AST3220 - Project 1

(Dated: March 8, 2023)

This abstract is abstract.

The scalar field has energy density and pressure

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2}$$

I. PROBLEM 1

Assuming the quintessence field follows the continuity equation

$$\dot{\rho}_\phi = -3H\left(\rho_\phi + p_\phi\right) \tag{3}$$

$$= -3\frac{\dot{a}}{a}\left(1 + w_\phi\right)\rho_\phi \tag{4}$$

we can begin solving for the density by separating the variables:

$$\frac{1}{\rho_\phi} d\rho_\phi = -\frac{3}{a}\left(1 + w_\phi\right) da \tag{5}$$

We then rewrite $a(t)$ in terms of the cosmological redshift of light emitted at some point t in the past:

$$1 + z = \frac{a_0}{a(t)} \Rightarrow a(t) = \frac{a_0}{1+z} \tag{6}$$

$$\Rightarrow da = -\frac{a_0}{(1+z)^2} dz \tag{7}$$

Inserting these into eq. 5 and integrating from some time t to today, we get:

$$\int_{\rho_\phi}^{\rho_{\phi 0}} d\rho_\phi \frac{1}{\rho_\phi} = \int_z^0 dz' \frac{3[1 + w_\phi(z')]}{(1+z')} \tag{8}$$

We then flip the integration limits on both sides, canceling the negatives. Computing the left hand integral and solving for rho_ϕ we then get the solution

$$\rho_\phi(z) = \rho_{\phi 0} \exp\left\{\int_0^z dz' \frac{3[1 + w_\phi(z')]}{(1+z')}\right\} \tag{9}$$

II. PROBLEM 2

Inserting equations for the density (1) and pressure (2) of the scalar field into the continuity equation we get

$$\dot{\rho}_\phi = -3H\left(\rho_\phi + p_\phi\right) \tag{10}$$

$$= -3H\dot{\phi}^2 \tag{11}$$

We can also find an expression for $\dot{\rho}_\phi$ directly, by taking the time derivative of equation (1):

$$\dot{\rho}_\phi = \frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{12}$$

$$= \dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi} \tag{13}$$

Equating the two expressions, we get the differential equation:

$$\ddot{\phi}\dot{\phi} + 3H\dot{\phi}^2 + V'(\phi)\dot{\phi} = 0 \tag{14}$$

Ignoring the boring case of a static $\phi(t) = \phi_0$, we have $\dot{\phi} \neq 0$. The equation can then be reduced to:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{15}$$

III. PROBLEM 3

Taking the time-derivative of the Hubble parameter

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \tag{16}$$

we recognize both terms on the right hand side from the Friedmann equations. For a flat ($k = 0$) universe the Friedmann equations read

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}\rho \tag{17}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}\left(\rho + 3p\right) \tag{18}$$

Where we have defined $\kappa^2 = 8\pi G$. For a universe with matter, radiation and quintessence we have

$$\rho = \rho_m + \rho_r + \rho_\phi \tag{19}$$

$$p = p_r + p_\phi \tag{20}$$

Inserting the Friedmann equations into (24), and writing the pressures in terms of the equation of state parameter, we get

$$\dot{H} = -\frac{\kappa^2}{6}\left(\rho + 3p\right) - \frac{\kappa^2}{3}\rho \tag{21}$$

$$= -\frac{\kappa^2}{2}\left[\rho + p\right] \tag{22}$$

$$= -\frac{\kappa^2}{2}\left[\rho_m + \rho_r(1 + w_r) + \dot{\phi}\right] \tag{23}$$

$$= -\frac{\kappa^2}{2}\left[\rho_m + \rho_r(1 + w_r) + \dot{\phi}\right] \tag{24}$$

IV. PROBLEM 4

We introduce the dimensionless variables

$$x_1 = \frac{\kappa\dot{\phi}}{\sqrt{6}H} \tag{25}$$

$$x_2 = \frac{\kappa\sqrt{V}}{\sqrt{3}H} \tag{26}$$

$$x_3 = \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H} \tag{27}$$

$$\tag{28}$$

and notice their squares can be rewritten them in terms of the critical density $\rho_c = \frac{3H^2}{\kappa^2}$:

$$x_1^2 = \frac{\frac{1}{2}\dot{\phi}^2}{\rho_c} \tag{29}$$

$$x_2^2 = \frac{V}{\rho_c} \tag{30}$$

$$x_3^2 = \frac{\rho_r}{\rho_c} \tag{31}$$

$$\tag{32}$$

The quintessence and radiation density parameters can then easily be found to be

$$\Omega_\phi = \frac{\frac{1}{2}\dot{\phi} + V}{\rho_c} = x_1^2 + x_2^2 \tag{33}$$

$$\Omega_r = \frac{\rho_r}{\rho_c} = x_3^2 \tag{34}$$

Lastly we want to find an expression for the density parameter for mass. We accomplish this by taking advantage of our model being spatially flat, meaning $\frac{\rho}{\rho_c} = 1$. Expanding ρ we get

$$1 = \frac{\rho_\phi + \rho_r + \rho_m}{\rho_c} \tag{35}$$

$$= \Omega_\phi + \Omega_r + \Omega_m \tag{36}$$

Which we can solve for Ω_m:

$$\Omega_m = 1 - \Omega_\phi - \Omega_m \tag{37}$$

$$= 1 - x_1^2 - x_2^2 - x_3^2 \tag{38}$$

V. PROBLEM 5

Next we want to rewrite equation (24) in terms of the dimensionless variables. Using the definitions of the density parameters and the dimensionless variables we get:

$$\dot{H} = -\frac{\kappa^2}{2}\left[\rho_m + \rho_r(1+w_r) + \dot{\phi}\right] \tag{39}$$

$$= -\frac{\kappa^2\rho_c}{2}\left[\Omega_m + \Omega_r(1+w_r) + 2\Omega_\phi - 2x_2^2\right] \tag{40}$$

$$= -\frac{3H^2}{2}\left[1 + x_1^2 - x_2^2 - x_3^2 + x_3^2(1+w_r)\right] \tag{41}$$

Simplyfing and inserting the radiation equation of state parameter $w_r = 1/3$, we get:

$$\frac{\dot{H}}{H^2} = -\frac{1}{2}\left[3 + 3x_1^2 - 3x_2^2 + x_3^2\right] \tag{42}$$

VI. PROBLEM 6

A. Equation of motion for x_1

Using the product rule:

$$\frac{dx_1}{dN} = \frac{1}{H}\frac{dx_1}{dt} \tag{43}$$

$$= \frac{\kappa}{\sqrt{6}H}\frac{d}{dt}\left(\frac{\dot{\phi}}{H}\right) \tag{44}$$

$$= \frac{\kappa\ddot{\phi}}{\sqrt{6}H^2} - \frac{\kappa\dot{\phi}}{\sqrt{6}H}\frac{\dot{H}}{H^2} \tag{45}$$

From equation (15) we have $\ddot{\phi} = -3H\dot{\phi} - V$. Inserting for $\ddot{\phi}$ and $\dot{H}/H^2$ (eq. 42) we then get:

$$\frac{dx_1}{dN} = -\frac{3\kappa\dot{\phi}}{\sqrt{6}H} - \frac{\kappa V'}{\sqrt{6}H} + \frac{\kappa\dot{\phi}}{\sqrt{6}H}\frac{\dot{H}}{H} \tag{46}$$

$$= -3x_1 + \frac{\sqrt{6}}{2}\lambda x_2^2 + \frac{1}{2}x_1(3 + 3x_1 - 3x_2^2 + 3x_3^3) \tag{47}$$

Where we have defined λ as

$$\lambda = -\frac{V'}{\kappa V} \tag{48}$$

B. Equation of motion for x_2

The equation for x_2 is found in a similar manner, we get:

$$\frac{dx_2}{dN} = \frac{\kappa}{\sqrt{3}H}\frac{d}{dt}\left(\frac{\sqrt{V}}{H}\right) \tag{49}$$

$$= \frac{\kappa}{\sqrt{3}H}\left(\frac{1}{2}\frac{V'\dot{\phi}}{\sqrt{\phi}H} - \sqrt{V}\frac{\dot{H}}{H^2}\right) \tag{50}$$

$$= \frac{\sqrt{6}}{2}\frac{V'}{\kappa V}\frac{\kappa\sqrt{V}}{\sqrt{3}H}\frac{\kappa\dot{\phi}}{\sqrt{6}H} - \frac{\kappa\sqrt{V}}{\sqrt{3}H}\frac{\dot{H}}{H^2} \tag{51}$$

$$= \frac{\sqrt{6}}{2}\lambda x_1 x_2 + \frac{1}{2}x_2(3 + 3x_1 - 3x_2^2 + 3x_3^3) \tag{52}$$

C. Equation of motion for x_3

Similarly, for x_3:

$$\frac{dx_3}{dN} = \frac{\kappa}{\sqrt{3}H}\frac{d}{dt}\left(\frac{\sqrt{\rho_r}}{H}\right) \tag{53}$$

$$= \frac{\kappa}{\sqrt{3}H}\left(\frac{1}{2}\frac{\dot{\rho}_r}{\sqrt{\rho_r}H} - \sqrt{\rho_r}\frac{\dot{H}}{H^2}\right) \tag{54}$$

$$\tag{55}$$

Where $\dot{\rho}_r$ is given by the first Friedmann equation:

$$\dot{\rho}_r = -3H(1 + w_r)\rho_r \tag{56}$$

$$= -4H\rho_r \tag{57}$$

Where we used that $w_r = 1/3$. Inserting the expressions for $\dot{\rho}_r$ and $\dot{H}/H^2$ we finally get:

$$\frac{dx_3}{dN} = -2x_3 + \frac{1}{2}x_3\left(3 + 3x_1 - 3x_2^2 + 3x_3^3\right) \tag{58}$$

VII. PROBLEM 7

Rewriting the expression for λ as a differential equation for V, we get:

$$V' + \kappa\lambda V = 0 \tag{59}$$

For a constant λ this has the solution

$$V = V_0 e^{-\kappa\lambda\phi} \tag{60}$$

We can then find the value of Γ by simple differentiation:

$$\Gamma = \frac{VV''}{(V')^2} \tag{61}$$

$$= \frac{\kappa^2\lambda^2V^2}{(-\kappa\lambda)^2V^2} \tag{62}$$

$$= 1 \tag{63}$$

VIII. PROBLEM 8

For a non-constant λ we have

$$\frac{d\lambda}{dN} = -\frac{1}{\kappa H}\frac{d}{dt}\left(\frac{V'}{V}\right) \tag{64}$$

$$= -\frac{1}{\kappa H}\left(\frac{V''}{V}\dot{\phi} - \frac{(V')^2}{V^2}\dot{\phi}\right) \tag{65}$$

$$= -\frac{\kappa\dot{\phi}}{\sqrt{6}H}\frac{\sqrt{6}}{\kappa^2}\left(\frac{V''}{V} - \frac{(V')^2}{V^2}\right) \tag{66}$$

$$= -x_1\sqrt{6}\left(\frac{V'}{\kappa V}\right)^2\left(\frac{VV''}{(V')^2} - 1\right) \tag{67}$$

$$= -\sqrt{6}\lambda^2(\Gamma - 1)x_1 \tag{68}$$

IX. PROBLEM 9

We see that the quintessence equation of state can be written in terms of the dimensionless variables as:

$$w_\phi = \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} \tag{69}$$

We can then find both the density parameters and the equations of state by integrating the equations of motion of the dimensionless variables. The density parameters plotted against redshift can be found in Figure (1), and the equations of state can be seen in Figure (2).

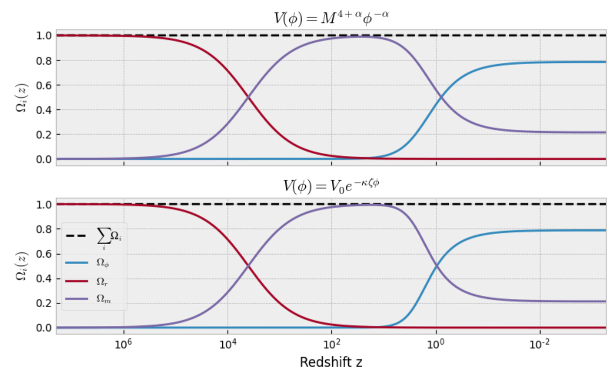

Figure 1. Density parameters for quintessence models using a power law potential (top) and an exponential potential (bottom), plotted against redshift.

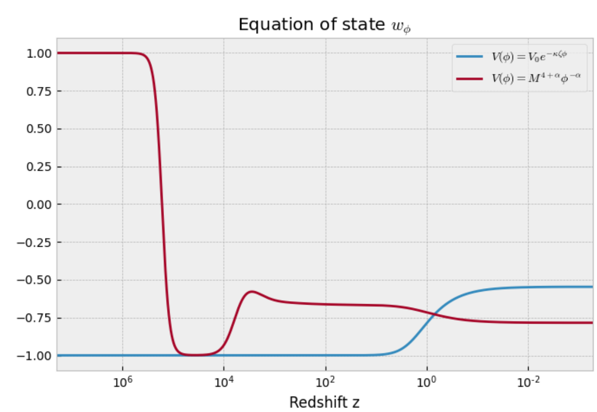

Figure 2. Equation of state for quintessence models using a power law potential and an exponential potential, plotted against redshift.

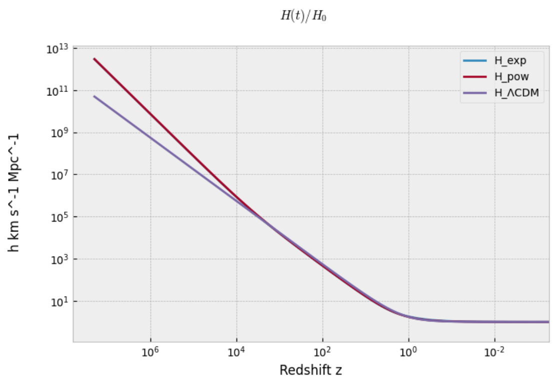

Figure 3. Hubble parameter for both quintessence models and the ΛCDM model plotted against redshift.

Model	$H_0 t_0$
V$_{exp}(z)$	0.973745
V$_{pow}(z)$	0.994003
ΛCDM	0.964101

Table I. The dimensionless age of both quintessence models and the ΛCDM model.

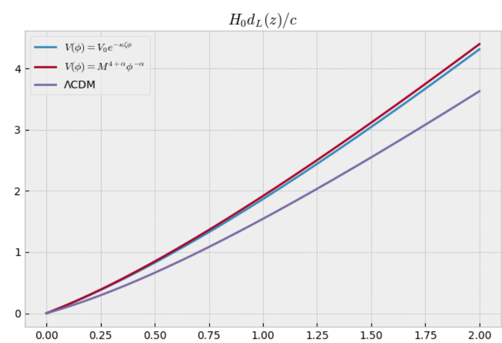

Figure 4. The dimensionless luminosity distance for the two quintessence models and the ΛCDM model, plotted against redshift.

X. PROBLEM 10

XI. PROBLEM 11

XII. PROBLEM 12

XIII. PROBLEM 13

XIV. PROBLEM 14

ACKNOWLEDGMENTS

I would like thank myself for writing this beautiful document.

Model	χ^2
Exponential	1.252e+06
Power law	1.29827e+06

Table II.

REFERENCES

- Reference 1

- Reference 2

Appendix A: Name of appendix

This will be the body of the appendix.

Appendix B: This is another appendix

Tada.