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1.1 Safety Stock

Theorem 1.1. *Let the lead-time T be normally distributed, i.e. $T \sim N(\mu_\ell, \sigma_\ell^2)$ and denote the set of demand of period i as $\{D_i\}_{i=1}^T$ where $D_i \sim N(\mu_d, \sigma_d^2)$ for each $1 \leqslant i \leqslant T$. Then the safety stock level, SS satisifies the following equation*

$$SS = z_\alpha \sqrt{\sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2},$$

where z_α is the Z-value of a desired α which is chosen.

Proof. Denote the demand between within lead-time as $D(T) := \sum_{i=1}^N D_i$. The safety stock level is then set at

$$SS = z_\alpha \sqrt{Var(D(T))}.$$

Thus we need only to prove $Var(D(T))$.

From Towering-property, note that

$$\begin{aligned}\mathbb{E}[D(T)] &= \mathbb{E}\big[\mathbb{E}[D(T)|T]\big] \\ &= \mathbb{E}\big[T\mu_d\big] = \mu_d \mathbb{E}[T] = \mu_d \mu_\ell.\end{aligned}$$

Furthemore, again with Towering-property, we have

$$\begin{aligned}\mathbb{E}[D^2(T)] &= \mathbb{E}\big[\mathbb{E}[D^2(T)|T]\big] \\ &= \mathbb{E}\Big[Var[D(T)|T] + \big(\mathbb{E}[D(T)|T]\big)^2\Big] \\ &= \mathbb{E}\Big[T\sigma_d^2 + \big(T\mu_d\big)^2\Big] \\ &= \sigma_d^2 \mathbb{E}[T] + \mu_d^2 \mathbb{E}[T^2] \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \left(Var(T) + (\mathbb{E}[T])^2\right) \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \left(\sigma_\ell^2 + \mu_\ell^2\right).\end{aligned}$$

Finally,

$$\begin{aligned}Var(D(T)) &= \mathbb{E}[D^2(T)] - \big(\mathbb{E}[D(T)]\big)^2 \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \left(\sigma_\ell^2 + \mu_\ell^2\right) - (\mu_d \mu_\ell)^2 \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2.\end{aligned}$$

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1.2 Binomal approx to Poisson

Theorem 1.2. *For* $\lambda = np$,

$$\lim_{n\to\infty} \binom{n}{k} p^k 1 - p^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}$$