1 Probability Theory

1.1 Some ole' Binomal

1.1.1 Expected value and variance of Binomial

Theorem 1.1. *Let $X \approx Bin(n,p)$, then the expected value and variance is given by*

1. $\mathbb{E}(X) = np$,
2. $\mathrm{Var}(X) = np(1-p)$.

1.1.2 Binomal approx to Poisson

Theorem 1.2. *For $\lambda = np$,*

$$\lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}$$

First, we need the following lemma

Lemma 1.1. *Let $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ be the Gamma function, then the following equation holds*

$$\lim_{n\to\infty} \frac{\binom{n}{k}}{n^k} = \frac{1}{k!}.$$

Proof of Lemma 1.1. Note first that

$$\Gamma(k+1) = k\Gamma(k) = \cdots = k!\Gamma(1) = k!.$$

and *Stirling's Formula for Gamma function* states that

$$\Gamma(x+1) = \sqrt{2\pi x}x^x e^{-x}\left(1 + \frac{1}{12x}\cdots\right).$$

Then we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\binom{n}{k}}{n^k} &= \lim_{n\to\infty} \frac{n!}{n^k(n-k)!k!} \\
&= \lim_{n\to\infty} \frac{\Gamma(n+1)}{n^k\Gamma(n-k+1)\Gamma(k+1)} \\
&= \frac{1}{\Gamma(k+1)} \lim_{n\to\infty} \sqrt{\frac{n}{n-k}} \left(\frac{n}{e}\right)^n \left(\frac{e}{n-k}\right)^{n-k} \frac{1}{n^k}.
\end{aligned}$$

Where we used the first order of Stirling's Formula for Gamma function in the last equality.

Next, we only need to prove the limit equals to 1:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{\frac{n}{n-k}} \left(\frac{n}{e}\right)^n \left(\frac{e}{n-k}\right)^{n-k} \frac{1}{n^k} &= \lim_{n\to\infty} \sqrt{\frac{n}{n-k}} \frac{1}{e^k} \frac{(1-k/n)^k}{(1-k/n)^n} \\
&= 1,
\end{aligned}$$

since

$$\begin{aligned}
\sqrt{\frac{n}{n-k}} &\to 1, \\
(1-k/n)^k &\to 1, \\
(1-k/n)^n &\to e^{-k},
\end{aligned}$$

as $n \to \infty$. ∎

Now we may prove the main theorem

Proof of Theorem 1.2. Let $p = \lambda/n$ and fix k, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\binom{n}{k}}{n^k} &= \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\gamma}{n}\right)^{n-k} \\
&= \lim_{n\to\infty} \frac{n^k}{k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \qquad \text{by Lemma 1.1} \\
&= \frac{\lambda^k}{k!} \left[\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n\right] \underbrace{\left[\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-k}\right]}_{=1} \\
&= \frac{\lambda^k}{k!} e^{-\lambda}.
\end{aligned}$$

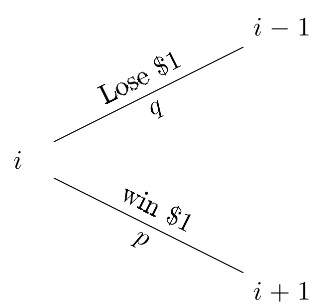
■

1.2 Markov Chain

Theorem 1.3 (Gambler's ruin)**.** *Let p is the probability of winning \$1 and $q = 1 - p$ is the probability of losing \$1. Suppose that a person starts with \$$i$ amount of money where $0 < i < N$ and $N \in \mathbb{N}$ is the number he would want to walk away with. The probability of him winning from this starting position is given by the following equation:*

$$a_i = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & \textit{where } p \neq q, \\ \frac{i}{N} & \textit{where } p = q = \frac{1}{2}. \end{cases}$$

Proof. Note that



Then, we observe

$$\begin{aligned}
a_i &= pa_{i+1} + qa_{i-1} \\
\underbrace{(p+q)}_{=1} a_i &= pa_{i+1} + qa_{i-1} \\
a_{i+1} - a_i &= \frac{q}{p}(a_i - a_{i-1}).
\end{aligned}$$

Denote $b_j := a_i - a_{i-1}$ and note that $b_1 = a_1$ since $a_0 = 0$ is the absorption edge, we deduce by recursion that

$$b_{i+1} = \left(\frac{q}{p}\right)^{i+1} a_1.$$

Moreover, note that

$$\begin{aligned}
a_{i+1} - a_1 &= \sum_{i}^{k=1} a_{k+1} - a_k \\
&= \sum_{k=1}^{i} \left(\frac{q}{p}\right)^k a_1. \\
\Rightarrow a_{i+1} &= a_1 \left(1 + \sum_{k=1}^{i} \left(\frac{q}{p}\right)^k\right) \\
&= a_1 \left(\sum_{k=0}^{i} \left(\frac{q}{p}\right)^k\right).
\end{aligned}$$

If $p \neq q$, by geometric series, we obtain

$$a_{i+1} = a_1 \frac{1 - \left(\frac{q}{p}\right)^{i+1}}{1 - \left(\frac{q}{p}\right)}.$$

If $p = q = 0.5$, then

$$a_{i+1} = a_1(i+1).$$

Now to solve for a_1, observe that since at absoprtion point $a_N = 1$, then letting $i + 1 = N$, we get

$$1 = a_N = \begin{cases} a_1 \frac{1-\left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)} & \text{where } p \neq q, \\ a_1 N & \text{where } p = q = \frac{1}{2}, \end{cases}$$

which implies

$$a_1 = \begin{cases} \frac{1-\left(\frac{q}{p}\right)}{1-\left(\frac{q}{p}\right)^N} & \text{where } p \neq q, \\ \frac{1}{N} & \text{where } p = q = \frac{1}{2}. \end{cases}$$

Substituting this back, we obtained the desired equation. ■

Corollary 1.1. *The expectation of stopping time with respect to Brownian motion, i.e.* $\tau = \inf\{t \geqslant 0 : B_\tau \in \{-\alpha, \beta\}\}$, *for any positive integer* α *and* β *is given by*

$$\mathbb{E}[\tau] = \alpha\beta.$$

Proof. From Theorem (1.3) we observe that $\Pr(B_\tau = b) = \frac{a}{a+b}$. Then, note that since $1 = \Pr(B_\tau = -\alpha) + \Pr(B_\tau = \beta)$, we have

$$\Pr(B_\tau = -\alpha) = \frac{b}{a+b}.$$

Since $(B_t^2 - t)_{t \geqslant 0}$ is a martingale, we by apply Optimal Stopping Theorem where

$$0 = \mathbb{E}[B_0^2 - 0] = \mathbb{E}[B_\tau^2 - \tau],$$

which implies

$$\mathbb{E}[\tau] = \mathbb{E}[B_\tau^2] = \alpha^2 \Pr(B_\tau = -\alpha) + \beta^2 \Pr(B_\tau = \beta) = \alpha\beta.$$

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2 Financial Mathematics

3 Time Series

3.1 Stationary and Autocorrelation

Definition 3.1. *The* ***autocovariance***

$$\gamma(s,t) := \mathbb{E}\left[(x_s - \mu_s)(x_t - \mu_t)\right],$$

and ***cross-covar.***

$$\rho_{x,y}(s,t) := \mathbb{E}\left[(x_s - \mu_{xs})(y_t - \mu_{ts})\right].$$

Definition 3.2. *The* ***autocorrelation function*** *(ACF) is defined as*

$$\rho_x(s,t) := \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}},$$

Notation: $\rho_x(t, t+h) \equiv \rho(h)$.

Definition 3.3 (Strictly stationary). *For all $k \in \mathbb{N}$, $t_k \in \mathbb{N}$ and $c_k \in \mathbb{R}$. Then the time series is strictly stationary if*

$$\mathbb{P}\big(x_{t_1} \leqslant c_1, \ldots x_{t_k} \leqslant c_k\big) = \mathbb{P}\big(x_{t_1+h} \leqslant c_1, \ldots x_{t_k+h} \leqslant c_k\big),$$

for given probability measure $\mathbb{P}$ and constant $h \in \mathbb{N}$.

Definition 3.4 (Weakly stationary). *The time series is strictly stationary if*

1. *$\mu_t = \mathbb{E}[x_t]$ is constant, and*
2. *$\gamma(s,t)$ depends only on $|s-t|$.*

Lemma 3.1. *Assume time series is weakly stationary. Using the notation $\gamma(t, t+h) \equiv \gamma(h)$, we have*

1. *$|\gamma(t)| \leqslant \gamma(0)$,*
2. *$\gamma(h) = \gamma(-h)$.*

Proof.

1. By Cauchy-Schwarz inequality $|\gamma(t,t+h)|^2 \leqslant \gamma(t,t)\gamma(t+h,t+h)$ by definition $|\gamma(h)|^2 \leqslant \gamma(0)\gamma(0)$.
2. From definition of covariance we have

$$\begin{aligned}\gamma(h) &= \gamma(t+h-t)\\ &= \mathbb{E}\left[(x_{t+h}-\mu)(x_t-\mu)\right]\\ &= \mathbb{E}\left[(x_t-\mu)(x_{t+h}-\mu)\right]\\ &= \gamma(t-(t+h)) = \gamma(-h)\end{aligned}$$

■

Example 3.1. Suppose w_t for all $t \in N$ is white noise process, i.e. iid $w_t \approx N(0, \sigma_w^2)$. Given two series

$$\begin{aligned}x_t &= w_t + w_{t-1},\\ y_t &= w_t - w_{t-1}.\end{aligned}$$

Then for any $h \in \mathbb{N}$, we get the following for (cross-)covariance:

1.

$$\begin{aligned}\gamma_x(0) &= \mathbb{E}[(w_t + w_{t-1})^2] = \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1}^2] + 2\underbrace{\mathbb{E}[w_t w_{t-1}]}_{=0} \\ &= 2\sigma_w^2, \\ \gamma_y(0) &= \mathbb{E}[(w_t - w_{t-1})^2] = \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1}^2] - 2\underbrace{\mathbb{E}[w_t w_{t-1}]}_{=0} \\ &= 2\sigma_w^2.\end{aligned}$$

2.

$$\begin{aligned}\gamma_x(1) &= \mathbb{E}\left[(w_t + w_{t-1})(w_{t+1} + w_t)\right] \\ &= \mathbb{E}[w_t w_{t+1}] + \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1} w_{t+1}] + \mathbb{E}[w_{t-1} w_t] \\ &= \sigma_w = \gamma_x(-1), \\ \gamma_y(1) &= \mathbb{E}[w_t w_{t+1}] - \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1} w_{t+1}] - \mathbb{E}[w_{t-1} w_t] \\ &= -\sigma_w = \gamma_y(-1).\end{aligned}$$

3. $\gamma_{xy}(0) = \mathbb{E}[w_t^2] - \mathbb{E}[w_{t-1}^2] = 0$ and $\gamma_{xy}(1) = cov(x_{t+1}, y_t) = -\sigma_w^2$ and $\gamma_{xy}(-1) = cov(x_t, y_{t-1}) = -\sigma_w^2$

4. ACF

$$\rho_{xy}(h) = \begin{cases} 0 & h = 0, \\ 1/2 & h = 1, \\ -1/2 & h = -1, \\ 0 & |h| \geqslant 2 \end{cases} \tag{3.1}$$

Thus the joint time series is stationary.

Definition 3.5 (Linear Process)**.** *$\{x_t\}$ is linear process if*

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty,$$

where w_t is white noise.

Definition 3.6 (Gaussian Process)**.** *$\{x_t\}$ is Gaussian process if the vector $x := (x_{t_1}, x_{t_2}, \ldots, x_{t_n})' \in \mathbb{R}^n$ has a multivariate normal distribution with density function*

$$f(x) = \frac{1}{(2\pi)^{n/2}} \det\left(\Gamma\right)^{-1/2} \exp\left(-\frac{1}{2}(x-\mu)'\Gamma^{-1}(x-\mu)\right),$$

where $\mu \in \mathbb{R}^n$ and $\Gamma := var(x) = \{\gamma(t_i, t_j);\ i, j = 1, \ldots, n\}$.

3.1.1 ARIMA

$ARMA(p, q)$:

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = w_t + \theta_t w_{t-1} + \cdots + \theta_q w_{t-q}.$$

3.2 Autoregressive Model

Definition 3.7 ($AR(p)$)**.** *Let w_t be white noise, $AR(p)$ is defined as*

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = w_t.$$

Definition 3.8 (Autoregressive Operator)**.** *Let B be time-lagged operator, i.e. $Bx_t = x_{t-1}$, the autoregressive operator is defined as*

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p.$$

Example 3.2. Assuming $\phi_j = \phi$ for all $j \in \mathbb{N}$, from AR(1)

$$\begin{aligned} x_t &= \phi x_{t-1} + w_t = \phi(\phi x_{t-2} + w_{t-1}) + w_t \\ &= \phi^2 x_{t-2} + \phi w_{t-1} + w_t \\ &\vdots \\ &= \phi^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j w_{t-j}. \end{aligned}$$

Thus, if $|\phi| < 1$ and x_t is sationary then for large $k \to \infty$, we have

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}.$$

Taking expectation value,

$$\mathbb{E}[x_t] = \sum_{j=0}^{\infty} \phi_j \mathbb{E}[w_t] = 0.$$

Moreover, for $h \geqslant$ the autocovriance function is

$$\begin{aligned} \gamma(h) &= \mathbb{E}\left[\left(\sum_{j=0}^{\infty} \phi^j w_{t+h-j} \right) \left(\sum_{j=0}^{\infty} \phi^k w_{t-k} \right) \right] \\ &= \mathbb{E}\big[(w_{t+h-1} + \phi w_{t+h-2} + \cdots \phi^h w_t + \phi^{h+1} w_{t-1} + \cdots)(w_t + \phi w_{t-1} + \cdots) \big] \\ &= \sigma_w^2 \sum_{j=0} \phi^{h+j} \phi^j = \sigma_w^2 \phi^h \sum_{j=0} \phi^{2j} \\ &= \frac{\sigma_w^2 \phi^h}{1 - \phi^2}. \end{aligned}$$

And ACF is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h.$$

As well as

$$\rho(h) = \phi \rho(h-1).$$

Definition 3.9. *The* ***moving average operator*** *is defined as*

$$\theta(B) := 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q.$$

4 Operational Research

4.1 Safety Stock

Theorem 4.1. *Let the lead-time T be normally distributed, i.e. $T \sim N(\mu_\ell, \sigma_\ell^2)$ and denote the set of demand of period i as $\{D_i\}_{i=1}^T$ where $D_i \sim N(\mu_d, \sigma_d^2)$ for each $1 \leqslant i \leqslant T$. Then the safety stock level, SS satisifies the following equation*

$$SS = z_\alpha \sqrt{\sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2},$$

where z_α is the Z-value of a desired α which is chosen.

Proof. Denote the demand between within lead-time as $D(T) := \sum_{i=1}^T D_i$. The safety stock level is then set at

$$SS = z_\alpha \sqrt{Var(D(T))}.$$

Thus we need only to prove $Var(D(T))$.

From Towering-property, note that

$$\begin{aligned} \mathbb{E}[D(T)] &= \mathbb{E}\big[\mathbb{E}[D(T)|T]\big] \\ &= \mathbb{E}\big[T\mu_d\big] = \mu_d \mathbb{E}[T] = \mu_d \mu_\ell. \end{aligned}$$

Furthemore, again with Towering-property, we have

$$\begin{aligned} \mathbb{E}[D^2(T)] &= \mathbb{E}\big[\mathbb{E}[D^2(T)|T]\big] \\ &= \mathbb{E}\big[Var[D(T)|T] + \big(\mathbb{E}[D(T)|T]\big)^2\big] \\ &= \mathbb{E}\big[T\sigma_d^2 + \big(T\mu_d\big)^2\big] \\ &= \sigma_d^2 \mathbb{E}[T] + \mu_d^2 \mathbb{E}[T^2] \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \left(Var(T) + (\mathbb{E}[T])^2\right) \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \left(\sigma_\ell^2 + \mu_\ell^2\right). \end{aligned}$$

Finally,

$$\begin{aligned} Var(D(T)) &= \mathbb{E}[D^2(T)] - \big(\mathbb{E}[D(T)]\big)^2 \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \left(\sigma_\ell^2 + \mu_\ell^2\right) - (\mu_d \mu_\ell)^2 \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2. \end{aligned}$$

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