Geometria 20231031

⇐ sottospazio vettoriale:

<u>Def</u>: $U \subset V$ svn, U è detto sottosp. vett. V (ssv V) se $U \neq \emptyset$ e valgano le propr.:

SS1) $\forall \underline{u}_1, \underline{u}_2 \in U \quad \underline{u}_1 + \underline{u}_2 \in U$
SS2) $\forall \underline{u}_1 \in U,\ \alpha \in \mathbb{R} \quad \alpha \underline{u} \in U$

} U è chiuso per vet + vet e scal · vett

↗ basta trovarne una errata per dimostrare non-SSV
$\forall \to \exists$

<u>Prop</u>: se U è ssv V allora $\underline{O}_V \in U$ condiz. necessaria allo SSV → quindi devono passare dall'O

↳ contiene ~~elem.~~ vett. nullo di V, quindi è non-vuoto

<u>Dim</u>: U è non-vuoto ↙ perché: dimostraz.

$\exists \underline{v} \in U \subset V$

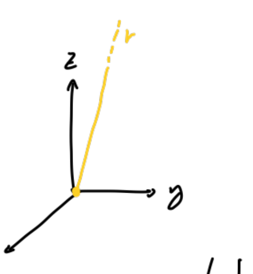
$0\underline{v} = \underline{0}_V$ } → per SS2 deve essere!

ma se $\underline{v} \in U$, allora $0\underline{v} \in U$ per SS2)
↓
$\underline{0}_V \in U$

quod erat demonstrandum

<u>Ossv.</u>: per qualunque V svn esistono 2 ssv banali
- V
- $\{\underline{0}_V\}$

<u>Prop.</u>: unici ssv di E_0^3 sono:
① $\{\underline{O}E_0\}$ ossia $\{O\}$
② Span $\underline{v}$, $\underline{v} \neq \underline{0}$ → tutte le rette pass. O
③ Span $\underline{u}, \underline{v}$, $\{\underline{u}, \underline{v}\}$ lin. indip. → tutti i piani pass. O
④ E_0^3



semiretta v
- ha vett. nullo
- verifica SS1
- non verifica SS2 (-1)

$\mathbb{R}^3 \Leftrightarrow E_0^3 \Leftrightarrow \{$

Esempio: ↙ entrate componenti coordinate

$U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x = 0 \right\}$ è ssv? di: $\mathbb{R}^3$

- ha vett. nullo? (SS0)
$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in U$? ✓

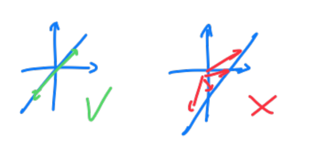
- SS1?
$\begin{pmatrix} 0 \\ a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ a+c \\ b+d \end{pmatrix}$ ✓ rimane in U
$a, b, c, d \in \mathbb{R}$

- SS2?
$\alpha \begin{pmatrix} 0 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha a \\ \alpha b \end{pmatrix}$ ✓ rimane in U
$\alpha, a, b \in \mathbb{R}$

$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mathbb{R}^4 \mid x_2 = 1 \right\}$ è ssv di $\mathbb{R}^4$? ✗

non ha nullo $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

← avevamo fatto l'esempio di: ssv che passano per O o no: somme di vettori sulle retta non pass. O portava fuori dalla retta!



$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_1^2 = x_2^2 \right\}$

2023

- SS0 ✓
- SS1
$\underset{\in U}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} + \underset{\in U}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} = \underset{\notin U}{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}}$ → non chiuso!

$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \mid \begin{matrix} x_1 + x_2 - x_3 = 0 \\ x_3 = x_1 + x_2 \end{matrix} \right\}$

- SS0 ✓
- SS1
$\begin{pmatrix} 1 \\ 3 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 6 \\ 0 \end{pmatrix}$ ✓
$\begin{pmatrix} a \\ b \\ a+b \\ 0 \end{pmatrix} + \begin{pmatrix} c \\ d \\ c+d \\ 0 \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \\ a+b+c+d \\ 0 \end{pmatrix}$ ✓
$a, b, c, d \in \mathbb{R}$

- SS2
$\alpha \begin{pmatrix} a \\ b \\ a+b \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \\ \alpha a + \alpha b \\ 0 \end{pmatrix}$ ✓
$\forall \alpha \in \mathbb{R}$
$a, b \in \mathbb{R}$

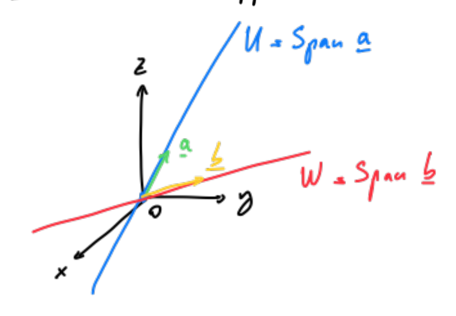
in $\mathbb{R}^n$, equazioni lineari omogenee ($G = 1$) per entrata di U sono sempre SSV!

<u>Operazioni tra SSV</u>:

Siano U, W ssv V svn.
$U \cap W$ intersez. → rimane SSV
$U \cup W$ unione → di solito non è SSV ma non sempre

<u>Def.</u>:
$U + W$ somma di sottospazi → rimane SSV
$U + W \underset{\text{def}}{:=} \{\underline{v} \in V \mid \underline{v} = \underline{u} + \underline{w},\ \underline{u} \in U,\ \underline{w} \in W\}$
tutte le possibili somme di $\underline{u}$ e $\underline{w}$ fra i vett di U e W

<u>Es.</u>: in E_0^3 (rappr.)



$\underline{a} \neq \underline{0}$ $\underline{b} \neq \underline{0}$
↓
$U + W = \text{Span}(\underline{a}, \underline{b})$

<u>Prop.</u>: Siano U, W ssv di V svn
1) $U \cap W$ ssv di V
2) $U + W$ ssv di V

- SS0: $\underline{0}_V \in (U \cap W)$? ✓ ↗ essendo entrambi SSV, passando entrambi per O, avranno perforza $\underline{0}_V$! quindi intersecando esso deve rimanere!

per esame finale!

- SS1:
$\forall \underline{v}_1, \underline{v}_2 \in (U \cap W)$
$\underline{v}_1 + \underline{v}_2 \in (U \cap W)$
→ $\underline{v}_1, \underline{v}_2 \in U$
$\underline{v}_1, \underline{v}_2 \in W$

$\underline{v}_1 + \underline{v}_2 = \underline{v}_{12}$
$\underline{v}_{12} \in U$ per $\underline{v}_1, \underline{v}_2 \in U$ e SS1
$\underline{v}_{12} \in V$ per $\underline{v}_1, \underline{v}_2 \in V$ e SS1
⇓
dev'essere nell'$\cap$! $\underline{v}_{12} \in (U \cap W)$ ✓

- SS2
$\underline{v} \in U \cap W$, $\lambda \in \mathbb{R}$
$\lambda \underline{v} \in U \cap W$?

→ $\underline{v} \in U \wedge \underline{v} \in W$
↓ per SS2
$\lambda \underline{v} \in U \wedge \lambda \underline{v} \in W$
↓ è in entrambi: $\cap$
$\lambda \underline{v} \in (U \cap W)$ ✓

funziona che siano U, W rette, piani, origine o l'intero E_0^3!

Ora la somma di sottospazi:

<u>OSSV.</u>:
siano U, W ssv V svn
$\underline{v} \in U + W \Leftrightarrow \exists \underline{u} \in U, \underline{w} \in W \mid \underline{u} + \underline{w} = \underline{v}$

<u>Dim.</u>: • SS0: $\underline{0}_V \in U + W$? ✓ $\underline{0}_V = \underset{\in U}{\underline{0}_V} + \underset{\in W}{\underline{0}_V}$

• SS1
$\forall \underline{v}_1, \underline{v}_2 \in U + W$
$\underline{v}_1 + \underline{v}_2 \in U + W$?

→ $\exists \underline{u}_1 + \underline{w}_1 = \underline{v}_1$ con $\underline{u}_1 \in U$, $\underline{w}_1 \in W$
" con $\underline{u}_2, \underline{w}_2$

↓
$\underline{v}_1 + \underline{v}_2 = (\underline{u}_1 + \underline{w}_1) + (\underline{u}_2 + \underline{w}_2)$ E_0^3?
↓ per associativa (per ?) E_0^3
↓ per commutativa (gr. abel.)
↓
$= \underset{\in U}{(\underline{u}_1 + \underline{u}_2)} + \underset{\in W}{(\underline{w}_1 + \underline{w}_2)}$ ✓

• SS2:
$\forall \underline{v} \in U + W$, $\forall \lambda \in \mathbb{R}$
⇓
$\exists \underline{u} \in U, \underline{w} \in W \mid \underline{u} + \underline{w} = \underline{v}$

$\lambda \underline{u} \in U \quad \lambda \underline{v} \in V$
↓
per distribut. scalare-vettore
↓
$\lambda \underline{v} = \lambda(\underline{u} + \underline{w}) = \underset{\in U}{\lambda \underline{u}} + \underset{\in W}{\lambda \underline{w}} \in U + W$ ✓
essendo sc. vett. chiuso nei SSV: dimostraz.
↓
$\lambda \underline{u} \in U$

<u>Ossv.</u>: $U + W = W + U$

<u>Def.</u>: $U_1, U_2, U_3 \cdots U_k$ SSV di V

$U_1 + U_2 + U_3 = (U_1 + U_2) + U_3$
$U_1 + U_2 + U_3 \cdots + U_k = ((\cdots(U_1 + U_2) + U_3)\overbrace{\cdots}^{+\cdots)+\cdots)}) + U_k$
↘ generiamo tutte le comb. di vettori possibili per ricavarne la somma tra SSV se lo facciamo per più di ~~tre~~ ~~ssv~~ due soli SSV,

<u>Prop</u>:
siano $\underline{v}_1, \underline{v}_2 \cdots \underline{v}_k \in V$ svn
Span $(\underline{v}_1, \underline{v}_2 \cdots \underline{v}_k)$ ssv di V

<u>Dim</u>: esplicita con $k = 2$, $k = 3$

$k = 1$
Span $(\underline{v}_1) = \{\underline{u} \in V \mid \underline{u} = \alpha \underline{v}_1,\ \alpha \in \mathbb{R}\}$
- SS0 $0\underline{v}_1 = \underline{0}$ ✓
- SS1 $\underline{u}, \underline{w} \in \text{Span}(\underline{v}_1)$
$\underline{u} + \underline{w} \in \text{Span}(\underline{v}_1)$?
$\begin{matrix} \underline{u} = \alpha \underline{v}_1 \\ \underline{w} = \beta \underline{v}_1 \end{matrix}$ } $\underline{u} + \underline{w} = \alpha \underline{v}_1 + \beta \underline{v}_1 = (\alpha + \beta)\underline{v}_1$
sc. per vett. → $\in \text{Span}(\underline{v}_1)$ ✓
- SS2 $\underline{u} = \alpha \underline{v}_1$
$\lambda \underline{u} = \lambda(\alpha \underline{v}_1) \xrightarrow{\text{omog. tra:}} = (\lambda \alpha)\underline{v}_1$ ✓

$k = 2$
- SS0 $\underline{0}_V \in \text{Span}(\underline{v}_1, \underline{v}_2)$
$\{\alpha \underline{v}_1 + \beta \underline{v}_2 \ \alpha \in \mathbb{R}, \beta \in \mathbb{R}\}$

con $\alpha = \beta = 0 \to 0\underline{v}_1 + 0\underline{v}_2 = \underline{0}_V + \underline{0}_V = \underline{0}_V$ ✓

- SS1: $\underline{u}, \underline{w} \in \text{Span}(\underline{v}_1, \underline{v}_2)$
$\exists \alpha, \beta \mid \underline{u} = \alpha \underline{v}_1 + \beta \underline{v}_2$
$\exists \gamma, \delta \mid \underline{w} = \gamma \underline{v}_1 + \delta \underline{v}_2$
$\underline{u} + \underline{w} = (\alpha + \gamma)\underline{v}_1 + (\beta + \delta)\underline{v}_2$
sc. vett $\in$ Span ✓

- SS2: $\forall \underline{u} \in \text{Span}(\underline{v}_1, \underline{v}_2)$ $\forall \lambda \in \mathbb{R}$
$\lambda \underline{u} = (\lambda \alpha)\underline{v}_1 + (\lambda \beta)\underline{v}_2 \in$ Span ✓