AST3220 - Project 1

(Dated: May 2, 2023)

This abstract is abstract.

1. PROBLEM A)

We can rewrite the number density n_i of species i in terms of the relative number density Y_i as:

$$Y_i = \frac{n_i}{n_b} \quad \Rightarrow \quad n_i = n_b Y_i \tag{1.1}$$

$$= \frac{n_{b0}}{a^3} Y_i \tag{1.2}$$

Where $n_b(t)$ is the baryon number density, n_{b0} is the baryon number density today, and $a(t)$ is the scale factor. Using the product rule for differentiation, we can then write

$$\frac{dn_i}{dt} = n_{b0} \frac{d}{dt} \left(Y_i a^{-3}\right) \tag{1.3}$$

$$= n_{b0} \left(\frac{1}{a^3} \frac{dY_i}{dt} - 3Y_i \frac{\dot{a}}{a^4} \right) \tag{1.4}$$

$$= n_b \frac{dY_i}{dt} - 3Y_i n_b H \tag{1.5}$$

$$= n_b \frac{dY_i}{dt} - 3n_i H \tag{1.6}$$

Next we want to switch from t to $\ln T$ as our time variable, where T is the temperature. Using $T = T_0 a^{-1}$ we get

$$\ln T = \ln T_0 - \ln a(t) \tag{1.7}$$

Then, using the chain rule of differentiation, we can rewrite

$$\frac{dY_i}{dt} = \frac{d(\ln T)}{dt} \frac{dY_i}{d(\ln T)} \tag{1.8}$$

$$= -\frac{\dot{a}}{a} \frac{dY_i}{d(\ln T)} \tag{1.9}$$

$$= -H \frac{dY_i}{d(\ln T)} \tag{1.10}$$

Inserting to equation (1.6) we get

$$\frac{dn_i}{dt} = -n_b H \frac{dY_i}{d(\ln T)} - 3n_i H \tag{1.11}$$

The equations for the evolution of the number densities of protons p and neutrons n are given as

$$\frac{dn_n}{dt} + 3Hn_n = n_p \Gamma_{p\to n} - n_n \Gamma_{n\to p} \tag{1.12}$$

$$\frac{dn_p}{dt} + 3Hn_p = n_n \Gamma_{n\to p} - n_p \Gamma_{p\to n} \tag{1.13}$$

$$= -\left(\frac{dn_n}{dt} + 3Hn_n \right) \tag{1.14}$$

And by inserting Eq. (1.1) and Eq. (1.6) we finally find the evolution of the relative number densities:

2. PROBLEM B)

The relation $T_\nu = \left(4/11\right)^{1/3} T$ can be derived from the conservation of entropy, which tells us that

$$g_{*s}(aT)^3 = const. \tag{2.1}$$

At the time where the universe had a temperature $k_B T$ 0.511 Mev, electrons and positrons were relativistic and the process

$$e^+ + e^- \gamma + \gamma \tag{2.2}$$

occured in both directions. However, as the temperature universe falls below the rest mass of the electron and positron $k_B T 0.511$, the average energy of a photon collision is too small for an electrons-positron pair to be created. Since electrons and positrons will still anihilate through the process

$$e^+ + e^- \to \gamma + \gamma \tag{2.3}$$

most of the positrons and electrons will then dissapear. Assuming this happened immediately, and that the universe is in thermal equilibrium ($T_i = T$), the effective number of degrees of freedom before and after can be written

$$g_{*s}^{before} = g_\nu + \frac{7}{8}(g_{e^-} + g_{e^+}) \tag{2.4}$$

$$= 2 + \frac{7}{8}4 \tag{2.5}$$

$$= \frac{11}{2} \tag{2.6}$$

$$g_{*s}^{after} = g_\nu \tag{2.7}$$

$$= 2 \tag{2.8}$$

If we also assume the scale factor a is the same before and after, the conservation of entropy gives us

$$\frac{11}{2}(aT)^3_{before} = 2(aT)^3_{after} \tag{2.9}$$

$$\Rightarrow T_{after} = (\frac{11}{4})^{1/3} T_{before} \tag{2.10}$$

Since neutrinos are decoupled, we then have

$$T_{\nu,after} = T_{\nu,before} = T_{before} = \left(\frac{4}{11}\right)^{1/3} T_{after} \quad (2.11)$$

Finally giving us

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T \quad (2.12)$$

3. PROBLEM C)

In the early universe, dominated by radiation, we have

$$\rho c^2 \approx \frac{\pi^2}{30} g_* \frac{(k_b T)^4}{(\hbar c)^3} \quad (3.1)$$

Where g_* is the effective number of relativistic degrees of freedom. Assuming all the radiation is composed of photons and N_eff number of neutrino species, g_* is

$$\begin{aligned} g_* &= 1 + N_{\text{eff}}\ g_\nu \left(\frac{T_i}{T}\right)^4 && (3.2) \\ &= 1 + N_{\text{eff}}\ \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} && (3.3) \end{aligned}$$

With $\rho_{c0} = \frac{3H_0^2}{8\pi G}$ as the critical density, we then find

$$\begin{aligned} \Omega_{r0} &= \frac{\rho_0}{\rho_{c0}} && (3.4) \\ &= \frac{1}{c^2}\left(\frac{\pi^2}{30} g_* \frac{(k_b T)^4}{(\hbar c)^3}\right) \cdot \left(\frac{8\pi G}{3H_0^2}\right) && (3.5) \\ &= \frac{4\pi^3}{45}\frac{G}{H_0^2}\frac{(k_b T_0)^4}{\hbar^3 c^5} g_* && (3.6) \\ &= \frac{4\pi^3}{45}\frac{G}{H_0^2}\frac{(k_b T_0)^4}{\hbar^3 c^5}\left[1 + N_{\text{eff}}\ \frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right] && (3.7) \end{aligned}$$

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4. PROBLEM D)

1. *Scale factor*

At the BBN, the Friedmann equations simplify to

$$\frac{1}{a} t = H_0 \sqrt{\Omega_{r0} a^{-2}} \quad (4.1)$$

With some rearranging we see this is a separable differential equation, which we solve for $a(t)$:

$$\begin{aligned} a\ \frac{da}{dt} &= H_0\sqrt{\Omega_{r0}} && (4.2) \\ \Rightarrow \int_0^a a'\ da' &= H_0\sqrt{\Omega_{r0}} \int_0^t dt' && (4.3) \\ \Rightarrow \frac{1}{2}a^2 &= H_0\sqrt{\Omega_{r0}} t && (4.4) \\ \Rightarrow a &= \sqrt{2H_0 t}\ \left(\Omega_{r0}\right)^{1/4} && (4.5) \end{aligned}$$

2. *Cosmic time*

To find the cosmic time as a function of the photon temperature, we use the relation

$$T = T_0 a^{-1} \quad \Rightarrow \quad a = \frac{T_0}{T} \quad (4.6)$$

Inserting this into eq. (4.5) and squaring both sides we get

$$\left(\frac{T_0}{T}\right)^2 = 2H_0 t\sqrt{(\Omega_{r0})} \quad (4.7)$$

$$(4.8)$$

Which is easily solved:

$$t(T) = \frac{1}{2H_0\sqrt{\Omega_{r0}}}\left(\frac{T_0}{T}\right)^2 \quad (4.9)$$

A table of this expression evaluated at temperatures 10^10, 10^9 and 10^8 is attached in table (I)

T [K]	$t(T)$ [s]
10^{10}	1.7774
10^9	1.7774×10^2
10^8	1.7774×10^4

Table I. Age of the universe at different temperatures.

ACKNOWLEDGMENTS

I would like thank myself for writing this beautiful document.

REFERENCES

- Reference 1
- Reference 2

Appendix A: Name of appendix

This will be the body of the appendix.

Appendix B: This is another appendix

Tada.