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1.1 Safety Stock

Theorem 1.1. *Let the lead-time T be normally distributed, i.e. $T \sim N(\mu_\ell, \sigma_\ell^2)$ and denote the set of demand of period i as $\{D_i\}_{i=1}^T$ where $D_i \sim N(\mu_d, \sigma_d^2)$ for each $1 \leq i \leq T$. Then the safety stock level, SS satisfies the following equation*

$$SS = z_\alpha \sqrt{\sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2},$$

where z_α is the Z-value of a desired α which is chosen.

Proof. Denote the demand between within lead-time as $D(T) := \sum_{i=1}^T D_i$. The safety stock level is then set at

$$SS = z_\alpha \sqrt{\text{Var}(D(T))}.$$

Thus we need only to prove $\text{Var}(D(T))$.

From Towering-property, note that

$$\begin{aligned} \mathbb{E}[D(T)] &= \mathbb{E}[\mathbb{E}[D(T)|T]] \\ &= \mathbb{E}[T\mu_d] = \mu_d \mathbb{E}[T] = \mu_d \mu_\ell. \end{aligned}$$

Furthemore, again with Towering-property, we have

$$\begin{aligned} \mathbb{E}[D^2(T)] &= \mathbb{E}[\mathbb{E}[D^2(T)|T]] \\ &= \mathbb{E}[\text{Var}[D(T)|T] + (\mathbb{E}[D(T)|T])^2] \\ &= \mathbb{E}[T\sigma_d^2 + (T\mu_d)^2] \\ &= \sigma_d^2 \mathbb{E}[T] + \mu_d^2 \mathbb{E}[T^2] \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 (\text{Var}(T) + (\mathbb{E}[T])^2) \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 (\sigma_\ell^2 + \mu_\ell^2). \end{aligned}$$

Finally,

$$\begin{aligned} \text{Var}(D(T)) &= \mathbb{E}[D^2(T)] - (\mathbb{E}[D(T)])^2 \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 (\sigma_\ell^2 + \mu_\ell^2) - (\mu_d \mu_\ell)^2 \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2. \end{aligned}$$

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