

1 Probability Theory

1.1 Some ole' Binomial

1.1.1 Expected value and variance of Binomial

Theorem 1.1. Let $X \approx \text{Bin}(n, p)$, then the expected value and variance is given by

1. $\mathbb{E}(X) = np$,
2. $\text{Var}(X) = np(1 - p)$.

1.1.2 Binomial approx to Poisson

Theorem 1.2. For $\lambda = np$,

$$\lim_{n \rightarrow \infty} \binom{n}{k} p^k (1-p)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}$$

First, we need the following lemma

Lemma 1.1. Let $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ be the Gamma function, then the following equation holds

$$\lim_{n \rightarrow \infty} \frac{\binom{n}{k}}{n^k} = \frac{1}{k!}.$$

Proof of Lemma 1.1. Note first that

$$\Gamma(k+1) = k\Gamma(k) = \cdots = k!\Gamma(1) = k!.$$

and *Stirling's Formula for Gamma function* states that

$$\Gamma(x+1) = \sqrt{2\pi x} x^x e^{-x} \left(1 + \frac{1}{12x} \cdots\right).$$

Then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\binom{n}{k}}{n^k} &= \lim_{n \rightarrow \infty} \frac{n!}{n^k (n-k)! k!} \\ &= \lim_{n \rightarrow \infty} \frac{\Gamma(n+1)}{n^k \Gamma(n-k+1) \Gamma(k+1)} \\ &= \frac{1}{\Gamma(k+1)} \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n-k}} \left(\frac{n}{e}\right)^n \left(\frac{e}{n-k}\right)^{n-k} \frac{1}{n^k}. \end{aligned}$$

Where we used the first order of Stirling's Formula for Gamma function in the last equality.

Next, we only need to prove the limit equals to 1:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n-k}} \left(\frac{n}{e}\right)^n \left(\frac{e}{n-k}\right)^{n-k} \frac{1}{n^k} &= \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n-k}} \frac{1}{e^k} \frac{(1-k/n)^k}{(1-k/n)^n} \\ &= 1, \end{aligned}$$

since

$$\begin{aligned} \sqrt{\frac{n}{n-k}} &\rightarrow 1, \\ (1-k/n)^k &\rightarrow 1, \\ (1-k/n)^n &\rightarrow e^{-k}, \end{aligned}$$

as $n \rightarrow \infty$. ■

Now we may prove the main theorem

Proof of Theorem 1.2. Let $p = \lambda/n$ and fix k , we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{\binom{n}{k}}{n^k} &= \lim_{n \rightarrow \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \lim_{n \rightarrow \infty} \frac{n^k}{k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \quad \text{by Lemma 1.1} \\
&= \frac{\lambda^k}{k!} \left[\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n \right] \underbrace{\left[\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^{-k} \right]}_{=1} \\
&= \frac{\lambda^k}{k!} e^{-\lambda}.
\end{aligned}$$

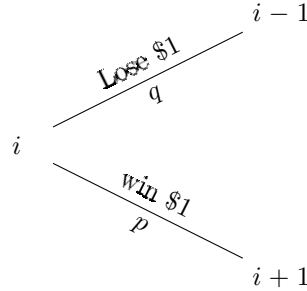
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1.2 Markov Chain

Theorem 1.3 (Gambler's ruin). *Let p is the probability of winning \$1 and $q = 1 - p$ is the probability of losing \$1. Suppose that a person starts with \$ i amount of money where $0 < i < N$ and $N \in \mathbb{N}$ is the number he would want to walk away with. The probability of him winning from this starting position is given by the following equation:*

$$a_i = \begin{cases} \frac{1 - \left(\frac{q}{p}\right)^i}{1 - \left(\frac{q}{p}\right)^N} & \text{where } p \neq q, \\ \frac{i}{N} & \text{where } p = q = \frac{1}{2}. \end{cases}$$

Proof. Note that



Then, we observe

$$\begin{aligned}
a_i &= pa_{i+1} + qa_{i-1} \\
\underbrace{(p+q)}_{=1} a_i &= pa_{i+1} + qa_{i-1} \\
a_{i+1} - a_i &= \frac{q}{p}(a_i - a_{i-1}).
\end{aligned}$$

Denote $b_j := a_i - a_{i-1}$ and note that $b_1 = a_1$ since $a_0 = 0$ is the absorption edge, we deduce by recursion that

$$b_{i+1} = \left(\frac{q}{p}\right)^{i+1} a_1.$$

Moreover, note that

$$\begin{aligned}
a_{i+1} - a_1 &= \sum_{k=1}^i a_{k+1} - a_k \\
&= \sum_{k=1}^i \left(\frac{q}{p}\right)^k a_1. \\
\Rightarrow a_{i+1} &= a_1 \left(1 + \sum_{k=1}^i \left(\frac{q}{p}\right)^k\right) \\
&= a_1 \left(\sum_{k=0}^i \left(\frac{q}{p}\right)^k\right).
\end{aligned}$$

If $p \neq q$, by geometric series, we obtain

$$a_{i+1} = a_1 \frac{1 - \left(\frac{q}{p}\right)^{i+1}}{1 - \left(\frac{q}{p}\right)}.$$

If $p = q = 0.5$, then

$$a_{i+1} = a_1(i+1).$$

Now to solve for a_1 , observe that since at absorption point $a_N = 1$, then letting $i+1 = N$, we get

$$1 = a_N = \begin{cases} a_1 \frac{1 - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)} & \text{where } p \neq q, \\ a_1 N & \text{where } p = q = \frac{1}{2}, \end{cases}$$

which implies

$$a_1 = \begin{cases} \frac{1 - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)} & \text{where } p \neq q, \\ \frac{1}{N} & \text{where } p = q = \frac{1}{2}. \end{cases}$$

Substituting this back, we obtained the desired equation. ■

Corollary 1.1. *The expectation of stopping time with respect to Brownian motion, i.e. $\tau = \inf\{t \geq 0 : B_t \in \{-\alpha, \beta\}\}$, for any positive integer α and β is given by*

$$\mathbb{E}[\tau] = \alpha\beta.$$

Proof. From Theorem (1.3) we observe that $\Pr(B_\tau = b) = \frac{a}{a+b}$. Then, note that since $1 = \Pr(B_\tau = -\alpha) + \Pr(B_\tau = \beta)$, we have

$$\Pr(B_\tau = -\alpha) = \frac{b}{a+b}.$$

Since $(B_t^2 - t)_{t \geq 0}$ is a martingale, we by apply Optimal Stopping Theorem where

$$0 = \mathbb{E}[B_0^2 - 0] = \mathbb{E}[B_\tau^2 - \tau],$$

which implies

$$\mathbb{E}[\tau] = \mathbb{E}[B_\tau^2] = \alpha^2 \Pr(B_\tau = -\alpha) + \beta^2 \Pr(B_\tau = \beta) = \alpha\beta. \quad \blacksquare$$

2 Financial Mathematics

Suppose

$$\begin{cases} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ S_t|_{t=0} &= S_0. \end{cases} \quad (2.1)$$

and risk free is

$$\begin{cases} dB_t &= r B_t dt \\ B_t|_{t=0} &= B_0. \end{cases}$$

Note that the solution to (2.1) is

$$\begin{aligned} d \log S_t &= \frac{1}{S_t} dS_t - \frac{1}{2} \frac{1}{S_t^2} (dS_t)^2 \\ &= \frac{1}{S_t} (\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2} \frac{1}{S_t^2} (\sigma^2 S_t^2 \underbrace{(dW_t)^2}_{dt} + O(dt)) \\ &= \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t. \\ \Rightarrow S_t &= S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t \right) \end{aligned}$$

2.1 Girsanov Theorem

Theorem 2.1. Denote

$$\eta_t := \exp \left(- \int_0^t \lambda(u) dW^{\mathbb{P}}(u) - \frac{1}{2} \int_0^t \lambda^2(u) du \right),$$

where $W^{\mathbb{P}}(t)$ is Wiener process under \mathbb{P} and $\lambda(t)$ is \mathcal{F}_t -measurable to be chosen. Then we can have the following Wiener process under arbitrage-free Wiener process $W^{\mathbb{Q}}(t)$ defined by

$$W^{\mathbb{Q}}(t) = W^{\mathbb{P}}(t) + \int_0^t \lambda(u) du.$$

The measure \mathbb{Q} is defined by

$$\mathbb{Q}[A] = \int_A \eta_t dP,$$

where $A \in \mathcal{F}_t$.

We may also write it as

$$dW^{\mathbb{Q}} = dW^{\mathbb{P}} + \lambda(t) dt. \quad (2.2)$$

Remark 2.1 (Risk neutral GBM). Choosing $\lambda(t) = \frac{\mu-r}{\sigma}$, the GBM process under risk-neutral measure can be written as

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW^{\mathbb{P}} \\ &= \mu S_t dt + \sigma S_t [dW^{\mathbb{Q}} - \lambda(t) dt] \\ &= r S_t dt + \sigma S_t dW^{\mathbb{Q}}. \end{aligned}$$

Remark 2.2 (Change of Numeraire). Choosing $\lambda(t) = \frac{\mu-r}{\sigma}$, the discounted risk neutral process can be expressed as

$$d \left(\frac{S_t}{B_t} \right) = \frac{S_t}{B_t} (\mu - r) dt + \sigma dW^{\mathbb{P}}$$

$$\begin{aligned}
&= \frac{S_t}{B_t}(\mu - r) dt + \sigma [dW^{\mathbb{Q}} - \lambda(t) dt] \\
&= \frac{S_t}{B_t} \sigma dW^{\mathbb{Q}},
\end{aligned}$$

which is a Martingale.

Remark 2.3. Since choice of $\lambda(t)$ is unique here to generate a Martingale process, \mathbb{Q} is known as unique equivalent martingale (EMM) which equals to say that \mathbb{Q} is a risk-neutral measure.

2.2 Black-Scholes

Suppose the call-option value process (admissible, BS is closed and whole other assumptinos)

$$C(S_t) = V_t((S_T - K)^+).$$

From Ito lemma, we have

$$\begin{aligned}
dC(S_t, t) &= \frac{\partial C(S_t)}{\partial t} dt + \frac{\partial C(S_t)}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 C(S_t)}{\partial S_t^2} (dS_t)^2 \\
&= \frac{\partial C(S_t)}{\partial t} dt + \frac{\partial C(S_t)}{\partial S_t} (rS_t dt + \sigma S_t dW_t) + \frac{1}{2} \frac{\partial^2 C(S_t)}{\partial S_t^2} \sigma^2 S_t^2 dt \\
&= \left(\frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} \right) dt + \sigma S_t \frac{\partial C(S_t)}{\partial S_t} dW_t.
\end{aligned}$$

Moreover, when we consider risk-free bond, we get

$$\begin{aligned}
d\left(\frac{C(S_t, t)}{B_t}\right) &= C(S_t, t) d\left(\frac{1}{B_t}\right) + \frac{1}{B_t} dC(S_t, t) \\
&= -C(S_t, t) \frac{1}{B_t^2} dB_t \frac{1}{B_t} + \frac{1}{B_t} dC(S_t, t) \\
&= -C(S_t, t) \frac{r}{B_t} dt + \frac{1}{B_t} dC(S_t, t) \\
&= \frac{1}{B_t} \left(\frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} - rC \right) dt + \sigma \frac{S_t}{B_t} \frac{\partial C(S_t)}{\partial S_t} dW_t
\end{aligned}$$

For no arbitrage, there is no drift term, so the Black-Scholes equation equals to

$$\frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} - rC = 0 \quad (2.3)$$

Remark 2.4. If there is a dividend d , the BS equation is

$$\frac{\partial C}{\partial t} + (r - d)S_t \frac{\partial C}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} - rC = 0$$

2.2.1 Solving Black-Scholes Equation

From (2.3), we denote the operator

$$L := - \left((r - d)S_t \frac{\partial}{\partial S_t} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2}{\partial S_t^2} - r \right),$$

where d is the dividend. Then, BS can be rewritten as

$$\begin{cases} (\partial_t + L)C(S_t, t) &= 0 \\ C(x, 0) &= (e^x - e^{x_0})^+, \end{cases} \quad (2.4)$$

where $x = \log S$.

Denote

$$\begin{aligned}\alpha &:= \frac{\sigma^2}{2} \\ \beta &:= (d - r) \\ \tau &:= T - t\end{aligned}$$

Then

$$L = -\alpha \partial_x^2 + \beta \partial_x + r.$$

We can calculate the heat-kernel of the BS-equation by using Fourier Transform, $U_{\mathbf{BS}}(\tau; x - y)$. Thus,

$$C_{\mathbf{BS}}(\tau, x) = \int_{x_0}^{\infty} U_{\mathbf{BS}}(\tau; x - y) C(y, 0) dy,$$

where the kernel is obtained by calculating

$$\begin{aligned}U_{\mathbf{BS}}(\tau; x - y) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-p(x-y)} e^{-\tau(\alpha p^2 + i\beta p + r)} dy \\ &= \frac{e^{-r\tau}}{2\pi} \int_{\mathbb{R}} e^{-\left[p\sqrt{\alpha\tau} - i\frac{x-y-\beta\tau}{2\sqrt{\alpha\tau}}\right]^2} dy \\ &= \frac{e^{-r\tau}}{\sqrt{4\pi\alpha\tau}} \exp\left(-\frac{(x-y-\beta\tau)^2}{4\alpha\tau}\right).\end{aligned}$$

2.2.2 Derivation of European Call Option

Proposition 2.1. *Let $g(0, \infty) \rightarrow \mathbb{R}_+$ be a claim such that $g(x) \leq c(1+x)$ for some $c \geq 0$ and $X = g(S_T)$. Then the value process $(V_t(X))_{t \in [0, T]}$ satisfies $V_t(X) = \nu(t, S_t)$ where*

$$\nu(t, x) = e^{-r(T-t)} \int_{\mathbb{R}} g(e^y) \phi_{\log(x) + (T-t)(r-\sigma^2/2), \sigma^2(T-t)}(y) dy, \quad (2.5)$$

and $\phi(\mu, \sigma^2)$ is the probability density function of normal distribution i.e.

$$\phi(\mu, \sigma^2)(y) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\mu - y)^2}{2\sigma^2}\right).$$

Proof. From discounted process we have

$$\begin{aligned}\hat{V}_t(X) &= \mathbb{E}^Q[\hat{X} | \mathcal{F}_t] \\ &= e^{-rT} \mathbb{E}^Q[g(S_T) | \mathcal{F}_t] \\ &= e^{-rT} \mathbb{E}^Q\left[g\left(S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right)\right) \mid \mathcal{F}_t\right] \\ &= e^{-rT} \mathbb{E}^Q\left[g\left(\exp\left(\log(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right)\right) \mid \mathcal{F}_t\right],\end{aligned}$$

where we used the solution to the Stochastic GBM. Note here that $(W_T - W_t) \sim \mathcal{N}(0, T - t)$. Thus, we have

$$\begin{aligned}\hat{V}_t(X) &= \mathbb{E}^Q[\hat{X} | \mathcal{F}_t] \\ &= e^{-rT} \int_{\mathbb{R}} g\left(\exp\left(\log(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma y\right)\right) \phi_{0, T-t}(y) dy \\ &= e^{-rT} \int_{\mathbb{R}} g(e^z) \phi_{\log(x) + (T-t)(r-\sigma^2/2), \sigma^2(T-t)}(z) dz.\end{aligned}$$

Finally, observe that the discounted value process is given by $\hat{V}_t(X) = e^{-rt} V_t(X)$ then we are done. \blacksquare

Theorem 2.2 (Call option pricing). *Let $C(S_t, t) := V_t((S_T - K)^+)$ value process of the call option, then we have*

$$C(S_t, t) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \quad (2.6)$$

where $\Phi(x)$ ($\mathbb{P}(X < x)$) is the distribution of standard normal distribution $\mathcal{N}(0, 1)$ and

$$\begin{aligned} d_1 &:= \frac{1}{\sigma \sqrt{T-t}} \left(\log \left(\frac{S_t}{K} \right) + (r + \sigma^2/2)(T-t) \right), \\ d_2 &:= d_1 - \sigma \sqrt{T-t}. \end{aligned}$$

Proof. Let $g(x) = (x - K)^+$ then from Proposition 2.1, we get

$$\begin{aligned} C(S_t, t) &= V_t((S_t - K)^+) \\ &= e^{-r(T-t)} \int_{\mathbb{R}} \underbrace{(e^y - K)^+}_{=g(e^y)} \phi_{\log(S_t) + (T-t)(r-\sigma^2/2), \sigma^2(T-t)}(y) dy \\ &= e^{-r(T-t)} \int_{\mathbb{R}} \left(x \exp \left(\left(r - \frac{\sigma}{2} \right) (T-t) + \sigma \sqrt{T-t} z \right) - K \right)^+ \phi_{0,1}(z) dz, \end{aligned}$$

where $x = S_t$.

Let

$$\begin{aligned} \tau &:= T - t, \\ \tilde{r} &:= r - \frac{\sigma^2}{2}, \\ \ell &:= \frac{\log(K/x) - \tilde{r}\tau}{\sigma \sqrt{\tau}}. \end{aligned}$$

Thus, we observe that $(x \exp((r - \frac{\sigma}{2})(T-t) + \sigma \sqrt{T-t} z) - K)^+$ is non-zero iff

$$\begin{aligned} x \exp \left(\left(r - \frac{\sigma}{2} \right) (T-t) + \sigma \sqrt{T-t} z \right) - K &> 0 \\ \Leftrightarrow y &> \ell \end{aligned}$$

Then

$$\begin{aligned} C(S_t, t) &= e^{-r\tau} \int_{\ell}^{\infty} (x \exp(\tilde{r}\tau + \sigma \sqrt{\tau} z) - K) \phi_{0,1}(z) dz \\ &= e^{-r\tau} \int_{\ell}^{\infty} (x \exp(\tilde{r}\tau + \sigma \sqrt{\tau} z) - K) \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \right) dz \\ &= x \int_{\ell}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{(z - \sigma \sqrt{\tau})^2}{2} \right) dz - e^{-r\tau} K \Phi(-\ell) \\ &= x \int_{\ell - \sigma \sqrt{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - e^{-r\tau} K \Phi(-\ell) \\ &= x \Phi(-\ell + \sigma \sqrt{\tau}) - e^{-r\tau} K \Phi(-\ell) \end{aligned}$$

■

Theorem 2.3 (Replicating strategies). *The replicating trading strategy $\varphi = (\varphi_t^0, \varphi_t^1)_{t \in [0, T]}$ of X is given by*

$$\begin{aligned} \varphi_t^0 &= e^{-rt} \left(\nu(S_t, t) - S_t \frac{\partial}{\partial x} \nu(S_t, t) \right), \\ \varphi_t^1 &= \frac{\partial}{\partial x} \nu(S_t, t), \end{aligned} \quad (2.7)$$

where ν is defined in (2.5)

2.2.3 Put-Call Parity

$$S_t - Ke^{-r(T-t)} = C(S_t, t) - P(S_t, t). \quad (2.8)$$

2.2.4 Barrier options

Suppose a down-and-out option

$$X := (S_t - K)^+ \mathbb{1}_{\min_{t \in [0, T]} S_t > \beta}$$

The value process is therefore,

$$V_t(X) = e^{-r(T-t)} \mathbb{E}^Q[X | \mathcal{F}_t]$$

2.3 Stochastic Volatility

3 Value at Risk

Definition

$$\alpha = \int_{-\text{VaR}}^{\infty} f(x) dx,$$

where f is a probability density function. If the probability density is well defined and the CDF has inverse, we may calculate the VaR as follows

$$\begin{aligned} \alpha &= \int_{-\text{VaR}}^{\infty} f(x) dx = 1 - \Phi(\text{VaR}) \\ \text{VaR} &= \Phi^{-1}(1 - \alpha). \end{aligned}$$

If f is probability density function of a normal distribution $\mathcal{N}(\mu, \sigma^2)$, then

$$\text{VaR} = z_{\alpha} \sigma.$$

If there is a time horizon T , then

$$\text{VaR}(T) = z_{\alpha} \sqrt{T} \sigma.$$

3.0.1 Expected shortfall

Expected shortfall (ES) is a risk measure—a concept used in the field of financial risk measurement to evaluate the market risk or credit risk of a portfolio. The "expected shortfall at $q\%$ level" is the expected return on the portfolio in the worst $q\%$ of cases. ES is an alternative to value at risk that is more sensitive to the shape of the tail of the loss distribution.

Expected shortfall is considered a more useful risk measure than VaR because it is a coherent spectral measure of financial portfolio risk. It is calculated for a given quantile-level q and is defined to be the mean loss of portfolio value given that a loss is occurring at or below the q -quantile.

Let $X \in L^p(\mathcal{F})$, the formula can be written as

$$\begin{aligned} \text{ES}_{\alpha}(X) &= \mathbb{E}[-X \mid X \geq -\text{VaR}_{\alpha}(X)] \\ &= -\frac{1}{\alpha} \int_0^{\alpha} \text{VaR}_{\gamma} d\gamma \\ &= -\frac{1}{\alpha} \int_{-\infty}^{-\text{VaR}_{\alpha}} x f(x) dx, \end{aligned}$$

where f is the probability density function.

4 Time Series

4.1 Stationary and Autocorrelation

Definition 4.1. *The autocovariance*

$$\gamma(s, t) := \mathbb{E}[(x_s - \mu_s)(x_t - \mu_t)],$$

and *cross-covar.*

$$\rho_{x,y}(s, t) := \mathbb{E}[(x_s - \mu_{xs})(y_t - \mu_{ts})].$$

Definition 4.2. *The autocorrelation function (ACF) is defined as*

$$\rho_x(s, t) := \frac{\gamma(s, t)}{\sqrt{\gamma(s, s)\gamma(t, t)}},$$

Notation: $\rho_x(t, t + h) \equiv \rho(h)$.

Definition 4.3 (Strictly stationary). *For all $k \in \mathbb{N}$, $t_k \in \mathbb{N}$ and $c_k \in \mathbb{R}$. Then the time series is strictly stationary if*

$$\mathbb{P}(x_{t_1} \leq c_1, \dots, x_{t_k} \leq c_k) = \mathbb{P}(x_{t_1+h} \leq c_1, \dots, x_{t_k+h} \leq c_k),$$

for given probability measure \mathbb{P} and constant $h \in \mathbb{N}$.

Definition 4.4 (Weakly stationary). *The time series is strictly stationary if*

1. $\mu_t = \mathbb{E}[x_t]$ is constant, and
2. $\gamma(s, t)$ depends only on $|s - t|$.

Lemma 4.1. *Assume time series is weakly stationary. Using the notation $\gamma(t, t + h) \equiv \gamma(h)$, we have*

1. $|\gamma(t)| \leq \gamma(0)$,
2. $\gamma(h) = \gamma(-h)$.

Proof. 1. By Cauchy-Schwarz inequality $|\gamma(t, t + h)|^2 \leq \gamma(t, t)\gamma(t + h, t + h)$ by definition $|\gamma(h)|^2 \leq \gamma(0)\gamma(0)$.

2. From definition of covariance we have

$$\begin{aligned} \gamma(h) &= \gamma(t + h - t) \\ &= \mathbb{E}[(x_{t+h} - \mu)(x_t - \mu)] \\ &= \mathbb{E}[(x_t - \mu)(x_{t+h} - \mu)] \\ &= \gamma(t - (t + h)) = \gamma(-h) \end{aligned}$$

■

Example 4.1. Suppose w_t for all $t \in \mathbb{N}$ is white noise process, i.e. iid $w_t \approx N(0, \sigma_w^2)$. Given two series

$$\begin{aligned} x_t &= w_t + w_{t-1}, \\ y_t &= w_t - w_{t-1}. \end{aligned}$$

Then for any $h \in \mathbb{N}$, we get the following for (cross-)covariance:

1.

$$\begin{aligned}
\gamma_x(0) &= \mathbb{E}[(w_t + w_{t-1})^2] = \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1}^2] + 2 \underbrace{\mathbb{E}[w_t w_{t-1}]}_{=0} \\
&= 2\sigma_w^2, \\
\gamma_y(0) &= \mathbb{E}[(w_t - w_{t-1})^2] = \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1}^2] - 2 \underbrace{\mathbb{E}[w_t w_{t-1}]}_{=0} \\
&= 2\sigma_w^2.
\end{aligned}$$

2.

$$\begin{aligned}
\gamma_x(1) &= \mathbb{E}[(w_t + w_{t-1})(w_{t+1} + w_t)] \\
&= \mathbb{E}[w_t w_{t+1}] + \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1} w_{t+1}] + \mathbb{E}[w_{t-1} w_t] \\
&= \sigma_w = \gamma_x(-1), \\
\gamma_y(1) &= \mathbb{E}[w_t w_{t+1}] - \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1} w_{t+1}] - \mathbb{E}[w_{t-1} w_t] \\
&= -\sigma_w = \gamma_y(-1).
\end{aligned}$$

3. $\gamma_{xy}(0) = \mathbb{E}[w_t^2] - \mathbb{E}[w_{t-1}^2] = 0$ and $\gamma_{xy}(1) = \text{cov}(x_{t+1}, y_t) = -\sigma_w^2$ and $\gamma_{xy}(-1) = \text{cov}(x_t, y_{t-1}) = -\sigma_w^2$

4. ACF

$$\rho_{xy}(h) = \begin{cases} 0 & h = 0, \\ 1/2 & h = 1, \\ -1/2 & h = -1, \\ 0 & |h| \geq 2 \end{cases} \quad (4.1)$$

Thus the joint time series is stationary.

Definition 4.5 (Linear Process). $\{x_t\}$ is linear process if

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}, \quad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty,$$

where w_t is white noise.

Definition 4.6 (Gaussian Process). $\{x_t\}$ is Gaussian process if the vector $x := (x_{t_1}, x_{t_2}, \dots, x_{t_n})' \in \mathbb{R}^n$ has a multivariate normal distribution with density function

$$f(x) = \frac{1}{(2\pi)^{n/2}} \det(\Gamma)^{-1/2} \exp\left(-\frac{1}{2}(x - \mu)' \Gamma^{-1} (x - \mu)\right),$$

where $\mu \in \mathbb{R}^n$ and $\Gamma := \text{var}(x) = \{\gamma(t_i, t_j); i, j = 1, \dots, n\}$.

4.1.1 ARIMA

ARMA(p, q):

$$x_t - \phi_1 x_{t-1} - \dots - \phi_p x_{t-p} = w_t + \theta_1 w_{t-1} + \dots + \theta_q w_{t-q}.$$

4.2 Autoregressive Model

Definition 4.7 ($AR(p)$). Let w_t be white noise, $AR(p)$ is defined as

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = w_t.$$

Definition 4.8 (Autoregressive Operator). Let B be time-lagged operator, i.e. $Bx_t = x_{t-1}$, the autoregressive operator is defined as

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p.$$

Example 4.2. Assuming $\phi_j = \phi$ for all $j \in \mathbb{N}$, from $AR(1)$

$$\begin{aligned} x_t &= \phi x_{t-1} + w_t = \phi(\phi x_{t-2} + w_{t-1}) + w_t \\ &= \phi^2 x_{t-2} + \phi w_{t-1} + w_t \\ &\vdots \\ &= \phi^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j w_{t-j}. \end{aligned}$$

Thus, if $|\phi| < 1$ and x_t is stationary then for large $k \rightarrow \infty$, we have

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}.$$

Taking expectation value,

$$\mathbb{E}[x_t] = \sum_{j=0}^{\infty} \phi^j \mathbb{E}[w_t] = 0.$$

Moreover, for $h \geq 0$ the autocovariance function is

$$\begin{aligned} \gamma(h) &= \mathbb{E} \left[\left(\sum_{j=0}^{\infty} \phi^j w_{t+h-j} \right) \left(\sum_{k=0}^{\infty} \phi^k w_{t-k} \right) \right] \\ &= \mathbb{E} [(w_{t+h-1} + \phi w_{t+h-2} + \cdots + \phi^h w_t + \phi^{h+1} w_{t-1} + \cdots)(w_t + \phi w_{t-1} + \cdots)] \\ &= \sigma_w^2 \sum_{j=0}^{\infty} \phi^{h+j} \phi^j = \sigma_w^2 \phi^h \sum_{j=0}^{\infty} \phi^{2j} \\ &= \frac{\sigma_w^2 \phi^h}{1 - \phi^2}. \end{aligned}$$

And ACF is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h.$$

As well as

$$\rho(h) = \phi \rho(h-1).$$

Definition 4.9. The *moving average operator* is defined as

$$\theta(B) := 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q.$$

5 Operational Research

5.1 Safety Stock

Theorem 5.1. *Let the lead-time T be normally distributed, i.e. $T \sim N(\mu_\ell, \sigma_\ell^2)$ and denote the set of demand of period i as $\{D_i\}_{i=1}^T$ where $D_i \sim N(\mu_d, \sigma_d^2)$ for each $1 \leq i \leq T$. Then the safety stock level, SS satisfies the following equation*

$$SS = z_\alpha \sqrt{\sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2},$$

where z_α is the Z-value of a desired α which is chosen.

Proof. Denote the demand between within lead-time as $D(T) := \sum_{i=1}^T D_i$. The safety stock level is then set at

$$SS = z_\alpha \sqrt{\text{Var}(D(T))}.$$

Thus we need only to prove $\text{Var}(D(T))$.

From Towering-property, note that

$$\begin{aligned} \mathbb{E}[D(T)] &= \mathbb{E}[\mathbb{E}[D(T)|T]] \\ &= \mathbb{E}[T\mu_d] = \mu_d \mathbb{E}[T] = \mu_d \mu_\ell. \end{aligned}$$

Furthermore, again with Towering-property, we have

$$\begin{aligned} \mathbb{E}[D^2(T)] &= \mathbb{E}[\mathbb{E}[D^2(T)|T]] \\ &= \mathbb{E}[\text{Var}[D(T)|T] + (\mathbb{E}[D(T)|T])^2] \\ &= \mathbb{E}[T\sigma_d^2 + (T\mu_d)^2] \\ &= \sigma_d^2 \mathbb{E}[T] + \mu_d^2 \mathbb{E}[T^2] \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 (\text{Var}(T) + (\mathbb{E}[T])^2) \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 (\sigma_\ell^2 + \mu_\ell^2). \end{aligned}$$

Finally,

$$\begin{aligned} \text{Var}(D(T)) &= \mathbb{E}[D^2(T)] - (\mathbb{E}[D(T)])^2 \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 (\sigma_\ell^2 + \mu_\ell^2) - (\mu_d \mu_\ell)^2 \\ &= \sigma_d^2 \mu_\ell + \mu_d^2 \sigma_\ell^2. \end{aligned}$$

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