

# 1 Probability Theory

## 1.1 Some ole' Binomial shits

### 1.1.1 Binomial approx to Poisson

**Theorem 1.1.** For  $\lambda = np$ ,

$$\lim_{n \rightarrow \infty} \binom{n}{k} p^k (1-p)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}$$

First, we need the following lemma

**Lemma 1.1.** Let  $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$  be the Gamma function, then the following equation holds

$$\lim_{n \rightarrow \infty} \frac{\binom{n}{k}}{n^k} = \frac{1}{k!}.$$

*Proof of Lemma 1.1.* Note first that

$$\Gamma(k+1) = k\Gamma(k) = \dots = k!\Gamma(1) = k!.$$

and Stirling's Formula for Gamma function states that

$$\Gamma(x+1) = \sqrt{2\pi x} x^x e^{-x} \left(1 + \frac{1}{12x} \dots\right).$$

Then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\binom{n}{k}}{n^k} &= \lim_{n \rightarrow \infty} \frac{n!}{n^k (n-k)! k!} \\ &= \lim_{n \rightarrow \infty} \frac{\Gamma(n+1)}{n^k \Gamma(n-k+1) \Gamma(k+1)} \\ &= \frac{1}{\Gamma(k+1)} \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n-k}} \left(\frac{n}{e}\right)^n \left(\frac{e}{n-k}\right)^{n-k} \frac{1}{n^k}. \end{aligned}$$

Where we used the first order of Stirling's Formula for Gamma function in the last equality.

Next, we only need to prove the limit equals to 1:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n-k}} \left(\frac{n}{e}\right)^n \left(\frac{e}{n-k}\right)^{n-k} \frac{1}{n^k} &= \sqrt{\frac{n}{n-k}} \frac{1}{e^k} \frac{(1-k/n)^k}{(1-k/n)^n} \\ &= 1, \end{aligned}$$

since

$$\begin{aligned} \sqrt{\frac{n}{n-k}} &\rightarrow 1, \\ (1-k/n)^k &\rightarrow 1, \\ \frac{(1-k/n)^k}{(1-k/n)^n} &\rightarrow e^{-k}, \end{aligned}$$

as  $n \rightarrow \infty$ . ■

Now we may prove the main theorem

*Proof of Theorem 1.1.* Let  $p = \gamma/n$  and fix  $k$ , we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{\binom{n}{k}}{n^k} &= \lim_{n \rightarrow \infty} \binom{n}{k} \left(\frac{\gamma}{k}\right)^k \left(1 - \frac{\gamma}{n}\right)^{n-k} \\
&= \lim_{n \rightarrow \infty} \frac{n^k}{k!} \left(\frac{\gamma}{k}\right)^k \left(1 - \frac{\gamma}{n}\right)^{n-k} \quad \text{by Lemma 1.1} \\
&= \frac{\gamma^k}{k!} \left[ \lim_{n \rightarrow \infty} \left(1 - \frac{\gamma}{n}\right)^n \right] \left[ \lim_{n \rightarrow \infty} \left(1 - \frac{\gamma}{n}\right)^k \right] \\
&= \frac{\gamma^k}{k!} e^{-\gamma} (1)
\end{aligned}$$

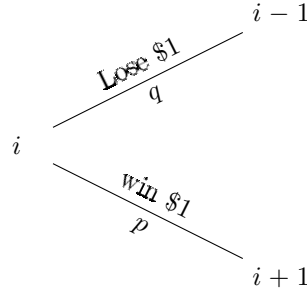
■

## 1.2 Gambler's Ruin

Let  $p$  is the probability of winning \$1 and  $q = 1 - p$  is the probability of losing \$1. Suppose that a person starts with \$ $i$  amount of money where  $0 < i < N$  and  $N \in \mathbb{N}$  is the number he would want to walk away with. The probability of him winning from this starting position is given by the following equation:

$$a_k = \begin{cases} \frac{1 - \left(\frac{q}{p}\right)^i}{1 - \left(\frac{q}{p}\right)^N} & \text{where } p \neq q, \\ \frac{i}{N} & \text{where } p = q = \frac{1}{2}. \end{cases}$$

*Proof.* Note that



Then, we observe

$$\begin{aligned}
a_i &= pa_{i+1} + qa_{i-1} \\
\underbrace{(p+q)}_{=1} a_i &= pa_{i+1} + qa_{i-1} \\
a_{i+1} - a_i &= \frac{p}{q} (a_i - a_{i-1}).
\end{aligned}$$

Denote  $b_j := a_i - a_{i-1}$  and note that  $b_1 = a_1$  since  $a_0 = 0$  is the absorption edge, we deduce by recursion that

$$b_{i+1} = \left(\frac{p}{q}\right)^{i+1} a_1.$$

Moreover, note that

$$a_{i+1} - a_1 = \sum_i^{k=1} a_{k+1} - a_k$$

$$\begin{aligned}
&= \sum_{k=1}^i \left(\frac{p}{q}\right)^k a_1. \\
\Rightarrow a_{i+1} &= a_1 \left(1 + \sum_{k=1}^i \left(\frac{p}{q}\right)^k\right) \\
&= a_1 \left(\sum_{k=0}^i \left(\frac{p}{q}\right)^k\right).
\end{aligned}$$

If  $p \neq q$ , by geometric series, we obtain

$$a_{i+1} = a_1 \frac{1 - \left(\frac{p}{q}\right)^{i+1}}{1 - \left(\frac{p}{q}\right)}.$$

If  $p = q = 0.5$ , then

$$a_{i+1} = a_1(i+1).$$

Now to solve for  $a_1$ , observe that since at absorption point  $a_N = 1$ , then letting  $i+1 = N$ , we get

$$1 = a_N = \begin{cases} \frac{1 - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)} & \text{where } p \neq q, \\ a_1 N & \text{where } p = q = \frac{1}{2}, \end{cases}$$

which implies

$$a_1 = \begin{cases} \frac{1 - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N} & \text{where } p \neq q, \\ \frac{1}{N} & \text{where } p = q = \frac{1}{2}. \end{cases}$$

Substituting this back, we obtained the desired equation. ■

## 2 Operational Research

### 2.1 Safety Stock

**Theorem 2.1.** *Let the lead-time  $T$  be normally distributed, i.e.  $T \sim N(\mu_\ell, \sigma_\ell^2)$  and denote the set of demand of period  $i$  as  $\{D_i\}_{i=1}^T$  where  $D_i \sim N(\mu_d, \sigma_d^2)$  for each  $1 \leq i \leq T$ . Then the safety stock level,  $SS$  satisfies the following equation*

$$SS = z_\alpha \sqrt{\sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2},$$

where  $z_\alpha$  is the Z-value of a desired  $\alpha$  which is chosen.

*Proof.* Denote the demand between within lead-time as  $D(T) := \sum_{i=1}^T D_i$ . The safety stock level is then set at

$$SS = z_\alpha \sqrt{\text{Var}(D(T))}.$$

Thus we need only to prove  $\text{Var}(D(T))$ .

From Towering-property, note that

$$\begin{aligned}
\mathbb{E}[D(T)] &= \mathbb{E}[\mathbb{E}[D(T)|T]] \\
&= \mathbb{E}[T\mu_d] = \mu_d \mathbb{E}[T] = \mu_d \mu_\ell.
\end{aligned}$$

Furthermore, again with Towering-property, we have

$$\mathbb{E}[D^2(T)] = \mathbb{E}[\mathbb{E}[D^2(T)|T]]$$

$$\begin{aligned}
&= \mathbb{E}[\text{Var}[D(T)|T] + (\mathbb{E}[D(T)|T])^2] \\
&= \mathbb{E}[T\sigma_d^2 + (T\mu_d)^2] \\
&= \sigma_d^2\mathbb{E}[T] + \mu_d^2\mathbb{E}[T^2] \\
&= \sigma_d^2\mu_\ell + \mu_d^2(\text{Var}(T) + (\mathbb{E}[T])^2) \\
&= \sigma_d^2\mu_\ell + \mu_d^2(\sigma_\ell^2 + \mu_\ell^2).
\end{aligned}$$

Finally,

$$\begin{aligned}
\text{Var}(D(T)) &= \mathbb{E}[D^2(T)] - (\mathbb{E}[D(T)])^2 \\
&= \sigma_d^2\mu_\ell + \mu_d^2(\sigma_\ell^2 + \mu_\ell^2) - (\mu_d\mu_\ell)^2 \\
&= \sigma_d^2\mu_\ell + \mu_d^2\sigma_\ell^2.
\end{aligned}$$

■