

# AST3220 - Project 3: Inflation without approximation

Candidate nr. 14  
(Dated: June 14, 2023)

## 1. PROBLEM A)

Using the convention of represent As  $H_i$  neccesarily has the same dimensions as  $H$  it is clear that  $h$  must be dimensionless:

$$[h] = [H][H_i]^{-1} = 1 \quad (1.1)$$

The hubble parameter has dimension velocity per distance, which can be written

$$[H] = (\text{LT}^{-1})\text{L}^{-1} = \text{T}^{-1} \quad (1.2)$$

so that  $\tau$  is dimensionless as well:

$$[\tau] = [H][t] = \text{T}^{-1}\text{T} = 1 \quad (1.3)$$

Both  $\phi$  and the Planck energy has units of energy, making  $\psi$  dimensionless as well:

$$[\psi] = [\phi][E_p]^{-1} = \text{ML}^2\text{T}^{-2}(\text{ML}^2\text{T}^{-2})^{-1} = 1 \quad (1.4)$$

Lastly, we rewrite the potential  $v$  using the definiton  $E_p^2 = \hbar c^5/G$ , so that

$$v = \frac{\hbar c^3}{H_i^2 E_p^2} = \frac{1}{H_i^2} \frac{G}{c^2} V \quad (1.5)$$

Using the definiton of  $H_i$ , we then see that

$$[v] = [H_i]^{-1}[Gc^{-2}V] = [H_i]^{-1}[H_i] = 1 \quad (1.6)$$

so  $v$  is dimensionless as well

## 2. PROBLEM B)

### A. Hubble parameter and scale factor

Using the definitions of the dimensionless variables and applying the chain rule, we see that

$$\frac{d}{d\tau} \left( \ln \frac{a}{a_i} \right) = \frac{dt}{d\tau} \frac{d}{dt} \left( \ln \frac{a}{a_i} \right) \quad (2.1)$$

$$= \frac{1}{H_i} \frac{\dot{a}}{a} \quad (2.2)$$

$$= \frac{H}{H_i} \quad (2.3)$$

$$= h \quad (2.4)$$

### B. Continuity equation (name???)

Similarly we continue using the chain rule to rewrite  $\dot{\phi}$ ,  $\ddot{\phi}$  and  $V'$  in terms of  $\tau$  and  $\psi$ :

$$\frac{d\phi}{dt} = \frac{d\tau}{dt} \frac{d\phi}{d\psi} \frac{d\psi}{d\tau} = H_i E_p \frac{d\psi}{d\tau} \quad (2.5)$$

$$\frac{d^2\phi}{dt^2} = \frac{d\tau}{dt} \frac{d}{d\tau} \frac{d\phi}{d\psi} \frac{d\psi}{d\tau} = H_i^2 E_p^2 \frac{d^2\psi}{d\tau^2} \quad (2.6)$$

$$\frac{dV}{d\phi} = \frac{d\psi}{d\phi} \frac{dV}{dv} \frac{dv}{d\psi} = \frac{1}{E_p} \frac{H_i^2 E_p^2}{\hbar c^3} \frac{dv}{d\psi} \quad (2.7)$$

Which can be inserted to the (???) equation, giving

$$H_i^2 E_p^2 \frac{d^2\psi}{d\tau^2} + 3H H_i E_p \frac{d\psi}{d\tau} + H_i^2 E_p^2 \frac{dv}{d\psi} = 0 \quad (2.8)$$

Dividing both sides by  $H_i^2 E_p^2$  and using the definition  $h = H/H_i$ , this reduces to

$$\frac{d^2\psi}{d\tau^2} + 3h \frac{d\psi}{d\tau} + \frac{dv}{d\psi} = 0 \quad (2.9)$$

#### 1. Hubble parameter

Lastly, we rewrite the Hubble parameter in terms of the dimensionless quantities. We have

$$H^2 = \frac{8\pi G}{3c^2} \left[ \frac{1}{2\hbar c^3} \dot{\phi}^2 + V(\phi) \right] \quad (2.10)$$

$$= \frac{8\pi G}{3c^2} \left[ \frac{H_i^2 E_p^2}{2\hbar c^3} \left( \frac{d\psi}{d\tau} \right)^2 + V(\phi) \right] \quad (2.11)$$

$$= \frac{8\pi G}{3c^2} \frac{H_i^2 E_p^2}{\hbar c^3} \left[ \frac{1}{2} \left( \frac{d\psi}{d\tau} \right)^2 + v(\psi) \right] \quad (2.12)$$

Where we used equation (2.5) and the definition of  $v$ . Next, inserting the definition of the planck energy

$$E_p^2 = \frac{\hbar c^5}{G} \quad (2.13)$$

and dividing both sides by  $H_i$ , we finally get:

$$h^2 = \frac{8\pi}{3} \left[ \frac{1}{2} \left( \frac{d\psi}{d\tau} \right)^2 + v(\psi) \right] \quad (2.14)$$

## 3. PROBLEM C)

Read up on this shit

#### 4. PROBLEM D)

During slow-roll, the number of remaining e-folds until inflation ends can be calculated as

$$N = \frac{8\pi}{E_p^2} \int_{\phi_{\text{end}}}^{\phi} \frac{V}{V'} d\phi \quad (4.1)$$

Where we make the substitution  $d\phi = E_p d\psi$  and have

$$\frac{V}{V'} = \frac{1}{2}\phi = \frac{E_p}{2}\psi \quad (4.2)$$

Giving

$$N = 4\pi \int_{\psi_{\text{end}}}^{\psi} \psi d\psi \quad (4.3)$$

$$= 2\pi(\psi^2 - \psi_{\text{end}}^2) \quad (4.4)$$

where  $\psi_{\text{end}}$  is given by taking  $\epsilon(\phi_{\text{end}}) = 1$  as the end of inflation:

$$\epsilon(\phi_{\text{end}}) = \frac{E_p^2}{16\pi} \left( \frac{V'}{V} \right)^2 = \frac{1}{16\pi} \left( \frac{dv}{d\psi} \right) \quad (4.5)$$

$$= \frac{1}{4\pi\psi_{\text{end}}^2} \quad (4.6)$$

$$= 1 \quad (4.7)$$

$$(4.8)$$

Solving for  $\psi_{\text{end}}$  this gives

$$\Rightarrow \psi_{\text{end}}^2 = \frac{1}{4\pi} \quad (4.9)$$

Inserting this, the number of remaining e-folds is

$$N = 2\pi\psi^2 - \frac{1}{2} \quad (4.10)$$

The number of remaining e-folds at the initial time  $t = t_i$  is the total number of e-folds  $N_{\text{tot}}$ . Evaluating  $N(t)$  at  $t_i$  and solving for the initial field value  $\psi_i$  we then get:

$$\psi_i = \sqrt{\frac{1}{2\pi} \left( N_{\text{tot}} + \frac{1}{2} \right)} \quad (4.11)$$

Which evaluates to  $\psi_i \approx 8.925$  for 500 total e-folds of inflation.

#### 5. PROBLEM E) & F)

To solve the equations of motion of the field, we rename  $\xi = \frac{d\psi}{d\tau}$  so equation 2.9 can instead be written as a set of two first order equations:

$$\frac{d\psi}{d\tau} = \xi \quad (5.1)$$

$$\frac{d\xi}{d\tau} = -3h(\xi, \psi)\xi - \frac{d}{d\psi}v(\psi) \quad (5.2)$$

where  $h$  is then given by

$$h^2 = \frac{8\pi}{3} \left[ \frac{1}{2}\xi^2 + v(\psi) \right] \quad (5.3)$$

and  $v(\psi)$  is given by our choice of potential. Equations 5.1 and 5.2 are then easily solved using a numerical integrator. A plot of the resulting field  $\psi$  plotted against  $\tau$  is attached in Figure (1), along with the slow-roll solution.

With the solutions  $\xi$  and  $\psi$  we can easily find  $\ln\left(\frac{a}{a_i}\right)$  by numerically evaluating the integral

$$\ln\left(\frac{a}{a_i}\right) = \int_0^t H dt \quad (5.4)$$

$$= \int_0^\tau h(\xi, \psi) d\tau \quad (5.5)$$

The results are plotted in Figure (5), along with the slow-roll solution

Based on the lectures, we expect the field to decay until it reaches  $\psi_{\text{end}}$ . When it reaches  $\psi_{\text{end}}$  it should then start behaving like a damped oscillator centered around  $\psi = 0$ . This is in agreement with the numerical solution (Figure 1), which starts oscillating at  $\tau \approx 1000$ . This is also when the slow-roll approximation starts deviating significantly from the exact solution, as it instead continues decaying at a constant rate.

This is around the same time the slow-roll approximation for the scale factor deviates from the exact solution (Figure 5), as one might expect. BLABLABLABLA

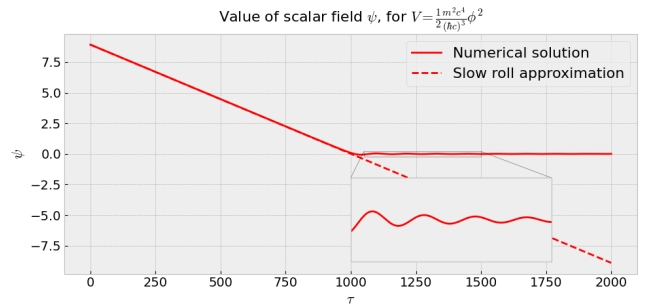


Figure 1. The dimensionless scalar field  $\psi$  for the quadratic potential, along with its approximation in the slow-roll regime. Plotted against the dimensionless time  $\tau$ .

#### 6. PROBLEM G)

SLOW ROLL PARAMETERS AGAINST TAU

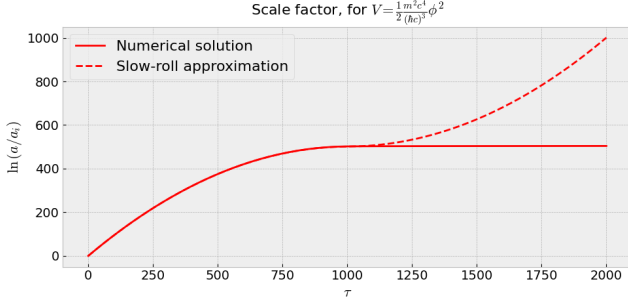


Figure 2. The logarithm of the scale factor  $\ln\left(\frac{a}{a_i}\right)$  for a quadratic potential, along with the slow-roll approximation. Plotted as a function of the dimensionless time  $\tau$ .

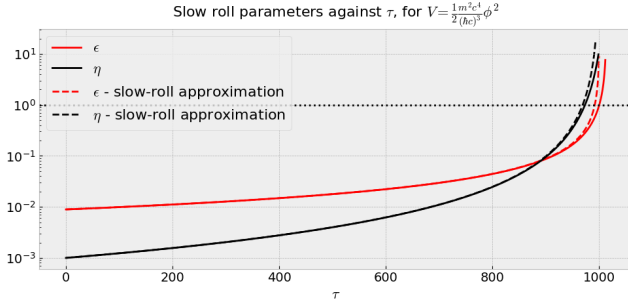


Figure 3. The slow-roll parameters  $\epsilon$  and  $\eta$  for a quadratic potential, plotted against dimensionless time  $\tau$ .

## 7. PROBLEM H)

We now want to express the equation of state parameter in terms of the dimensionless variables. The equation of state of the scalar field is given by

$$w_\phi = \frac{p_\phi}{\rho_\phi c^2} = \frac{\frac{1}{2\hbar c^3} \left(\frac{d\phi}{dt}\right)^2 - V}{\frac{1}{2\hbar c^3} \left(\frac{d\phi}{dt}\right)^2 + V} \quad (7.1)$$

From before, we know that

$$\frac{d\phi}{dt} = H_i E_p \frac{d\psi}{d\tau} \quad (7.2)$$

and by definition we have

$$V = \frac{H_i^2 E_p^2}{\hbar c^3} v \quad (7.3)$$

Inserting these into the equation of state, we then get

$$w_\phi = \frac{\frac{H_i^2 E_p^2}{2\hbar c^3} \frac{1}{2} \left(\frac{d\psi}{d\tau}\right)^2 - v}{\frac{H_i^2 E_p^2}{2\hbar c^3} \frac{1}{2} \left(\frac{d\psi}{d\tau}\right)^2 + v} = \frac{\frac{1}{2} \left(\frac{d\psi}{d\tau}\right)^2 - v}{\frac{1}{2} \left(\frac{d\psi}{d\tau}\right)^2 + v} \quad (7.4)$$

## 8. PROBLEM I)

In the slow-roll regime we expect a small  $\frac{d\psi}{d\tau}$ , giving  $\left(\frac{d\psi}{d\tau}\right)^2 \approx 0$ . The equation of state then becomes

$$w_\phi \approx \frac{-v}{v} = -1 \quad (8.1)$$

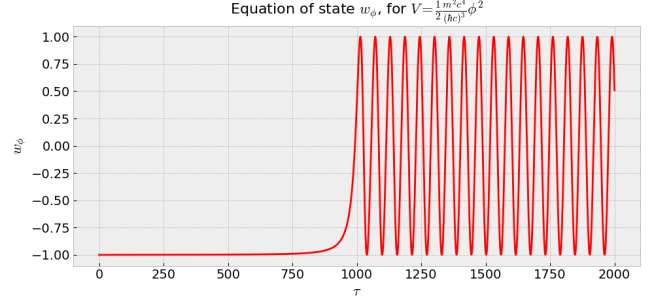


Figure 4. The scalar field  $\phi$ 's equation of state, given a quadratic potential, plotted against the dimensionless time  $\tau$ .

## 9. PROBLEM J)

Plotting the slowroll parameters against  $N$  we get the plot shown in Figure (5).

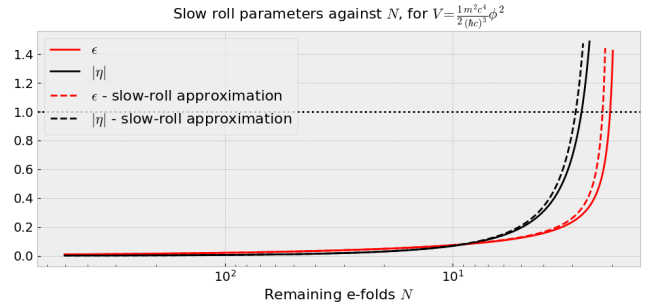


Figure 5.

## 10. PROBLEM K)

The predicted curve in the  $n$ - $r$  plane for  $N \in [50, 60]$  can be seen in Figure (6)

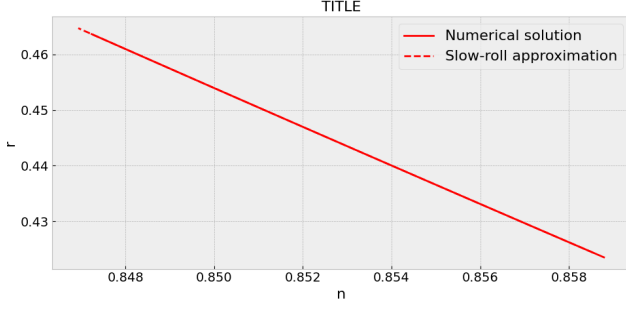


Figure 6. The spectral index  $n$  and the tensor-to-scalar ratio of the quadratic potential, for  $N \in [50, 60]$ .

### 11. PROBLEM L)

Substituting  $y = -\sqrt{\frac{2}{3}} \frac{\phi}{M_p}$ , we can write

$$V(\phi) = \frac{3M^2 M_p^2}{4} (1 - e^y)^2 \quad (11.1)$$

We also have

$$y' = -\sqrt{\frac{2}{3}} \frac{1}{M_p}, \quad y'' = 0 \quad (11.2)$$

We then get

$$V' = -\frac{3M^2 M_p^2}{2} (1 - e^y) \cdot e^y \cdot y' \quad (11.3)$$

$$= \sqrt{\frac{2}{3}} \frac{2}{M_p} \frac{e^y}{1 - e^y} V \quad (11.4)$$

And, differentiating equation (11.3), we find

$$V'' = y' \frac{3M^2 M_p^2}{2} \frac{d}{d\phi} (e^{2y} - e^y) \quad (11.5)$$

$$= y'^2 \frac{3M^2 M_p^2}{2} (2e^{2y} - e^y) \quad (11.6)$$

$$= \frac{2}{3} \frac{3M^2}{2} (2e^{2y} - e^y) \quad (11.7)$$

$$= \frac{4}{3} \frac{1}{M_p^2} \frac{(2e^{2y} - e^y)}{(1 - e^y)^2} V \quad (11.8)$$

We can then find the slow roll parameters. Using  $M_p = \frac{E_p^2}{8\pi}$ , these can be written

$$\epsilon = \frac{M_p^2}{2} \left( \frac{V'}{V} \right)^2 \quad (11.9)$$

$$\eta = M_p^2 \frac{V''}{V} \quad (11.10)$$

By insertion, we then get

$$\epsilon = \frac{4}{3} \frac{e^{2y}}{(1 - e^y)^2} \quad (11.11)$$

$$\eta = \frac{4}{3} \frac{(2e^{2y} - e^y)}{(1 - e^y)^2} \quad (11.12)$$

### 12. PROBLEM M)

The governing equations are then solved just like before, only exchanging the quadratic potential with the Starobinsky potential. The resulting dimensionless field  $\psi$  is then shown in Figure (7), and the scale factor can be seen in Figure (8).

From Figure (8) we note that the logarithm of the scale factor goes as

$$\ln \left( \frac{a}{a_i} \right) = \tau \quad (12.1)$$

Giving an exponential scale factor during inflation

$$a = a_i e^\tau = a_i e^{H_i t} \quad (12.2)$$

In other words the Starobinsky potential gives rise to inflation following the de Sitter model.

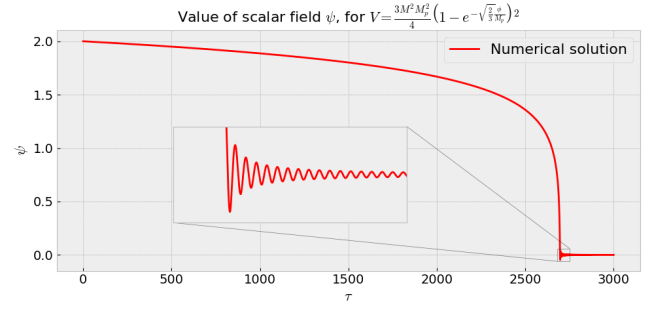


Figure 7. The value of the dimensionless field  $\psi$  for Starobinsky inflation, plotted as a function of the dimensionless time  $\tau$ .

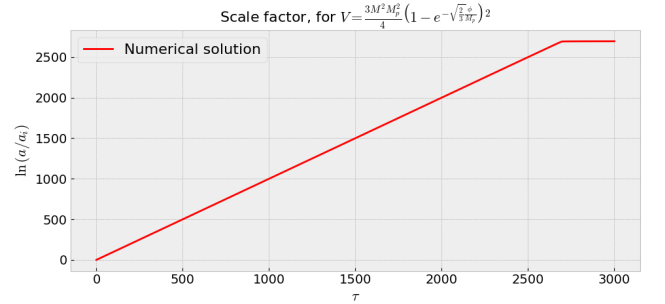


Figure 8. The logarithm of the scale factor  $\ln \left( \frac{a}{a_i} \right)$  for Starobinsky inflation. Plotted against the dimensionless time  $\tau$ .

### 13. PROBLEM N)

Plotting the slowroll parameters against  $N$  we get the plot shown in Figure (9). The predicted curve in the  $n$ - $r$  plane for  $N \in [50, 60]$  can be seen in Figure (10)

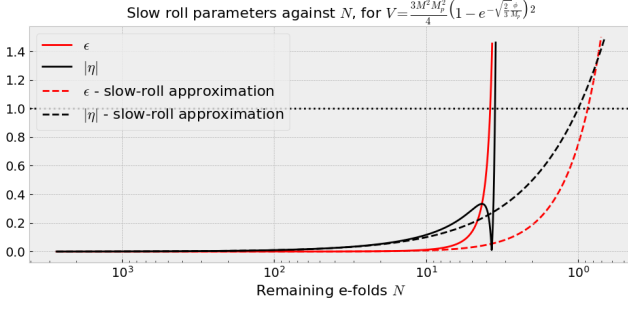


Figure 9. The slow-roll parameters  $\epsilon$  and  $\eta$  plotted against  $N$ , the number  $e$ -folds remaining before the end of inflation.

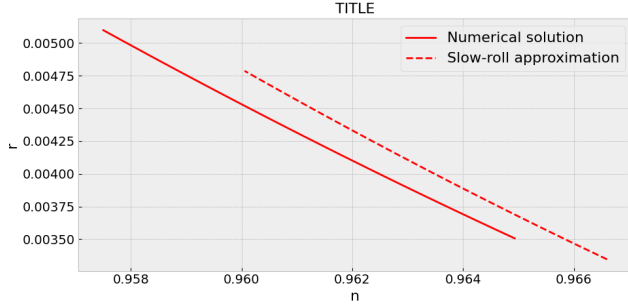


Figure 10. The spectral index  $n$  and the tensor-to-scalar ratio of the Starobinsky potential, for  $N \in [50, 60]$ .

## 14. PROBLEM O

### A. Remaining e-folds $N$

From before, we have that

$$\frac{V}{V'} = \sqrt{\frac{3}{2}} \frac{M_p}{2} \frac{1 - e^y}{e^y} \quad (14.1)$$

$$= \sqrt{\frac{3}{16\pi}} \frac{E_p}{2} (e^{-y} - 1) \quad (14.2)$$

$$(14.3)$$

Using  $y = -\sqrt{\frac{16\pi}{3}} \frac{\phi}{E_p}$ , the remaining number of  $e$ -foldings becomes

$$N = \frac{8\pi}{E_p^2} \int_{\phi_{end}}^{\phi} \frac{V}{V'} d\phi \quad (14.4)$$

$$= -\frac{8\pi}{E_p} \sqrt{\frac{3}{16\pi}} \int_{y_{end}}^y \frac{V}{V'} d\tilde{y} \quad (14.5)$$

$$= -\frac{3}{4} \int_{y_{end}}^y e^{-\tilde{y}} - 1 d\tilde{y} \quad (14.6)$$

$$= \frac{3}{4} \left[ e^{-\tilde{y}} + \tilde{y} \right]_{\tilde{y}=y_{end}}^y \quad (14.7)$$

Where we on the last line applied the slow-roll criterion

$$\phi \gg \phi_{end} \Rightarrow y \gg y_{end} \quad (14.8)$$

Such that

$$N \approx \frac{3}{4} (e^{-y} + y) \quad (14.9)$$

Assuming the field is on the order  $\phi \sim \phi_i = 2$ , we have  $y \approx -8$ , giving

$$e^{-y} \gg y \quad (14.10)$$

With which we use to further approximate  $N$  as

$$N \approx \frac{3}{4} e^{-y} \quad (14.11)$$

### B. Slow-roll parameters

The slow-roll parameters are found in a similar manner. Using

$$e^y \ll 1 \quad (14.12)$$

we have that  $1 - e^y \approx 1$ , so that  $\epsilon$  can be approximated as

$$\epsilon = \frac{4}{3} \frac{e^{2y}}{(1 - e^y)^2} \approx \frac{4}{3} e^{2y} = \frac{3}{4} \frac{1}{N^2} \quad (14.13)$$

Similarly,  $\eta$  becomes

$$\eta = \frac{4}{3} \frac{(2e^{2y} - e^y)}{(1 - e^y)^2} \approx \frac{4}{3} (2e^{2y} - e^y) \quad (14.14)$$

We multiplying equation (14.12) by  $e^y$  on both sides, we get

$$e^{2y} \ll e^y \quad (14.15)$$

and  $\eta$  is further simplified to

$$\eta \approx -\frac{4}{3} e^y = -\frac{1}{N} \quad (14.16)$$

### C. Tensor-to-scalar ratio and spectral index

To approximate the tensor-to-scalar ratio  $r$  we simply insert our approximation of  $\epsilon$ , giving

$$r = 16\epsilon \approx \frac{12}{N^2} \quad (14.17)$$

Similarly, the spectral index is approximated

$$n = 1 - 6\epsilon + 2\eta \approx 1 - \frac{9}{2} \frac{1}{N^2} - \frac{2}{N} \quad (14.18)$$

$$\approx 1 - \frac{2}{N} \quad (14.19)$$

where we used that  $N$  will be a large number during slow-roll, making the second order term negligible.

**15. PROBLEM P)****ACKNOWLEDGMENTS**

I would like thank myself for writing this beautiful document.

**REFERENCES**

READ STUFF

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name reference