AST3220 - Project 2:
Big Bang Nuclesynthesis

Candidate nr. 14

(Dated: May 10, 2023)

1. PROBLEM A)

We can rewrite the number density n_i of species i in terms of the relative number density Y_i as:

$$Y_i = \frac{n_i}{n_b} \quad \Rightarrow \quad n_i = n_b Y_i \tag{1.1}$$

$$= \frac{n_{b0}}{a^3} Y_i \tag{1.2}$$

Where $n_b(t)$ is the baryon number density, n_{b0} is the baryon number density today, and $a(t)$ is the scale factor. Using the product rule for differentiation, we can then write

$$\frac{dn_i}{dt} = n_{b0} \frac{d}{dt} \left(Y_i a^{-3}\right) \tag{1.3}$$

$$= n_{b0} \left(\frac{1}{a^3}\frac{dY_i}{dt} - 3Y_i \frac{\dot{a}}{a^4}\right) \tag{1.4}$$

$$= n_b \frac{dY_i}{dt} - 3Y_i n_b H \tag{1.5}$$

$$= n_b \frac{dY_i}{dt} - 3n_i H \tag{1.6}$$

Next we want to switch from t to $\ln T$ as our time variable, where T is the temperature. Using $T = T_0 a^{-1}$ we get

$$\ln T = \ln T_0 - \ln a(t) \tag{1.7}$$

Then, using the chain rule of differentiation, we can rewrite

$$\frac{dY_i}{dt} = \frac{d(\ln T)}{dt}\frac{dY_i}{d(\ln T)} \tag{1.8}$$

$$= -\frac{\dot{a}}{a}\frac{dY_i}{d(\ln T)} \tag{1.9}$$

$$= -H \frac{dY_i}{d(\ln T)} \tag{1.10}$$

Inserting to equation (1.6) we get

$$\frac{dn_i}{dt} = -n_b H \frac{dY_i}{d(\ln T)} - 3n_i H \tag{1.11}$$

The equations for the evolution of the number densities of protons p and neutrons n are given as

$$\frac{dn_n}{dt} + 3Hn_n = n_p \Gamma_{p\to n} - n_n \Gamma_{n\to p} \tag{1.12}$$

$$\frac{dn_p}{dt} + 3Hn_p = n_n \Gamma_{n\to p} - n_p \Gamma_{p\to n} \tag{1.13}$$

$$= -\left(\frac{dn_n}{dt} + 3Hn_n\right) \tag{1.14}$$

Inserting equation 1.11 into these, we get

$$-n_b H \frac{dY_n}{d(\ln T)} = n_p \Gamma_{p\to n} - n_n \Gamma_{n\to p} \tag{1.15}$$

$$-n_b H \frac{dY_p}{d(\ln T)} = n_n \Gamma_{n\to p} - n_p \Gamma_{p\to n} \tag{1.16}$$

Dividing by $-n_b H$ and using Eq. (1.1) we finally get:

$$\frac{dY_n}{d(\ln T)} = -\frac{1}{H}\left[Y_p \Gamma_{p\to n} - Y_n \Gamma_{n\to p}\right] \tag{1.17}$$

$$\frac{dY_p}{d(\ln T)} = -\frac{1}{H}\left[Y_n \Gamma_{n\to p} - Y_p \Gamma_{p\to n}\right] \tag{1.18}$$

2. PROBLEM B)

The relation $T_\nu = \left(4/11\right)^{1/3} T$ can be derived from the conservation of entropy, which tells us that

$$g_{*s}(aT)^3 = const. \tag{2.1}$$

At the time where the universe had a temperature $k_B T > 0.511$ Mev, electrons and positrons were relativistic and the process

$$e^+ + e^- \rightleftharpoons \gamma + \gamma \tag{2.2}$$

occured in both directions. However, as the temperature universe falls below the rest mass of the electron and positron $k_B T < 0.511$, the average energy of a photon collision is too small for an electrons-positron pair to be created. Since electrons and positrons will still anihilate through the process

$$e^+ + e^- \to \gamma + \gamma \tag{2.3}$$

and most of the positrons and electrons will then dissapear. Assuming this happened immediately, and that the universe is in thermal equilibrium ($T_i = T$), the effective number of degrees of freedom before and after can be written

$$g_{*s}^{\text{before}} = g_\nu + \frac{7}{8}(g_{e^-} + g_{e^+}) \tag{2.4}$$

$$= 2 + \frac{7}{8}4 \tag{2.5}$$

$$= \frac{11}{2} \tag{2.6}$$

$$g_{*s}^{\text{after}} = g_\nu \tag{2.7}$$

$$= 2 \tag{2.8}$$

If we also assume the scale factor a is the same before and after, the conservation of entropy gives us

$$\frac{11}{2}(aT)^3_{\text{before}} = 2(aT)^3_{\text{after}} \tag{2.9}$$

$$\Rightarrow T_{\text{after}} = (\frac{11}{4})^{1/3} T_{\text{before}} \tag{2.10}$$

Assuming neutrinos were already decoupled before this time, the neutrino temperature would remain unaffected:

$$T_{\nu,\text{after}} = T_{\nu,\text{before}} \tag{2.11}$$

If we then choose $T_{\nu,\text{before}}$ to be some time before neutrino decoupling, we expect neutrinos and photons to be in a state of thermal equilibrium, such that the neutron temperature and photon temperature are equal

$$T_{\nu,\text{before}} = T_{\text{before}} \tag{2.12}$$

We then find that

$$T_{\nu,\text{after}} = T_{\nu,\text{before}} = T_{\text{before}} = \left(\frac{4}{11}\right)^{1/3} T_{\text{after}} \tag{2.13}$$

Finally giving us an expression for the neutrino temperature today

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T \tag{2.14}$$

Where we have renamed $T_{\nu,\text{after}} = T_{\nu,\text{before}} = T_\nu$, as well as $T_{\text{after}} = T$.

3. PROBLEM C)

In the early universe, dominated by radiation, we have

$$\rho c^2 \approx \frac{\pi^2}{30} g_* \frac{(k_b T)^4}{(\hbar c)^3} \tag{3.1}$$

Where g_* is the effective number of relativistic degrees of freedom. Assuming all the radiation is composed of photons and N_{eff} number of neutrino species, g_* is

$$g_* = 1 + N_{\text{eff}}\ g_\nu \left(\frac{T_i}{T}\right)^4 \tag{3.2}$$

$$= 1 + N_{\text{eff}}\ \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \tag{3.3}$$

With $\rho_{c0} = \frac{3H_0^2}{8\pi G}$ as the critical density, we then find

$$\Omega_{r0} = \frac{\rho_0}{\rho_{c0}} \tag{3.4}$$

$$= \frac{1}{c^2}\left(\frac{\pi^2}{30} g_* \frac{(k_b T)^4}{(\hbar c)^3}\right) \cdot \left(\frac{8\pi G}{3H_0^2}\right) \tag{3.5}$$

$$= \frac{4\pi^3}{45}\frac{G}{H_0^2}\frac{(k_b T_0)^4}{\hbar^3 c^5} g_* \tag{3.6}$$

$$= \frac{4\pi^3}{45}\frac{G}{H_0^2}\frac{(k_b T_0)^4}{\hbar^3 c^5}\left[1 + N_{\text{eff}}\ \frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right] \tag{3.7}$$

4. PROBLEM D)

1. *Scale factor*

At the time of the BBN, the Friedmann equations simplify to

$$\frac{1}{a}\frac{da}{dt} = \frac{1}{a^2} H_0 \sqrt{\Omega_{r0}} \tag{4.1}$$

With some rearranging we see this is a separable differential equation, which we solve for $a(t)$:

$$a\ \frac{da}{dt} = H_0\sqrt{\Omega_{r0}} \tag{4.2}$$

$$\Rightarrow \int_0^a a'\ da' = H_0\sqrt{\Omega_{r0}}\int_0^t dt' \tag{4.3}$$

$$\Rightarrow \frac{1}{2}a^2 = H_0\sqrt{\Omega_{r0}}t \tag{4.4}$$

$$\Rightarrow a = \sqrt{2H_0 t}\ (\Omega_{r0})^{1/4} \tag{4.5}$$

2. *Cosmic time*

To find the cosmic time as a function of the photon temperature, we use the relation

$$T = T_0 a^{-1} \quad \Rightarrow \quad a = \frac{T_0}{T} \tag{4.6}$$

Inserting this into equation (4.5) and squaring both sides we get

$$\left(\frac{T_0}{T}\right)^2 = 2H_0 t\sqrt{(\Omega_{r0})} \tag{4.7}$$

$$\tag{4.8}$$

Which is easily solved:

$$t(T) = \frac{1}{2H_0\sqrt{\Omega_{r0}}}\left(\frac{T_0}{T}\right)^2 \tag{4.9}$$

A table of this expression evaluated at temperatures 10^{10}, 10^9 and 10^8 is attached in table (I)

T [K]	$t(T)$
10^{10}	1.7774 Sec
10^9	2 Min, 57.7400 Sec
10^8	4 Hr, 56 Min, 14.0000 Sec

Table I. Age of the universe at different temperatures.

5. PROBLEM E)

Assuming protons and neutrons are non relativistic at this point, they follow the Maxwell Boltzmann distribuiton. At equilibruium we then have

$$\frac{n_n^{(0)}}{n_p^{(0)}} = \frac{Y_n^{(0)}}{Y_p^{(0)}} \tag{5.1}$$
$$= \left(\frac{m_p}{m_n}\right) e^{-(m_n-m_p)c^2/k_B T_i} \tag{5.2}$$
$$\approx e^{-(m_n-m_p)c^2/k_B T_i} \tag{5.3}$$

where we have used that $m_p/m_n \approx 1$. Also assuming protons and neutrons make up all the baryonic mass, we have

$$Y_p + Y_n = \frac{n_p + n_n}{n_b} = 1 \quad \Rightarrow \quad Y_p = 1 - Y_n \tag{5.4}$$

Such that

$$\frac{Y_n}{Y_p} = \frac{Y_n}{1-Y_n} = e^{-(m_n-m_p)c^2/k_B T_i} \tag{5.5}$$

Which can be solved for $Y_n(T_i)$:

$$Y_n(T_i)\ e^{(m_n-m_p)c^2/k_B T_i} = 1 - Y_n \tag{5.6}$$
$$\Rightarrow\ Y_n(T_i)\left[1 + e^{-(m_n-m_p)c^2/k_B T_i}\right] = 1 \tag{5.7}$$
$$\Rightarrow\ Y_n(T_i) = \left[1 + e^{-(m_n-m_p)c^2/k_B T_i}\right]^{-1} \tag{5.8}$$

6. PROBLEM F)

At first, we try to solve the integral for the decay rates $\Gamma_{n\to p}$ and $\Gamma_{n\to p}$ using scipy's *quad* integrator, but this turned out to be very slow. As one can see the integral should converge at a reasonable pace due to the exponential terms in the demoninators, we instead approximated the integral by making a cut-off at $x = 250$. This was performed with an implementation of simpsons method, using a step size of $dx \approx 0.024$. The resulting plot can be seen in Fig. (1)

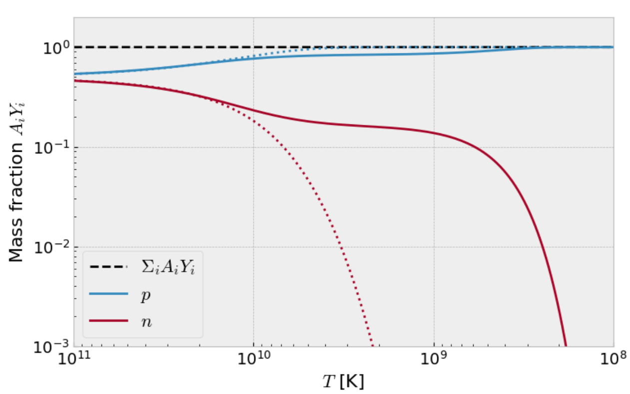

Figure 1. Caption

7. PROBLEM G)

From Problem a), we recall that

$$\frac{dn_i}{dt} + 3Hn_i = n_b H \frac{dY_i}{d\ln T} \tag{7.1}$$

Inserting the given expression

$$\frac{dn_i}{dt} + 3Hn_i \quad = \quad \sum_{j\neq i}[n_j\Gamma_{j\to i} - n_i\Gamma_{i\to j}] \tag{7.2}$$
$$+ \sum_{jkl}[n_k n_l \gamma_{kl\to ij} - n_i n_j \gamma_{ij\to kl}] \tag{7.3}$$

and using the definition $\Gamma_{ij\to kl} = n_b\gamma_{ij\to kl}$, we see that

$$\frac{dY_i}{d\ln T} = \frac{1}{H}\Bigg\{ \sum_{j\neq i}[\frac{n_j}{n_b}\Gamma_{j\to i} - \frac{n_i}{n_b}\Gamma_{i\to j}] \tag{7.4}$$
$$+ \sum_{jkl}[\frac{n_k}{n_b}\frac{n_l}{n_b} n_b\gamma_{kl\to ij} - \frac{n_i}{n_b}\frac{n_j}{n_b} n_b\gamma_{ij\to kl}]\Bigg\} \tag{7.5}$$
$$= \frac{1}{H}\Bigg\{ \sum_{j\neq i}[Y_j\Gamma_{j\to i} - Y_i\Gamma_{i\to j}] \tag{7.6}$$
$$+ \sum_{jkl}[Y_k Y_l \Gamma_{kl\to ij} - Y_i Y_j \Gamma_{ij\to kl}]\Bigg\} \tag{7.7}$$

Which is what we wanted to show.

8. PROBLEM H)

Implementing the additional reactions results in Fig. (2). The plot shows a drop in proton and neutron mass fraction as the temperature decreases past $\sim 10^9$K. This is due to the formation of deuterium, which has a sharp increase in its mass fraction before it stabilizes at $A_D Y_D \sim \frac{1}{4}$ due to exhausting the neutron supply.

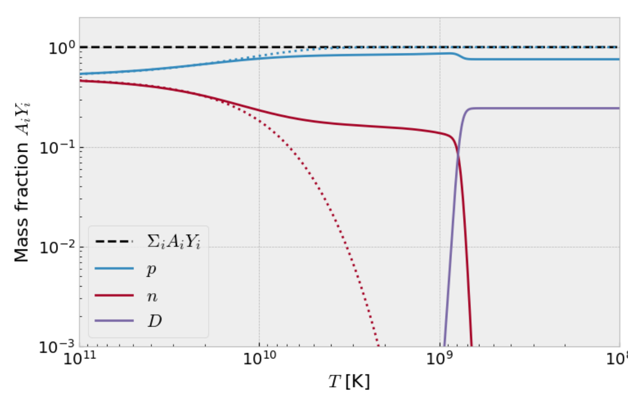

Figure 2. Caption.

9. PROBLEM I)

Implementing the remaining reactions we get the plot shown in Fig. (3). Ignoring neutrons and protons, the plot shows the production of most elements happens in the range $\sim 10^{10} - 10^9$ K with most of the mass fractions peaking at $\sim 10^9$K (i.e. ~ 3 minutes after the big bang).

The big exception to this is Be^7 which has sharply increases a little bit after this, while most other elements (an exception being He^3) quickly drops in their mass fractions. This is presumably due to being too large, and possibly requiring sufficient amounts of other elements to form.

We also note that by the time the temperature has decreased to 10^7K, the production of elements seems to have halted completely.

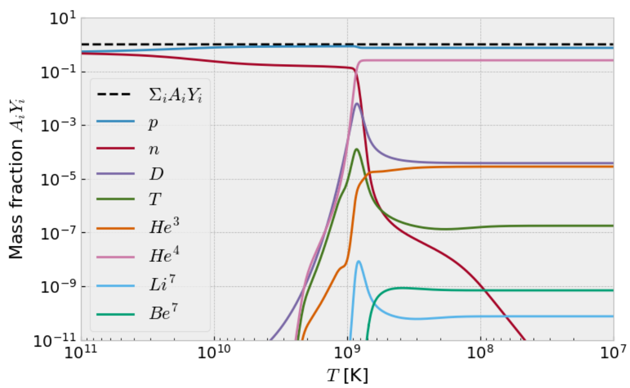

Figure 3. Caption.

10. PROBLEM J)

To compute the relic abundances from BBN, we simply pick out the last values from the simulation in the previous task. In order to find the optimal value for the baryonic matter content Ω_{b0}, we compute the relic abundances for $\Omega_{b0} \in [0.01, 1]$, and use the chi square method to find the best fit given the data. This gives an optimal value of $\Omega_{b0} = 0.05$. A plot of the results is attached in Fig. (4).

Comparing the results with the total matter content $\Omega_{m0} \approx 0.3$, we see that baryonic matter only makes up around $\Omega_{b0}/\Omega_{m0} = 16.7\%$ of total matter in the universe. Given that most of Ω_{m0} is in the form of dark matter, this indicates dark matter is a sort of non-baryonic matter. For example, one possibility is that dark matter is composed of some unknown elementary particles.

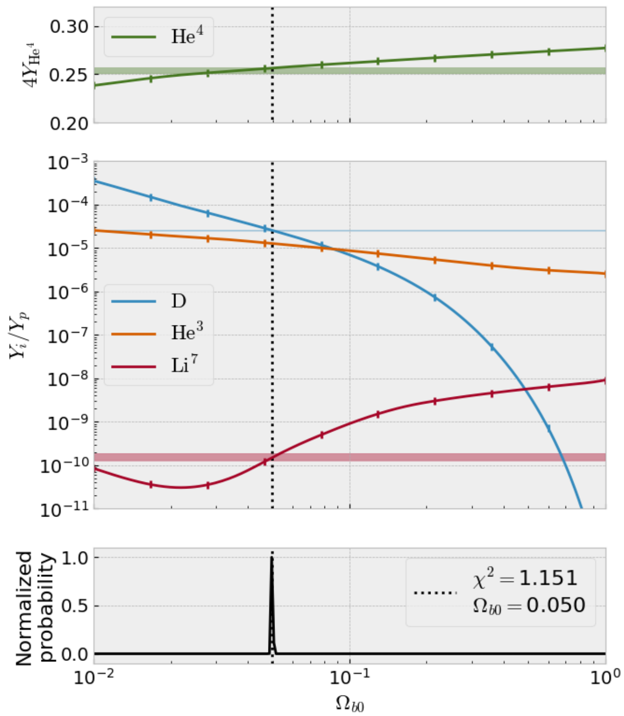

Figure 4. Relic abundances of elements produced in BBN as function of the baryonic matter content Ω_{b0}, as well as the likelihood of different Ω_{b0} given the data. The relic abundances are calculated for 10 different values of Ω_{b0} (indicated by notches), which are interpolated in order to create smooth functions of Ω_{b0}.

11. PROBLEM K)

Computing the relic abundances for $N_{\text{eff}} \in [1, 5]$ we find that the best fit given the data is $N_{\text{eff}} = 3.002$ effective number of neutrino species (Fig. 5). This matches well with what we expect from the standard model, which predicts three neutrino species from its three generations of elementary particles.

ACKNOWLEDGMENTS

I would like thank myself for writing this beautiful document.

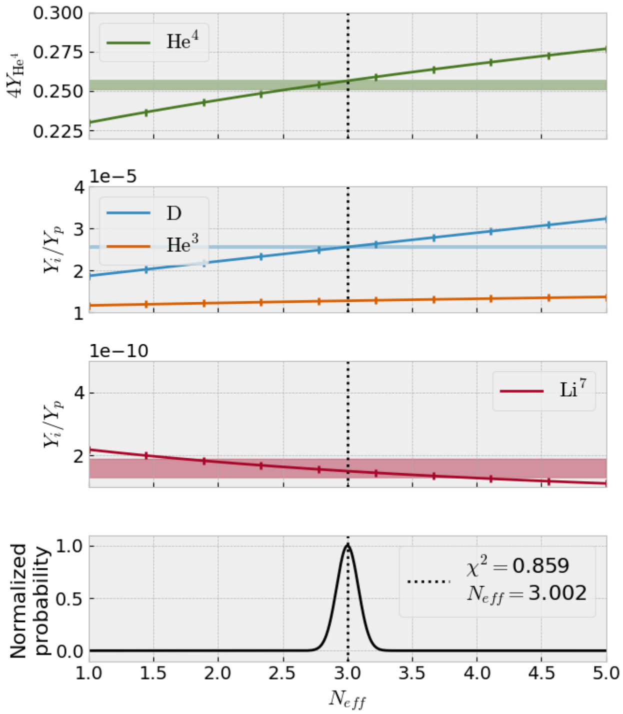

Figure 5. Relic abundances of elements produced in BBN as function of the effective number of neutrino species N_{eff}, and the likelihood of different values for N_{eff} given the data. The relic abundances are calculated for 10 different values of N_{eff} (indicated by notches), which are interpolated in order to produce smooth functions of N_{eff}.