1 2023

1.1 Safety Stock

Theorem 1.1. *Let the lead-time T be normally distributed, i.e. $T \sim N(\mu_\ell, \sigma_\ell^2)$ and denote the set of demand of period i as $\{D_i\}_{i=1}^T$ where $D_i \sim N(\mu_d, \sigma_d^2)$ for each $1 \leqslant i \leqslant T$. Then the safety stock level, SS satisifies the following equation*

$$SS = z_\alpha \sqrt{\sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2},$$

where z_α is the Z-value of a desired α which is chosen.

Proof. Denote the demand between within lead-time as $D(T) := \sum_{i=1}^T D_i$. The safety stock level is then set at

$$SS = z_\alpha \sqrt{Var(D(T))}.$$

Thus we need only to prove $Var(D(T))$.

From Towering-property, note that

$$\begin{aligned}\mathbb{E}[D(T)] &= \mathbb{E}\big[\mathbb{E}[D(T)|T]\big] \\ &= \mathbb{E}\big[T\mu_d\big] = \mu_d \mathbb{E}[T] = \mu_d \mu_\ell.\end{aligned}$$

Furthemore, again with Towering-property, we have

$$\begin{aligned}\mathbb{E}[D^2(T)] &= \mathbb{E}\big[\mathbb{E}[D^2(T)|T]\big] \\ &= \mathbb{E}\Big[Var[D(T)|T] + \big(\mathbb{E}[D(T)|T]\big)^2\Big] \\ &= \mathbb{E}\Big[T\sigma_d^2 + \big(T\mu_d\big)^2\Big] \\ &= \sigma_d^2 \mathbb{E}[T] + \mu_d^2 \mathbb{E}[T^2] \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \left(Var(T) + (\mathbb{E}[T])^2\right) \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \left(\sigma_\ell^2 + \mu_\ell^2\right).\end{aligned}$$

Finally,

$$\begin{aligned}Var(D(T)) &= \mathbb{E}[D^2(T)] - \big(\mathbb{E}[D(T)]\big)^2 \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \left(\sigma_\ell^2 + \mu_\ell^2\right) - (\mu_d \mu_\ell)^2 \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2.\end{aligned}$$

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1.2 Binomal approx to Poisson

Theorem 1.2. *For $\lambda = np$,*

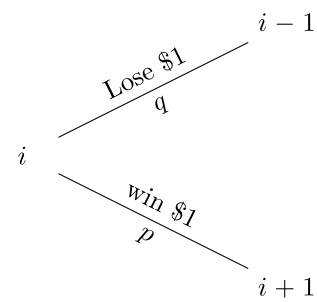
$$\lim_{n\to\infty} \binom{n}{k} p^k 1 - p^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}$$

1.3 Gambler's Ruin

Let p is the probability of winning \$1 and $q = 1 - p$ is the probability of losing \$1. Suppose that a person starts with $\$i$ amount of money where $0 < i < N$ and $N \in \mathbb{N}$ is the number he would want to walk away with. The probability of him winning from this starting position is given by the following equation:

$$a_k = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & \text{where } p \neq q, \\ \frac{i}{N} & \text{where } p = q = \frac{1}{2}. \end{cases}$$

Proof. Note that



Then, we observe

$$\begin{aligned} a_i &= pa_{i+1} + qa_{i-1} \\ \underbrace{(p+q)}_{=1} a_i &= pa_{i+1} + qa_{i-1} \\ a_{i+1} - a_i &= \frac{p}{q}(a_i - a_{i-1}). \end{aligned}$$

Denote $b_j := a_i - a_{i-1}$ and note that $b_1 = a_1$ since $a_0 = 0$ is the absorption edge, we deduce by recursion that

$$b_{i+1} = \left(\frac{p}{q}\right)^{i+1} a_1.$$

Moreover, note that

$$\begin{aligned} a_{i+1} - a_1 &= \sum_{i}^{k=1} a_{k+1} - a_k \\ &= \sum_{k=1}^{i} \left(\frac{p}{q}\right)^k a_1. \\ \Rightarrow a_{i+!} &= a_1 \left(1 + \sum_{k=1}^{i} \left(\frac{p}{q}\right)^k\right) \\ &= a_1 \left(\sum_{k=0}^{i} \left(\frac{p}{q}\right)^k\right). \end{aligned}$$

If $p \neq q$, by geometric series, we obtain

$$a_{i+1} = a_1 \frac{1 - \left(\frac{p}{q}\right)^{i+1}}{1 - \left(\frac{p}{q}\right)}.$$

If $p = q = 0.5$, then

$$a_{i+1} = a_1(i+1).$$

Now to solve for a_1, observe that since at absoprtion point $a_N = 1$, then letting $i + 1 = N$, we get

$$1 = a_N = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)} & \text{where } p \neq q, \\ a_1 N & \text{where } p = q = \frac{1}{2}, \end{cases}$$

which implies

$$a_1 = \begin{cases} \frac{1-\left(\frac{q}{p}\right)}{1-\left(\frac{q}{p}\right)^N} & \text{where } p \neq q, \\ \frac{1}{N} & \text{where } p = q = \frac{1}{2}. \end{cases}$$

Substituting this back, we obtained the desired equation. ∎