Project 4

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Task 1

$$u_{xx} + u_{yy} = f(x, y), \qquad (x, y) \in \Omega$$

Were Ω is the domain given by, $x \geq 0$, $y \geq 0$ and $y \leq 1 - x^2$.

With boundary contitions $u(x, 0) = u(0, x) = 0$ and $\frac{\partial u}{\partial n} = 0$ at the $y = 1 - x^2$ boundary.

Using central differences we get,

$$\frac{\delta_x^2 u}{h^2} + \frac{\delta_y^2 u}{h^2} = f(x, y)$$

wich is the classic 5-point formula.

$$\frac{U_{m+1,n} - 2U_{m,n} + U_{m-1,n}}{h^2} + \frac{U_{m,n+1} - 2U_{m,n} + U_{m,n-1}}{h^2} = f(x, y)$$

Rewriting using the four diretions north (n), south (s), east (e) and west (w), and p for the central point, while moving the h we get,

$$U_n + U_s + U_e + U_w - 4U_p = hf_p$$

We handle the boundary at $y = 1 - x^2$ as outlined on page 70 of the note by BO, we make a point Q inbetween the gridpoints at the itersection between the grid and the normal n. We aproximate the point by linear interpolation between the two closest gridpoints, called S and R

We aproximate $\frac{\partial u}{\partial n}$ by,

$$\frac{U_p - U_Q}{d}$$

where d is the distance between U_p and U_Q, the aproximation for Q is,

$$U_Q = U_R\frac{h'}{h} + U_S\frac{h - h'}{h}$$

where h' is the distance between U_Q and U_S

We insert the aproximation for Q in the aproximation of $\frac{\partial u}{\partial n} = 0$

$$\frac{1}{d}(U_p - (U_R\frac{h'}{h} + U_S\frac{h - h'}{h})) = 0$$

solving for U_P becomes,

$$U_p = U_R\frac{h'}{h} + U_S\frac{h - h'}{h}$$

We also need to handle the uneven stepsizes between the points next to the boundary $y = 1 - x^2$ and the pointsn

Lets say we have a boundary point U_n north of a point U_p, with distance h_n between them. When you have variable stepsizes you get a lower error if the points on opisite sides of U_p are equidistant. We therefore aproximate a point with a distance h_n from U_p south of U_p called $U_s^{'}$

In the scheme would then become,

$$\frac{U_n - 2U_p + U_s^{'}}{h_n^2} + \frac{U_w - 2U_p + U_e}{h^2} = f(x, y)$$

The aproximation of $U_s^{'}$ is

$$U_s^{'} = U_s \frac{h_n^{'}}{h} + U_p \frac{h - h_n^{'}}{h}$$

Inserting into the scheme we get,

$$\frac{U_n - U_p(1 + \frac{h_n^{'}}{h}) + U_s \frac{h_n^{'}}{h}}{h_n^2} + \frac{U_w - 2U_p + U_e}{h^2} = f(x, y)$$

Task 2

$$u_t - au_x = 0, \qquad 0 \leq x \leq 1, \qquad a > 0$$

With initial and boundary contitions,

$$u(x, 0) = f(x), \qquad u(1, t) = 0$$

a)

The Lax-Wendroff method is,

$$U_m^{n+1} = U_m^n - \frac{-ap}{2}(U_{m+1}^n - U_{m-1}^n) + \frac{(-ap)^2}{2}(U_{m+1}^n - 2U_m^n + U_{m-1}^n)$$

This method breaks down at the boudary as it requires a point to the left of the boundary. We can solve this issue many different ways, one way is to use a different method at the boundary that only uses points to the right. Or you could use ghost points outside the domain, you could for instance set the ghost point $U_{-1}^n = U_0^n$.

I am going to use linear extrapolation to set the value of the ghost point,

$$U_{-1}^n = 2U_0^n - U_1^n$$

Using this point in the Lax-Wendroff method at the point U_0^{n+1} we get,

$$U_0^{n+1} = U_0^n - (-ap)(U_1^n - U_0^n)$$

Wich is a 1st order method.

Since the $u_t - au_x = 0$ has a linear characteristic with a incline of $-a$ and the dependence intervall for Lax-Wendroff is given by $[x* - t*/p, x* + t*/p]$ the CFL-codition becomes,

$$a \leq \frac{1}{p}$$

b)

Von Neumann stability:

By aplying $U_m^n = \xi^n e^{i\beta x_m}$ to the method we get,

$$\xi^{n+1} e^{i\beta x_m} = \xi^n e^{i\beta x_m} - \frac{-ap}{2}(\xi^n e^{i\beta x_{m+1}} - \xi^n e^{i\beta x_{m-1}}) + \frac{(-ap)^2}{2}(\xi^n e^{i\beta x_{m+1}} - 2\xi^n e^{i\beta x_m} + \xi^n e^{i\beta x_{m-1}})$$

Dividing by $\xi^n e^{i\beta x_m}$ we get

$$\xi = 1 - \frac{-ap}{2}(e^{i\beta h} - e^{-i\beta h}) + \frac{(-ap)^2}{2}(e^{i\beta h} - 2 + e^{-i\beta h})$$

Using Eulers formula and setting $r = -ap$ we get,

$$\xi = 1 - risin(\beta h) + r^2(cos(\beta h) - 1)$$

Finding $|\xi|^2 = Re(\xi)^2 + Im(\xi)^2$

$$|\xi|^2 = (1 + r^2(cos(\beta h) - 1)^2 + r^2 sin^2(\beta h) = (1 - r^2(cos(\beta h) - 1)^2 + r^2(1 - (cos^2(\beta h)))$$

Using the triginometric identety $cos(\beta h) = 1 - sin^2(\frac{\beta h}{2})$ and setting $q = sin(\frac{\beta h}{2})$

$$|\xi|^2 = (1 - r^2 q^2)^2 + r^2 + r^2(1 - (1 - q^2)^2) = 1 - 4r^2(1 - r^2)q^4$$

The Von Neumann stability condition $|\xi| \leq 1$ becomes,

$$1 - 4r^2(1 - r^2)q^4 \leq 1$$

$$r^2(1 - r^2)q^4 \geq 0$$

Given that $r^2, q^4 \geq 0$ for all r, q we get

$$|r| \leq 1$$

Consistency:

The Taylor aproximation of u_m^{n+1} is,

$$u_m^{n+1} = u_m^n + ku_t(x_m, t_n) + k^2\frac{u_{tt}(x_m, t_n)}{2} + k^3\frac{u_{ttt}(x_m, \xi_1)}{6} \qquad \xi_1 \in [t_n, t_{n+1}]$$

From the problem we have $u_t = -au_x$, using this we get,

$$u_m^{n+1} = u_m^n + k(-a)u_x(x_m, t_n) + k^2\frac{a^2 u_{xx}(x_m, t_n)}{2} + k^3\frac{u_{ttt}(x_m, \xi_1)}{6}$$

By taylor expantion we find,

$$u_x(x_m, t_n) = \frac{u_{m+1}^n - u_{m-1}^n}{2h} + h^2\frac{u_{xxx}(\xi_2, t_n)}{6}$$

and

$$u_{xx}(x_m, t_n) = \frac{u^n_{m+1} - 2u^n_m + u^n_{m-1}}{h} + h^2 \frac{u_{xxxx}(\xi_2, t_n)}{24}$$

Setting this in for $u_x(x_m, t_n)$ and $u_{xx}(x_m, t_n)$ we get,

$$u^{n+1}_m = u^n_m + k(-a)\frac{u^n_{m+1} - u^n_{m-1}}{2h} + k^2 a^2 \frac{u^n_{m+1} - 2u^n_m + u^n_{m-1}}{2h} + k^3 \frac{u_{ttt}(x_m, \xi_1)}{6} + h^2 k(-a)\frac{u_{xxx}(\xi_2, t_n)}{12} + h^2 k^2 a^2 \frac{u_{xxxx}(\xi_2}{24}$$

By subtracting the numeical method from the exact we find the local truncation error τ,

$$\tau = k^3 \frac{u_{ttt}(x_m, \xi_1)}{6} + h^2 k(-a)\frac{u_{xxx}(\xi_2, t_n)}{12} + h^2 k^2 a^2 \frac{u_{xxxx}(\xi_2, t_n)}{24}$$

We see that all the terms in τ depend on k or h, and thus,

$$k, h \to 0 \Rightarrow \tau \to 0$$

Since τ goes to zero when h, h goes to zero, we know that the method is consistent.

c)

A better method for finding the value could be to swap the centeral differences in Lax-Wendroff with forward differences,

$$U^{n+1}_m = U^n_m - (-ap)(U^n_{m+1} - U^n_m) + \frac{(-ap)^2}{2}(U^n_{m+2} - 2U^n_{m+1} + U^n_m)$$

Since this method is of second order.

Task 3

Dissipation:

By aplying $U_m^n = \xi^n e^{i\beta x_m}$ to the method

$$U_m^{n+1} = \frac{1}{2}(U_{m+1}^n + U_{m-1}^n) + \frac{-ak}{2h}(U_{m+1}^n - U_{m-1}^n)$$

we get,

$$\xi^{n+1} e^{i\beta x_m} = \frac{1}{2}(\xi^n e^{i\beta x_{m+1}} + \xi^n e^{i\beta x_{m-1}}) + \frac{-ak}{2h}(\xi^n e^{i\beta x_{m+1}} - \xi^n e^{i\beta x_{m-1}})$$

Dividing by $\xi^n e^{i\beta x_m}$ we get

$$\xi = \frac{1}{2}(e^{i\beta h} + e^{-i\beta h}) + \frac{-ak}{2h}(e^{i\beta h} - e^{-i\beta h})$$

Using Eulers formula we get,

$$\xi = \frac{1}{2}(cos(\beta h) + isin(\beta h) + cos(-\beta h) + isin(-\beta h)) + \frac{-ak}{2h}(cos(\beta h) + isin(\beta h) - cos(-\beta h) - isin(-\beta h))$$

Wich simplefies to,

$$\xi = cos(\beta h) + i\frac{-ak}{h}sin(\beta h)$$

This gives us,

$$|\xi|^2 = cos^2(\beta h) + r^2 sin^2(\beta h)$$

$$|\xi|^2 = 1 - (1 - r^2)sin^2(\beta h)$$

Because $\sqrt{1-x} \leq 1 - \frac{x}{2}$ we get,

$$|\xi| \leq 1 - \frac{1-r^2}{2}sin^2(\beta h) = 1 - \frac{1-r^2}{2}\left(\frac{sin(\beta h)}{\beta h}\right)^2(\beta h)^2$$

The largest value $\frac{sin(\beta h)}{\beta h}$ can have on the interval $[-\pi, \pi]$ is 1,

$$|\xi| \leq 1 - \frac{1-r^2}{2}(\beta h)^2$$

From this we see that the method is dissapative of order 2.

Dispersion:

Using the formula,

$$\alpha = \frac{1}{\beta k}arctan\left(\frac{-Im(\xi)}{Re(\xi)}\right)$$

we get,

$$\alpha = \frac{1}{\beta k} arctan\left(\frac{\frac{ak}{h}sin(\beta h)}{cos(\beta h)}\right) = \frac{1}{\beta h} arctan(r * tan(\beta h))$$

We see that the method is generaly disspersive. α depends on β for all values of r exept for when$r = 1$, in wich case α is constant equal too $\frac{h}{k}$.

Task 4

$$-u_{xx} + cu = f, \qquad u(0) = u(1) = 0,$$

Find $u \in V$ such that $a(u, v) = F(v)$, for all $v \in V$.

a)

We set up the variational form by multipyling by the test function v and integating over the domain;

$$-\int_0^1 u''(x)v(x)dx + c\int_0^1 u(x)v(x)dx = \int_0^1 f(x)v(x)dx$$

By partial integating the first integral we get,

$$\int_0^1 u'(x)v'(x)dx + c\int_0^1 u(x)v(x)dx = \int_0^1 f(x)v(x)dx$$

since the term $[u'(x)v(x)]_0^1$ becomes zero.

We find,

$$a(u, v) = \int_0^1 u'(x)v'(x)dx + c\int_0^1 u(x)v(x)dx,$$

$$F(v) = \int_0^1 f(x)v(x)dx,$$

and V is the function space of square-integrable functions on $(0, 1)$ $L^2((0, 1))$

Letting $H^1((0, 1))$ be the space of functions $v \in L^2$ for which a weak derivative $v \in L^2$ exists. H^1 is a hilbert space.

Defining innerproducts and norms for H^1 and L^2:

$$< f, g >_{L^2} = \int_0^1 f(x)g(x)dx$$

$$< f, g >_{H^1} = \int_0^1 f(x)g(x)dx + \int_0^1 f'(x)g'(x)dx$$

$$||f||_{L^2}^2 = \int_0^1 f(x)^2 dx$$

$$||f||_{H^1}^2 = \int_0^1 f(x)^2 dx + \int_0^1 f'(x)^2 dx$$

From the Lax-Milgram theorem we know that the weak formulation has a unique solution if F is bounded on H^1, and a is continuous and coercive.

The condition for continuity is,

$$|a(u, v)| \leq M||u||_{H^1}||v||_{H^1}$$

and the condition for coercivity is,

$$|a(u, u)| \geq N||u||_{H^1}^2$$

F is bounded on H^1 if,

$$||F||_{H^1} = \sup_{v \neq 0} \frac{|F(v)|}{||v||_{H^1}^2} < \infty$$

Finding when F is bound by using the Cauchy-Schwarz inequality in L^2

$$|F(v)| = | < f, g >_{L^2}| \leq ||f||_{L^2}||v||_{L^2} \leq ||f||_{L^2}||v||_{H^1}$$

$$\Downarrow$$

$$||F||_{H^1} \leq ||f||_{L^2} < \infty$$

We need $f \in L^2$, and bounded on $(0, 1)$ for the weak formulation to have a unique solution.

Now we check for continuity, from the triangle inequality we have,

$$|a(u, v)| = | < u, v >_{H^1} + (c-1) < u, v >_{L^2}| \leq | < u, v >_{H^1}| + |(c-1)|| < u, v >_{L^2}|$$

From the Cauchy-Schwarz inequality we have,

$$\leq ||u||_{H^1}||v||_{H^1} + |c-1|||u||_{L^2}||v||_{L^2} \leq ||u||_{H^1}||v||_{H^1} + |c-1|||u||_{H^1}||v||_{H^1} = |c|||u||_{H^1}||v||_{H}$$

We find that the continuity condtiton is kept for all c

$$|a(u, v)| \leq M||u||_{H^1}||v||_{H^1}, \qquad M = |c|$$

Now we check for coercivity,

$$|a(u, u)| = |||u||^2_{H^1} + (c-1)||u||^2_{L^2}|$$

We see that,

$$|a(u, u)| \geq N||u||^2_{H^1}$$

when $c \geq 1$.

When using the Poincar´e inequality when $c = 0$ we find,

$$|a(u, u)| = |u|^2_{H^1} \geq C^2||u||^2_{H^1}$$

We have corcivity when $c = 0$ and when $c \geq 1$, since we have conituity for all c we know there is a unique soulution for of the weak formulation when $c = 0$ and $c \geq 1$.

b)

Let $V_h = span\{\varphi_i\}_{i=1}^N$ be some n-dimensional subspace of V

In a Galerkin method you restrict the functions u, v to lie in a finite dimentional subspace of V. The natural basis for this subspace is $\{\varphi_i\}_{i=1}^N$, where φ_i is given by,

$$\varphi_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & x_i \leq x \leq x_{i+1} \\ 0 & otherwise \end{cases}$$

If we write $u(x)$ and $v(x)$ as,

$$u(x) = \Sigma_i u_i \varphi_i(x), \qquad v(x) = \Sigma_i v_i \varphi_i(x)$$

the weak formulation from earlier becomes,

$$\sum_{i,j} u_i v_j \int_0^1 \varphi_i'(x)\varphi_j'(x)dx + c\sum_{i,j} u_i v_j \int_0^1 \varphi_i(x)\varphi_j(x)dx = \sum_i v_j \int_0^1 \varphi_i(x) f(x)dx$$

Now we define the stiffness matrix B, mass matrix M and load vector F,

$$A_{i,j} = \int_0^1 \varphi_i'(x)\varphi_j'(x)dx, \qquad M_{i,j} = \int_0^1 \varphi_i(x)\varphi_j(x)dx, \qquad F_j = \int_0^1 \varphi_i(x) f(x)dx$$

Using this we can rewrite the weak formulation as,

$$v^T A u + c v^T M u = v^T F$$

Since this must hold for all v this is equivelent to,

$$Au + cMu = F$$

$$(A + cM)u = F$$

Here $A + cM$ is the A and F is b in the system of linear eqations outlined in the task.

Since φ_i-s are zero everywere exept around x_i we can assebele A and M from 2×2 matrices,

$$\int_k^{k+1} \varphi_k'(x)\varphi_k'(x)dx = \int_k^{k+1} \frac{1}{(x_{k+1}-x_k)^2}dx = \frac{1}{x_{k+1}-x_k}$$

$$\int_k^{k+1} \varphi_k'(x)\varphi_{k+1}'(x)dx = \int_k^{k+1} -\frac{1}{(x_{k+1}-x_k)^2}dx = -\frac{1}{x_{k+1}-x_k}$$

If we have a eqidistant grid $\frac{1}{x_{k+1}-x_k}$ is h and the 2×2 matrices for A become,

$$\frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The complete matrix becomes,

$$A = \frac{1}{h}\begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{bmatrix}$$

Doing the same for M:

$$\int_k^{k+1} \varphi_k(x)\varphi_k(x)dx = \int_k^{k+1} \frac{(x_{k+1}-x)^2}{(x_{k+1}-x_k)^2}dx = \frac{x_{k+1}-x_k}{3}$$

$$\int_k^{k+1} \varphi_k(x)\varphi_k(x)dx = \int_k^{k+1} \frac{(x_{k+1}-x)(x-x_k)}{(x_{k+1}-x_k)^2}dx = \frac{x_{k+1}-x_k}{6}$$

If we have a eqidistant grid $\frac{1}{x_{k+1}-x_k}$ is h and the 2×2 matrices for B become,

$$\frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The complete matrix becomes,

$$A = \frac{1}{h}\begin{bmatrix} 2 & 1 & 0 & \dots & 0 & 0 \\ 1 & 4 & 1 & \dots & 0 & 0 \\ 0 & 1 & 4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4 & 1 \\ 0 & 0 & 0 & \dots & 1 & 2 \end{bmatrix}$$

We enforce the boundary conditions by revoving the first and last rows and coulums, the matrices become,

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & \dots & 0 \\ -1 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 2 \end{bmatrix}$$

$$M = \frac{h}{6}\begin{bmatrix} 4 & 1 & \dots & 0 \\ 1 & 4 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 4 \end{bmatrix}$$

For F the 2-dimentional vector becomes,

$$\frac{1}{x_{k+1}-x_k}\begin{pmatrix}\int_{x_k}^{x_k+1}(x_{k+1}-x)f(x)dx\\ \int_{x_k}^{x_k+1}(x-x_k)f(x)dx\end{pmatrix}$$

For most $f(x)$ we need to use a quadrature rule like $\int_a^b g(x)dx \simeq (b-a)\frac{g(a)+g(b)}{2}$ to slove the intergal.By doing this we get,

$$\frac{1}{2}\begin{pmatrix}f(x_{x_k})\\ f(x_{x_k+1})\end{pmatrix}$$

Thus making the full vector,

$$F=\frac{1}{2}\begin{pmatrix}f(x_0) & 2f(x_1 & 2f(x_2) & \ldots & 2f(x_M) & f(x_{M+1}))\end{pmatrix}$$

Enforcing the boundary we get,

$$F=\begin{pmatrix}f(x_1 & f(x_2) & \ldots & f(x_M))\end{pmatrix}$$

After the system of linear equations is solved, the expretion for the numerical sulution is simply,

c)

When we have $V_h = span\{sin(\pi x), sin(2\pi x), sin(3\pi x)\}$, we get that,

$$A_{i,j} = \int_0^1 \varphi_i'(x)\varphi_j'(x)dx, \qquad F_j = \int_0^1 \varphi_i(x)f(x)dx$$

By sloving the intergals we get the linear system of equations,

$$\begin{bmatrix} \frac{1}{2}\pi^2 & 0 & 0 \\ 0 & 2\pi^2 & 0 \\ 0 & 0 & \frac{9}{2}\pi^2 \end{bmatrix} u = \begin{bmatrix} \frac{1}{\pi} \\ -\frac{1}{2\pi} \\ \frac{1}{3\pi} \end{bmatrix}$$

solving we get,

$$u = \begin{bmatrix} \frac{2}{\pi^3} \\ -\frac{1}{\pi^3} \\ \frac{2}{27\pi^3} \end{bmatrix}$$

Setting this back in to $U(x) = \Sigma_i u_i \varphi_i(x)$ we get,

$$U(x) = \frac{2}{\pi^3} sin(\pi x) - \frac{1}{\pi^3} sin(2\pi x) + \frac{2}{27\pi^3} sin(3\pi x)$$