

AST3220 - Project 1

(Dated: February 28, 2023)

This abstract is abstract.

The scalar field has energy density and pressure

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad (1)$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \quad (2)$$

I. PROBLEM 1

Assuming the quintessence field follows the continuity equation

$$\dot{\rho}_\phi = -3H(\rho_\phi + p_\phi) \quad (3)$$

$$= -3\frac{\dot{a}}{a}(1 + w_\phi)\rho_\phi \quad (4)$$

we can begin solving for the density by separating the variables:

$$\frac{1}{\rho_\phi}d\rho_\phi = -\frac{3}{a}(1 + w_\phi)da \quad (5)$$

We then rewrite $a(t)$ in terms of the cosmological redshift of light emitted at some point t in the past:

$$1 + z = \frac{a_0}{a(t)} \Rightarrow a(t) = \frac{a_0}{1 + z} \quad (6)$$

$$\Rightarrow da = -\frac{a_0}{(1 + z)^2}dz \quad (7)$$

Inserting these into eq. 5 and integrating from some time t to today, we get:

$$\int_{\rho_\phi}^{\rho_{\phi 0}} d\rho_\phi \frac{1}{\rho_\phi} = \int_z^0 dz' \frac{3[1 + w_\phi(z')]}{(1 + z')^2} \quad (8)$$

We then flip the integration limits on both sides, canceling the negatives. Computing the left hand integral and solving for ρ_ϕ we then get the solution

$$\rho_\phi(z) = \rho_{\phi 0} \exp \left\{ \int_0^z dz' \frac{3[1 + w_\phi(z')]}{(1 + z')^2} \right\} \quad (9)$$

II. PROBLEM 2

Inserting equations for the density (1) and pressure (2) of the scalar field into the continuity equation we get

$$\dot{\rho}_\phi = -3H(\rho_\phi + p_\phi) \quad (10)$$

$$= -3H\dot{\phi}^2 \quad (11)$$

We can also find an expression for $\dot{\rho}_\phi$ directly, by taking the time derivative of equation (1):

$$\dot{\rho}_\phi = \frac{d}{dt} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) \quad (12)$$

$$= \dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi} \quad (13)$$

Equating the two expressions, we get the differential equation:

$$\dot{\phi}\ddot{\phi} + 3H\dot{\phi}^2 + V'(\phi)\dot{\phi} = 0 \quad (14)$$

Ignoring the boring case of a static $\phi(t) = \phi_0$, we have $\dot{\phi} \neq 0$. The equation can then be reduced to:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \quad (15)$$

III. PROBLEM 3

Taking the time-derivative of the Hubble parameter

$$\dot{H} = \frac{d}{dt} \left(\frac{\dot{a}}{a} \right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2 \quad (16)$$

we recognize both terms on the right hand side from the Friedmann equations. For a flat ($k = 0$) universe the Friedmann equations read

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{\kappa^2}{3}\rho \quad (17)$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}(\rho + 3p) \quad (18)$$

Where we have defined $\kappa^2 = 8\pi G$. For a universe with matter, radiation and quintessence we have

$$\rho = \rho_m + \rho_r + \rho_\phi \quad (19)$$

$$p = p_r + p_\phi \quad (20)$$

Inserting the Friedmann equations into (24), and writing the pressures in terms of the equation of state parameter, we get

$$\dot{H} = -\frac{\kappa^2}{6}(\rho + 3p) - \frac{\kappa^2}{3}\rho \quad (21)$$

$$= -\frac{\kappa^2}{2}[\rho + p] \quad (22)$$

$$= -\frac{\kappa^2}{2}[\rho_m + \rho_r(1 + w_r) + \dot{\phi}] \quad (23)$$

$$= -\frac{\kappa^2}{2}[\rho_m + \rho_r(1 + w_r) + \dot{\phi}] \quad (24)$$

IV. PROBLEM 4

We introduce the dimensionless variables

$$x_1 = \frac{\kappa \dot{\phi}}{\sqrt{6}H} \quad (25)$$

$$x_2 = \frac{\kappa \sqrt{V}}{\sqrt{3}H} \quad (26)$$

$$x_3 = \frac{\kappa \sqrt{\rho_r}}{\sqrt{3}H} \quad (27)$$

$$(28)$$

and notice their squares can be rewritten them in terms of the critical density $\rho_c = \frac{3H^2}{\kappa^2}$:

$$x_1^2 = \frac{\frac{1}{2}\dot{\phi}^2}{\rho_c} \quad (29)$$

$$x_2^2 = \frac{V}{\rho_c} \quad (30)$$

$$x_3^2 = \frac{\rho_r}{\rho_c} \quad (31)$$

$$(32)$$

The quintessence and radiation density parameters can then easily be found to be

$$\Omega_\phi = \frac{\frac{1}{2}\dot{\phi}^2 + V}{\rho_c} = x_1^2 + x_2^2 \quad (33)$$

$$\Omega_r = \frac{\rho_r}{\rho_c} = x_3^2 \quad (34)$$

Lastly we want to find an expression for the density parameter for mass. We accomplish this by taking advantage of our model being spatially flat, meaning $\frac{\rho}{\rho_c} = 1$. Expanding ρ we get

$$1 = \frac{\rho_\phi + \rho_r + \rho_m}{\rho_c} \quad (35)$$

$$= \Omega_\phi + \Omega_r + \Omega_m \quad (36)$$

Which we can solve for Ω_m :

$$\Omega_m = 1 - \Omega_\phi - \Omega_r \quad (37)$$

$$= 1 - x_1^2 - x_2^2 - x_3^2 \quad (38)$$

V. PROBLEM 5

Next we want to rewrite equation (24) in terms of the dimensionless variables. Using the definitions of the density parameters and the dimensionless variables we get:

$$\dot{H} = -\frac{\kappa^2}{2} [\rho_m + \rho_r(1 + w_r) + \dot{\phi}] \quad (39)$$

$$= -\frac{\kappa^2 \rho_c}{2} [\Omega_m + \Omega_r(1 + w_r) + 2\Omega_\phi - 2x_2^2] \quad (40)$$

$$= -\frac{3H^2}{2} [1 + x_1^2 - x_2^2 - x_3^2 + x_3^2(1 + w_r)] \quad (41)$$

Simplifying and inserting the radiation equation of state parameter $w_r = 1/3$, we get:

$$\frac{\dot{H}}{H^2} = -\frac{1}{2} [3 + 3x_1^2 - 3x_2^2 + x_3^2] \quad (42)$$

VI. PROBLEM 6

A. Equation of motion for x_1

Using the product rule:

$$\frac{dx_1}{dN} = \frac{1}{H} \frac{dx_1}{dt} \quad (43)$$

$$= \frac{\kappa}{\sqrt{6}H} \frac{d}{dt} \left(\frac{\dot{\phi}}{H} \right) \quad (44)$$

$$= \frac{\kappa \ddot{\phi}}{\sqrt{6}H^2} - \frac{\kappa \dot{\phi}}{\sqrt{6}H} \frac{\dot{H}}{H^2} \quad (45)$$

From equation (15) we have $\ddot{\phi} = -3H\dot{\phi} - V$. Inserting for $\dot{\phi}$ and \dot{H}/H^2 (eq. 42) we then get:

$$\frac{dx_1}{dN} = -\frac{3\kappa\dot{\phi}}{\sqrt{6}H} - \frac{\kappa V'}{\sqrt{6}H} + \frac{\kappa\dot{\phi}}{\sqrt{6}H} \frac{\dot{H}}{H} \quad (46)$$

$$= -3x_1 + \frac{\sqrt{6}}{2} \lambda x_2^2 + \frac{1}{2} x_1 (3 + 3x_1 - 3x_2^2 + 3x_3^2) \quad (47)$$

Where we have defined λ as

$$\lambda = -\frac{V'}{\kappa V} \quad (48)$$

B. Equation of motion for x_2

The equation for x_2 is found in a similar manner, we get:

$$\frac{dx_2}{dN} = \frac{\kappa}{\sqrt{3}H} \frac{d}{dt} \left(\frac{\sqrt{V}}{H} \right) \quad (49)$$

$$= \frac{\kappa}{\sqrt{3}H} \left(\frac{1}{2} \frac{V'\dot{\phi}}{\sqrt{\phi}H} - \sqrt{V} \frac{\dot{H}}{H^2} \right) \quad (50)$$

$$= \frac{\sqrt{6}}{2} \frac{V'}{\kappa V} \frac{\kappa \sqrt{V}}{\sqrt{3}H} \frac{\kappa \dot{\phi}}{\sqrt{6}H} - \frac{\kappa \sqrt{V}}{\sqrt{3}H} \frac{\dot{H}}{H^2} \quad (51)$$

$$= \frac{\sqrt{6}}{2} \lambda x_1 x_2 + \frac{1}{2} x_2 (3 + 3x_1 - 3x_2^2 + 3x_3^2) \quad (52)$$

C. Equation of motion for x_3

Similarly, for x_3 :

$$\frac{dx_3}{dN} = \frac{\kappa}{\sqrt{3}H} \frac{d}{dt} \left(\frac{\sqrt{\rho_r}}{H} \right) \quad (53)$$

$$= \frac{\kappa}{\sqrt{3}H} \left(\frac{1}{2} \frac{\dot{\rho}_r}{\sqrt{\rho_r}H} - \sqrt{\rho_r} \frac{\dot{H}}{H^2} \right) \quad (54)$$

$$(55)$$

Where $\dot{\rho}_r$ is given by the first Friedmann equation:

$$\dot{\rho}_r = -3H(1 + w_r)\rho_r \quad (56)$$

$$= -4H\rho_r \quad (57)$$

Where we used that $w_r = 1/3$. Inserting the expressions for $\dot{\rho}_r$ and \dot{H}/H^2 we finally get:

$$\frac{dx_3}{dN} = -2x_3 + \frac{1}{2}x_3(3 + 3x_1 - 3x_2^2 + 3x_3^2) \quad (58)$$

VII. PROBLEM 7

Rewriting the expression for λ as a differential equation for V , we get:

$$V' + \kappa\lambda V = 0 \quad (59)$$

For a constant λ this has the solution

$$V = V_0 e^{-\kappa\lambda\phi} \quad (60)$$

We can then find the value of Γ by simple differentiation:

$$\Gamma = \frac{VV''}{(V')^2} \quad (61)$$

$$= \frac{\kappa^2\lambda^2 V^2}{(-\kappa\lambda)^2 V^2} \quad (62)$$

$$= 1 \quad (63)$$

VIII. PROBLEM 8

For a non-constant λ we have

$$\frac{d\lambda}{dN} = -\frac{1}{\kappa H} \frac{d}{dt} \left(\frac{V'}{V} \right) \quad (64)$$

$$= -\frac{1}{\kappa H} \left(\frac{V''}{V} \dot{\phi} - \frac{(V')^2}{V^2} \dot{\phi} \right) \quad (65)$$

$$= -\frac{\kappa\dot{\phi}}{\sqrt{6}H} \frac{\sqrt{6}}{\kappa^2} \left(\frac{V''}{V} - \frac{(V')^2}{V^2} \right) \quad (66)$$

$$= -x_1 \sqrt{6} \left(\frac{V'}{\kappa V} \right)^2 \left(\frac{VV''}{(V')^2} - 1 \right) \quad (67)$$

$$= -\sqrt{6}\lambda^2(\Gamma - 1)x_1 \quad (68)$$

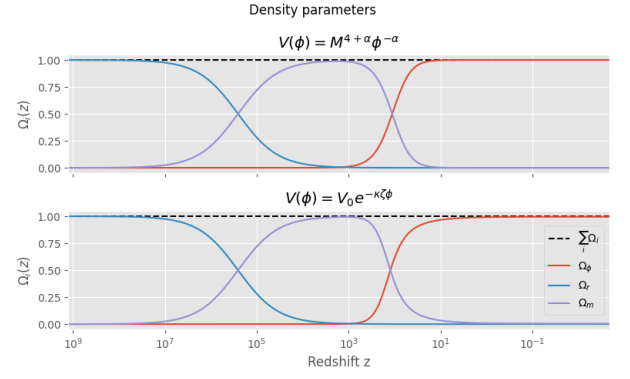


Figure 1. Density parameters

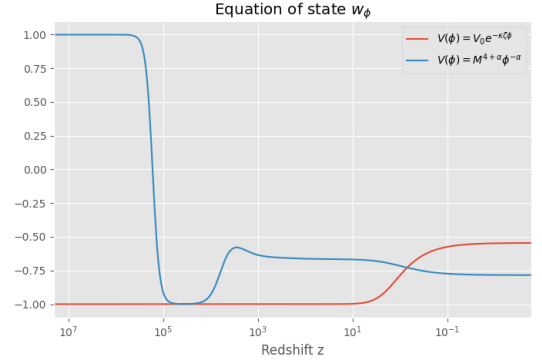


Figure 2. Equation of state

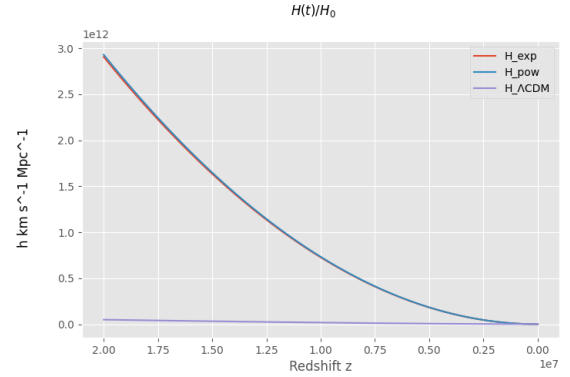


Figure 3. Hubble param

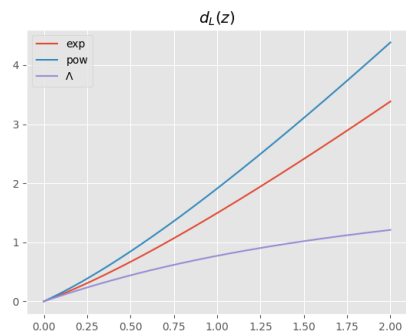


Figure 4.

IX. PROBLEM 9

X. PROBLEM 10

XI. PROBLEM 11

XII. PROBLEM 12

XIII. PROBLEM 13

XIV. PROBLEM 14

ACKNOWLEDGMENTS

I would like thank myself for writing this beautiful document.

REFERENCES

- Reference 1
- Reference 2

Appendix A: Name of appendix

This will be the body of the appendix.

Appendix B: This is another appendix

Tada.