1 Probability Theory

1.1 Some ole' Binomal

1.1.1 Expected value and variance of Binomial

Theorem 1.1. *Let $X \approx Bin(n,p)$, then the expected value and variance is given by*

1. $\mathbb{E}(X) = np$,
2. $\mathrm{Var}(X) = np(1-p)$.

1.1.2 Binomal approx to Poisson

Theorem 1.2. *For $\lambda = np$,*

$$\lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}$$

First, we need the following lemma

Lemma 1.1. *Let $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t$ be the Gamma function, then the following equation holds*

$$\lim_{n\to\infty} \frac{\binom{n}{k}}{n^k} = \frac{1}{k!}.$$

Proof of Lemma 1.1. Note first that

$$\Gamma(k+1) = k\Gamma(k) = \cdots = k!\Gamma(1) = k!.$$

and *Stirling's Formula for Gamma function* states that

$$\Gamma(x+1) = \sqrt{2\pi x}\, x^x e^{-x} \left(1 + \frac{1}{12x} \cdots\right).$$

Then we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\binom{n}{k}}{n^k} &= \lim_{n\to\infty} \frac{n!}{n^k (n-k)!k!} \\
&= \lim_{n\to\infty} \frac{\Gamma(n+1)}{n^k \Gamma(n-k+1)\Gamma(k+1)} \\
&= \frac{1}{\Gamma(k+1)} \lim_{n\to\infty} \sqrt{\frac{n}{n-k}} \left(\frac{n}{e}\right)^n \left(\frac{e}{n-k}\right)^{n-k} \frac{1}{n^k}.
\end{aligned}$$

Where we used the first order of Stirling's Formula for Gamma function in the last equality.

Next, we only need to prove the limit equals to 1:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{\frac{n}{n-k}} \left(\frac{n}{e}\right)^n \left(\frac{e}{n-k}\right)^{n-k} \frac{1}{n^k} &= \lim_{n\to\infty} \sqrt{\frac{n}{n-k}} \frac{1}{e^k} \frac{(1-k/n)^k}{(1-k/n)^n} \\
&= 1,
\end{aligned}$$

since

$$\begin{aligned}
\sqrt{\frac{n}{n-k}} &\to 1, \\
(1-k/n)^k &\to 1, \\
(1-k/n)^n &\to e^{-k},
\end{aligned}$$

as $n \to \infty$. ∎

Now we may prove the main theorem

Proof of Theorem 1.2. Let $p = \lambda/n$ and fix k, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\binom{n}{k}}{n^k} &= \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\gamma}{n}\right)^{n-k} \\
&= \lim_{n\to\infty} \frac{n^k}{k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \qquad \text{by Lemma 1.1} \\
&= \frac{\lambda^k}{k!} \left[\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n\right] \underbrace{\left[\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-k}\right]}_{=1} \\
&= \frac{\lambda^k}{k!} e^{-\lambda}.
\end{aligned}$$

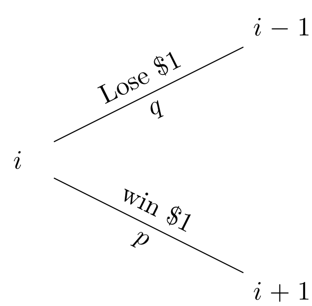
■

1.2 Markov Chain

Theorem 1.3 (Gambler's ruin)**.** *Let p is the probability of winning \$1 and $q = 1 - p$ is the probability of losing \$1. Suppose that a person starts with \$$i$ amount of money where $0 < i < N$ and $N \in \mathbb{N}$ is the number he would want to walk away with. The probability of him winning from this starting position is given by the following equation:*

$$a_i = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & \text{where } p \neq q, \\ \frac{i}{N} & \text{where } p = q = \frac{1}{2}. \end{cases}$$

Proof. Note that



Then, we observe

$$\begin{aligned}
a_i &= pa_{i+1} + qa_{i-1} \\
\underbrace{(p+q)}_{=1} a_i &= pa_{i+1} + qa_{i-1} \\
a_{i+1} - a_i &= \frac{q}{p}(a_i - a_{i-1}).
\end{aligned}$$

Denote $b_j := a_i - a_{i-1}$ and note that $b_1 = a_1$ since $a_0 = 0$ is the absorption edge, we deduce by recursion that

$$b_{i+1} = \left(\frac{q}{p}\right)^{i+1} a_1.$$

Moreover, note that

$$
\begin{aligned}
a_{i+1} - a_1 &= \sum_{i}^{k=1} a_{k+1} - a_k \\
&= \sum_{k=1}^{i} \left(\frac{q}{p}\right)^k a_1. \\
\Rightarrow a_{i+1} &= a_1 \left(1 + \sum_{k=1}^{i} \left(\frac{q}{p}\right)^k\right) \\
&= a_1 \left(\sum_{k=0}^{i} \left(\frac{q}{p}\right)^k\right).
\end{aligned}
$$

If $p \neq q$, by geometric series, we obtain

$$
a_{i+1} = a_1 \frac{1 - \left(\frac{q}{p}\right)^{i+1}}{1 - \left(\frac{q}{p}\right)}.
$$

If $p = q = 0.5$, then

$$
a_{i+1} = a_1(i+1).
$$

Now to solve for a_1, observe that since at absoprtion point $a_N = 1$, then letting $i + 1 = N$, we get

$$
1 = a_N = \begin{cases} a_1 \frac{1-\left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)} & \text{where } p \neq q, \\ a_1 N & \text{where } p = q = \frac{1}{2}, \end{cases}
$$

which implies

$$
a_1 = \begin{cases} \frac{1-\left(\frac{q}{p}\right)}{1-\left(\frac{q}{p}\right)^N} & \text{where } p \neq q, \\ \frac{1}{N} & \text{where } p = q = \frac{1}{2}. \end{cases}
$$

Substituting this back, we obtained the desired equation. ■

Corollary 1.1. *The expectation of stopping time with respect to Brownian motion, i.e.* $\tau = \inf\{t \geqslant 0 : B_\tau \in \{-\alpha, \beta\}\}$, *for any positive integer* α *and* β *is given by*

$$
\mathbb{E}[\tau] = \alpha\beta.
$$

Proof. From Theorem (1.3) we observe that $\Pr(B_\tau = b) = \frac{a}{a+b}$. Then, note that since $1 = \Pr(B_\tau = -\alpha) + \Pr(B_\tau = \beta)$, we have

$$
\Pr(B_\tau = -\alpha) = \frac{b}{a+b}.
$$

Since $(B_t^2 - t)_{t \geqslant 0}$ is a martingale, we by apply Optimal Stopping Theorem where

$$
0 = \mathbb{E}[B_0^2 - 0] = \mathbb{E}[B_\tau^2 - \tau],
$$

which implies

$$
\mathbb{E}[\tau] = \mathbb{E}[B_\tau^2] = \alpha^2 \Pr(B_\tau = -\alpha) + \beta^2 \Pr(B_\tau = \beta) = \alpha\beta.
$$

■

2 Financial Mathematics

Suppose

$$\begin{cases} \mathrm{d}S_t &= \mu S_t\,\mathrm{d}t + \sigma S_t dW_t \\ S_t\big|_{t=0} &= S_0. \end{cases} \tag{2.1}$$

and risk free is

$$\begin{cases} \mathrm{d}B_t &= rB_t\,\mathrm{d}t \\ B_t\big|_{t=0} &= B_0. \end{cases}$$

Note that the solution to (2.1) is

$$\begin{aligned} \mathrm{d}\log S_t &= \frac{1}{S_t}\,\mathrm{d}S_t - \frac{1}{2}\frac{1}{S_t^2}\,(\mathrm{d}S_t)^2 \\ &= \frac{1}{S_t}\,(\mu S_t\,\mathrm{d}t + \sigma S_t dW_t) - \frac{1}{2}\frac{1}{S_t^2}\big(\sigma^2 S_t^2 \underbrace{(\mathrm{d}W_t)^2}_{\mathrm{d}t} + O(dt)\big) \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\,\mathrm{d}W_t\,. \\ \Rightarrow S_t &= S_0 \exp\!\left((\mu - \frac{1}{2}\sigma^2)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\right) \end{aligned}$$

2.1 Girsanov Theorem

Theorem 2.1. *Denote*

$$\eta_t := \exp\!\left(-\int_0^t \lambda(u)\,\mathrm{d}W^{\mathbb{P}}(u) - \frac{1}{2}\int_0^t \lambda^2(u)\,\mathrm{d}u\right),$$

where $W^{\mathbb{P}}(t)$ is Wiener process under $\mathbb{P}$ and $\lambda(t)$ is $\mathcal{F}_t$-measurable to be chosen. Then we can have the following Wiener process under arbitrage-free Wiener process $W^{\mathbb{Q}}(t)$ defined by

$$W^{\mathbb{Q}}(t) = W^{\mathbb{P}}(t) + \int_0^t \lambda(u)\,\mathrm{d}u\,.$$

The measure $\mathbb{Q}$ is defined by

$$\mathbb{Q}[A] = \int_A \eta_t\,\mathrm{d}P\,,$$

where $A \in \mathcal{F}_t$.

We may also write it as

$$\mathrm{d}W^{\mathbb{Q}} = \mathrm{d}W^{\mathbb{P}} + \lambda(t)\,\mathrm{d}t\,. \tag{2.2}$$

Remark 2.1 (**Risk neutral GBM**). Choosing $\lambda(t) = \frac{\mu - r}{\sigma}$, the GBM process under risk-neutral measure can be written as

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W^{\mathbb{P}} \\ &= \mu S_t\,\mathrm{d}t + \sigma S_t\big[\,\mathrm{d}W^{\mathbb{Q}} - \lambda(t)\,\mathrm{d}t\,\big] \\ &= rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W^{\mathbb{Q}}\,. \end{aligned}$$

Remark 2.2 (**Change of Numeraire**). Choosing $\lambda(t) = \frac{\mu - r}{\sigma}$, the discounted risk neutral process can be expressed as

$$\mathrm{d}\left(\frac{S_t}{B_t}\right) = \frac{S_t}{B_t}(\mu - r)\,\mathrm{d}t + \sigma\,\mathrm{d}W^{\mathbb{P}}$$

$$
\begin{aligned}
&= \frac{S_t}{B_t}(\mu - r)\,\mathrm{d}t + \sigma\big[\,\mathrm{d}W^{\mathbb{Q}} - \lambda(t)\,\mathrm{d}t\,\big] \\
&= \frac{S_t}{B_t}\sigma\,\mathrm{d}W^{\mathbb{Q}}\,,
\end{aligned}
$$

which is a Martingale.

Remark 2.3. Since choice of $\lambda(t)$ is unique here to generate a Martingle process, $\mathbb{Q}$ is known as unique equivalent martingale (EMM) which equals to say that $\mathbb{Q}$ is a risk-neutral measure.

2.2 Black-Scholes

Suppose the call-option value process (admissible, BS is closed and whole other assumptinos)

$$
C(S_t) = V_t((S_T - K)^+).
$$

From Ito lemma, we have

$$
\begin{aligned}
\mathrm{d}C(S_t,t) &= \frac{\partial C(S_t)}{\partial t}\,\mathrm{d}t + \frac{\partial C(S_t)}{\partial S_t}\,\mathrm{d}S_t + \frac{1}{2}\frac{\partial^2 C(S_t)}{\partial S_t^2}\,(\mathrm{d}S_t)^2 \\
&= \frac{\partial C(S_t)}{\partial t}\,\mathrm{d}t + \frac{\partial C(S_t)}{\partial S_t}\,(rS_t\,\mathrm{d}t + \sigma S_t dW_t) + \frac{1}{2}\frac{\partial^2 C(S_t)}{\partial S_t^2}\sigma^2 S_t^2\,\mathrm{d}t \\
&= \left(\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2}\right)\mathrm{d}t + \sigma S_t\frac{\partial C(S_t)}{\partial S_t}\,\mathrm{d}W_t\,.
\end{aligned}
$$

Moreover, when we consider risk-free bond, we get

$$
\begin{aligned}
\mathrm{d}\left(\frac{C(S_t,t)}{B_t}\right) &= C(S_t,t)\,\mathrm{d}\left(\frac{1}{B_t}\right) + \frac{1}{B_t}\,\mathrm{d}C(S_t,t) \\
&= -C(S_t,t)\frac{1}{B_t^2}\,\mathrm{d}B_t\,\frac{1}{B_t} + \frac{1}{B_t}\,\mathrm{d}C(S_t,t) \\
&= -C(S_t,t)\frac{r}{B_t}\,\mathrm{d}t + \frac{1}{B_t}\,\mathrm{d}C(S_t,t) \\
&= \frac{1}{B_t}\left(\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2} - rC\right)\mathrm{d}t + \sigma\frac{S_t}{B_t}\frac{\partial C(S_t)}{\partial S_t}\,\mathrm{d}W_t
\end{aligned}
$$

For no arbitrage, there is no drift term, so the Black-Scholes equation equals to

$$
\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2} - rC = 0 \tag{2.3}
$$

Remark 2.4. If there is a dividend d, the BS equation is

$$
\frac{\partial C}{\partial t} + (r - d)S_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2} - rC = 0
$$

2.2.1 Solving Black-Scholes Equation

From (2.3), we denote the operator

$$
L := -\left((r-d)S_t\frac{\partial}{\partial S_t} + \frac{\sigma^2}{2}S_t^2\frac{\partial^2}{\partial S_t^2} - r\right),
$$

where d is the dividend. Then, BS can be rewritten as

$$
\begin{cases}
(\partial_t + L)C(S_t,t) &= 0 \\
C(x,0) &= (e^x - e^{x_0})^+,
\end{cases} \tag{2.4}
$$

where $x = \log S$.

Denote

$$\alpha := \frac{\sigma^2}{2}$$
$$\beta := (d - r)$$
$$\tau := T - t$$

Then

$$L = -\alpha\partial_x^2 + \beta\partial_x + r.$$

We can calculate the heat-kernel of the BS-equation by using Fourier Transform, $U_{\mathbf{BS}}(\tau; x - y)$. Thus,

$$C_{\mathbf{BS}}(\tau, x) = \int_{x_0}^{\infty} U_{\mathbf{BS}}(\tau; x - y) C(y, 0)\,\mathrm{d}y\,,$$

where the kernel is obtained by calculating

$$\begin{aligned}
U_{\mathbf{BS}}(\tau; x - y) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-p(x-y)} e^{-\tau(\alpha p^2 + i\beta p + r)}\,\mathrm{d}y \\
&= \frac{e^{-r\tau}}{2\pi}\int_{\mathbb{R}} e^{-\left[p\sqrt{\alpha\tau} - i\frac{x-y-\beta\tau}{2\sqrt{\alpha\tau}}\right]^2}\,\mathrm{d}y \\
&= \frac{e^{-r\tau}}{\sqrt{4\pi\alpha\tau}}\exp\left(-\frac{(x-y-\beta\tau)^2}{4\alpha\tau}\right).
\end{aligned}$$

2.2.2 Derivation of European Call Option

Proposition 2.1. *Let $g(0,\infty) \to \mathbb{R}_+$ be a claim such that $g(x) \leqslant c(1+x)$ for some $c \geqslant 0$ and $X = g(S_T)$. Then the value process $(V_t(X))_{t\in[0,T]}$ satisfies $V_t(X) = \nu(t, S_t)$ where*

$$\nu(t, x) = e^{-r(T-t)}\int_{\mathbb{R}} g(e^y)\phi_{\log(x)+(T-t)(r-\sigma^2/2),\ \sigma^2(T-t)}(y)\,\mathrm{d}y\,, \tag{2.5}$$

and $\varphi(\mu, \sigma^2)$ is the probability density function of normal distribution i.e.

$$\phi(\mu, \sigma^2)(y) := \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(\mu - y)^2}{2\sigma^2}\right).$$

Proof. From discounted process we have

$$\begin{aligned}
\hat{V}_t(X) &= \mathbb{E}^Q[\hat{X}|\mathcal{F}_t] \\
&= e^{-rT}\mathbb{E}^Q[g(S_T)|\mathcal{F}_t] \\
&= e^{-rT}\mathbb{E}^Q\left[g\left(S_t\exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right)\right) \mid \mathcal{F}_t\right] \\
&= e^{-rT}\mathbb{E}^Q\left[g\left(\exp\left(\log(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right)\right) \mid \mathcal{F}_t\right],
\end{aligned}$$

where we used the solution to the Stochastic GBM. Note here that $(W_T - W_t) \sim \mathcal{N}(0, T-t)$. Thus, we have

$$\begin{aligned}
\hat{V}_t(X) &= \mathbb{E}^Q[\hat{X}|\mathcal{F}_t] \\
&= e^{-rT}\int_{\mathbb{R}} g\left(\exp\left(\log(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma y\right)\right)\phi_{0,T-t}(y)\,\mathrm{d}y \\
&= e^{-rT}\int_{\mathbb{R}} g(e^z)\phi_{\log(x)+(T-t)(r-\sigma^2/2),\ \sigma^2(T-t)}(z)\,\mathrm{d}z\,.
\end{aligned}$$

Finally, observe that the discounted value process is given by $\hat{V}_t(X) = e^{-rt}V_t(X)$ then we are done. ∎

Theorem 2.2 (Call option pricing)**.** *Let $C(S_t, t) := V_t((S_T - K)^+)$ value process of the call option, then we have*

$$C(S_t, t) = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \tag{2.6}$$

where $\Phi(x)$ ($\mathbb{P}(X < x)$) is the distribution of standard normal distribution $\mathcal{N}(0, 1)$ and

$$\begin{aligned} d_1 &:= \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{S_t}{K}\right) + (r + \sigma^2/2)(T-t)\right), \\ d_2 &:= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

Proof. Let $g(x) = (x - K)^+$ then from Proposition 2.1, we get

$$\begin{aligned} C(S_t, t) &= V_t((S_t - K)^+) \\ &= e^{-r(T-t)}\int_{\mathbb{R}} \underbrace{(e^y - K)^+}_{=g(e^y)} \phi_{\log(S_t)+(T-t)(r-\sigma^2/2),\ \sigma^2(T-t)}(y)\,\mathrm{d}y \\ &= e^{-r(T-t)}\int_{\mathbb{R}} \left(x\exp\left(\left(r - \frac{\sigma}{2}\right)(T-t) + \sigma\sqrt{T-t}z\right) - K\right)^+ \phi_{0,1}(z)\,\mathrm{d}z\,, \end{aligned}$$

where $x = S_t$.

Let

$$\begin{aligned} \tau &:= T - t, \\ \widetilde{r} &:= r - \frac{\sigma^2}{2}, \\ \ell &:= \frac{\log(K/x) - \widetilde{r}\tau}{\sigma\sqrt{\tau}}. \end{aligned}$$

Thus, we observe that $\left(x\exp\left(\left(r - \frac{\sigma}{2}\right)(T-t) + \sigma\sqrt{T-t}z\right) - K\right)^+$ is non-zero iff

$$\begin{aligned} x\exp\left(\left(r - \frac{\sigma}{2}\right)(T-t) + \sigma\sqrt{T-t}z\right) - K > 0 \\ \Leftrightarrow y > \ell \end{aligned}$$

Then

$$\begin{aligned} C(S_t, t) &= e^{-r\tau}\int_\ell^\infty \left(x\exp\left(\widetilde{r}\tau + \sigma\sqrt{\tau}z\right) - K\right)\phi_{0,1}(z)\,\mathrm{d}z \\ &= e^{-r\tau}\int_\ell^\infty \left(x\exp\left(\widetilde{r}\tau + \sigma\sqrt{\tau}z\right) - K\right)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\right)\mathrm{d}z \\ &= x\int_\ell^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(z - \sigma\sqrt{\tau})}{2}\right)\mathrm{d}z - e^{-r\tau}K\Phi(-\ell) \\ &= x\int_{\ell-\sigma\sqrt{\tau}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,\mathrm{d}z - e^{-r\tau}K\Phi(-\ell) \\ &= x\Phi(-\ell + \sigma\sqrt{\tau}) - e^{-r\tau}K\Phi(-\ell) \end{aligned}$$

■

Theorem 2.3 (Replicating strategies)**.** *The replicating trading strategy $\varphi = (\varphi_t^0, \varphi_t^1)_{t\in[0,T]}$ of X is given by*

$$\begin{aligned} \varphi_t^0 &= e^{-rt}\left(\nu(S_t, t) - S_t\frac{\partial}{\partial x}\nu(S_t, t)\right), \\ \varphi_t^1 &= \frac{\partial}{\partial x}\nu(S_t, t), \end{aligned} \tag{2.7}$$

where ν is defined in (2.5)

2.2.3 Put-Call Parity

$$S_t - Ke^{-r(T-t)} = C(S_t, t) - P(S_t, t). \tag{2.8}$$

2.2.4 Barrier options

Suppose a down-and-out option

$$X := (S_t - K)^+ \mathbb{1}_{\min_{t\in[0,T]} S_t > \beta}$$

The value process is therefore,

$$V_t(X) = e^{-r(T-t)} \mathbb{E}^Q[X|\mathcal{F}_t]$$

3 Stochastic Volatility

4 Time Series

4.1 Stationary and Autocorrelation

Definition 4.1. *The **autocovariance***

$$\gamma(s,t) := \mathbb{E}\left[(x_s - \mu_s)(x_t - \mu_t)\right],$$

*and **cross-covar.***

$$\rho_{x,y}(s,t) := \mathbb{E}\left[(x_s - \mu_{xs})(y_t - \mu_{ts})\right].$$

Definition 4.2. *The **autocorrelation function** (ACF) is defined as*

$$\rho_x(s,t) := \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}},$$

Notation: $\rho_x(t,t+h) \equiv \rho(h)$.

Definition 4.3 (Strictly stationary)**.** *For all $k \in \mathbb{N}$, $t_k \in \mathbb{N}$ and $c_k \in \mathbb{R}$. Then the time series is strictly stationary if*

$$\mathbb{P}\big(x_{t_1} \leqslant c_1, \ldots x_{t_k} \leqslant c_k\big) = \mathbb{P}\big(x_{t_1+h} \leqslant c_1, \ldots x_{t_k+h} \leqslant c_k\big),$$

for given probability measure $\mathbb{P}$ and constant $h \in \mathbb{N}$.

Definition 4.4 (Weakly stationary)**.** *The time series is strictly stationary if*

1. *$\mu_t = \mathbb{E}[x_t]$ is constant, and*
2. *$\gamma(s,t)$ depends only on $|s-t|$.*

Lemma 4.1. *Assume time series is weakly stationary. Using the notation $\gamma(t,t+h) \equiv \gamma(h)$, we have*

1. *$|\gamma(t)| \leqslant \gamma(0)$,*
2. *$\gamma(h) = \gamma(-h)$.*

Proof.

1. By Cauchy-Schwarz inequality $|\gamma(t,t+h)|^2 \leqslant \gamma(t,t)\gamma(t+h,t+h)$ by definition $|\gamma(h)|^2 \leqslant \gamma(0)\gamma(0)$.
2. From definition of covariance we have

$$\begin{aligned}
\gamma(h) &= \gamma(t+h-t) \\
&= \mathbb{E}\left[(x_{t+h} - \mu)(x_t - \mu)\right] \\
&= \mathbb{E}\left[(x_t - \mu)(x_{t+h} - \mu)\right] \\
&= \gamma(t - (t+h)) = \gamma(-h)
\end{aligned}$$

∎

Example 4.1. Suppose w_t for all $t \in N$ is white noise process, i.e. iid $w_t \approx N(0, \sigma_w^2)$. Given two series

$$\begin{aligned}
x_t &= w_t + w_{t-1}, \\
y_t &= w_t - w_{t-1}.
\end{aligned}$$

Then for any $h \in \mathbb{N}$, we get the following for (cross-)covariance:

1.

$$\begin{aligned}\gamma_x(0) &= \mathbb{E}[(w_t + w_{t-1})^2] = \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1}^2] + 2\underbrace{\mathbb{E}[w_t w_{t-1}]}_{=0} \\ &= 2\sigma_w^2, \\ \gamma_y(0) &= \mathbb{E}[(w_t - w_{t-1})^2] = \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1}^2] - 2\underbrace{\mathbb{E}[w_t w_{t-1}]}_{=0} \\ &= 2\sigma_w^2.\end{aligned}$$

2.

$$\begin{aligned}\gamma_x(1) &= \mathbb{E}\left[(w_t + w_{t-1})(w_{t+1} + w_t)\right] \\ &= \mathbb{E}[w_t w_{t+1}] + \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1} w_{t+1}] + \mathbb{E}[w_{t-1} w_t] \\ &= \sigma_w = \gamma_x(-1), \\ \gamma_y(1) &= \mathbb{E}[w_t w_{t+1}] - \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1} w_{t+1}] - \mathbb{E}[w_{t-1} w_t] \\ &= -\sigma_w = \gamma_y(-1).\end{aligned}$$

3. $\gamma_{xy}(0) = \mathbb{E}[w_t^2] - \mathbb{E}[w_{t-1}^2] = 0$ and $\gamma_{xy}(1) = cov(x_{t+1}, y_t) = -\sigma_w^2$ and $\gamma_{xy}(-1) = cov(x_t, y_{t-1}) = -\sigma_w^2$

4. ACF

$$\rho_{xy}(h) = \begin{cases} 0 & h = 0, \\ 1/2 & h = 1, \\ -1/2 & h = -1, \\ 0 & |h| \geqslant 2 \end{cases} \tag{4.1}$$

Thus the joint time series is stationary.

Definition 4.5 (Linear Process)**.** *$\{x_t\}$ is linear process if*

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty,$$

where w_t is white noise.

Definition 4.6 (Gaussian Process)**.** *$\{x_t\}$ is Gaussian process if the vector $x := (x_{t_1}, x_{t_2}, \dots, x_{t_n})' \in \mathbb{R}^n$ has a multivariate normal distribution with density function*

$$f(x) = \frac{1}{(2\pi)^{n/2}} \det\left(\Gamma\right)^{-1/2} \exp\left(-\frac{1}{2}(x-\mu)'\Gamma^{-1}(x-\mu)\right),$$

where $\mu \in \mathbb{R}^n$ and $\Gamma := var(x) = \{\gamma(t_i, t_j);\ i, j = 1, \dots, n\}$.

4.1.1 ARIMA

ARMA(p, q):

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = w_t + \theta_t w_{t-1} + \cdots + \theta_q w_{t-q}.$$

4.2 Autoregressive Model

Definition 4.7 ($AR(p)$)**.** *Let w_t be white noise, $AR(p)$ is defined as*

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = w_t.$$

Definition 4.8 (Autoregressive Operator)**.** *Let B be time-lagged operator, i.e. $Bx_t = x_{t-1}$, the autoregressive operator is defined as*

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p.$$

Example 4.2. Assuming $\phi_j = \phi$ for all $j \in \mathbb{N}$, from AR(1)

$$\begin{aligned} x_t &= \phi x_{t-1} + w_t = \phi(\phi x_{t-2} + w_{t-1}) + w_t \\ &= \phi^2 x_{t-2} + \phi w_{t-1} + w_t \\ &\vdots \\ &= \phi^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j w_{t-j}. \end{aligned}$$

Thus, if $|\phi| < 1$ and x_t is sationary then for large $k \to \infty$, we have

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}.$$

Taking expectation value,

$$\mathbb{E}[x_t] = \sum_{j=0}^{\infty} \phi_j \mathbb{E}[w_t] = 0.$$

Moreover, for $h \geqslant$ the autocovriance function is

$$\begin{aligned} \gamma(h) &= \mathbb{E}\left[\left(\sum_{j=0}^{\infty} \phi^j w_{t+h-j} \right) \left(\sum_{j=0}^{\infty} \phi^k w_{t-k} \right) \right] \\ &= \mathbb{E}\big[(w_{t+h-1} + \phi w_{t+h-2} + \cdots \phi^h w_t + \phi^{h+1} w_{t-1} + \cdots)(w_t + \phi w_{t-1} + \cdots) \big] \\ &= \sigma_w^2 \sum_{j=0} \phi^{h+j} \phi^j = \sigma_w^2 \phi^h \sum_{j=0} \phi^{2j} \\ &= \frac{\sigma_w^2 \phi^h}{1 - \phi^2}. \end{aligned}$$

And ACF is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h.$$

As well as

$$\rho(h) = \phi \rho(h-1).$$

Definition 4.9. *The **moving average operator** is defined as*

$$\theta(B) := 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q.$$

5 Operational Research

5.1 Safety Stock

Theorem 5.1. *Let the lead-time T be normally distributed, i.e. $T \sim N(\mu_\ell, \sigma_\ell^2)$ and denote the set of demand of period i as $\{D_i\}_{i=1}^T$ where $D_i \sim N(\mu_d, \sigma_d^2)$ for each $1 \leqslant i \leqslant T$. Then the safety stock level, SS satisifies the following equation*

$$SS = z_\alpha \sqrt{\sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2},$$

where z_α is the Z-value of a desired α which is chosen.

Proof. Denote the demand between within lead-time as $D(T) := \sum_{i=1}^T D_i$. The safety stock level is then set at

$$SS = z_\alpha \sqrt{Var(D(T))}.$$

Thus we need only to prove $Var(D(T))$.

From Towering-property, note that

$$\begin{aligned}\mathbb{E}[D(T)] &= \mathbb{E}\big[\mathbb{E}[D(T)|T]\big] \\ &= \mathbb{E}\big[T\mu_d\big] = \mu_d \mathbb{E}[T] = \mu_d \mu_\ell.\end{aligned}$$

Furthemore, again with Towering-property, we have

$$\begin{aligned}\mathbb{E}[D^2(T)] &= \mathbb{E}\big[\mathbb{E}[D^2(T)|T]\big] \\ &= \mathbb{E}\big[Var[D(T)|T] + \big(\mathbb{E}[D(T)|T]\big)^2\big] \\ &= \mathbb{E}\big[T\sigma_d^2 + \big(T\mu_d\big)^2\big] \\ &= \sigma_d^2 \mathbb{E}[T] + \mu_d^2 \mathbb{E}[T^2] \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \left(Var(T) + (\mathbb{E}[T])^2\right) \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \left(\sigma_\ell^2 + \mu_\ell^2\right).\end{aligned}$$

Finally,

$$\begin{aligned}Var(D(T)) &= \mathbb{E}[D^2(T)] - \big(\mathbb{E}[D(T)]\big)^2 \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \left(\sigma_\ell^2 + \mu_\ell^2\right) - (\mu_d \mu_\ell)^2 \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2.\end{aligned}$$

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