AST3220 - Project 3:
Inflation without approximation

Candidate nr. 14
(Dated: June 12, 2023)

1. PROBLEM A)

Using the convention of represent As H_i neccesarily has the same dimensions as H it is clear that h must be dimensionless:

$$[h] = [H][H_i]^{-1} = 1 \tag{1.1}$$

The hubble parameter has dimension velocity per distance, which can be written

$$[H] = (\mathsf{L}\mathsf{T}^{-1})\mathsf{L}^{-1} = \mathsf{T}^{-1} \tag{1.2}$$

so that τ is dimensionless as well:

$$[\tau] = [H][t] = \mathsf{T}^{-1}\mathsf{T} = 1 \tag{1.3}$$

Both ϕ and the Planck energy has units of energy, making ψ dimensionless as well:

$$[\psi] = [\phi][E_p]^{-1} = \mathsf{M}\mathsf{L}^2\mathsf{T}^{-2}(\mathsf{M}\mathsf{L}^2\mathsf{T}^{-2})^{-1} = 1 \tag{1.4}$$

Lastly, we rewrite the potential v using the definiton $E_p^2 = \hbar c^5/G$, so that

$$v = \frac{\hbar c^3}{H_i^2 E_p^2} = \frac{1}{H_i^2}\frac{G}{c^2}V \tag{1.5}$$

Using the definiton of H_i, we then see that

$$[v] = [H_i]^{-1}[Gc^{-2}V] = [H_i]^{-1}[H_i] = 1 \tag{1.6}$$

so v is dimensionless as well

2. PROBLEM B)

A. Hubble parameter and scale factor

Using the definitons of the dimensionless variables and applying the chain rule, we see that

$$\frac{d}{d\tau}\left(\ln\frac{a}{a_i}\right) = \frac{dt}{d\tau}\frac{d}{dt}\left(\ln\frac{a}{a_i}\right) \tag{2.1}$$
$$= \frac{1}{H_i}\frac{\dot{a}}{a} \tag{2.2}$$
$$= \frac{H}{H_i} \tag{2.3}$$
$$= h \tag{2.4}$$

B. Continuity equation (name???)

Similarly we continue using the chain rule to rewrite $\dot{\phi}$, $\ddot{\phi}$ and V' in terms of τ and ψ:

$$\frac{d\phi}{dt} = \frac{d\tau}{dt}\frac{d\phi}{d\psi}\frac{d\psi}{d\tau} = H_i E_p \frac{d\psi}{d\tau} \tag{2.5}$$
$$\frac{d^2\phi}{dt^2} = \frac{d\tau}{dt}\frac{d}{d\tau}\frac{d\phi}{dt} = H_i^2 E_p \frac{d^2\psi}{d\tau^2} \tag{2.6}$$
$$\frac{dV}{d\phi} = \frac{d\psi}{d\phi}\frac{dV}{dv}\frac{dv}{d\psi} = \frac{1}{E_p}\frac{H_i^2 E_p^2}{\hbar c^3}\frac{dv}{d\psi} \tag{2.7}$$

Which can be inserted to the (????) equation, giving

$$H_i^2 E_p \frac{d^2\psi}{d\tau^2} + 3HH_iE_p\frac{d\psi}{d\tau} + H_i^2 E_p \frac{dv}{d\psi} = 0 \tag{2.8}$$

Dividing both sides by $H_i^2 E_p$ and using the definition $h = H/H_i$, this reduces to

$$\frac{d^2\psi}{d\tau^2} + 3h\frac{d\psi}{d\tau} + \frac{dv}{d\psi} = 0 \tag{2.9}$$

1. *Hubble parameter*

afgasfas

$$h^2 = \frac{8\pi}{3}\left[\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v(\psi)\right] \tag{2.10}$$

3. PROBLEM C)

Read up on this shit

4. PROBLEM D)

During slow-roll, the number of remaining e-folds until inflation ends can be calculated as

$$N(t) = \frac{8\pi}{E_p^2}\int_{\phi_{\text{end}}}^{\phi}\frac{V}{V'}d\phi \tag{4.1}$$
$$\tag{4.2}$$

Where we easily find

$$\frac{V}{V'} = \frac{1}{2}\phi \tag{4.3}$$

and ψ_{end} is given by requiring $\epsilon(\phi_{\text{end}}) = 1$:

$$\epsilon(\phi_{\text{end}}) = \frac{E_p^2}{16\pi}\left(\frac{V'}{V}\right)^2 \tag{4.4}$$
$$= \frac{E_p^2}{4\pi\phi_{\text{end}}^2} \tag{4.5}$$
$$= 1 \tag{4.6}$$
$$\Rightarrow \qquad \phi_{\text{end}} = \frac{E_p}{\sqrt{4\pi}} \tag{4.7}$$

We can then calculate the integral:

$$N(t) = \frac{8\pi}{E_p^2}\int_{\phi_{\text{end}}}^{\phi} \frac{1}{2}\phi d\phi \tag{4.8}$$
$$= \frac{2\pi}{E_p^2}\left(\phi - \frac{E_p^2}{4\pi}\right) \tag{4.9}$$
$$= \frac{2\pi}{E_p^2}\phi^2 - \frac{1}{2} \tag{4.10}$$

By definition, the number of remaining e-folds at the initial time $t = t_i$ is the total number of e-folds N_tot. Evaluating $N(t)$ at t_i and solving for the initial field value ϕ_i we then get:

$$\phi_i = \frac{E_p}{\sqrt{2\pi}}\sqrt{N_{tot} + \frac{1}{2}} \tag{4.11}$$

or in terms of the dimensionless field:

$$\psi_i = \sqrt{\frac{1}{2\pi}\left(N_{tot} + \frac{1}{2}\right)} \tag{4.12}$$

Which evaluates to $\psi_i \approx 8.925$ for 500 e-folds of inflation.

5. PROBLEM E) & F)

To solve the equations of motion of the field, we rename $\xi = \frac{d\psi}{d\tau}$ so equation 2.9 can instead be written as a set of two first order equations:

$$\frac{d\psi}{d\tau} = \xi \tag{5.1}$$
$$\frac{d\xi}{d\tau} = -3h(\xi,\psi)\xi - \frac{d}{d\psi}v(\psi) \tag{5.2}$$

where h is then given by

$$h^2 = \frac{8\pi}{3}\left[\frac{1}{2}\xi^2 + v(\psi)\right] \tag{5.3}$$

and $v(\psi)$ is given by our choice of potential. Equations 5.1 and 5.2 are then easily solved using a numerical integrator. A plot of the resulting field ψ plotted against τ is attached in Figure (1), along with the slow-roll solution.

With the solutions ξ and ψ we can easily find $\ln\left(\frac{a}{a_i}\right)$ by numerically evaluating the integral

$$\ln\left(\frac{a}{a_i}\right) = \int_0^t H dt \tag{5.4}$$
$$= \int_0^\tau h(\xi,\psi)d\tau \tag{5.5}$$

The results are plotted in Figure (2), along with the slow-roll solution

Based on the lectures, we expect the field to decay until it reaches ψ_{end}. When it reaches ψ_{end} it should then start behaving like a damped oscillator centered around $\psi = 0$. This is in agreement with the numerical solution (Figure 1), which starts oscillating at $\tau \approx 1000$. This is also when the slow-roll approximation starts deviating significantly from the exact solution, as it instead continues decaying at a constant rate.

This is around the same time the slow-roll approximation for the scale factor deviates from the exact solution (Figure 2), as one might expect. BLABLABLABLA

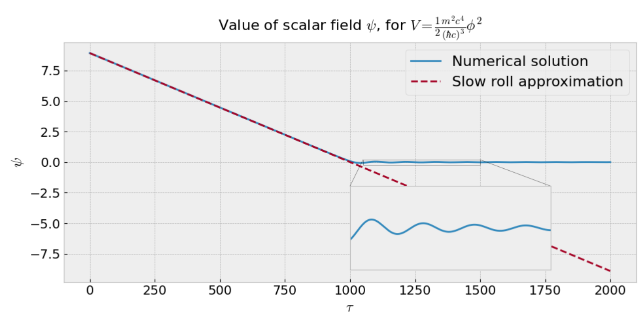

Figure 1.

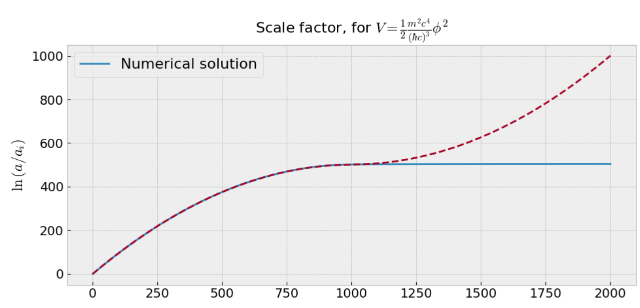

Figure 2.

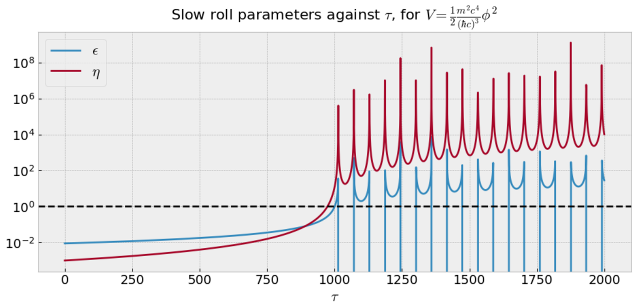

Figure 3.

6. PROBLEM G)

7. PROBLEM H)

SHOW CALCULATION

8. PROBLEM I)

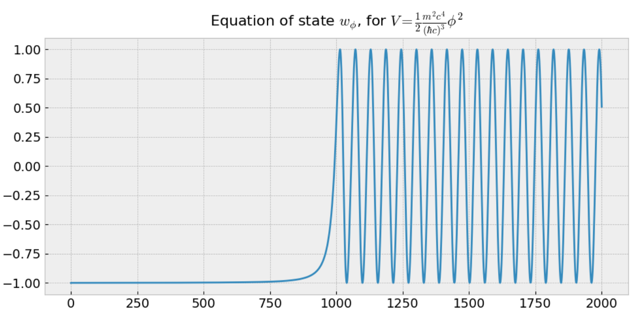

Figure 4.

9. PROBLEM J)

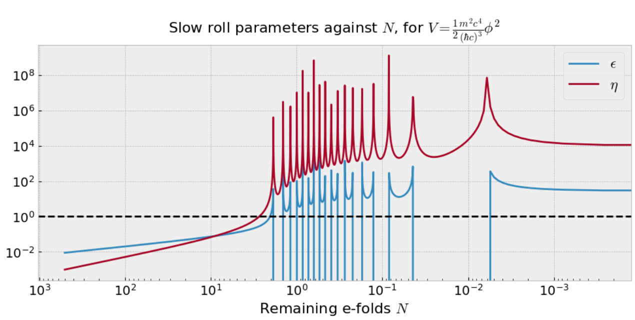

Figure 5.

10. PROBLEM K)

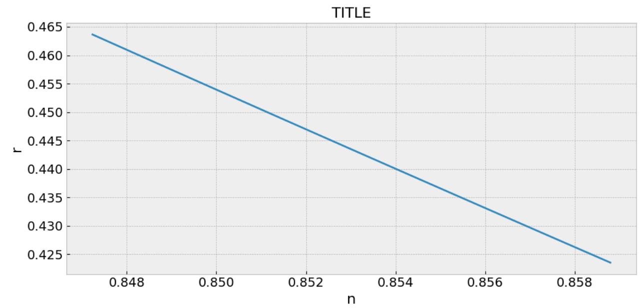

Figure 6.

11. PROBLEM L)

SHOW CALCULATION

12. PROBLEM M)

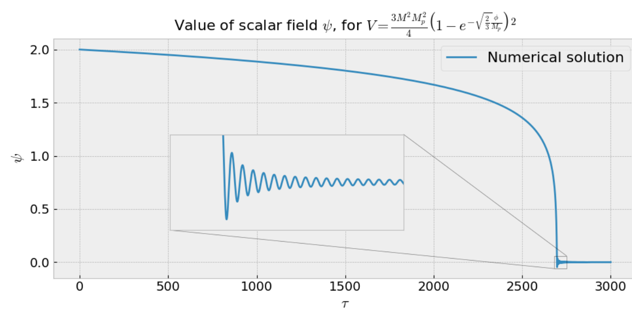

Figure 7.

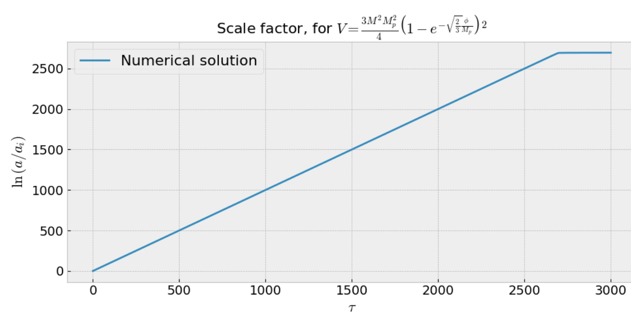

Figure 8.

13. PROBLEM N)

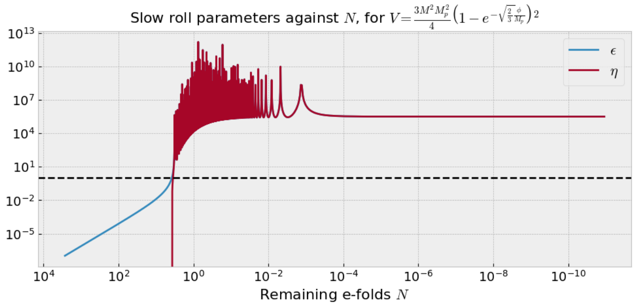

Figure 9.

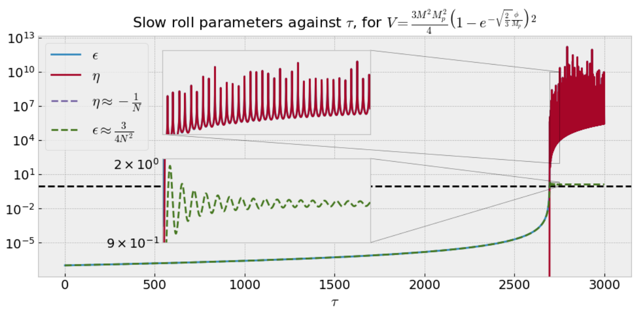

Figure 10.

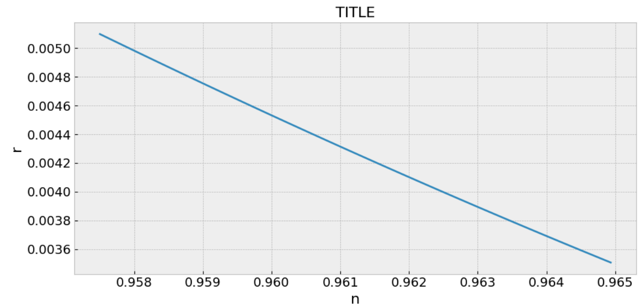

Figure 11.

14. PROBLEM O)

SHOW CALCULATION

15. PROBLEM P)

READ STUFF

ACKNOWLEDGMENTS

I would like thank myself for writing this beautiful document.