} u_i(A_j)$$

An allocation A is said to be α-MMS if it guarantees $u_i(A_i) \geq \alpha \cdot \mu_i$ for $\mu_i \geq 0$ and $u_i(A_i) \geq \frac{1}{\alpha} \cdot \mu_i$ when $\mu_i \leq 0$, where $\alpha \in (0, 1]$.

Definition 6 (Pareto-Optimal (PO)). *An allocation A is PO if $\nexists\ A'$ s.t., $\forall i \in N$, $u_i(A'_i) \geq u_i(A_i)$ and $\exists i \in N$, $u_i(A'_i) > u_i(A_i)$.*

We also consider efficiency notions like Maximum Utilitarian Welfare (MUW), that maximizes the sum of agent utilities. Maximum Nash Welfare (MNW) maximizes the product of agent utilities and Maximum Egalitarian Welfare (MEW) maximizes the minimum agent utility.

In the next section, we define a transformation from 2-D to 1-D that plays a major role in adaptation of existing algorithms for ensuring desirable properties.

[2] Beyond 2-D, one must include the concept of swapping bundles in EF [6, 41].

3 Reduction from 2-D to 1-D

We define a transformation $\mathfrak{T} : \mathcal{V} \rightarrow \mathcal{W}$, where $\mathcal{V}$ is the valuations in 2-D, i.e., $\mathcal{V} = \{V_1, V_2, \ldots, V_n\}$ and $\mathcal{W}$ is the valuations in 1-D, i.e., $\mathcal{W} = \{w_1, w_2, \ldots, w_n\}$.

Definition 7 (Transformation $\mathfrak{T}$). *Given a resource allocation problem $(N, M, \mathcal{V})$ we obtain the corresponding 1-D valuations denoted by $\mathcal{W} = \mathfrak{T}(\mathcal{V}(\cdot))$ as follows,*

$$\forall i \in N, w_i(A_i) = \mathfrak{T}(V_i(A_i)) = v_i(A_i) + v_i'(A_{-i}) - v_i'(M) \tag{1}$$

When valuations are additive, we obtain $w_i(A_i) = v_i(A_i) - v_i'(A_i)$. An agent's utility in 2-D is $u_i(A_i)$ and the corresponding utility in 1-D is $w_i(A_i)$.

Lemma 1. *For goods (chores), under monotonicity (anti-monotonicity) of $\mathcal{V}$, $\mathcal{W} = \mathfrak{T}(\mathcal{V}(\cdot))$ is normalized, monotonic (anti-monotonic), and non-negative (negative).*

Proof. We assume monotonicity of utility for goods in 2-D. Therefore, $\forall S \subseteq M, w_i(S)$ is also monotone. And $w_i(\emptyset) = v_i(\emptyset) + v_i'(M) - v_i'(M) = 0$ is normalized. Since $w_i(\cdot)$ is monotone and normalized, it is non negative for goods. Similarly we can prove that $w_i(\cdot)$ is normalized, anti-monotonic and non-negative for chores.

Theorem 1. *An allocation A is $\mathfrak{F}$-Fair and $\mathfrak{E}$-Efficient in $\mathcal{V}$* iff *A is $\mathfrak{F}$-Fair and $\mathfrak{E}$-Efficient in the transformed 1-D, $\mathcal{W}$, where $\mathfrak{F} \in \{$EF, EF1, EFX, PROP-E, PROP1-E, PROPX-E, MMS$\}$ and $\mathfrak{E} \in \{$PO, MUW$\}$.*

Proof Sketch. We first consider $\mathfrak{F} =$ EF. Let allocation A be EF in $\mathcal{W}$ then,

$$\begin{aligned} \forall i,\ \forall j,\ w_i(A_i) &\geq w_i(A_j) \\ v_i(A_i) + v_i'(A_{-i}) - v_i'(M) &\geq v_i(A_j) + v_i'(A_{-j}) - v_i'(M) \\ u_i(A_i) &\geq u_i(A_j) \end{aligned}$$

We can proof the rest in a similar manner.
From Lemma 1 and Theorem 1, we obtain the following.

Corollary 1. *To determine $\{$EF, EF1, EFX, MMS$\}$ fairness and $\{$PO, MUW$\}$ efficiency, we can apply existing algorithms to the transformed $\mathcal{W} = \mathfrak{T}(V(\cdot))$ for general valuations.*

Existing algorithms cannot be directly applied. Modified *leximin* algorithm gives PROP1 and PO for chores for 3 or 4 agents in [17], but it is not PROP1-E (or PROP1) and PO in 2-D when applied on utilities. The following example demonstrates the same,

Example 1. Consider 3 agents $\{1, 2, 3\}$ and 4 chores $\{c_1, c_2, c_3, c_4\}$ with positive externality. The 2-D valuation profile is as follows, $V_{1c_1} = (-30, 1)$, $V_{1c_2} = (-20, 1)$, $V_{1c_3} = (-30, 1)$, $V_{1c_4} = (-30, 1)$, $V_{3c_1} = (-1, 40)$, $V_{3c_2} = (-1, 40)$, $V_{3c_3} = (-1, 40)$, and $V_{3c_4} = (-1, 40)$. The valuation profile of agent 2 is identical to agent 1. Allocation $\{\emptyset, \emptyset, (c_1, c_2, c_3, c_4)\}$ is leximin allocation, which is not PROP1-E. However, allocation $\{c_3, (c_2, c_4), (c_1)\}$ is leximin on transformed valuations; it is PROP1 and PO in $\mathcal{W}$ and it is PROP1-E and PO in $\mathcal{V}$.

For chores, the authors in [31] showed that any PROPX allocation ensures 2-MMS for symmetric agents. This result also doesn't extend to 2-D. For e.g., consider two agents $\{1,2\}$ having additive identical valuations for six chores $\{c_1, c_2, c_3, c_4,\ c_5, c_6\}$, given as $V_{1_{c1}} = (-9,1)$, $V_{1_{c2}} = (-11,1)$, $V_{1_{c3}} = (-12,1)$, $V_{1_{c4}} = (-13,1)$, $V_{1_{c5}} = (-9,1)$, and $V_{1_{c6}} = (-1,38)$. Allocation $A = \{(c_1, c_2, c_3, c_4, c_5), (c_6)\}$ is PROPX-E, but is not 2-MMS in $\mathcal{V}$. Further, adapting certain fairness or efficiency criteria to 2-D is not straightforward. E.g., MNW cannot be defined in 2-D because agents can have positive or negative utilities. Hence MNW implies EF1 and PO doesn't extend to 2-D. The authors proved that MNW allocation gives at least $\frac{2}{1+\sqrt{4n-3}}$-MMS value to each agent in [15], which doesn't imply for 2-D. Similarly, we show that approximation to MMS, α-MMS, does not exist in the presence of externalities (Section 5).

4 Proportionality in 2-D

We remark that PROP (Def. 2) is too strict in 2-D. As a result, we introduce PROP-E and its additive relaxations in Defs. 3 and 4 for general valuations.

Proposition 1. *For additive 2-D, we can adapt the existing algorithms of PROP and its relaxations to 2-D using $\mathfrak{T}$.*

Proof. In the absence of externalities, for additive valuations, PROP-E is equivalent to PROP. From Theorem 1, we know that $\mathfrak{T}$ retains PROP-E and its relaxations, and hence all existing algorithms of 1-D is applicable using $\mathfrak{T}$.

It is known that EF $\implies$ PROP for sub-additive valuation in 1-D. In the case of PROP-E, $\forall i, j \in N$, $u_i(A_i) \geq u_i(A_j) \implies u_i(A_i) \geq \frac{1}{n} \cdot \sum_{j=1}^{n} u_i(A_j)$.

Corollary 2. *EF $\implies$ PROP-E for arbitrary valuations with full externalities.*

We now compare PROP-E with existing PROP extensions for capturing externalities. We consider two definitions stated in literature from [37] (Average Share) and [6] (General Fair Share). Note that both these definitions are applicable when agents have additive valuations, while PROP-E applies for any general arbitrary valuations. In [6], the authors proved that Average Share $\implies$ General Fair Share, i.e., if an allocation guarantees all agents their average share value, it also guarantees general fair share value. With that, we state the definition of Average Share (in 2-D) and compare it with PROP-E.

Definition 8 (Average Share [37]). *In $\mathcal{V}$, the average value of item k for agent i, denoted by $avg[v_{ik}] = \frac{1}{n} \cdot [v_{ik} + (n-1)v'_{ik}]$. The average share of agent i, $\overline{v_i(M)} = \sum_{k \in M} avg[v_{ik}]$. An allocation A is said to ensure average share if $\forall i, u_i(A_i) \geq \overline{v_i(M)}$.*

Proposition 2. *PROP-E is equivalent to Average Share in 2-D, for additive valuations.*

Proof Sketch. $\forall i \in N$,

$$u_i(A_i) \geq \frac{1}{n} \cdot \sum_{j \in N} u_i(A_j) = \frac{1}{n} \cdot \sum_{j \in N} v_i(A_j) - v'_i(M \setminus A_j)$$
$$= \frac{1}{n} \cdot \sum_{k \in M} v_{ik} - \frac{1}{n} \cdot \sum_{k \in M} (n-1) v'_{ik}$$

Next, we briefly state the relation of EF, PROP-E, and Average Share beyond 2-D and omit the details due to space constraints.

Remark 1. In case of full externality, EF $\not\Longrightarrow$ Average Share [6].

Proposition 3. *Beyond 2-D, PROP-E* $\not\Longrightarrow$ *Average Share and Average Share* $\not\Longrightarrow$ *PROP-E.*

To conclude this section, we state that for the special case of 2-D externalities with additive valuations, we can adapt existing algorithms to 2-D, and further analysis is required for the general setting.

We now provide analysis of MMS for 2-D valuations in the next section.

5 Approximate MMS in 2-D

From Theorem 1, we showed that transformation $\mathfrak{T}$ retains MMS property, i.e., an allocation A guarantees MMS in 1-D *iff* A guarantees MMS in 2-D. We draw attention to the point that,

$$\mu_i = \mu_i^{\mathcal{W}} + v'_i(M) \tag{2}$$

where $\mu_i^{\mathcal{W}}$ and μ_i are the MMS value of agent i in 1-D and 2-D, respectively. [30] proved that MMS allocation may not exist even for additive valuations, but α-MMS always exists in 1-D. The current best approximation results on MMS allocation are $3/4 + 1/(12n)$-MMS for goods [25] and 11/9-MMS for chores [27] for additive valuations. We are interested in finding multiplicative approximation to MMS in 2-D. Note that we only study α-MMS for complete goods or chores in 2-D, as [28] proved the non-existence of α-MMS for combination of goods and chores in 1-D. From Eq. 5 for $\alpha \in (0, 1]$, if μ_i is positive, we consider α-MMS, and if it is negative, then $1/\alpha$-MMS.
We categorize externalities in two ways for better analysis 1) Correlated Externality 2) Inverse Externality. In the correlated setting, we study goods with positive externality and chores with negative externality. And in the inverse externality, we study goods with negative externality and chores with positive externality. Next, we investigate α-MMS guarantees for correlated externality.

5.1 α-MMS for Correlated Externality

Proposition 4. *For correlated externality, if an allocation A is α-MMS in $\mathcal{W}$, A is α-MMS in $\mathcal{V}$, but need not vice versa.*

Proof. **Part-1.** Let A be α-MMS in $\mathcal{W}$,

$$\forall i \in N, w_i(A_i) \geq \alpha\mu_i^{\mathcal{W}} \implies u_i(A_i) - v_i'(M) \geq -\alpha v_i'(M) + \alpha\mu_i \quad \textit{for goods}$$
$$\forall i \in N, w_i(A_i) \geq \frac{1}{\alpha}\mu_i^{\mathcal{W}} \implies u_i(A_i) - v_i'(M) \geq -\frac{1}{\alpha}v_i'(M) + \frac{1}{\alpha}\mu_i \quad \textit{for chores}$$

For goods with positive externalities, $\mu_i > 0$, $\alpha \in (0, 1]$, and $\forall S \subseteq M$, $v_i'(S) \geq 0$. We derive $v_i'(M) \geq \alpha v_i'(M)$, and hence we can say $u_i(A_i) \geq \alpha\mu_i$. For chores with negative externalities, $\mu_i < 0$, $1/\alpha \geq 1$, and $\forall S \subseteq M, v'(S) \leq 0$. Similarly to the previous point, we derive $v'(M) \geq \frac{1}{\alpha}v'(M)$ and thus $u_i(A_i) \geq \frac{1}{\alpha}\mu_i$.
Part-2. We now prove A is α-MMS in $\mathcal{V}$ but not in $\mathcal{W}$.

Example. Consider $N = \{1, 2\}$ both have additive identical valuations for goods $\{g_1, g_2, g_3, g_4, g_5, g_6\}$, $V_{ig_1} = (0.5, 0.1)$, $V_{ig_2} = (0.5, 0.1)$, $V_{ig_3} = (0.3, 0.1)$, $V_{ig_4} = (0.5, 0.1)$, $V_{ig_5} = (0.5, 0.1)$, and $V_{ig_6} = (0.5, 0.1)$. After transformation, we get $\mu_i^{\mathcal{W}} = 1$ and in 2-D $\mu_i = 1.6$. Allocation, $A = \{\{g_1\}, \{g_2, g_3, g_4, g_5, g_6\}\}$ is 1/2-MMS in $\mathcal{V}$, but not in $\mathcal{W}$. Similarly, it is easy to verify the same for chores.

Corollary 3. *We can adapt the existing α-MMS algorithms using $\mathfrak{T}$ for correlated externality for* general valuations.

Corollary 4. *For correlated 2-D externality, we can always obtain* $3/4+1/(12n)$*-MMS for goods and* $11/9$*-MMS for chores for* additive.

5.2 α-MMS for Inverse Externality

Motivated by the example given in [30] for non-existence of MMS allocation for 1-D valuations, we adapted it to construct the following instance in 2-D to prove the impossibility of α-MMS in 2-D. We show that for any $\alpha \in (0, 1]$, an α-MMS or $1/\alpha$-MMS allocation may not exist for inverse externality in this section. We construct an instance V^g such that α-MMS exists in V^g only if MMS allocation exists in $W = \mathfrak{T}(V^g)$. Note that W is exactly the instance of the example in [30]. Hence the contradiction.

Non-existence of α-MMS in Goods. Consider the following example.

Example 2. We consider a problem of allocating 12 goods among three agents, and represent valuation profile as V^g. The valuation profile V^g is equivalent to $10^3 \times V$ given in Table 1. We set $\epsilon_1 = 10^{-4}$ and $\epsilon_2 = 10^{-3}$. We transform these valuations in 1-D using $\mathfrak{T}$, and the valuation profile $\mathfrak{T}(V^g)$ is the same as the instance in [30] that proves the non-existence of MMS for goods. Note that $\forall i, v_i'(M) = -4055000 + 10^3\epsilon_1$ The MMS value of every agent in $\mathfrak{T}(V^g)$ is 4055000 and from Eq. 2, the MMS value of every agent in V^g is $10^3\epsilon_1$.

Recall that $\mathfrak{T}$ retains MMS property (Theorem 1) and thus we can say that MMS allocation doesn't exist in V^g.

Lemma 2. *There is no α-MMS allocation for the valuation profile V^g of Example 2 for any $\alpha \in [0, 1]$.*

Table 1. Additive 2-D Valuation Profile (V)

Item	Agent 1 (v_1, v_1')	Agent 2 (v_2, v_2')	Agent 3 (v_3, v_3')
k_1	($3\epsilon_2$, -1017+$3\epsilon_1$-$3\epsilon_2$)	($3\epsilon_2$, -1017+$3\epsilon_1$-$3\epsilon_2$)	($3\epsilon_2$, -1017+$3\epsilon_1$-$3\epsilon_2$)
k_2	($2\epsilon_1$, -1025+$2\epsilon_1$+ϵ_2)	($2\epsilon_1$, -1025+$2\epsilon_1$+ϵ_2)	($1025 - \epsilon_1$, -ϵ_1)
k_3	($2\epsilon_1$, -1012+$2\epsilon_1$+ϵ_2)	($1012 - \epsilon_1$, -ϵ_1)	($2\epsilon_1$, -1012+$2\epsilon_1$+ϵ_2)
k_4	($2\epsilon_1$, -1001+$2\epsilon_1$+ϵ_2)	($1001 - \epsilon_1$, -ϵ_1)	($1001 - \epsilon_1$, -ϵ_1)
k_5	($1002 - \epsilon_1$, -ϵ_1)	($2\epsilon_1$, -1002+$2\epsilon_1$+ϵ_2)	($1002 - \epsilon_1$, -ϵ_1)
k_6	($1022 - \epsilon_1$, -ϵ_1)	($1022 - \epsilon_1$, -ϵ_1)	($1022 - \epsilon_1$, -ϵ_1)
k_7	($1003 - \epsilon_1$, -ϵ_1)	($1003 - \epsilon_1$, -ϵ_1)	($2\epsilon_1$, -1003+$2\epsilon_1$+ϵ_2)
k_8	($1028 - \epsilon_1$, -ϵ_1)	($1028 - \epsilon_1$, -ϵ_1)	($1028 - \epsilon_1$, -ϵ_1)
k_9	($1011 - \epsilon_1$, -ϵ_1)	($2\epsilon_1$, -1011+$2\epsilon_1$+ϵ_2)	($1011 - \epsilon_1$, -ϵ_1)
k_{10}	($1000 - \epsilon_1$, -ϵ_1)	($1000 - \epsilon_1$, -ϵ_1)	($1000 - \epsilon_1$, -ϵ_1)
k_{11}	($1021 - \epsilon_1$, -ϵ_1)	($1021 - \epsilon_1$, -ϵ_1)	($1021 - \epsilon_1$, -ϵ_1)
k_{12}	($1023 - \epsilon_1$, -ϵ_1)	($1023 - \epsilon_1$, -ϵ_1)	($2\epsilon_1$, -1023+$2\epsilon_1$+ϵ_2)

Proof. An allocation A is α-MMS for $\alpha \geq 0$ *iff* $\forall i, u_i(A_i) \geq \alpha\mu_i \geq 0$ when $\mu_i > 0$. Note that the transformed valuations $w_i(A_i) = \mathfrak{T}(V_i^g(A_i))$. From Eq. 1, $u_i(A_i) \geq 0$, *iff* $w_i(A_i) \geq -v_i'(M)$, which gives us $w_i(A_i) \geq 4055000 - 0.1$. For this to be true, we need $w_i(A_i) \geq 4055000$ since $\mathfrak{T}(V^g)$ has all integral values. We know that such an allocation doesn't exist [30]. Hence for any $\alpha \in [0, 1]$, α-MMS does not exist for V^g.

Non-existence of $1/\alpha$-MMS in Chores. Consider the following example.

Example 3. We consider a problem of allocating 12 chores among three agents. The valuation profile V^c is equivalent to $-10^3 V$ given in Table 1. We set $\epsilon_2 = -10^{-3}$. We transform these valuations in 1-D, and $\mathfrak{T}(V^c)$ is the same as the instance in [5] that proves the non-existence of MMS for chores. Note that $v_i'(M) = 4055000 - 10^3\epsilon_1$. The MMS value of every agent in $\mathfrak{T}(V^c)$ and V^c is -4055000 and $-10^3\epsilon_1$, respectively.

Lemma 3. *There is no $1/\alpha$-MMS allocation for the valuation profile V^c of Example 3 with $\epsilon_1 \in (0, 10^{-4}]$ for any $\alpha > 0$.*

Proof. An allocation A is $1/\alpha$-MMS for $\alpha > 0$ *iff* $\forall i, u_i(A_i) \geq \frac{1}{\alpha}\mu_i$ when $\mu_i < 0$. We set $\epsilon_1 \leq 10^{-4}$ in V^c. When $\alpha \geq 10^3\epsilon_1$ $\forall i$ then $u_i(A_i) \geq -1$. From Eq. 1,

$u_i(A_i) \geq -1$ *iff* $w_i(A_i) \geq -4055001+10^3\epsilon_1$. Note that $0 < 10^3\epsilon_1 \leq 0.1$ and since $w_i(A_i)$ has only integral values, we need $\forall i, w_i(A_i) \geq -4055000$. Such A does not exist [5]. As ϵ_1 decreases, $1/\epsilon_1$ increases, and even though approximation guarantees weakens, it still does not exist for V^c.

From Lemma 2 and 3 we conclude the following theorem,

Theorem 2. *There may not exist α-MMS for any $\alpha \in [0,1]$ for $\mu_i > 0$ or $1/\alpha$-MMS allocation for any $\alpha \in (0,1]$ for $\mu_i < 0$ in the presence of externalities.*

Interestingly, in 1-D, α-MMS's non-existence is known for α value close to 1 [30, 19], while in 2-D, it need not exist even for $\alpha = 0$. It follows because α-MMS could not lead to any relaxation in the presence of inverse externalities. Consider the situation of goods having negative externalities, where MMS share μ_i comprises of the positive value from the assigned bundle A_i and negative value from the unassigned bundles A_{-i}. We re-write μ_i as follows, $\mu_i = \mu_i^+ + \mu_i^-$ where μ_i^+ corresponds to utility from assigned goods/unassigned chores and μ_i^- corresponds to utility from unassigned goods/assigned chores. When $\mu_i \geq 0$, applying $\alpha\mu_i$ is not only relaxing positive value $\alpha\mu_i^+$, but also requires $\alpha\mu_i^-$ which is stricter than μ_i^- since $\mu_i^- < 0$. Hence, the impossibility of α-MMS in 2-D. Similar argument holds for chores. Next, we explore MMS relaxation such that it exists in 2-D.

5.3 Re-defining Approximate MMS

In this section, we define *Shifted α-MMS* that guarantees a fraction of MMS share shifted by certain value, such that it always exist in 2-D. We also considered intuitive ways of approximating MMS in 2-D. These ways are based on relaxing the positive value obtained from MMS allocation μ^+ and the negative value μ^-, $\mu = \mu^+ + \mu^-$. In other words, we look for allocations that guarantee $\alpha\mu^+$ and $(1+\alpha)$ or $1/\alpha$ of μ^-. Unfortunately, such approximations may not always exist. We skip the details due to space constraints.

Definition 9 (Shifted α-MMS). *An allocation A guarantees shifted α-MMS if $\forall i \in N, \alpha \in (0,1]$*

$$\left.u_i(A_i) \geq \alpha\mu_i + (1-\alpha)v_i'(M)\right\} \qquad \textit{for goods}$$
$$\left.u_i(A_i) \geq \frac{1}{\alpha}\mu_i + \frac{\alpha-1}{\alpha}v_i'(M)\right\} \qquad \textit{for chores}$$

Proposition 5. *An allocation A is α-MMS in $\mathcal{W}$* iff *A is shifted α-MMS in $\mathcal{V}$.*

Proof. For goods, if allocation A is shifted α-MMS, $\forall i, u_i(A_i) \geq \alpha\mu_i + (1-\alpha)v_i'(M)$. Applying $\mathfrak{T}$, we get $w_i(A_i) + v_i'(M) \geq \alpha\mu_i^{\mathcal{W}} + \alpha v_i'(M) + (1-\alpha)v_i'(M)$ which gives $w_i(A_i) \geq \alpha\mu_i^{\mathcal{W}}$. For chores, if A is shifted $1/\alpha$-MMS, $\forall i, u_i(A_i) \geq \frac{1}{\alpha}\mu_i + \frac{(\alpha-1)}{\alpha}v_i'(M)$, which gives $w_i(A_i) \geq \frac{1}{\alpha}\mu_i^{\mathcal{W}}$. Similarly we can prove vice versa.

Corollary 5. *We can adapt all the existing algorithms for α-MMS in $\mathcal{W}$ to get shifted α-MMS in $\mathcal{V}$.*

We use $\mathfrak{T}$ and apply the existing algorithms and obtain the corresponding shifted multiplicative approximations. In the next section, we examine the additive relaxation of MMS since a multiplicative approximation need not exist in the presence of externalities.

Additive Relaxation of MMS

Definition 10 (MMS relaxations). *An allocation A that satisfies, $\forall i, j \in N$,*

$$\left.\begin{array}{l} \forall\ k \in \{M \setminus A_i\}, v_{ik} > 0, u_i(A_i \cup \{k\}) \geq \mu_i \\ \forall\ k \in A_i, v_{ik} < 0, u_i(A_i \setminus \{k\}) \geq \mu_i \end{array}\right\} \quad \textit{MMSX, MMS upto any item}$$
$$\left.\begin{array}{l} \exists\ k \in \{M \setminus A_i\}, u_i(A_i \cup \{k\}) \geq \mu_i,\ \ or, \\ \exists\ k \in A_i, u_i(A_i \setminus \{k\}) \geq \mu_i \end{array}\right\} \quad \textit{MMS1, MMS upto an item} \tag{3}$$

Proposition 6. *From Lemma 1 and Theorem 1, we conclude that MMS1 and MMSX are preserved after transformation.*

EF1 is a stronger fairness notion than MMS1 and can be computed in polynomial time. On the other hand, PROPX might not exist for goods [4]. Since PROPX implies MMSX, it is interesting to settle the existence of MMSX for goods. Note that MMSX and Shifted α-MMS are not related. It is interesting to study these relaxations further, even in full externalities.

6 Conclusion

In this paper, we conducted a study on indivisible item allocation with special externalities – 2-D externalities. We proposed a simple yet compelling transformation from 2-D to 1-D to employ existing algorithms to ensure many fairness and efficiency notions. We can adapt existing fair division algorithms via the transformation in such settings. We proposed proportionality extension in the presence of externalities and studied its relation with other fairness notions. For MMS fairness, we proved the impossibility of multiplicative approximation of MMS in 2-D, and we proposed Shifted α-MMS instead. There are many exciting questions here which we leave for future works. (i) It might be impossible to have fairness-preserving valuation transformation for general externalities. However, what are some interesting domains where such transformations exist? (ii) What are interesting approximations to MMS in 2-D as well as in general externalities?

References

1. Amanatidis, G., Markakis, E., Nikzad, A., Saberi, A.: Approximation algorithms for computing maximin share allocations. ACM Transactions on Algorithms (TALG) **13**(4), 1–28 (2017)

2. Aziz, H., Caragiannis, I., Igarashi, A.: Fair allocation of combinations of indivisible goods and chores. CoRR **abs/1807.10684** (2018), http://arxiv.org/abs/1807.10684
3. Aziz, H., Huang, X., Mattei, N., Segal-Halevi, E.: Computing fair utilitarian allocations of indivisible goods. CoRR **abs/2012.03979** (2020), https://arxiv.org/abs/2012.03979
4. Aziz, H., Moulin, H., Sandomirskiy, F.: A polynomial-time algorithm for computing a pareto optimal and almost proportional allocation. Operations Research Letters **48**(5), 573–578 (2020)
5. Aziz, H., Rauchecker, G., Schryen, G., Walsh, T.: Algorithms for max-min share fair allocation of indivisible chores. p. 335–341. AAAI'17, AAAI Press (2017)
6. Aziz, H., Suksompong, W., Sun, Z., Walsh, T.: Fairness concepts for indivisible items with externalities (2021)
7. Barman, S., Biswas, A., Krishnamurthy, S., Narahari, Y.: Groupwise maximin fair allocation of indivisible goods. In: Proceedings of the AAAI Conference on Artificial Intelligence. vol. 32 (2018)
8. Barman, S., Ghalme, G., Jain, S., Kulkarni, P., Narang, S.: Fair division of indivisible goods among strategic agents. In: Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems. p. 1811–1813. AAMAS '19, International Foundation for Autonomous Agents and Multiagent Systems, Richland, SC (2019)
9. Barman, S., Krishnamurthy, S.K., Vaish, R.: Finding fair and efficient allocations. In: Proceedings of the 2018 ACM Conference on Economics and Computation. pp. 557–574 (2018)
10. Bei, X., Huzhang, G., Suksompong, W.: Truthful fair division without free disposal. Social Choice and Welfare **55**(3), 523–545 (2020)
11. Bhaskar, U., Sricharan, A.R., Vaish, R.: On approximate envy-freeness for indivisible chores and mixed resources. CoRR **abs/2012.06788** (2020), https://arxiv.org/abs/2012.06788
12. Brams, S.J., Brams, S.J., Taylor, A.D.: Fair Division: From cake-cutting to dispute resolution. Cambridge University Press (1996)
13. Brânzei, S., Procaccia, A., Zhang, J.: Externalities in cake cutting. In: Twenty-Third International Joint Conference on Artificial Intelligence (2013)
14. Budish, E.: The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. Journal of Political Economy **119**(6), 1061–1103 (2011)
15. Caragiannis, I., Kurokawa, D., Moulin, H., Procaccia, A.D., Shah, N., Wang, J.: The unreasonable fairness of maximum nash welfare. ACM Transactions on Economics and Computation (TEAC) **7**(3), 1–32 (2019)
16. Chaudhury, B.R., Garg, J., Mehlhorn, K.: Efx exists for three agents. In: Proceedings of the 21st ACM Conference on Economics and Computation. pp. 1–19 (2020)
17. Chen, X., Liu, Z.: The fairness of leximin in allocation of indivisible chores. arXiv preprint arXiv:2005.04864 (2020)
18. De Keijzer, B., Bouveret, S., Klos, T., Zhang, Y.: On the complexity of efficiency and envy-freeness in fair division of indivisible goods with additive preferences. In: International Conference on Algorithmic Decision Theory. pp. 98–110. Springer (2009)
19. Feige, U., Sapir, A., Tauber, L.: A tight negative example for mms fair allocations. arXiv preprint arXiv:2104.04977 (2021)

20. Foley, D.K.: Resource allocation and the public sector. Yale University (1966)
21. Freeman, R., Sikdar, S., Vaish, R., Xia, L.: Equitable allocations of indivisible goods. In: IJCAI (2019)
22. Garg, D., Narahari, Y., Gujar, S.: Foundations of mechanism design: A tutorial part 2-advanced concepts and results. Sadhana **33**(2), 131–174 (2008)
23. Garg, D., Narahari, Y., Gujar, S.: Foundations of mechanism design: A tutorial part 1-key concepts and classical results. Sadhana **33**(2), 83–130 (2008)
24. Garg, J., McGlaughlin, P., Taki, S.: Approximating maximin share allocations. Open access series in informatics **69** (2019)
25. Garg, J., Taki, S.: An improved approximation algorithm for maximin shares. Artificial Intelligence p. 103547 (2021)
26. Ghodsi, M., HajiAghayi, M., Seddighin, M., Seddighin, S., Yami, H.: Fair allocation of indivisible goods: Improvements and generalizations. In: Proceedings of the 2018 ACM Conference on Economics and Computation. pp. 539–556 (2018)
27. Huang, X., Lu, P.: An algorithmic framework for approximating maximin share allocation of chores. In: Proceedings of the 22nd ACM Conference on Economics and Computation. pp. 630–631 (2021)
28. Kulkarni, R., Mehta, R., Taki, S.: Indivisible mixed manna: On the computability of mms+ po allocations. In: Proceedings of the 22nd ACM Conference on Economics and Computation. pp. 683–684 (2021)
29. Kurokawa, D., Procaccia, A.D., Wang, J.: When can the maximin share guarantee be guaranteed? In: Thirtieth AAAI Conference on Artificial Intelligence (2016)
30. Kurokawa, D., Procaccia, A.D., Wang, J.: Fair enough: Guaranteeing approximate maximin shares. Journal of the ACM (JACM) **65**(2), 1–27 (2018)
31. Li, B., Li, Y., Wu, X.: Almost (weighted) proportional allocations for indivisible chores. arXiv preprint arXiv:2103.11849 (2021)
32. Lipton, R.J., Markakis, E., Mossel, E., Saberi, A.: On approximately fair allocations of indivisible goods. In: Proceedings of the 5th ACM Conference on Electronic Commerce. pp. 125–131 (2004)
33. Moulin, H.: Fair division and collective welfare. MIT press (2004)
34. Padala, M., Gujar, S.: Mechanism design without money for fair allocations (2021)
35. Plaut, B., Roughgarden, T.: Almost envy-freeness with general valuations. SIAM Journal on Discrete Mathematics **34**(2), 1039–1068 (2020)
36. Procaccia, A.D., Wang, J.: Fair enough: Guaranteeing approximate maximin shares. In: Proceedings of the fifteenth ACM conference on Economics and computation. pp. 675–692 (2014)
37. Seddighin, M., Saleh, H., Ghodsi, M.: Externalities and fairness. In: The World Wide Web Conference. pp. 538–548 (2019)
38. Segal-Halevi, E.: Cake-cutting with different entitlements: How many cuts are needed? Journal of Mathematical Analysis and Applications **480**(1), 123382 (2019)
39. Steihaus, H.: The problem of fair division. Econometrica **16**, 101–104 (1948)
40. Su, F.E.: Cake-cutting algorithms: Be fair if you can. by jack robertson and william webb. The American Mathematical Monthly **107**(2), 185–188 (2000)
41. Velez, R.A.: Fairness and externalities. Theoretical Economics **11**(1), 381–410 (2016)