# AST3220 - Project 1

(Dated: February 20, 2023)

This abstract is abstract.

The scalar field has energy density and pressure

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1}$$
$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2}$$

## I. PROBLEM 1

Assuming the quintessence field follows the continuity equation

$$\dot{\rho}_\phi = -3H\left(\rho_\phi + p_\phi\right) \tag{3}$$
$$= -3\frac{\dot{a}}{a}\left(1 + w_\phi\right)\rho_\phi \tag{4}$$

we can begin solving for the density by separating the variables:

$$\frac{1}{\rho_\phi}d\rho_\phi = -\frac{3}{a}\left(1 + w_\phi\right)da \tag{5}$$

We then rewrite $a(t)$ in terms of the cosmological redshift of light emitted at some point $t$ in the past:

$$1 + z = \frac{a_0}{a(t)} \quad \Rightarrow a(t) = \frac{a_0}{1+z} \tag{6}$$
$$\Rightarrow da = -\frac{a_0}{(1+z)^2}dz \tag{7}$$

Inserting these into eq. 5 and integrating from some time $t$ to today, we get:

$$\int_{\rho_\phi}^{\rho_{\phi 0}} d\rho_\phi \frac{1}{\rho_\phi} = \int_z^0 dz' \frac{3[1 + w_\phi(z')]}{(1+z')} \tag{8}$$

We then flip the integration limits on both sides, canceling the negatives. Computing the left hand integral and solving for $rho_\phi$ we then get the solution

$$\rho_\phi(z) = \rho_{\phi 0}\,\exp\left\{\int_0^z dz' \frac{3[1 + w_\phi(z')]}{(1+z')}\right\} \tag{9}$$

## II. PROBLEM 2

Inserting equations for the density (1) and pressure (2) of the scalar field into the continuity equation we get

$$\dot{\rho}_\phi = -3H\left(\rho_\phi + p_\phi\right) \tag{10}$$
$$= -3H\dot{\phi}^2 \tag{11}$$

We can also find an expression for $\dot{\rho}_\phi$ directly, by taking the time derivative of equation (1):

$$\dot{\rho}_\phi = \frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{12}$$
$$= \dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi} \tag{13}$$

Equating the two expressions, we get the differential equation:

$$\ddot{\phi}\dot{\phi} + 3H\dot{\phi}^2 + V'(\phi)\dot{\phi} = 0 \tag{14}$$

Ignoring the boring case of a static $\phi(t) = \phi_0$, we have $\dot{\phi} \neq 0$. The equation can then be reduced to:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{15}$$

## III. PROBLEM 3

Taking the time-derivative of the Hubble parameter

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \tag{16}$$

we recognize both terms on the right hand side from the Friedmann equations. For a flat ($k = 0$) universe the Friedmann equations read

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}\rho \tag{17}$$
$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}\left(\rho + 3p\right) \tag{18}$$

Where we have defined $\kappa^2 = 8\pi G$. For a universe with matter, radiation and quintessence we have

$$\rho = \rho_m + \rho_r + \rho_\phi \tag{19}$$
$$p = p_r + p_\phi \tag{20}$$

Inserting the Friedmann equations into (24), and writing the pressures in terms of the equation of state parameter, we get

$$\dot{H} = -\frac{\kappa^2}{6}\left(\rho + 3p\right) - \frac{\kappa^2}{3}\rho \tag{21}$$
$$= -\frac{\kappa^2}{2}\left[\rho + p\right] \tag{22}$$
$$= -\frac{\kappa^2}{2}\left[\rho_m + \rho_r(1 + w_r) + \dot{\phi}\right] \tag{23}$$
$$= -\frac{\kappa^2}{2}\left[\rho_m + \rho_r(1 + w_r) + \dot{\phi}\right] \tag{24}$$

## IV. PROBLEM 4

We introduce the dimensionless variables

$$x_1 = \frac{\kappa\dot{\phi}}{\sqrt{6}H} \tag{25}$$
$$x_2 = \frac{\kappa\sqrt{V}}{\sqrt{3}H} \tag{26}$$
$$x_3 = \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H} \tag{27}$$
$$\tag{28}$$

and notice their squares can be rewritten them in terms of the critical density $\rho_c = \frac{3H^2}{\kappa^2}$:

$$x_1^2 = \frac{\frac{1}{2}\dot{\phi}^2}{\rho_c} \tag{29}$$
$$x_2^2 = \frac{V}{\rho_c} \tag{30}$$
$$x_3^2 = \frac{\rho_r}{\rho_c} \tag{31}$$
$$\tag{32}$$

The quintessence and radiation density parameters can then easily be found to be

$$\Omega_\phi = \frac{\frac{1}{2}\dot{\phi} + V}{\rho_c} = x_1^2 + x_2^2 \tag{33}$$
$$\Omega_r = \frac{\rho_r}{\rho_c} = x_3^2 \tag{34}$$

Lastly we want to find an expression for the density parameter for mass. We accomplish this by taking advantage of our model being spatially flat, meaning $\frac{\rho}{\rho_c} = 1$. Expanding $\rho$ we get

$$1 = \frac{\rho_\phi + \rho_r + \rho_m}{\rho_c} \tag{35}$$
$$= \Omega_\phi + \Omega_r + \Omega_m \tag{36}$$

Which we can solve for $\Omega_m$:

$$\Omega_m = 1 - \Omega_\phi - \Omega_m \tag{37}$$
$$= 1 - x_1^2 - x_2^2 - x_3^2 \tag{38}$$

## V. PROBLEM 5

Next we want to rewrite equation (24) in terms of the dimensionless variables. Using the definitions of the density parameters and the dimensionless variables we get:

$$\dot{H} = -\frac{\kappa^2}{2}\left[\rho_m + \rho_r(1+w_r) + \dot{\phi}\right] \tag{39}$$
$$= -\frac{\kappa^2\rho_c}{2}\left[\Omega_m + \Omega_r(1+w_r) + 2\Omega_\phi - 2x_2^2\right] \tag{40}$$
$$= -\frac{3H^2}{2}\left[1 + x_1^2 - x_2^2 - x_3^2 + x_3^2(1+w_r)\right] \tag{41}$$

Simplyfing and inserting the radiation equation of state parameter $w_r = 1/3$, we get:

$$\frac{\dot{H}}{H^2} = -\frac{1}{2}\left[3 + 3x_1^2 - 3x_2^2 + x_3^2\right] \tag{42}$$

## VI. PROBLEM 6

### A. Equation of motion for $x_1$

Using the product rule:

$$\frac{dx_1}{dN} = \frac{1}{H}\frac{dx_1}{dt} \tag{43}$$
$$= \frac{\kappa}{\sqrt{6}H}\frac{d}{dt}\left(\frac{\dot{\phi}}{H}\right) \tag{44}$$
$$= \frac{\kappa\ddot{\phi}}{\sqrt{6}H^2} - \frac{\kappa\dot{\phi}}{\sqrt{6}H}\frac{\dot{H}}{H^2} \tag{45}$$

From equation (15) we have $\ddot{\phi} = -3H\dot{\phi} - V$. Inserting for $\ddot{\phi}$ and $\dot{H}/H^2$ (eq. 42) we then get:

$$\frac{dx_1}{dN} = -\frac{3\kappa\dot{\phi}}{\sqrt{6}H} - \frac{\kappa V'}{\sqrt{6}H} + \frac{\kappa\dot{\phi}}{\sqrt{6}H}\frac{\dot{H}}{H} \tag{46}$$
$$= -3x_1 + \frac{\sqrt{6}}{2}\lambda x_2^2 + \frac{1}{2}x_1(3 + 3x_1 - 3x_2^2 + 3x_3^3) \tag{47}$$

Where we have defined $\lambda$ as

$$\lambda = -\frac{V'}{\kappa V} \tag{48}$$

### B. Equation of motion for $x_2$

The equation for $x_2$ is found in a similar manner, we get:

$$\frac{dx_2}{dN} = \frac{\kappa}{\sqrt{3}H}\frac{d}{dt}\left(\frac{\sqrt{V}}{H}\right) \tag{49}$$
$$= \frac{\kappa}{\sqrt{3}H}\left(\frac{1}{2}\frac{V'\dot{\phi}}{\sqrt{\phi}H} - \sqrt{V}\frac{\dot{H}}{H^2}\right) \tag{50}$$
$$= \frac{\sqrt{6}}{2}\frac{V'}{\kappa V}\frac{\kappa\sqrt{V}}{\sqrt{3}H}\frac{\kappa\dot{\phi}}{\sqrt{6}H} - \frac{\kappa\sqrt{V}}{\sqrt{3}H}\frac{\dot{H}}{H^2} \tag{51}$$
$$= \frac{\sqrt{6}}{2}\lambda x_1 x_2 + \frac{1}{2}x_2(3 + 3x_1 - 3x_2^2 + 3x_3^3) \tag{52}$$

### C. Equation of motion for $x_3$

Similarly, for $x_3$:

$$\frac{dx_3}{dN} = \frac{\kappa}{\sqrt{3}H}\frac{d}{dt}\left(\frac{\sqrt{\rho_r}}{H}\right) \tag{53}$$

$$= \frac{\kappa}{\sqrt{3}H}\left(\frac{1}{2}\frac{\dot{\rho}_r}{\sqrt{\rho_r}H} - \sqrt{\rho_r}\frac{\dot{H}}{H^2}\right) \tag{54}$$

$$\tag{55}$$

Where $\dot{\rho}_r$ is given by the first Friedmann equation:

$$\dot{\rho}_r = -3H(1+w_r)\rho_r \tag{56}$$

$$= -4H\rho_r \tag{57}$$

Where we used that $w_r = 1/3$. Inserting the expressions for $\dot{\rho}_r$ and $\dot{H}/H^2$ we finally get:

$$\frac{dx_3}{dN} = -2x_3 + \frac{1}{2}x_3\left(3 + 3x_1 - 3x_2^2 + 3x_3^3\right) \tag{58}$$

## VII. PROBLEM 7

Rewriting the expression for $\lambda$ as a differential equation for V, we get:

$$V' + \kappa\lambda V = 0 \tag{59}$$

For a constant $\lambda$ this has the solution

$$V = V_0 e^{-\kappa\lambda\phi} \tag{60}$$

We can then find the value of $\Gamma$ by simple differentiation:

$$\Gamma = \frac{VV''}{(V')^2} \tag{61}$$

$$= \frac{\kappa^2\lambda^2V^2}{(-\kappa\lambda)^2V^2} \tag{62}$$

$$= 1 \tag{63}$$

## VIII. PROBLEM 8

## IX. PROBLEM 9

CALCULATE SUM OF DENSITY PARAMETERS FOR QUALITY CONTROL!

## ACKNOWLEDGMENTS

I would like thank myself for writing this beautiful document.

## REFERENCES

- Reference 1
- Reference 2

### Appendix A: Name of appendix

This will be the body of the appendix.

### Appendix B: This is another appendix

Tada.