Theoretical Time-Series Notes

For myself and other mathematicians

Matthew Liew

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Chapter 1

Introduction

1.1 Stationary and Autocorrelation

Definition 1.1.1. *The* ***autocovariance***

$$\gamma(s,t) := \mathbb{E}\left[(x_s-\mu_s)(x_t-\mu_t)\right],$$

and ***cross-covar.***

$$\rho_{x,y}(s,t) := \mathbb{E}\left[(x_s-\mu_{xs})(y_t-\mu_{ts})\right].$$

Definition 1.1.2. *The* ***autocorrelation function*** *(ACF) is defined as*

$$\rho_x(s,t) := \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}},$$

Notation: $\rho_x(t,t+h) \equiv \rho(h)$.

Definition 1.1.3 (Strictly stationary)**.** *For all $k \in \mathbb{N}$, $t_k \in \mathbb{N}$ and $c_k \in \mathbb{R}$. Then the time series is strictly stationary if*

$$\mathbb{P}\big(x_{t_1} \leqslant c_1, \ldots x_{t_k} \leqslant c_k\big) = \mathbb{P}\big(x_{t_1+h} \leqslant c_1, \ldots x_{t_k+h} \leqslant c_k\big),$$

for given probability measure $\mathbb{P}$ and constant $h \in \mathbb{N}$.

Definition 1.1.4 (Weakly stationary). *The time series is strictly stationary if*

1. $\mu_t = \mathbb{E}[x_t]$ *is constant, and*
2. $\gamma(s,t)$ *depends only on* $|s-t|$.

Lemma 1.1.1. *Assume time series is weakly stationary. Using the notation* $\gamma(t, t+h) \equiv \gamma(h)$, *we have*

1. $|\gamma(t)| \leqslant \gamma(0)$,
2. $\gamma(h) = \gamma(-h)$.

Proof. 1. By Cauchy-Schwarz inequality $|\gamma(t,t+h)|^2 \leqslant \gamma(t,t)\gamma(t+h,t+h)$ by definition $|\gamma(h)|^2 \leqslant \gamma(0)\gamma(0)$.

2. From definition of covariance we have

$$\begin{aligned}\gamma(h) &= \gamma(t+h-t) \\ &= \mathbb{E}\left[(x_{t+h}-\mu)(x_t-\mu)\right] \\ &= \mathbb{E}\left[(x_t-\mu)(x_{t+h}-\mu)\right] \\ &= \gamma(t-(t+h)) = \gamma(-h)\end{aligned}$$

∎

Example 1.1.1. Suppose w_t for all $t \in N$ is white noise process, i.e. iid $w_t \approx N(0, \sigma_w^2)$. Given two series

$$\begin{aligned}x_t &= w_t + w_{t-1}, \\ y_t &= w_t - w_{t-1}.\end{aligned}$$

Then for any $h \in \mathbb{N}$, we get the following for (cross-)covariance:

1. $$\begin{aligned}\gamma_x(0) &= \mathbb{E}[(w_t + w_{t-1})^2] = \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1}^2] + 2\underbrace{\mathbb{E}[w_t w_{t-1}]}_{=0} \\ &= 2\sigma_w^2, \\ \gamma_y(0) &= \mathbb{E}[(w_t - w_{t-1})^2] = \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1}^2] - 2\underbrace{\mathbb{E}[w_t w_{t-1}]}_{=0} \\ &= 2\sigma_w^2.\end{aligned}$$

2.

$$
\begin{aligned}
\gamma_x(1) &= \mathbb{E}\left[(w_t + w_{t-1})(w_{t+1} + w_t)\right] \\
&= \mathbb{E}[w_t w_{t+1}] + \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1} w_{t+1}] + \mathbb{E}[w_{t-1} w_t] \\
&= \sigma_w = \gamma_x(-1), \\
\gamma_y(1) &= \mathbb{E}[w_t w_{t+1}] - \mathbb{E}[w_t^2] + \mathbb{E}[w_{t-1} w_{t+1}] - \mathbb{E}[w_{t-1} w_t] \\
&= -\sigma_w = \gamma_y(-1).
\end{aligned}
$$

3. $\gamma_{xy}(0) = \mathbb{E}[w_t^2] - \mathbb{E}[w_{t-1}^2] = 0$ and $\gamma_{xy}(1) = cov(x_{t+1}, y_t) = -\sigma_w^2$ and $\gamma_{xy}(-1) = cov(x_t, y_{t-1}) = -\sigma_w^2$

4. ACF

$$
\rho_{xy}(h) = \begin{cases} 0 & h = 0, \\ 1/2 & h = 1, \\ -1/2 & h = -1, \\ 0 & |h| \geqslant 2 \end{cases} \tag{1.1.1}
$$

Thus the joint time series is stationary.

Definition 1.1.5 (Linear Process)**.** *$\{x_t\}$ is linear process if*

$$
x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty,
$$

where w_t is white noise.

Definition 1.1.6 (Gaussian Process)**.** *$\{x_t\}$ is Gaussian process if the vector $x := (x_{t_1}, x_{t_2}, \ldots, x_{t_n})' \in \mathbb{R}^n$ has a multivariate normal distribution with density function*

$$
f(x) = \frac{1}{(2\pi)^{n/2}} \det\left(\Gamma\right)^{-1/2} \exp\left(-\frac{1}{2}(x-\mu)'\Gamma^{-1}(x-\mu)\right),
$$

where $\mu \in \mathbb{R}^n$ and $\Gamma := var(x) = \{\gamma(t_i, t_j);\ i, j = 1, \ldots, n\}$.

Chapter 2

ARIMA

$ARMA(p, q)$:

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = w_t + \theta_t w_{t-1} + \cdots + \theta_q w_{t-q}.$$

2.1 Autoregressive Model

Definition 2.1.1 ($AR(p)$)**.** *Let w_t be white noise, $AR(p)$ is defined as*

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = w_t.$$

Definition 2.1.2 (Autoregressive Operator)**.** *Let B be time-lagged operator, i.e. $Bx_t = x_{t-1}$, the autoregressive operator is defined as*

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p.$$

Example 2.1.1. Assuming $\phi_j = \phi$ for all $j \in \mathbb{N}$, from AR(1)

$$\begin{aligned} x_t &= \phi x_{t-1} + w_t = \phi(\phi x_{t-2} + w_{t-1}) + w_t \\ &= \phi^2 x_{t-2} + \phi w_{t-1} + w_t \\ &\vdots \\ &= \phi^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j w_{t-j}. \end{aligned}$$

Thus, if $|\phi| < 1$ and x_t is sationary then for large $k \to \infty$, we have

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}.$$

Taking expectation value,

$$\mathbb{E}[x_t] = \sum_{j=0}^{\infty} \phi_j \mathbb{E}[w_t] = 0.$$

Moreover, for $h \geqslant$ the autocovriance function is

$$\begin{aligned}
\gamma(h) &= \mathbb{E}\left[\left(\sum_{j=0}^{\infty} \phi^j w_{t+h-j} \right) \left(\sum_{j=0}^{\infty} \phi^k w_{t-k} \right) \right] \\
&= \mathbb{E}\big[(w_{t+h-1} + \phi w_{t+h-2} + \cdots \phi^h w_t + \phi^{h+1} w_{t-1} + \cdots)(w_t + \phi w_{t-1} + \cdots) \big] \\
&= \sigma_w^2 \sum_{j=0} \phi^{h+j} \phi^j = \sigma_w^2 \phi^h \sum_{j=0} \phi^{2j} \\
&= \frac{\sigma_w^2 \phi^h}{1 - \phi^2}.
\end{aligned}$$

And ACF is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h.$$

As well as

$$\rho(h) = \phi \rho(h-1).$$

Definition 2.1.3. *The* ***moving average operator*** *is defined as*

$$\theta(B) := 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q.$$