Geometria 20231122

$A = (A^1 | \dots | A^n)$ $\quad \mathcal{B} = \{A^1, \dots, A^n\}$

$A[\underline{v}]_{\mathcal{B}} = [\underline{v}]_{canonica} = \underline{v}$ $\quad \rightsquigarrow \underset{\text{da} \quad \text{a}}{M_{\mathcal{B}\,can}}\, [\underline{v}]_{\mathcal{B}} = [\underline{v}]_{canonica}$

$\mathcal{D} = \{B^1 \dots B^n\}$

$B[\underline{v}]_{\mathcal{D}} = [\underline{v}]_{canonica} = \underline{v}$

$A[\underline{v}]_{\mathcal{B}} = B[\underline{v}]_{\mathcal{D}}$

$A^{-1}A[\underline{v}]_{\mathcal{B}} = A^{-1}B[\underline{v}]_{\mathcal{D}}$

$[\underline{v}]_{\mathcal{B}} = \underbrace{A^{-1}}_{M_{\mathcal{B}\mathcal{D}}} B[\underline{v}]_{\mathcal{D}}$

$\underset{\text{da} \quad \text{a}}{M_{\mathcal{B}\mathcal{D}}}$

in E_O^2 base canonica $\{\hat{\underline{i}}, \hat{\underline{j}}\}$

$\mathcal{B} = \{\underset{\underline{v}_1}{\hat{\underline{i}} + \hat{\underline{j}}}, \underset{\underline{v}_2}{2\hat{\underline{i}} - \hat{\underline{j}}}\}$

$\mathcal{D} = \{\underset{\underline{u}_1}{\hat{\underline{i}}}, \underset{\underline{u}_2}{\hat{\underline{i}} + \hat{\underline{j}}}\}$

$A = M_{\mathcal{B}can\,\mathcal{B}}$

$B = M_{\mathcal{B}can\,\mathcal{D}}$

$M_{\mathcal{B}\mathcal{D}} = A^{-1}B$

$\underline{w} = 3\hat{\underline{i}} + \hat{\underline{j}} = \alpha \underline{v}_1 + \beta \underline{v}_2$

$[\underline{w}]_{\mathcal{B}} = (\alpha + 2\beta)\hat{\underline{i}} + (\alpha - \beta)\hat{\underline{j}}$

$\begin{cases} \alpha + 2\beta = 3 \\ \alpha - \beta = 1 \end{cases}$ $\quad \alpha = 5/3$, $\beta = 2/3$

$\downarrow$

$[\underline{w}]_{\mathcal{B}} = \begin{pmatrix} 5/3 \\ 2/3 \end{pmatrix}$

$[\underline{w}]_{\mathcal{D}} = \alpha \hat{\underline{i}} + (\hat{\underline{i}} + \hat{\underline{j}})\beta$

$\{ \quad \dots$

inv: $\quad \mathbb{R}^2$

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & a+2c & b+2d \\ 1 & -1 & a-c & b-d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\begin{cases} a + 2c = 1 \\ b + 2d = 0 \\ a - c = 0 \\ b - d = 1 \end{cases} \rightarrow \begin{matrix} a = 1/3 \\ b = 2/3 \\ c = 1/3 \\ d = -1/3 \end{matrix} \rightarrow$ mat. inv. $\begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix}$

$\downarrow$

A = passa da base B (di A) alla canonica

A^{-1} = fa l'opposto

inv:

$\rightarrow A^{-1}B$

Def.: $A \in M_{\mathbb{R}}(k, n)$ chiamiamo matrice trasposta A^T la matrice $M_{\mathbb{R}}(n, k)$ tale che $(A^T)^j$ con $j = 1 \dots k$ $= (A_j)^T$ ossia la j-esima riga messa su colonna

$$\underset{3\times4}{A = \begin{pmatrix} 1 & 0 & 5 & 1 \\ 2 & 1 & -1 & 0 \\ 4 & 1 & 3 & 2 \end{pmatrix}} \qquad \underset{4\times3}{A^T = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 5 & -1 & 3 \\ 1 & 0 & 2 \end{pmatrix}}$$

$\underline{v}$ vett. riga / colonna $\Leftrightarrow$ $\underline{v}^T$ vett. colonna / riga

Ossv.: $(A^T)_i = (A^i)^T$

È L'INVERSIONE LUNGO LA DIAGONALE

$A = \begin{pmatrix} 1 & 4 \\ 2 & -3 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 3 & 7 \\ 0 & 4 & \boxed{7} \\ -1 & 5 & 2 \end{pmatrix}$ b_{23} (riga, colonna) $\quad B^T = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 7 & 7 & 2 \end{pmatrix}$

$(A^T)_{ij} = a_{ji}$

Tr. ha proprietà utili:

$\forall A, B \in M_{\mathbb{R}}(k, n) \quad \forall \lambda \in \mathbb{R}$

$((A+B)^T)^j = ((A+B)_j)^T = (A_j + B_j)^T = A_j^T + B_j^T$

posso scegliere se sommare o trasporre prima

$(A+B)^T = A^T + B^T$

$((\lambda A)^T)_{ij} = (\lambda A)_{ji}$

$(\lambda A)^T = \lambda(A^T)$

stessa cosa, fare prima o dopo scal. non cambia!

← T: trasposizione / posizione? → la trasposta di matrice rettangolare è in uno ≠ spazio!

Def.: Sia $A \in M_{\mathbb{R}}(n)$ con $A^T \in M_{\mathbb{R}}(n)$

A "antisimmetrica" se $A^T = -A$

A "simmetrica" se $A^T = A$

simm.: $A = \begin{pmatrix} 5 & 2 \\ 2 & -1 \end{pmatrix} \quad A^T = \begin{pmatrix} 5 & 2 \\ 2 & -1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & -3 \\ 2 & -3 & 0 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & -3 \\ 2 & -3 & 0 \end{pmatrix}$

antisim.: $W = \begin{pmatrix} 0 & -7 \\ 7 & 0 \end{pmatrix} \quad W^T = \begin{pmatrix} 0 & 7 \\ -7 & 0 \end{pmatrix}$

$W = \begin{pmatrix} 0 & 5 & -1 \\ -5 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix} \quad W^T = \begin{pmatrix} 0 & -5 & 1 \\ 5 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}$

Propr. della mat.²: invertibilità?

Def.: si chiama "Traccia" di A la somma degli a_{ij} lungo la diag. principale.

ossia: $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$

→ antisimm. sempre traccia nulla!

simm. qualunque traccia

$O_n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ unica mat. contemporaneam. simm. ed antisimm.

simbolo:
- det A
- $|A|$ o $\|A\|$ nei libri

Def.: "Determinante" di mat² SOLO MATRICI QUADRATE

$A \in M_{\mathbb{R}}(n)$

per induzione:

$A_{[i,j]}$ cancello riga i, colonna j

$\downarrow \quad A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad A_{[1,1]} = (4)$

n = 1

$\rightarrow \underset{1\times1}{A} = (a)$ $\qquad \det A = a$

n = 2

$\rightarrow \underset{2\times2}{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\qquad \det A = a_{11} + \det A_{[1,1]} - a_{21} \det A_{[2,1]} = a_{11}a_{22} - a_{21}a_{12}$

$A_{[1,1]}$: sottomat. che cancella riga 1 e colonna 1

$A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ $\qquad |A| = \det A = \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -1 - 2 = -3$

$B = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ $\qquad |B| = \det B = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 4 - 4 = 0$

↘ lin. dip. quindi non invertibile

↘ il det delle non invertibili è = 0...

$\downarrow$

lin. dip → 0

n = 3

$\rightarrow A \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$\det A = a_{11} \det A_{[1,1]} - a_{21} \det A_{[2,1]} + a_{31} \det A_{[3,1]}$

n qualunque

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{pmatrix}$

formula in n×n

$$\det A = \sum_{i=1}^{n} (-1)^{n-1} a_{i1} \det A_{[i,1]}$$

ci sarà met. più veloce!

Esempi:

$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 4 \end{pmatrix}$

$\det A = 1 \det_{[1,1]} - 2 \det_{[2,1]} - 1 \det_{[3,1]}$

$= 1 \begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$

$= 1(4 - 0) - 2(0 - 1) - 1(1 - 0)$

$= 4 + 2 + 1 = 7$

$B = \begin{pmatrix} 1 & 0 & 1 & 1 \\ -2 & 1 & 2 & 1 \\ 1 & 0 & 3 & 1 \\ 1 & 1 & 4 & 1 \end{pmatrix}$

$\det B = 1 \det B_{[1,1]} + 2 \det B_{[2,1]} + 1 \det B_{[3,1]} - 1 \det B_{[4,1]}$

$= \begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 1 & 4 & 1 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 4 & 1 \end{vmatrix} + [illegible] \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{vmatrix}$

$= \left(\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 2 & [illegible] \end{vmatrix} [illegible] \right) + 2\left(\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \right) + [illegible] \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \Big) - \left(\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \right)$

$= (-1 [illegible]) + [illegible]) + (3) - 2$

$= - [illegible] \cdot 2$

$= - [illegible]$

[illegible]