AST3220 - Project 3:
Inflation without approximation

Candidate nr. 14
(Dated: June 15, 2023)

1. PROBLEM A)

We will in this section let L, T and M represent dimensions length, time and mass respectively.

As H_i necessarily has the same dimensions as H it is immediately clear that h must be dimensionless:

$$[h] = [H][H_i]^{-1} = 1 \tag{1.1}$$

To check the units of τ we first examine the dimensions of the Hubble parameter. The Hubble parameter has dimension velocity per distance, which can be written

$$[H] = (\mathsf{L}\mathsf{T}^{-1})\mathsf{L}^{-1} = \mathsf{T}^{-1} \tag{1.2}$$

so that τ is dimensionless as well:

$$[\tau] = [H][t] = \mathsf{T}^{-1}\mathsf{T} = 1 \tag{1.3}$$

Both ϕ and the Planck energy E_p has units of energy, making ψ dimensionless as well:

$$[\psi] = [\phi][E_p]^{-1} = \mathsf{M}\mathsf{L}^2\mathsf{T}^{-2}(\mathsf{M}\mathsf{L}^2\mathsf{T}^{-2})^{-1} = 1 \tag{1.4}$$

Lastly, we rewrite the potential v using the definition $E_p^2 = \hbar c^5/G$, so that

$$v = \frac{\hbar c^3}{H_i^2 E_p^2} = \frac{1}{H_i^2}\frac{G}{c^2}V \tag{1.5}$$

Using the definition of H_i, we then see that

$$[v] = [H_i]^{-2}[Gc^{-2}V] = [H_i]^{-2}[H_i]^2 = 1 \tag{1.6}$$

so v is dimensionless as well

2. PROBLEM B)

1. Hubble parameter and scale factor

Using the definitions of the dimensionless variables and applying the chain rule, we see that

$$\frac{d}{d\tau}\left(\ln\frac{a}{a_i}\right) = \frac{dt}{d\tau}\frac{d}{dt}\left(\ln\frac{a}{a_i}\right) \tag{2.1}$$

$$= \frac{1}{H_i}\frac{\dot{a}}{a} \tag{2.2}$$

$$= \frac{H}{H_i} \tag{2.3}$$

$$= h \tag{2.4}$$

2. Equation of motion

Similarly, we continue using the chain rule to rewrite $\dot{\phi}$, $\ddot{\phi}$ and V' in terms of τ and ψ:

$$\frac{d\phi}{dt} = \frac{d\tau}{dt}\frac{d\phi}{d\psi}\frac{d\psi}{d\tau} = H_i E_p\frac{d\psi}{d\tau} \tag{2.5}$$

$$\frac{d^2\phi}{dt^2} = \frac{d\tau}{dt}\frac{d}{d\tau}\frac{d\phi}{dt} = H_i^2 E_p\frac{d^2\psi}{d\tau^2} \tag{2.6}$$

$$\frac{dV}{d\phi} = \frac{d\psi}{d\phi}\frac{dV}{dv}\frac{dv}{d\psi} = \frac{1}{E_p}\frac{H_i^2 E_p^2}{\hbar c^3}\frac{dv}{d\psi} \tag{2.7}$$

Which can be inserted into the equation of motion, giving

$$H_i^2 E_p\frac{d^2\psi}{d\tau^2} + 3HH_iE_p\frac{d\psi}{d\tau} + H_i^2E_p\frac{dv}{d\psi} = 0 \tag{2.8}$$

Dividing both sides by $H_i^2 E_p$ and using the definition $h = H/H_i$, this reduces to

$$\frac{d^2\psi}{d\tau^2} + 3h\frac{d\psi}{d\tau} + \frac{dv}{d\psi} = 0 \tag{2.9}$$

3. First Friedmann equation

Lastly, we rewrite the first Friedmann equation in terms of the dimensionless quantities. We have

$$H^2 = \frac{8\pi G}{3c^2}\left[\frac{1}{2\hbar c^3}\dot{\phi} + V(\phi)\right] \tag{2.10}$$

$$= \frac{8\pi G}{3c^2}\left[\frac{H_i^2E_p^2}{2\hbar c^3}\left(\frac{d\psi}{d\tau}\right)^2 + V(\phi)\right] \tag{2.11}$$

$$= \frac{8\pi G}{3c^2}\frac{H_i^2E_p^2}{\hbar c^3}\left[\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v(\psi)\right] \tag{2.12}$$

Where we used equation (2.5) and the definition of v. Next, inserting the definition of the Planck energy

$$E_p^2 = \frac{\hbar c^5}{G} \tag{2.13}$$

and dividing both sides by H_i, we finally get:

$$h^2 = \frac{8\pi}{3}\left[\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v(\psi)\right] \tag{2.14}$$

3. PROBLEM C)

In the slow-roll regime, $\frac{d^2\psi}{d\tau^2}$ is small

$$\frac{d^2\psi}{d\tau^2} \ll \frac{dv}{d\psi} \tag{3.1}$$

Inserting, the equation of motion can then be approximated as

$$3h\frac{d\psi}{d\tau} + \frac{dv}{d\psi} = 0 \tag{3.2}$$

This is a first-order differential equation, which only requires an initial value for ψ. Thus, when in the slow-roll approximation, the initial value of $\frac{d\psi}{d\tau}$ does not matter as long as it fulfils the slow-roll conditions.

4. PROBLEM D)

During slow-roll, the number of remaining e-folds until inflation ends can be calculated as

$$N = \frac{8\pi}{E_p^2}\int_{\phi_{\text{end}}}^{\phi} \frac{V}{V'}\, d\phi \tag{4.1}$$

Substituting $d\phi = E_p d\psi$, we then have

$$\frac{V}{V'} = \frac{1}{2}\phi = \frac{E_p}{2}\psi \tag{4.2}$$

Giving

$$\begin{aligned} N &= 4\pi\int_{\psi_{\text{end}}}^{\psi} \psi\, d\psi && (4.3)\\ &= 2\pi\left(\psi^2 - \psi_{\text{end}}^2\right) && (4.4) \end{aligned}$$

where ψ_{end} is given by taking $\epsilon(\phi_{\text{end}}) = 1$ as the end of inflation:

$$\begin{aligned} \epsilon(\phi_{\text{end}}) = \frac{E_p^2}{16\pi}\left(\frac{V'}{V}\right)^2 &= \frac{1}{16\pi}\left(\frac{\frac{dv}{d\psi}}{v}\right) && (4.5)\\ &= \frac{1}{4\pi\psi_{\text{end}}^2} && (4.6)\\ &= 1 && (4.7)\\ & && (4.8) \end{aligned}$$

Solving for ψ_{end} this gives

$$\Rightarrow \quad \psi_{\text{end}}^2 = \frac{1}{4\pi} \tag{4.9}$$

Inserting this, the number of remaining e-folds is

$$N = 2\pi\psi^2 - \frac{1}{2} \tag{4.10}$$

The number of remaining e-folds at the initial time $t = t_i$ is the total number of e-folds N_{tot}. Evaluating $N(t)$ at t_i and solving for the initial field value ψ_i we then get:

$$\psi_i = \sqrt{\frac{1}{2\pi}\left(N_{\text{tot}} + \frac{1}{2}\right)} \tag{4.11}$$

Which evaluates to $\psi_i \approx 8.925$ for 500 total e-folds of inflation.

5. PROBLEM E) & F)

To solve the equations of motion of the field, we rename $\xi = \frac{d\psi}{d\tau}$ so equation 2.9 can instead be written as a set of two first-order equations:

$$\begin{aligned} \frac{d\psi}{d\tau} &= \xi && (5.1)\\ \frac{d\xi}{d\tau} &= -3h(\xi,\psi)\xi - \frac{d}{d\psi}v(\psi) && (5.2) \end{aligned}$$

where h is again given by the first Friedmann equation

$$h^2 = \frac{8\pi}{3}\left[\frac{1}{2}\xi^2 + v(\psi)\right] \tag{5.3}$$

and $v(\psi)$ is given by our choice of potential. Equations 5.1 and 5.2 are then easily solved using a numerical integrator. A plot of the resulting field ψ plotted against τ is attached in Figure (1), along with the slow-roll solution.

With the solutions ξ and ψ we can find $\ln\left(\frac{a}{a_i}\right)$ by numerically evaluating the integral

$$\begin{aligned} \ln\left(\frac{a}{a_i}\right) &= \int_0^t H dt && (5.4)\\ &= \int_0^\tau h(\xi,\psi)d\tau && (5.5) \end{aligned}$$

The results of which are plotted in Figure (2), along with the slow-roll solution.

Based on the lectures, we expect the field to decay until it reaches ψ_{end}. When it reaches ψ_{end} it should then start behaving like a damped oscillator centred around $\psi = 0$. This is in agreement with the numerical solution (Figure 1), which shows the field decaying linearly until it enters the oscillating phase at $\tau \approx 1000$, indicating the end of inflation. This is also when the slow-roll approximation starts deviating significantly from the exact solution, as it instead continues decaying at a constant rate.

This is around the same time the slow-roll approximation for the scale factor deviates from the exact solution (Figure 2), as one expects. While the exact numerical solution shows the expansion halting as inflation ends, the slow-roll approximation continues expanding shortly after.

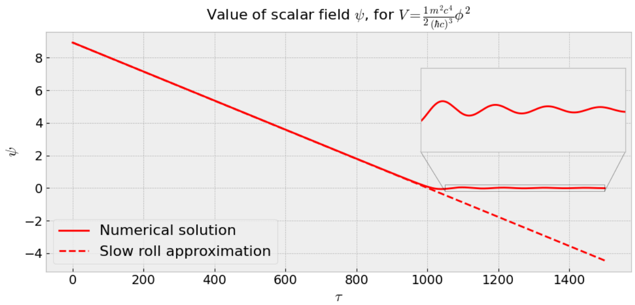

Figure 1. The dimensionless scalar field ψ for the quadratic potential, along with its approximation in the slow-roll regime. Plotted against the dimensionless time τ.

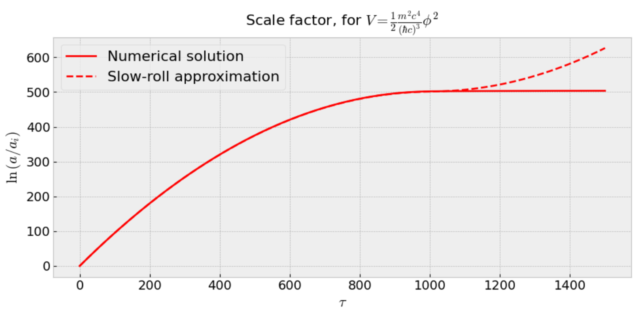

Figure 2. The logarithm of the scale factor $\ln\left(\frac{a}{a_i}\right)$ for a quadratic potential, along with the slow-roll approximation. Plotted as a function of the dimensionless time τ.

6. PROBLEM G)

The slow-roll parameters are plotted against τ in Figure (3), along with the slow-roll approximations. For the quadratic potential, the slow-roll parameters are

$$\epsilon = \eta = \frac{1}{4\pi}\frac{1}{\psi^2} \tag{6.1}$$

To find the total number of e-folds, we numerically integrated

$$N_{\text{tot}} = \int_{t_{\text{end}}}^{t_i} H\ dt = \int_{\tau_{\text{end}}}^{\tau_i} h\ d\tau \tag{6.2}$$

where the boundary τ_{end} is taken care of by cutting off the solution arrays τ and h where $\epsilon = 1$. This gives a total

$$N_{\text{tot}} = 501.4 \tag{6.3}$$

e-foldings produced during inflation, slightly more than the 500 e-foldings predicted by the slow-roll approximation.

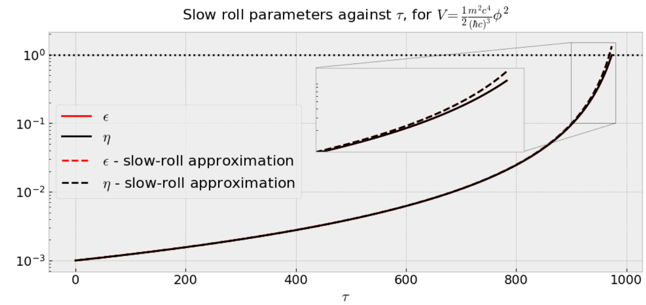

Figure 3. The slow-roll parameters ϵ and η for a quadratic potential, plotted against dimensionless time τ.

7. PROBLEM H)

We now want to express the equation of state parameter in terms of the dimensionless variables. The equation of state of the scalar field is given by

$$w_\phi = \frac{p_\phi}{\rho_\phi c^2} = \frac{\frac{1}{2\hbar c^3}\left(\frac{d\phi}{dt}\right)^2 - V}{\frac{1}{2\hbar c^3}\left(\frac{d\phi}{dt}\right)^2 + V} \tag{7.1}$$

Earlier, we found that

$$\frac{d\phi}{dt} = H_i E_p \frac{d\psi}{d\tau} \tag{7.2}$$

and by definition we have

$$V = \frac{H_i^2 E_p^2}{\hbar c^3} v \tag{7.3}$$

Inserting these into the equation of state, we then get

$$w_\phi = \frac{\frac{H_i^2 E_p^2}{2\hbar c^3}}{\frac{H_i^2 E_p^2}{2\hbar c^3}} \frac{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 - v}{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v} = \frac{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 - v}{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v} \tag{7.4}$$

which is what we wanted to show.

8. PROBLEM I)

In the slow-roll regime we expect a small $\frac{d\psi}{d\tau} \ll 1$, giving $\left(\frac{d\psi}{d\tau}\right)^2 \approx 0$. The equation of state then becomes

$$w_\phi \approx \frac{-v}{v} = -1 \tag{8.1}$$

However, this should change when the field enters the oscillating phase. Initially, when the field reaches $\psi = 0$, the potential becomes $v(0) = 0$, which gives an equation of state

$$w_\phi = \frac{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2}{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2} = 1 \tag{8.2}$$

However, as the field oscillates it will at some point reach a peak, where $\frac{d\psi}{d\tau}$ has to be zero, again giving an equation of state

$$w_\phi = \frac{-v}{v} = -1 \tag{8.3}$$

With the field oscillating back and forth, this behaviour repeats and we should get an equation of state oscillating between $w_\phi = -1$ and $w_\phi = 1$. This is exactly what we see in the numerical results seen in Figure (4).

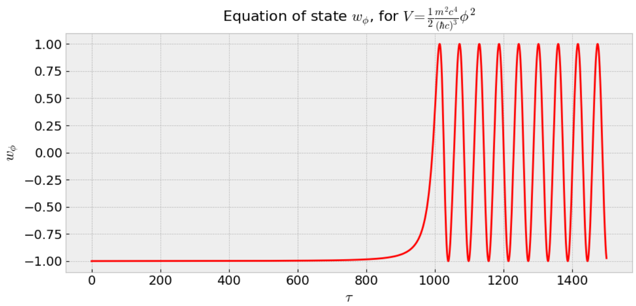

Figure 4. The scalar field ϕ's equation of state, given a quadratic potential, plotted against the dimensionless time τ.

9. PROBLEM J)

Plotting the slow-roll parameters against N we get the plot shown in Figure (5). As expected we see that the exact numerical solution of ϵ $(= \eta)$ approaches $\epsilon = 1$ as N approaches zero and inflation ends.

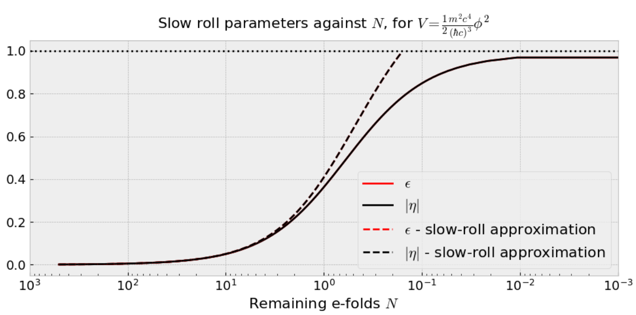

Figure 5.

10. PROBLEM K)

The predicted curve in the n-r plane for $N \in [50, 60]$ can be seen in Figure (6). The prediction shows a spectral index n in the approximate range of 0.960 to 0.966, and a tensor-to-scalar ratio r in the approximate range 0.135 to 0.160.

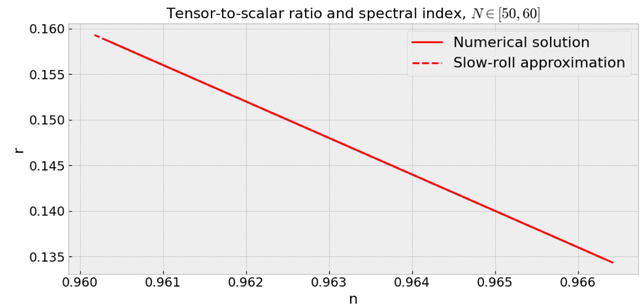

Figure 6. The spectral index n and the tensor-to-scalar ratio of the quadratic potential, for $N \in [50, 60]$.

11. PROBLEM L)

Substituting $y = -\sqrt{\frac{2}{3}}\frac{\phi}{M_p}$, the potential can be written

$$V(\phi) = \frac{3M^2M_P^2}{4}\left(1 - e^y\right)^2 \tag{11.1}$$

We also have

$$y' = -\sqrt{\frac{2}{3}}\frac{1}{M_p}, \quad y'' = 0 \tag{11.2}$$

So that the derivative of the potential $\frac{dV}{d\phi}$ becomes

$$V' = -\frac{3M^2M_p^2}{2}\left(1 - e^y\right) \cdot e^y \cdot y' \tag{11.3}$$

$$= \sqrt{\frac{2}{3}}\frac{2}{M_p}\frac{e^y}{1 - e^y}V \tag{11.4}$$

And, differentiating equation (11.3) once again, we find

$$V'' = y'\frac{3M^2M_p^2}{2}\frac{d}{d\phi}\left(e^{2y} - e^y\right) \tag{11.5}$$

$$= y'^2\frac{3M^2M_p^2}{2}\left(2e^{2y} - e^y\right) \tag{11.6}$$

$$= \frac{2}{3}\frac{3M^2}{2}\left(2e^{2y} - e^y\right) \tag{11.7}$$

$$= \frac{4}{3}\frac{1}{M_p^2}\frac{\left(2e^{2y} - e^y\right)}{\left(1 - e^y\right)^2}V \tag{11.8}$$

We can use these to find the slow roll parameters. Using $M_p = \frac{E_p^2}{8\pi}$, the slow-roll parameters are

$$\epsilon = \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2 \tag{11.9}$$

$$\eta = M_p^2\frac{V''}{V} \tag{11.10}$$

By insertion, we then get

$$\epsilon = \frac{4}{3}\frac{e^{2y}}{\left(1-e^{y}\right)^2} \tag{11.11}$$

$$\eta = \frac{4}{3}\frac{\left(2e^{2y}-e^{y}\right)}{\left(1-e^{y}\right)^2} \tag{11.12}$$

12. PROBLEM M)

The governing equations are then solved just like before, only switching the quadratic potential with the Starobinsky potential. The resulting dimensionless field ψ is then shown in Figure (7), and the scale factor can be seen in Figure (8). Calculating the total number of e-folds, we found that the Starobinsky inflation gave about $N_{\text{tot}} = 2690$ e-folds of inflation.

From Figure (8) we note that the logarithm of the scale factor goes as

$$\ln\left(\frac{a}{a_i}\right) = \tau \tag{12.1}$$

Giving an exponential scale factor during inflation

$$a = a_i e^{\tau} = a_i e^{H_i t} \tag{12.2}$$

In other words, the Starobinsky potential gives rise to inflation following the de Sitter model.

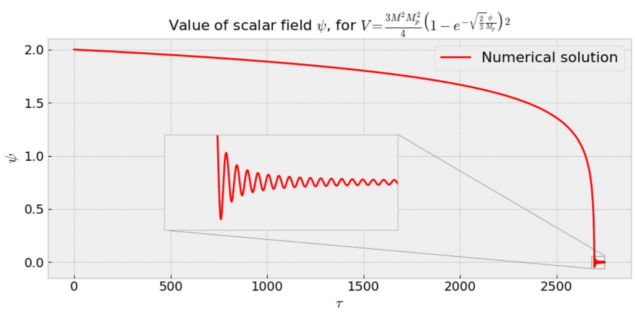

Figure 7. The value of the dimensionless field ψ for Starobinsky inflation, plotted as a function of the dimensionless time τ.

13. PROBLEM N)

The slow-roll parameters plotted against N are shown in Figure (9), along with the slow-roll approximations calculated in problem o). The predicted curve in the n-r plane for $N \in [50, 60]$ can be seen in Figure (10), also with the slow-roll approximations.

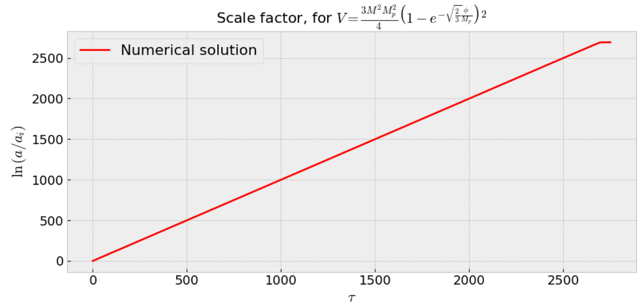

Figure 8. The logarithm of the scale factor $\ln\left(\frac{a}{a_i}\right)$ for Starobinsky inflation. Plotted against the dimensionless time τ.

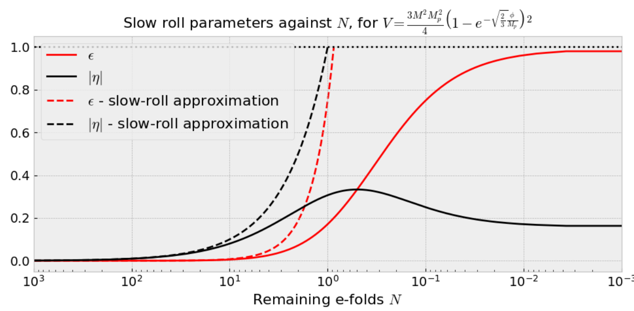

Figure 9. The slow-roll parameters ϵ and η plotted against N, the number e-folds remaining before the end of inflation.

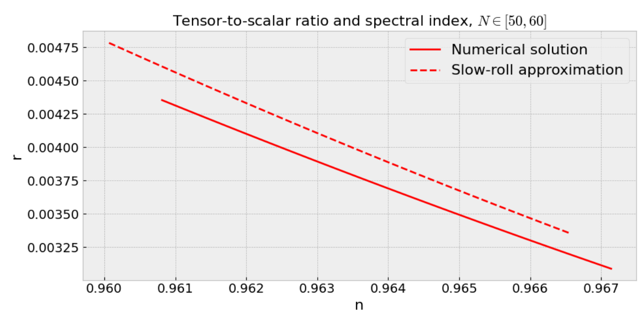

Figure 10. The spectral index n and the tensor-to-scalar ratio of the Starobinsky potential, for $N \in [50, 60]$.

14. PROBLEM O)

1. *Remaining e-folds N*

From earlier, we have that

$$\frac{V}{V'} = \sqrt{\frac{3}{2}}\frac{M_p}{2}\frac{1-e^{y}}{e^{y}} \tag{14.1}$$

$$= \sqrt{\frac{3}{16\pi}}\frac{E_p}{2}\left(e^{-y}-1\right) \tag{14.2}$$

$$\tag{14.3}$$

Using $y = -\sqrt{\frac{16\pi}{3}}\frac{\phi}{E_p}$, the remaining number of e-foldings becomes

$$N = \frac{8\pi}{E_p^2}\int_{\phi_{\text{end}}}^{\phi}\frac{V}{V'}\,d\phi \tag{14.4}$$
$$= -\frac{8\pi}{E_p}\sqrt{\frac{3}{16\pi}}\int_{y_{\text{end}}}^{y}\frac{V}{V'}\,d\tilde{y} \tag{14.5}$$
$$= -\frac{3}{4}\int_{y_{\text{end}}}^{y} e^{-\tilde{y}} - 1\,d\tilde{y} \tag{14.6}$$
$$= \frac{3}{4}\left[e^{-\tilde{y}} + \tilde{y}\right]_{\tilde{y}=y_{\text{end}}}^{y} \tag{14.7}$$

This can be further simplified by applying slow-roll criterion

$$\phi \gg \phi_{\text{end}} \quad \Rightarrow \quad -y \gg -y_{\text{end}} \tag{14.8}$$

(Remember $y \propto -\phi$). Thus, the exponential term e^{-y} will dominate, giving

$$N \approx \frac{3}{4}e^{-y} \tag{14.9}$$

2. *Slow-roll parameters ϵ & η*

The slow-roll parameters are found similarly. Using

$$e^y \ll 1 \tag{14.10}$$

we have that $1 - e^y \approx 1$, so that ϵ can be approximated as

$$\epsilon = \frac{4}{3}\frac{e^{2y}}{(1-e^y)^2} \approx \frac{4}{3}e^{2y} = \frac{3}{4}\frac{1}{N^2} \tag{14.11}$$

Similarly, η becomes

$$\eta = \frac{4}{3}\frac{(2e^{2y} - e^y)}{(1-e^y)^2} \approx \frac{4}{3}(2e^{2y} - e^y) \tag{14.12}$$

Multiplying equation (14.10) by e^y on both sides, we get

$$e^{2y} \ll e^y \tag{14.13}$$

and η is further simplified to

$$\eta \approx -\frac{4}{3}e^y = -\frac{1}{N} \tag{14.14}$$

3. *Tensor-to-scalar ratio r & spectral index n*

To approximate the tensor-to-scalar ratio r we insert our approximation of ϵ, giving

$$r = 16\epsilon \approx \frac{12}{N^2} \tag{14.15}$$

Similarly, the spectral index is approximated

$$n = 1 - 6\epsilon + 2\eta \approx 1 - \frac{9}{2}\frac{1}{N^2} - \frac{2}{N} \tag{14.16}$$
$$\approx 1 - \frac{2}{N} \tag{14.17}$$

where we used that N will be a large number during slow-roll, making the second-order term negligible.

15. PROBLEM P)

Analysis of the 2018 Planck CMB anisotropy measurements (Ref. [1]) gives constraints on the values of the spectral index and the tensor-to-scalar ratio. Specifically, they suggest a spectral index $n = 0.9649 \pm 0.0042$ and put an upper limit $r < 0.056 <<$ on the tensor-to-scalar ration.

While the predicted spectral index of the ϕ^2-potential (Figure 6) is consistent with this, the predicted values of the tensor-to-scalar ratio are far above the upper limit, ruling this potential out.

On the other hand, the predictions made by the Starobinsky model (Figure 10) are fully consistent with these constraints. We see that this model predicts a spectral index within the range observed range, and tensor-to-scalar ratios well below the upper limit.

REFERENCES

[1] Planck Collaboration, 2020, *Planck 2018 results - X. Constraints on inflation.* *A&A* **641** A10 https://doi.org/10.1051/0004-6361/201833887