Theoretical Time-Series Notes

For myself and other mathematicians

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Chapter 1

ARIMA

$ARMA(p,q)$:

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = w_t + \theta_t w_{t-1} + \cdots + \theta_q w_{t-q}.$$

1.1 Stationary and Autocorrelation

Definition 1.1.1. *The **autocovariance***

$$\gamma(s,t) := \mathbb{E}\left[(x_s - \mu_s)(x_t - \mu_t)\right],$$

*and **cross-covar.***

$$\rho_{x,y}(s,t) := \mathbb{E}\left[(x_s - \mu_{xs})(y_t - \mu_{ts})\right].$$

Definition 1.1.2. *The **autocorrelation function** (ACF) is defined as*

$$\rho_x(s,t) := \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}},$$

Notation: $\rho_x(t, t+h) \equiv \rho(h)$.

Definition 1.1.3 (Strictly stationary)**.** *For all $k \in \mathbb{N}$, $t_k \in \mathbb{N}$ and $c_k \in \mathbb{R}$. Then the time series is strictly stationary if*

$$\mathbb{P}\big(x_{t_1} \leqslant c_1, \ldots x_{t_k} \leqslant c_k\big) = \mathbb{P}\big(x_{t_1+h} \leqslant c_1, \ldots x_{t_k+h} \leqslant c_k\big),$$

for given probability measure $\mathbb{P}$ and constant $h \in \mathbb{N}$.

Definition 1.1.4 (Weakly stationary). *The time series is strictly stationary if*

1. $\mu_t = \mathbb{E}[x_t]$ *is constant, and*

2. $\gamma(s,t)$ *depends only on* $|s-t|$.

Lemma 1.1.1. *Assume time series is weakly stationary. Using the notation* $\gamma(t, t+h) \equiv \gamma(h)$, *we have*

1. $|\gamma(t)| \leqslant \gamma(0)$,

2. $\gamma(h) = \gamma(-h)$.

Proof. 1. By Cauchy-Schwarz inequality $|\gamma(t,t+h)|^2 \leqslant \gamma(t,t)\gamma(t+h,t+h)$ by definition $|\gamma(h)|^2 \leqslant \gamma(0)\gamma(0)$.

2. From definition of covariance we have

$$\begin{aligned}
\gamma(h) &= \gamma(t+h-t) \\
&= \mathbb{E}\left[(x_{t+h}-\mu)(x_t-\mu)\right] \\
&= \mathbb{E}\left[(x_t-\mu)(x_{t+h}-\mu)\right] \\
&= \gamma(t-(t+h)) = \gamma(-h)
\end{aligned}$$

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Example 1.1.1. Suppose w_t for all $t \in N$ is white noise process, i.e. iid $w_t \approx N(0, \sigma_w^2)$. Given two series

$$\begin{aligned}
x_t &= w_t + w_{t-1}, \\
y_t &= w_t - w_{t-1}.
\end{aligned}$$

Then for any $h \in \mathbb{N}$, we get the following for (cross-)covariance:

1.

$$\begin{aligned}
\gamma_x(0) &= \mathbb{E}[(w_t + w_{t-1})^2] = \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1}^2] + 2\underbrace{\mathbb{E}[w_t w_{t-1}]}_{=0} \\
&= 2\sigma_w^2, \\
\gamma_y(0) &= \mathbb{E}[(w_t - w_{t-1})^2] = \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1}^2] - 2\underbrace{\mathbb{E}[w_t w_{t-1}]}_{=0} \\
&= 2\sigma_w^2.
\end{aligned}$$

2.

$$\begin{aligned}
\gamma_x(1) &= \mathbb{E}\left[(w_t + w_{t-1})(w_{t+1} + w_t)\right] \\
&= \mathbb{E}[w_t w_{t+1}] + \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1} w_{t+1}] + \mathbb{E}[w_{t-1} w_t] \\
&= \sigma_w = \gamma_x(-1), \\
\gamma_y(1) &= \mathbb{E}[w_t w_{t+1}] - \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1} w_{t+1}] - \mathbb{E}[w_{t-1} w_t] \\
&= -\sigma_w = \gamma_y(-1).
\end{aligned}$$

3. $\gamma_{xy}(0) = \mathbb{E}[w_t^2] - \mathbb{E}[w_{t-1}^2]0$ and $\gamma_{xy}(1) = cov(x_{t+1}, y_t) = -\sigma_w^2$ and $\gamma_{xy}(-1) = cov(x_t, y_{t-1}) = -\sigma_w^2$

4. ACF

$$\rho_{xy}(h) = \begin{cases} 0 & h = 0, \\ 1/2 & h = 1, \\ -1/2 & h = -1, \\ 0 & |h| \geqslant 2 \end{cases} \tag{1.1.1}$$

Thus the joint time series is stationary.

Definition 1.1.5 (Linear Process)**.** *$\{x_t\}$ is linear process if*

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty,$$

where w_t is white noise.

Definition 1.1.6 (Gaussian Process)**.** *$\{x_t\}$ is Gaussian process if the vector $x := (x_{t_1}, x_{t_2}, \ldots, x_{t_n})' \in \mathbb{R}^n$ has a multivariate normal distribution with density function*

$$f(x) = \frac{1}{(2\pi)^{n/2}} \det\left(\Gamma\right)^{-1/2} \exp\left(-\frac{1}{2}(x-\mu)'\Gamma^{-1}(x-\mu)\right),$$

where $\mu \in \mathbb{R}^n$ and $\Gamma := var(x) = \{\gamma(t_i, t_j);\ i, j = 1, \ldots, n\}$.