1 Probability Theory

1.1 Some ole' Binomal shits

1.1.1 Expected value and variance of Binomial

Theorem 1.1. *Let $X \approx Bin(n,p)$, then the expected value and variance is given by*

1. $\mathbb{E}(X) = np$,
2. $Var(X) = np(1-p)$.

1.1.2 Binomal approx to Poisson

Theorem 1.2. *For $\lambda = np$,*

$$\lim_{n\to\infty} \binom{n}{k} p^k 1 - p^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}$$

First, we need the following lemma

Lemma 1.1. *Let $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t$ be the Gamma function, then the following equation holds*

$$\lim_{n\to\infty} \frac{\binom{n}{k}}{n^k} = \frac{1}{k!}.$$

Proof of Lemma 1.1. Note first that

$$\Gamma(k+1) = k\Gamma(k) = \cdots = k!\Gamma(1) = k!.$$

and *Stirling's Formula for Gamma function* states that

$$\Gamma(x+1) = \sqrt{2\pi x} x^x e^{-x} \left(1 + \frac{1}{12x} \cdots\right).$$

Then we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\binom{n}{k}}{n^k} &= \lim_{n\to\infty} \frac{n!}{n^k (n-k)! k!} \\
&= \lim_{n\to\infty} \frac{\Gamma(n+1)}{n^k \Gamma(n-k+1)\Gamma(k+1)} \\
&= \frac{1}{\Gamma(k+1)} \lim_{n\to\infty} \sqrt{\frac{n}{n-k}} \left(\frac{n}{e}\right)^n \left(\frac{e}{n-k}\right)^{n-k} \frac{1}{n^k}.
\end{aligned}$$

Where we used the first order of Stirling's Formula for Gamma function in the last equality.

Next, we only need to prove the limit equals to 1:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{\frac{n}{n-k}} \left(\frac{n}{e}\right)^n \left(\frac{e}{n-k}\right)^{n-k} \frac{1}{n^k} &= \lim_{n\to\infty} \sqrt{\frac{n}{n-k}} \frac{1}{e^k} \frac{(1-k/n)^k}{(1-k/n)^n} \\
&= 1,
\end{aligned}$$

since

$$\begin{aligned}
\sqrt{\frac{n}{n-k}} &\to 1, \\
(1-k/n)^k &\to 1, \\
\frac{(1-k/n)^k}{(1-k/n)^n} &\to e^{-k},
\end{aligned}$$

as $n \to \infty$. ∎

Now we may prove the main theorem

Proof of Theorem 1.2. Let $p = \gamma/n$ and fix k, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\binom{n}{k}}{n^k} &= \lim_{n\to\infty} \binom{n}{k} \left(\frac{\gamma}{k}\right)^k \left(1 - \frac{\gamma}{n}\right)^{n-k} \\
&= \lim_{n\to\infty} \frac{n^k}{k!} \left(\frac{\gamma}{k}\right)^k \left(1 - \frac{\gamma}{n}\right)^{n-k} \qquad \text{by Lemma 1.1} \\
&= \frac{\gamma^k}{k!} \left[\lim_{n\to\infty} \left(1 - \frac{\gamma}{n}\right)^n\right] \left[\lim_{n\to\infty} \left(1 - \frac{\gamma}{n}\right)^k\right] \\
&= \frac{\gamma^k}{k!} e^{-\gamma}(1)
\end{aligned}$$

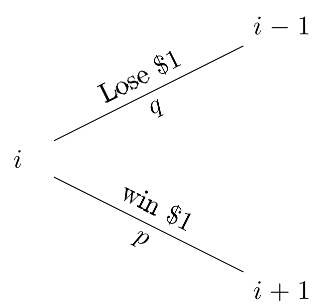
■

1.2 Gambler's Ruin

Let p is the probability of winning \$1 and $q = 1 - p$ is the probability of losing \$1. Suppose that a person starts with \$$i$ amount of money where $0 < i < N$ and $N \in \mathbb{N}$ is the number he would want to walk away with. The probability of him winning from this starting position is given by the following equation:

$$a_k = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & \text{where } p \neq q, \\ \frac{i}{N} & \text{where } p = q = \frac{1}{2}. \end{cases}$$

Proof. Note that



Then, we observe

$$\begin{aligned}
a_i &= pa_{i+1} + qa_{i-1} \\
\underbrace{(p+q)}_{=1} a_i &= pa_{i+1} + qa_{i-1} \\
a_{i+1} - a_i &= \frac{p}{q}(a_i - a_{i-1}).
\end{aligned}$$

Denote $b_j := a_i - a_{i-1}$ and note that $b_1 = a_1$ since $a_0 = 0$ is the absorption edge, we deduce by recursion that

$$b_{i+1} = \left(\frac{p}{q}\right)^{i+1} a_1.$$

Moreover, note that

$$a_{i+1} - a_1 = \sum_{i}^{k=1} a_{k+1} - a_k$$

$$
\begin{aligned}
&= \sum_{k=1}^{i} \left(\frac{p}{q}\right)^k a_1. \\
\Rightarrow a_{i+!} &= a_1 \left(1 + \sum_{k=1}^{i} \left(\frac{p}{q}\right)^k\right) \\
&= a_1 \left(\sum_{k=0}^{i} \left(\frac{p}{q}\right)^k\right).
\end{aligned}
$$

If $p \neq q$, by geometric series, we obtain

$$
a_{i+1} = a_1 \frac{1 - \left(\frac{p}{q}\right)^{i+1}}{1 - \left(\frac{p}{q}\right)}.
$$

If $p = q = 0.5$, then

$$
a_{i+1} = a_1(i+1).
$$

Now to solve for a_1, observe that since at absoprtion point $a_N = 1$, then letting $i + 1 = N$, we get

$$
1 = a_N = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)} & \text{where } p \neq q, \\ a_1 N & \text{where } p = q = \frac{1}{2}, \end{cases}
$$

which implies

$$
a_1 = \begin{cases} \frac{1-\left(\frac{q}{p}\right)}{1-\left(\frac{q}{p}\right)^N} & \text{where } p \neq q, \\ \frac{1}{N} & \text{where } p = q = \frac{1}{2}. \end{cases}
$$

Substituting this back, we obtained the desired equation. ■

2 Operational Research

2.1 Safety Stock

Theorem 2.1. *Let the lead-time T be normally distributed, i.e. $T \sim N(\mu_\ell, \sigma_\ell^2)$ and denote the set of demand of period i as $\{D_i\}_{i=1}^T$ where $D_i \sim N(\mu_d, \sigma_d^2)$ for each $1 \leqslant i \leqslant T$. Then the safety stock level, SS satisifies the following equation*

$$
SS = z_\alpha \sqrt{\sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2},
$$

where z_α is the Z-value of a desired α which is chosen.

Proof. Denote the demand between within lead-time as $D(T) := \sum_{i=1}^T D_i$. The safety stock level is then set at

$$
SS = z_\alpha \sqrt{Var(D(T))}.
$$

Thus we need only to prove $Var(D(T))$.

From Towering-property, note that

$$
\begin{aligned}
\mathbb{E}[D(T)] &= \mathbb{E}\big[\mathbb{E}[D(T)|T]\big] \\
&= \mathbb{E}\big[T\mu_d\big] = \mu_d \mathbb{E}[T] = \mu_d \mu_\ell.
\end{aligned}
$$

Furthemore, again with Towering-property, we have

$$
\mathbb{E}[D^2(T)] = \mathbb{E}\big[\mathbb{E}[D^2(T)|T]\big]
$$

$$
\begin{aligned}
&= \mathbb{E}\big[Var[D(T)|T] + \big(\mathbb{E}[D(T)|T]\big)^2\big] \\
&= \mathbb{E}\big[T\sigma_d^2 + \big(T\mu_d\big)^2\big] \\
&= \sigma_d^2\mathbb{E}[T] + \mu_d^2\mathbb{E}[T^2] \\
&= \sigma_d^2\mu_\ell + \mu_d^2\left(Var(T) + (\mathbb{E}[T])^2\right) \\
&= \sigma_d^2\mu_\ell + \mu_d^2\left(\sigma_\ell^2 + \mu_\ell^2\right).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
Var(D(T)) &= \mathbb{E}[D^2(T)] - \big(\mathbb{E}[D(T)]\big)^2 \\
&= \sigma_d^2\mu_\ell + \mu_d^2\left(\sigma_\ell^2 + \mu_\ell^2\right) - (\mu_d\mu_\ell)^2 \\
&= \sigma_d^2\mu_\ell + \mu_d^2\sigma_\ell^2.
\end{aligned}
$$

$\blacksquare$