AST3220 - Project 2:
Big Bang Nuclesynthesis

Candidate nr. 14

(Dated: May 3, 2023)

1. PROBLEM A)

We can rewrite the number density n_i of species i in terms of the relative number density Y_i as:

$$Y_i = \frac{n_i}{n_b} \quad \Rightarrow \quad n_i = n_b Y_i \tag{1.1}$$
$$= \frac{n_{b0}}{a^3} Y_i \tag{1.2}$$

Where $n_b(t)$ is the baryon number density, n_{b0} is the baryon number density today, and $a(t)$ is the scale factor. Using the product rule for differentiation, we can then write

$$\frac{dn_i}{dt} = n_{b0} \frac{d}{dt} \left(Y_i a^{-3} \right) \tag{1.3}$$
$$= n_{b0} \left(\frac{1}{a^3} \frac{dY_i}{dt} - 3Y_i \frac{\dot{a}}{a^4} \right) \tag{1.4}$$
$$= n_b \frac{dY_i}{dt} - 3Y_i n_b H \tag{1.5}$$
$$= n_b \frac{dY_i}{dt} - 3n_i H \tag{1.6}$$

Next we want to switch from t to $\ln T$ as our time variable, where T is the temperature. Using $T = T_0 a^{-1}$ we get

$$\ln T = \ln T_0 - \ln a(t) \tag{1.7}$$

Then, using the chain rule of differentiation, we can rewrite

$$\frac{dY_i}{dt} = \frac{d(\ln T)}{dt} \frac{dY_i}{d(\ln T)} \tag{1.8}$$
$$= -\frac{\dot{a}}{a} \frac{dY_i}{d(\ln T)} \tag{1.9}$$
$$= -H \frac{dY_i}{d(\ln T)} \tag{1.10}$$

Inserting to equation (1.6) we get

$$\frac{dn_i}{dt} = -n_b H \frac{dY_i}{d(\ln T)} - 3n_i H \tag{1.11}$$

The equations for the evolution of the number densities of protons p and neutrons n are given as

$$\frac{dn_n}{dt} + 3Hn_n = n_p \Gamma_{p\to n} - n_n \Gamma_{n \to p} \tag{1.12}$$
$$\frac{dn_p}{dt} + 3Hn_p = n_n \Gamma_{n\to p} - n_p \Gamma_{p \to n} \tag{1.13}$$
$$= -\left(\frac{dn_n}{dt} + 3Hn_n \right) \tag{1.14}$$

And by inserting Eq. (1.1) and Eq. (1.6) we finally find the evolution of the relative number densities:

2. PROBLEM B)

The relation $T_\nu = \left(4/11\right)^{1/3} T$ can be derived from the conservation of entropy, which tells us that

$$g_{*s}(aT)^3 = const. \tag{2.1}$$

At the time where the universe had a temperature $k_B T > 0.511$ Mev, electrons and positrons were relativistic and the process

$$e^+ + e^- \rightleftharpoons \gamma + \gamma \tag{2.2}$$

occured in both directions. However, as the temperature universe falls below the rest mass of the electron and positron $k_B T < 0.511$, the average energy of a photon collision is too small for an electrons-positron pair to be created. Since electrons and positrons will still anihilate through the process

$$e^+ + e^- \to \gamma + \gamma \tag{2.3}$$

most of the positrons and electrons will then dissapear. Assuming this happened immediately, and that the universe is in thermal equilibrium ($T_i = T$), the effective number of degrees of freedom before and after can be written

$$g_{*s}^{before} = g_\nu + \frac{7}{8}(g_{e^-} + g_{e^+}) \tag{2.4}$$
$$= 2 + \frac{7}{8} 4 \tag{2.5}$$
$$= \frac{11}{2} \tag{2.6}$$
$$g_{*s}^{after} = g_\nu \tag{2.7}$$
$$= 2 \tag{2.8}$$

If we also assume the scale factor a is the same before and after, the conservation of entropy gives us

$$\frac{11}{2}(aT)^3_{before} = 2(aT)^3_{after} \tag{2.9}$$
$$\Rightarrow T_{after} = (\frac{11}{4})^{1/3} T_{before} \tag{2.10}$$

Since neutrinos are decoupled, we then have

$$T_{\nu,after} = T_{\nu,before} = T_{before} = \left(\frac{4}{11}\right)^{1/3} T_{after} \quad (2.11)$$

Finally giving us

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T \quad (2.12)$$

3. PROBLEM C)

In the early universe, dominated by radiation, we have

$$\rho c^2 \approx \frac{\pi^2}{30} g_* \frac{(k_b T)^4}{(\hbar c)^3} \quad (3.1)$$

Where g_* is the effective number of relativistic degrees of freedom. Assuming all the radiation is composed of photons and N_eff number of neutrino species, g_* is

$$\begin{aligned} g_* &= 1 + N_{\text{eff}}\ g_\nu \left(\frac{T_i}{T}\right)^4 & (3.2)\\ &= 1 + N_{\text{eff}}\ \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} & (3.3)\end{aligned}$$

With $\rho_{c0} = \frac{3H_0^2}{8\pi G}$ as the critical density, we then find

$$\begin{aligned} \Omega_{r0} &= \frac{\rho_0}{\rho_{c0}} & (3.4)\\ &= \frac{1}{c^2}\left(\frac{\pi^2}{30} g_* \frac{(k_b T)^4}{(\hbar c)^3}\right) \cdot \left(\frac{8\pi G}{3H_0^2}\right) & (3.5)\\ &= \frac{4\pi^3}{45}\frac{G}{H_0^2}\frac{(k_b T_0)^4}{\hbar^3 c^5} g_* & (3.6)\\ &= \frac{4\pi^3}{45}\frac{G}{H_0^2}\frac{(k_b T_0)^4}{\hbar^3 c^5}\left[1 + N_{\text{eff}}\ \frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right] & (3.7)\end{aligned}$$

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4. PROBLEM D)

1. *Scale factor*

At the BBN, the Friedmann equations simplify to

$$\frac{1}{a} t = H_0 \sqrt{\Omega_{r0} a^{-2}} \quad (4.1)$$

With some rearranging we see this is a separable differential equation, which we solve for $a(t)$:

$$\begin{aligned} a\ \frac{da}{dt} &= H_0\sqrt{\Omega_{r0}} & (4.2)\\ \Rightarrow \quad \int_0^a a'\ da' &= H_0\sqrt{\Omega_{r0}} \int_0^t dt' & (4.3)\\ \Rightarrow \quad \frac{1}{2}a^2 &= H_0\sqrt{\Omega_{r0}}t & (4.4)\\ \Rightarrow \quad a &= \sqrt{2H_0 t}\ \ (\Omega_{r0})^{1/4} & (4.5)\end{aligned}$$

2. *Cosmic time*

To find the cosmic time as a function of the photon temperature, we use the relation

$$T = T_0 a^{-1} \quad \Rightarrow \quad a = \frac{T_0}{T} \quad (4.6)$$

Inserting this into eq. (4.5) and squaring both sides we get

$$\left(\frac{T_0}{T}\right)^2 = 2H_0 t\sqrt{(\Omega_{r0})} \quad (4.7)$$

$$(4.8)$$

Which is easily solved:

$$t(T) = \frac{1}{2H_0\sqrt{\Omega_{r0}}}\left(\frac{T_0}{T}\right)^2 \quad (4.9)$$

A table of this expression evaluated at temperatures 10^{10}, 10^9 and 10^8 is attached in table (I)

T [K]	$t(T)$
10^{10}	1.7774 Sec
10^9	2 Min, 57.7400 Sec
10^8	5 Hr, 56 Min, 14.0000 Sec

Table I. Age of the universe at different temperatures.

5. PROBLEM E)

Assuming protons and neutrons are non relativistic at this point, they follow the Maxwell Boltzmann distribuiton. At equilibruium we then have

$$\begin{aligned} \frac{n_n^{(0)}}{n_p^{(0)}} &= \frac{Y_n^{(0)}}{Y_p^{(0)}} & (5.1)\\ &= \left(\frac{m_p}{m_n}\right) e^{-(m_n - m_p)c^2/k_B T_i} & (5.2)\\ &\approx e^{-(m_n - m_p)c^2/k_B T_i} & (5.3)\end{aligned}$$

where we have used that $m_p/m_n \approx 1$. Also assuming protons and neutrons make up all the baryonic mass, we have

$$Y_p + Y_n = \frac{n_p + n_n}{n_b} = 1 \quad \Rightarrow \quad Y_p = 1 - Y_n \quad (5.4)$$

Such that

$$\frac{Y_n}{Y_p} = \frac{Y_n}{1 - Y_n} = e^{-(m_n - m_p)c^2/k_B T_i} \quad (5.5)$$

Which can be solved for $Y_n(T_i)$:

$$Y_n(T_i)\ e^{(m_n-m_p)c^2/k_BT_i} = 1 - Y_n \quad (5.6)$$

$$\Rightarrow\ Y_n(T_i)\left[1+e^{-(m_n-m_p)c^2/k_BT_i}\right] = 1 \quad (5.7)$$

$$\Rightarrow\ Y_n(T_i) = \left[1+e^{-(m_n-m_p)c^2/k_BT_i}\right]^{-1} \quad (5.8)$$

6. PROBLEM F)

At first, we try to solve the integral for the decay rates $\Gamma_{n\to p}$ and $\Gamma_{n\to p}$ using scipy's *quad* integrator, but this turned out to be very slow. As one can see the integral should converge at a reasonable pace due to the exponential terms in the demoninators, we instead approximated the integral by making a cut-off at $x = 250$. This was performed with an implementation of simpsons method, using a step size of $dx \approx 0.024$. The resulting plot can be seen in Fig. (1)

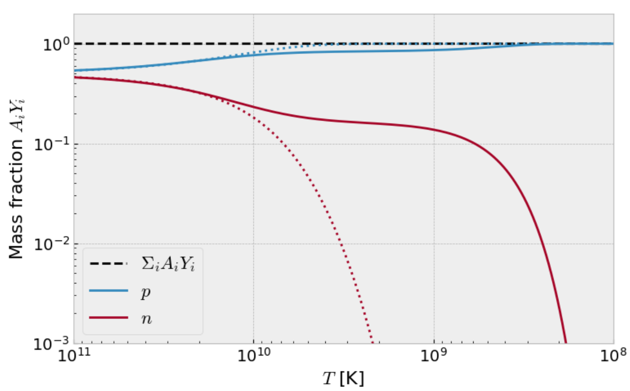

Figure 1. Caption

7. PROBLEM G)

From Problem a), we recall that

$$\frac{dn_i}{dt} + 3Hn_i = n_b H \frac{dY_i}{d\ln T} \quad (7.1)$$

Inserting the given expression

$$\frac{dn_i}{dt} + 3Hn_i = \sum_{j\neq i}[n_j\Gamma_{j\to i} - n_i\Gamma_{i\to j}] \quad (7.2)$$

$$+\sum_{jkl}[n_k n_l \gamma_{kl\to ij} - n_i n_j \gamma_{ij\to kl}] \quad (7.3)$$

and using the definition $\Gamma_{ij\to kl} = n_b\gamma_{ij\to kl}$, we see that

$$\frac{dY_i}{d\ln T} = \frac{1}{H}\Bigg\{\sum_{j\neq i}[\frac{n_j}{n_b}\Gamma_{j\to i} - \frac{n_i}{n_b}\Gamma_{i\to j}] \quad (7.4)$$

$$+\sum_{jkl}[\frac{n_k}{n_b}\frac{n_l}{n_b}n_b\gamma_{kl\to ij} - \frac{n_i}{n_b}\frac{n_j}{n_b}n_b\gamma_{ij\to kl}]\Bigg\} \quad (7.5)$$

$$= \frac{1}{H}\Bigg\{\sum_{j\neq i}[Y_j\Gamma_{j\to i} - Y_i\Gamma_{i\to j}] \quad (7.6)$$

$$+\sum_{jkl}[Y_kY_l\Gamma_{kl\to ij} - Y_iY_j\Gamma_{ij\to kl}]\Bigg\} \quad (7.7)$$

Which is what we wanted to show.

8. PROBLEM H)

Implementing the additional reactions results in Fig. (2). The plot shows a drop in proton and neutron mass fraction as the temperature decreases past $\sim 10^9$K. This is due to the formation of deuterium, which has a sharp increase in its mass fraction before it stabilizes at $A_DY_D \sim \frac{1}{4}$ due to neutrons supply being exhausted.

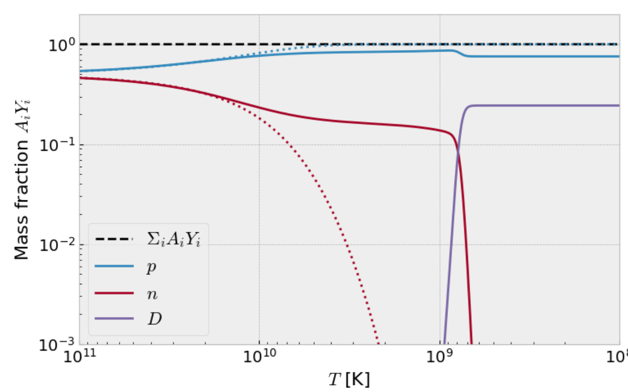

Figure 2. Caption.

9. PROBLEM I)

Implementing the remaining reactions we get the plot shown in Fig. (3). Ignoring neutrons and protons, the plot shows the production of most elements happens in the range $\sim 10^{10} - 10^9$ K with most of the mass fractions peaking at $\sim 10^9$K (i.e. ~ 3 minutes after the big bang).

The big exception to this is Be^7 which has sharply increases a little bit after this, while most other elements (an exception being He^3) quickly drops in their mass fractions. This is presumably due to being too large, and possibly requiring sufficient amounts of other elements to form.

We also note that by the time the temperature has decreased to 10^7K, the production of elements seems to have halted completely.

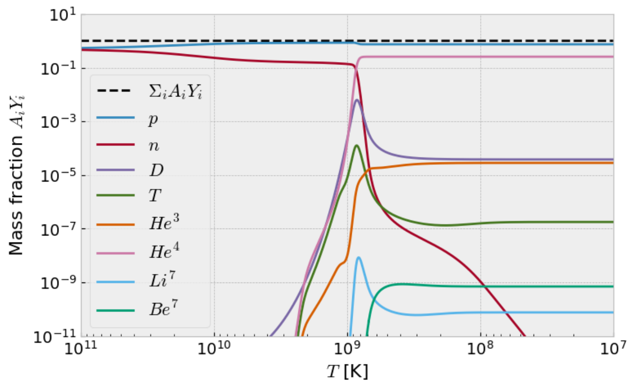

Figure 3. Caption.

10. PROBLEM J)

Blablabla logspace blabla. Plot in Fig. (4).

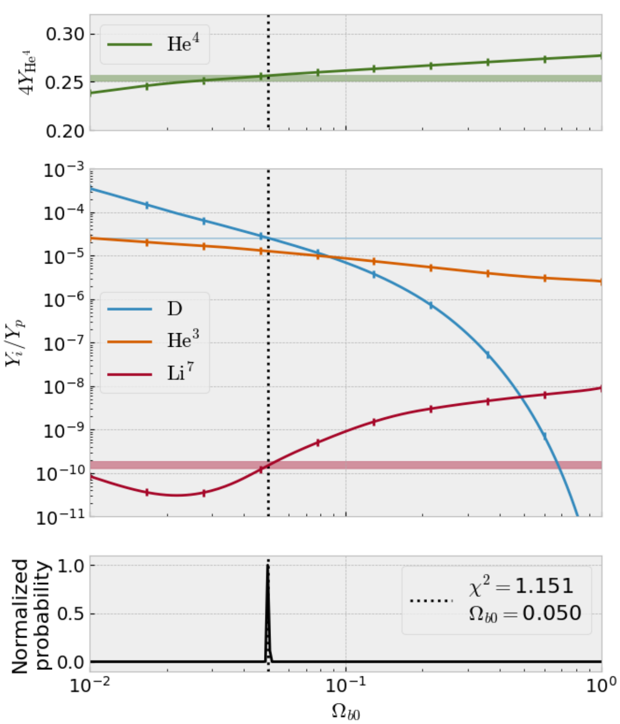

Figure 4. Caption.

11. PROBLEM K)

Blabla 3 neutrinos species (wow!) blablabla. Plot in Fig. (5).

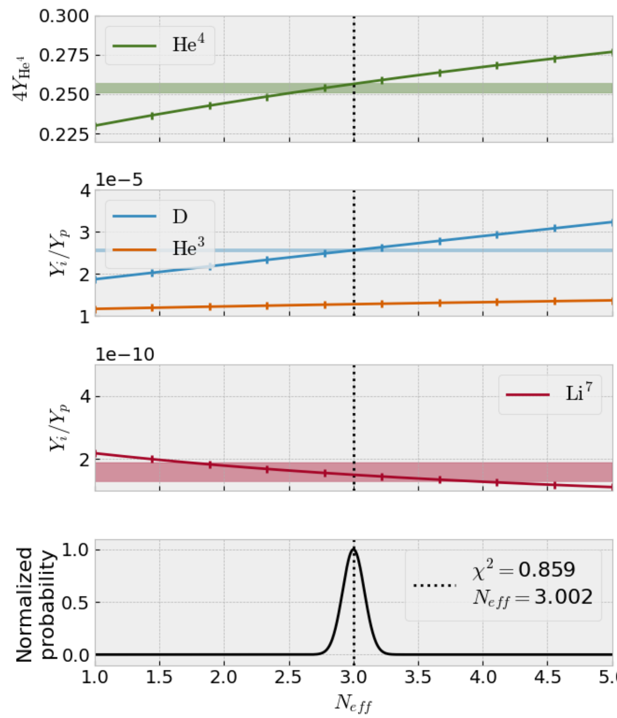

Figure 5. Caption.

ACKNOWLEDGMENTS

I would like thank myself for writing this beautiful document.

REFERENCES

- Reference 1
- Reference 2

Appendix A: Name of appendix

This will be the body of the appendix.

Appendix B: This is another appendix

Tada.