

AST3220 - Project 1

(Dated: May 2, 2023)

This abstract is abstract.

1. PROBLEM A)

We can rewrite the number density n_i of species i in terms of the relative number density Y_i as:

$$Y_i = \frac{n_i}{n_b} \Rightarrow n_i = n_b Y_i \quad (1.1)$$

$$= \frac{n_{b0}}{a^3} Y_i \quad (1.2)$$

Where $n_b(t)$ is the baryon number density, n_{b0} is the baryon number density today, and $a(t)$ is the scale factor. Using the product rule for differentiation, we can then write

$$\frac{dn_i}{dt} = n_{b0} \frac{d}{dt} (Y_i a^{-3}) \quad (1.3)$$

$$= n_{b0} \left(\frac{1}{a^3} \frac{dY_i}{dt} - 3Y_i \frac{\dot{a}}{a^4} \right) \quad (1.4)$$

$$= n_b \frac{dY_i}{dt} - 3Y_i n_b H \quad (1.5)$$

$$= n_b \frac{dY_i}{dt} - 3n_i H \quad (1.6)$$

Next we want to switch from t to $\ln T$ as our time variable, where T is the temperature. Using $T = T_0 a^{-1}$ we get

$$\ln T = \ln T_0 - \ln a(t) \quad (1.7)$$

Then, using the chain rule of differentiation, we can rewrite

$$\frac{dY_i}{dt} = \frac{d(\ln T)}{dt} \frac{dY_i}{d(\ln T)} \quad (1.8)$$

$$= -\frac{\dot{a}}{a} \frac{dY_i}{d(\ln T)} \quad (1.9)$$

$$= -H \frac{dY_i}{d(\ln T)} \quad (1.10)$$

Inserting to equation (1.6) we get

$$\frac{dn_i}{dt} = -n_b H \frac{dY_i}{d(\ln T)} - 3n_i H \quad (1.11)$$

The equations for the evolution of the number densities of protons p and neutrons n are given as

$$\frac{dn_n}{dt} + 3Hn_n = n_p \Gamma_{p \rightarrow n} - n_n \Gamma_{n \rightarrow p} \quad (1.12)$$

$$\frac{dn_p}{dt} + 3Hn_p = n_n \Gamma_{n \rightarrow p} - n_p \Gamma_{p \rightarrow n} \quad (1.13)$$

$$= - \left(\frac{dn_n}{dt} + 3Hn_n \right) \quad (1.14)$$

And by inserting Eq. (1.1) and Eq. (1.6) we finally find the evolution of the relative number densities:

2. PROBLEM B)

The relation $T_\nu = (4/11)^{1/3} T$ can be derived from the conservation of entropy, which tells us that

$$g_{*s}(aT)^3 = \text{const.} \quad (2.1)$$

At the time where the universe had a temperature $k_B T$ 0.511 MeV, electrons and positrons were relativistic and the process

$$e^+ + e^- \rightarrow \gamma + \gamma \quad (2.2)$$

occurred in both directions. However, as the temperature universe falls below the rest mass of the electron and positron $k_B T$ 0.511, the average energy of a photon collision is too small for an electrons-positron pair to be created. Since electrons and positrons will still annihilate through the process

$$e^+ + e^- \rightarrow \gamma + \gamma \quad (2.3)$$

most of the positrons and electrons will then disappear. Assuming this happened immediately, and that the universe is in thermal equilibrium ($T_i = T$), the effective number of degrees of freedom before and after can be written

$$g_{*s}^{\text{before}} = g_\nu + \frac{7}{8}(g_{e^-} + g_{e^+}) \quad (2.4)$$

$$= 2 + \frac{7}{8}4 \quad (2.5)$$

$$= \frac{11}{2} \quad (2.6)$$

$$g_{*s}^{\text{after}} = g_\nu \quad (2.7)$$

$$= 2 \quad (2.8)$$

If we also assume the scale factor a is the same before and after, the conservation of entropy gives us

$$\frac{11}{2}(aT)_{\text{before}}^3 = 2(aT)_{\text{after}}^3 \quad (2.9)$$

$$\Rightarrow T_{\text{after}} = \left(\frac{11}{4}\right)^{1/3} T_{\text{before}} \quad (2.10)$$

Since neutrinos are decoupled, we then have

$$T_{\nu,after} = T_{\nu,before} = T_{before} = \left(\frac{4}{11}\right)^{1/3} T_{after} \quad (2.11)$$

Finally giving us

$$T_{\nu} = \left(\frac{4}{11}\right)^{1/3} T \quad (2.12)$$

3. PROBLEM C)

In the early universe, dominated by radiation, we have

$$\rho c^2 \approx \frac{\pi^2}{30} g_* \frac{(k_b T)^4}{(\hbar c)^3} \quad (3.1)$$

Where g_* is the effective number of relativistic degrees of freedom. Assuming all the radiation is composed of photons and N_{eff} number of neutrino species, g_* is

$$g_* = 1 + N_{eff} g_{\nu} \left(\frac{T_i}{T}\right)^4 \quad (3.2)$$

$$= 1 + N_{eff} \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} \quad (3.3)$$

With $\rho_{c0} = \frac{3H_0^2}{8\pi G}$ as the critical density, we then find

$$\Omega_{r0} = \frac{\rho_0}{\rho_{c0}} \quad (3.4)$$

$$= \frac{1}{c^2} \left(\frac{\pi^2}{30} g_* \frac{(k_b T)^4}{(\hbar c)^3} \right) \cdot \left(\frac{8\pi G}{3H_0^2} \right) \quad (3.5)$$

$$= \frac{4\pi^3}{45} \frac{G}{H_0^2} \frac{(k_b T_0)^4}{\hbar^3 c^5} g_* \quad (3.6)$$

$$= \frac{4\pi^3}{45} \frac{G}{H_0^2} \frac{(k_b T_0)^4}{\hbar^3 c^5} \left[1 + N_{eff} \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} \right] \quad (3.7)$$

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4. PROBLEM D)

1. Scale factor

At the BBN, the Friedmann equations simplify to

$$\frac{1}{a} \frac{da}{dt} = H_0 \sqrt{\Omega_{r0} a^{-2}} \quad (4.1)$$

With some rearranging we see this is a separable differential equation, which we solve for $a(t)$:

$$a \frac{da}{dt} = H_0 \sqrt{\Omega_{r0}} \quad (4.2)$$

$$\Rightarrow \int_0^a a' da' = H_0 \sqrt{\Omega_{r0}} \int_0^t dt' \quad (4.3)$$

$$\Rightarrow \frac{1}{2} a^2 = H_0 \sqrt{\Omega_{r0}} t \quad (4.4)$$

$$\Rightarrow a = \sqrt{2H_0 t} (\Omega_{r0})^{1/4} \quad (4.5)$$

2. Cosmic time

To find the cosmic time as a function of the photon temperature, we use the relation

$$T = T_0 a^{-1} \Rightarrow a = \frac{T_0}{T} \quad (4.6)$$

Inserting this into eq. (4.5) and squaring both sides we get

$$\left(\frac{T_0}{T}\right)^2 = 2H_0 t \sqrt{\Omega_{r0}} \quad (4.7)$$

$$(4.8)$$

Which is easily solved:

$$t(T) = \frac{1}{2H_0 \sqrt{\Omega_{r0}}} \left(\frac{T_0}{T}\right)^2 \quad (4.9)$$

A table of this expression evaluated at temperatures 10^{10} , 10^9 and 10^8 is attached in table (I)

T [K]	$t(T)$ [s]
10^{10}	1.7774
10^9	1.7774×10^2
10^8	1.7774×10^4

Table I. Age of the universe at different temperatures.

5. PROBLEM E)

ACKNOWLEDGMENTS

I would like thank myself for writing this beautiful document.

REFERENCES

- Reference 1
- Reference 2

Appendix A: Name of appendix

This will be the body of the appendix.

Appendix B: This is another appendix

Tada.