AST3220, spring 2023: Project 3

Read this before you start

This project consists of a set of tasks, some analytical, some numerical. There is no need to structure your answers as a report with an introduction, methods, results, discussion and conclusion, you can just answer the questions, one by one. In fact, we prefer it that way. It is important that you explain how you think, just writing down a bunch of equations with no explanations will not give you a maximum score. You should write your report/answers using LaTeX. Posting handwritten lecture notes and solutions to problems is a privilege that belongs to the lecturer alone. Your figures should have a clear layout with proper axis labels and units, and with a caption explaining what the figure shows. The figures should be referenced in the main text. You are also required to hand in your source code in a separate file. Please don't write your name or other facts which may disclose your identity anywhere.

Inflation without approximation

In the lectures and in the problems we have studied inflation analytically with the slow-roll approximation. If we forego nice, closed expressions we can however, solve the full equations numerically, and this is what you will do in this project.

I remind you that the Planck energy, Planck mass, and Planck length are defined by, respectively

$$E_{\mathrm{P}}^2 = \frac{\hbar c^5}{G}, \;\; m_{\mathrm{P}}^2 = \frac{\hbar c}{G}, \;\; l_{\mathrm{P}}^2 = \frac{\hbar G}{c^3}. \tag{1}$$

Assuming spatial flatness and that the scalar field dominates the energy density, the equations governing the evolution of the scalar field and the scale factor are

$$\ddot{\phi} + 3H\dot{\phi} + \hbar c^3 V'(\phi) = 0 \tag{2}$$

$$H^2 = \frac{8\pi G}{3c^2}\left[\frac{1}{2\hbar c^3}\dot{\phi}^2 + V(\phi)\right]. \tag{3}$$

Before solving these equations numerically it is useful to rewrite them in terms of dimensionless quantities. First, define

$$H_i^2 \equiv \frac{8\pi G}{3c^2} V(\phi_i), \tag{4}$$

where ϕ_i is the initial value of the field, and then introduce the variables

$$\begin{aligned}
\tau &= H_i t && (5)\\
h &= \frac{H}{H_i} && (6)\\
\psi &= \frac{\phi}{E_{\rm P}} && (7)\\
v &= \frac{\hbar c^3}{H_i^2 E_{\rm P}^2} V && (8)
\end{aligned}$$

a)(2 points) Check that these variables are dimensionless.

b) (5 points) Show that

$$\frac{d}{d\tau}\left[\ln\left(\frac{a}{a_i}\right)\right] = h(\tau) \tag{9}$$

and that equations (2) and (3) can be rewritten as

$$\begin{aligned}
\frac{d^2\psi}{d\tau^2} + 3h\frac{d\psi}{d\tau} + \frac{dv}{d\psi} &= 0 && (10)\\
h^2 &= \frac{8\pi}{3}\left[\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v(\psi)\right] && (11)
\end{aligned}$$

We need to think about the initial conditions. It is convenient to shift the origin of the time coordinate so we can start at $\tau = 0$. The condition for h is trivial: $h(0) = H(0)/H_i = 1$. Also, by definition $\ln(a(0)/a_i) = \ln(a_i/a_i) = 0$. For $\psi = \phi/E_{\rm P}$ we should choose a value that makes sure that we get inflation, and that means that the slow-roll conditions should be fulfilled. But since the equation for the scalar field is a second-order equation, we also seem to need an initial value for $d\psi/d\tau$.

c) (3 points) Explain why, in the slow-roll approximation, the initial value for $d\psi/d\tau$ is irrelevant as long as it doesn't violate the slow-roll conditions.

You can in the following choose $d\psi/d\tau = 0$ as the initial value.

We are now ready to look at specific models. Let's try

$$V(\phi) = \frac{1}{2}\frac{m^2c^4}{(\hbar c)^3}\phi^2, \tag{12}$$

from the example starting on page 103 in the lecture notes.

d) (3 points) Use the slow-roll conditions to choose an initial value for the field that will give 500 e-folds of inflation.

e)(20 points) Solve equations (9), (10) and (10) numerically and plot the results. Based on the lectures, how would you expect ψ to behave? Does the numerical solution conform with your expectation?

f) (2 points) In the same plot, plot the slow-roll solution from the lecture notes. When does it start to deviate significantly from the exact, numerical solution?

g) (5 points) Plot the slow-roll parameter ϵ as a function of τ. Taking $\epsilon = 1$ as the end of inflation, find numerically N_{tot}, the total number of e-foldings produced during inflation and compare with the prediction from the slow-roll approximation.

h) (5 points) Show that in terms of the dimensionless variables

$$\frac{p_\phi}{\rho_\phi c^2} = \frac{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 - v}{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v} \tag{13}$$

i) (2 points) What would you expect the ratio in equation (13) to be in the slow-roll regime? And in the oscillating phase? Plot the numerical result and compare.

j) (5 points) Plot the slow-roll parameters ϵ and η as functions of N, the number of e-foldings remaining before the end of inflation. (Hint: If a_f denotes the value of a when $\epsilon = 1$, $N = \ln(a_f/a) = \ln(a_f a_i/(a_i a)) = \ln(a_f/a_i) + \ln(a_i/a) = N_{\text{tot}} - \ln(a/a_i)$)

The density perturbations set up by quantum fluctuations in the inflaton field are usually characterized by their power spectrum $P(k)$, which measures the amplitude of the fluctuations on a scale given by the wave number k. It is well approximated by a power-law $P(k) \propto k^{n-4}$, where n is called the scalar spectral index, and $n = 1$ corresponds to a scale-invariant spectrum $k^3 P(k) =$ constant. In AST5220 you will show this in detail, and also show that n is given in terms of the slow-roll parameters as

$$n = 1 - 6\epsilon + 2\eta \tag{14}$$

where ϵ and η are evaluated at the time when scales that are observable today crossed the horizon during inflation, which was 50-60 e-folds before the end of inflation. The scalar spectral index can be measured by observing the CMB temperature anisotropies, and the same goes for another important quantity, the tensor-to-scalar ratio, which we found to be

$$r = 16\epsilon \tag{15}$$

where again ϵ is to be evaluated 50-60 e-foldings before the end of inflation.

k) (5 points) Plot the curve in the n-r plane which corresponds to the predictions of the ϕ^2-model of inflation when we let N vary from 50 to 60.

Alexei Starobinsky is one of the "founding fathers" of inflation. Back in 1980 he studied a model based on quantum corrections to general relativity. It was later shown that one can transform coordinates so that this model looks like normal general relativity with the addition of a scalar field moving in a potential

$$V(\phi) = \frac{3M^2M_p^2}{4}\left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_\mathrm{P}}}\right)^2 \tag{16}$$

I have now, to save some writing, switched to units where $\hbar = c = 1$, so that $E_\mathrm{P}^2 = \frac{1}{G}$. Note that M_P here is the so-called reduced Planck mass $M_\mathrm{P}^2 = \frac{1}{8\pi G} = \frac{E_\mathrm{P}^2}{8\pi}$.

l) (3 points) Show that the slow-roll parameters in this model are given by

$$\epsilon = \frac{4}{3}\frac{e^{2y}}{(1-e^y)^2} \tag{17}$$

$$\eta = \frac{4}{3}\frac{(2e^{2y}-e^y)}{(1-e^y)^2} \tag{18}$$

where $y = -\sqrt{\frac{2}{3}}\frac{\phi}{M_\mathrm{P}} = -\sqrt{\frac{16\pi}{3}}\psi$.

m) (15 points) Set up and solve numerically for $\psi(\tau)$ and $\ln(a(\tau)/a_i)$. Use $\psi(0) = 2$ as the initial condition on the field. Plot the results.

n) (15 points) Repeat steps j) and k) for the Starobinsky model.

o) (5 points) Show that the slow-roll approximation applied to this model gives $N \approx \frac{3}{4}e^{-y}$, $\epsilon \approx \frac{3}{4N^2}$, $\eta \approx -\frac{1}{N}$, $n \approx 1 - \frac{2}{N}$, and $r \approx \frac{12}{N^2}$. Compare with the numerical results.

p) (5 points) Do a literature search and find the constraints ESA's CMB satellite Planck provided for n and r. Is the ϕ^2-model consistent with them? What about the Starobinsky model?