AST3220 - Project 1

(Dated: March 15, 2023)

The scalar field has energy density and pressure

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{0.1}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{0.2}$$

1. PROBLEM 1

Assuming the quintessence field follows the continuity equation

$$\dot{\rho}_\phi = -3H\left(\rho_\phi + p_\phi\right) \tag{1.1}$$

$$= -3\frac{\dot{a}}{a}\left(1 + w_\phi\right)\rho_\phi \tag{1.2}$$

we can begin solving for the density by separating the variables:

$$\frac{1}{\rho_\phi}d\rho_\phi = -\frac{3}{a}\left(1 + w_\phi\right)da \tag{1.3}$$

We then rewrite $a(t)$ in terms of the cosmological redshift of light emitted at some point t in the past:

$$1 + z = \frac{a_0}{a(t)} \quad \Rightarrow a(t) = \frac{a_0}{1+z} \tag{1.4}$$

$$\Rightarrow da = -\frac{a_0}{(1+z)^2}dz \tag{1.5}$$

Inserting these into eq. 1.3 and integrating from some time t to today, we get:

$$\int_{\rho_\phi}^{\rho_{\phi 0}} d\rho_\phi \frac{1}{\rho_\phi} = \int_z^0 dz' \frac{3[1 + w_\phi(z')]}{(1+z')} \tag{1.6}$$

We then flip the integration limits on both sides, canceling the negatives. Computing the left hand integral and solving for ρ_ϕ we then get the solution

$$\rho_\phi(z) = \rho_{\phi 0}\, \exp\left\{\int_0^z dz' \frac{3[1 + w_\phi(z')]}{(1+z')}\right\} \tag{1.7}$$

2. PROBLEM 2

Inserting equations for the density (0.1) and pressure (0.2) of the scalar field into the continuity equation we get

$$\dot{\rho}_\phi = -3H\left(\rho_\phi + p_\phi\right) \tag{2.1}$$

$$= -3H\dot{\phi}^2 \tag{2.2}$$

We can also find an expression for $\dot{\rho}_\phi$ directly, by taking the time derivative of equation (0.1):

$$\dot{\rho}_\phi = \frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{2.3}$$

$$= \dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi} \tag{2.4}$$

Equating the two expressions, we get the differential equation:

$$\ddot{\phi}\dot{\phi} + 3H\dot{\phi}^2 + V'(\phi)\dot{\phi} = 0 \tag{2.5}$$

Ignoring the boring case of a static $\phi(t) = \phi_0$, we have $\dot{\phi} \neq 0$. The equation can then be reduced to:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{2.6}$$

3. PROBLEM 3

Taking the time-derivative of the Hubble parameter

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \tag{3.1}$$

we recognize both terms on the right hand side from the Friedmann equations. For a flat ($k = 0$) universe the Friedmann equations read

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}\rho \tag{3.2}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}\left(\rho + 3p\right) \tag{3.3}$$

Where we have defined $\kappa^2 = 8\pi G$. For a universe with matter, radiation and quintessence we have

$$\rho = \rho_m + \rho_r + \rho_\phi \tag{3.4}$$

$$p = p_m + p_r + p_\phi \tag{3.5}$$

$$= w_r\rho_r + p_\phi \tag{3.6}$$

Since the equation of state parameter for non-relativistic matter is $w_m = 0$. Inserting the Friedmann equations into (3.1), and using $\rho_\phi + p_\phi = \dot{\phi}$, we then get:

$$\dot{H} = -\frac{\kappa^2}{6}\left(\rho + 3p\right) - \frac{\kappa^2}{3}\rho \tag{3.7}$$

$$= -\frac{\kappa^2}{2}\left[\rho + p\right] \tag{3.8}$$

$$= -\frac{\kappa^2}{2}\left[\rho_m + \rho_r + \rho_r w_r + \rho_\phi + p_\phi\right] \tag{3.9}$$

$$= -\frac{\kappa^2}{2}\left[\rho_m + \rho_r(1 + w_r) + \dot{\phi}\right] \tag{3.10}$$

4. PROBLEM 4

Introducing the dimensionless variables

$$x_1 = \frac{\kappa\dot{\phi}}{\sqrt{6}H} \tag{4.1}$$

$$x_2 = \frac{\kappa\sqrt{V}}{\sqrt{3}H} \tag{4.2}$$

$$x_3 = \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H} \tag{4.3}$$

$$\tag{4.4}$$

we notice their squares can be rewritten them in terms of the critical density $\rho_c = \frac{3H^2}{\kappa^2}$:

$$x_1^2 = \frac{\frac{1}{2}\dot{\phi}^2}{\rho_c} \tag{4.5}$$

$$x_2^2 = \frac{V}{\rho_c} \tag{4.6}$$

$$x_3^2 = \frac{\rho_r}{\rho_c} \tag{4.7}$$

$$\tag{4.8}$$

Using the definition $\Omega_i = \frac{\rho_i}{\rho_c}$, the density parameters for quintessence and radiation can be expressed as:

$$\Omega_\phi = \frac{\frac{1}{2}\dot{\phi} + V}{\rho_c} = x_1^2 + x_2^2 \tag{4.9}$$

$$\Omega_r = \frac{\rho_r}{\rho_c} = x_3^2 \tag{4.10}$$

Lastly we want to find an expression for the density parameter for mass. We accomplish this by taking advantage of our model being spatially flat, meaning $\frac{\rho}{\rho_c} = 1$. Expanding ρ we get

$$1 = \frac{\rho_\phi + \rho_r + \rho_m}{\rho_c} \tag{4.11}$$

$$= \Omega_\phi + \Omega_r + \Omega_m \tag{4.12}$$

Which we can solve for Ω_m:

$$\Omega_m = 1 - \Omega_\phi - \Omega_m \tag{4.13}$$

$$= 1 - x_1^2 - x_2^2 - x_3^2 \tag{4.14}$$

5. PROBLEM 5

Next we want to rewrite equation (3.1) in terms of the dimensionless variables. Using the definitions of the density parameters and the dimensionless variables we get:

$$\dot{H} = -\frac{\kappa^2}{2}\left[\rho_m + \rho_r(1+w_r) + \dot{\phi}\right] \tag{5.1}$$

$$= -\frac{\kappa^2\rho_c}{2}\left[\Omega_m + \Omega_r(1+w_r) + 2\Omega_\phi - 2x_2^2\right] \tag{5.2}$$

$$= -\frac{3H^2}{2}\left[1 + x_1^2 - x_2^2 - x_3^2 + x_3^2(1+w_r)\right] \tag{5.3}$$

Simplyfing and inserting the radiation equation of state parameter $w_r = 1/3$, we get:

$$\frac{\dot{H}}{H^2} = -\frac{1}{2}\left[3 + 3x_1^2 - 3x_2^2 + x_3^2\right] \tag{5.4}$$

6. PROBLEM 6

A. Equation of motion for x_1

Using the product rule:

$$\frac{dx_1}{dN} = \frac{1}{H}\frac{dx_1}{dt} \tag{6.1}$$

$$= \frac{\kappa}{\sqrt{6}H}\frac{d}{dt}\left(\frac{\dot{\phi}}{H}\right) \tag{6.2}$$

$$= \frac{\kappa\ddot{\phi}}{\sqrt{6}H^2} - \frac{\kappa\dot{\phi}}{\sqrt{6}H}\frac{\dot{H}}{H^2} \tag{6.3}$$

From equation (2.6) we have $\ddot{\phi} = -3H\dot{\phi} - V$. Inserting for $\ddot{\phi}$ and $\dot{H}/H^2$ (eq. 5.4) we then get:

$$\frac{dx_1}{dN} = -\frac{3\kappa\dot{\phi}}{\sqrt{6}H} - \frac{\kappa V'}{\sqrt{6}H} + \frac{\kappa\dot{\phi}}{\sqrt{6}H}\frac{\dot{H}}{H} \tag{6.4}$$

$$= -3x_1 + \frac{\sqrt{6}}{2}\lambda x_2^2 + \frac{1}{2}x_1(3 + 3x_1 - 3x_2^2 + 3x_3^3) \tag{6.5}$$

Where we have defined λ as

$$\lambda = -\frac{V'}{\kappa V} \tag{6.6}$$

B. Equation of motion for x_2

The equation for x_2 is found in a similar manner, we get:

$$\frac{dx_2}{dN} = \frac{\kappa}{\sqrt{3}H}\frac{d}{dt}\left(\frac{\sqrt{V}}{H}\right) \tag{6.7}$$

$$= \frac{\kappa}{\sqrt{3}H}\left(\frac{1}{2}\frac{V'\dot{\phi}}{\sqrt{\phi}H} - \sqrt{V}\frac{\dot{H}}{H^2}\right) \tag{6.8}$$

$$= \frac{\sqrt{6}}{2}\frac{V'}{\kappa V}\frac{\kappa\sqrt{V}}{\sqrt{3}H}\frac{\kappa\dot{\phi}}{\sqrt{6}H} - \frac{\kappa\sqrt{V}}{\sqrt{3}H}\frac{\dot{H}}{H^2} \tag{6.9}$$

$$= \frac{\sqrt{6}}{2}\lambda x_1 x_2 + \frac{1}{2}x_2(3 + 3x_1 - 3x_2^2 + 3x_3^3) \tag{6.10}$$

C. Equation of motion for x_3

Similarly, for x_3:

$$\frac{dx_3}{dN} = \frac{\kappa}{\sqrt{3}H}\frac{d}{dt}\left(\frac{\sqrt{\rho_r}}{H}\right) \tag{6.11}$$

$$= \frac{\kappa}{\sqrt{3}H}\left(\frac{1}{2}\frac{\dot{\rho}_r}{\sqrt{\rho_r}H} - \sqrt{\rho_r}\frac{\dot{H}}{H^2}\right) \tag{6.12}$$

$$\tag{6.13}$$

Where $\dot{\rho}_r$ is given by the continuity equation:

$$\dot{\rho}_r = -3H(1+w_r)\rho_r \tag{6.14}$$

$$= -4H\rho_r \tag{6.15}$$

Where we used that $w_r = 1/3$. Inserting the expressions for $\dot{\rho}_r$ and $\dot{H}/H^2$ we finally get:

$$\frac{dx_3}{dN} = -2x_3 + \frac{1}{2}x_3\left(3 + 3x_1 - 3x_2^2 + 3x_3^3\right) \tag{6.16}$$

7. PROBLEM 7

Rewriting the expression for λ as a differential equation for V, we get:

$$V' + \kappa\lambda V = 0 \tag{7.1}$$

For a constant λ this has the solution

$$V = V_0 e^{-\kappa\lambda\phi} \tag{7.2}$$

We can then find the value of Γ by simple differentiation:

$$\Gamma = \frac{VV''}{(V')^2} \tag{7.3}$$

$$= \frac{\kappa^2\lambda^2V^2}{(-\kappa\lambda)^2V^2} \tag{7.4}$$

$$= 1 \tag{7.5}$$

8. PROBLEM 8

For a non-constant λ we differentiate $V'V^{-1}$ using the product rule and chain rule. Massaging the equation a bit, we then get:

$$\frac{d\lambda}{dN} = -\frac{1}{\kappa H}\frac{d}{dt}\left(\frac{V'}{V}\right) \tag{8.1}$$

$$= -\frac{1}{\kappa H}\left(\frac{V''}{V}\dot{\phi} - \frac{(V')^2}{V^2}\dot{\phi}\right) \tag{8.2}$$

$$= -\frac{\kappa\dot{\phi}}{\sqrt{6}H}\frac{\sqrt{6}}{\kappa^2}\left(\frac{V''}{V} - \frac{(V')^2}{V^2}\right) \tag{8.3}$$

$$= -x_1\sqrt{6}\left(\frac{V'}{\kappa V}\right)^2\left(\frac{VV''}{(V')^2} - 1\right) \tag{8.4}$$

$$= -\sqrt{6}\lambda^2(\Gamma - 1)x_1 \tag{8.5}$$

9. PROBLEM 9

At long last, we have amassed, after being algebraically harassed, the tools to numerically unsurpassed blast these equations off the jiggermast[1].

We note that we can also express the equation of state parameter for quintessence in terms of the dimensionless variables:

$$w_\phi = \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} \tag{9.1}$$

We can then find both the density parameters (using our results from Problem 4) and the equation of state parameters by integrating the equations of motion of the dimensionless variables numerically. The density parameters plotted against redshift can be found in Figure (1), and the equations of state can be seen in Figure (2).

The density parameters match quite well with what we expect; the early universe is radiation-dominated, while the universe today is governed primarily by quintessence (”dark energy”), with smaller contributions from ordinary matter.

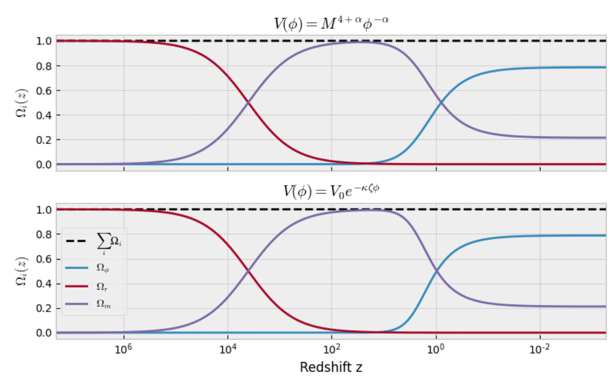

Figure 1. Density parameters for quintessence models using a power law potential (top) and an exponential potential (bottom), plotted against redshift. The sum of all density parameters (which should sum to 1) is plotted as a quality control.

[1] **Jiggermast - noun**: Any small mast on a sailing vessel; especially the mizzenmast of a yawl.

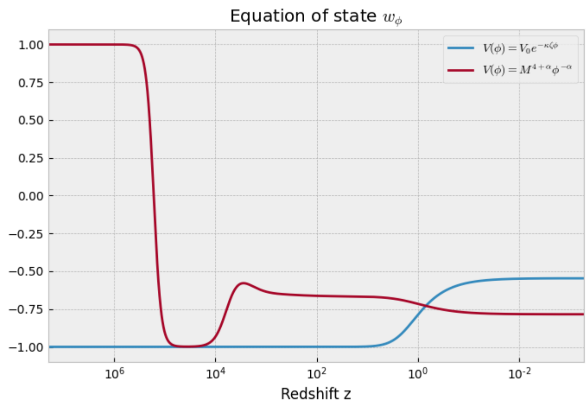

Figure 2. Equation of state for quintessence models using a power law potential and an exponential potential, plotted against redshift.

10. PROBLEM 10

A. Quintessence models

We know that the dimensionless Hubble parameter for the quintessence model can be weritten

$$\frac{H^2}{H_0^2} = \left[\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{\phi 0}e^{I(z)}\right] \tag{10.1}$$

Where $I(z)$ is the integral

$$I(z) = \int_0^z dz' \frac{3[1+w_\phi(z')]}{(1+z')} \tag{10.2}$$

Which can be rewritten in terms of the dimensionless variable $N = -\ln(1+z)$. Using $dz = -(1+z)\ dN$, we get:

$$I(N) = \int_0^N dN'(1+z')\frac{3[1+w_\phi(N')]}{(1+z')} \tag{10.3}$$

$$= -3\int_N^0 dN'[1+w_\phi(N')] \tag{10.4}$$

Inserting this into Eq. (10.1), as well as substituting $(1+z) = e^{-N}$, we get:

$$\frac{H^2}{H_0^2} = \left[\Omega_{m0}e^{-3N} + \Omega_{r0}e^{-4N} + \Omega_{\phi 0}e^{-I(N)}\right] \tag{10.5}$$

Which we evaluate numerically.

B. ΛCDM model

For the ΛCDM model, we use the formula

$$\frac{H^2}{H_0^2} = \Omega_{m0}(1+z)^3 + (1-\Omega_{m0}) \tag{10.6}$$

Model	H_0t_0
$\mathrm{V}_{exp}(z)$	0.973745
$\mathrm{V}_{pow}(z)$	0.994003
ΛCDM	0.964101

Table I. The dimensionless age of both quintessence models and the ΛCDM model.

The dimensionless Hubble parameters for both quintessence models and the ΛCDM model are plotted in Figure (3). The most surprising result here is probably that the two quintessence models seem to have such identical Hubble parameters, in spite of their equations of state parameters being quite different. We also see that the quintessence models match ΛCDM quite well for low redshift, which is a good sign. They do however start deviatiing noticably for redshift $z \sim 10^4$ and greater.

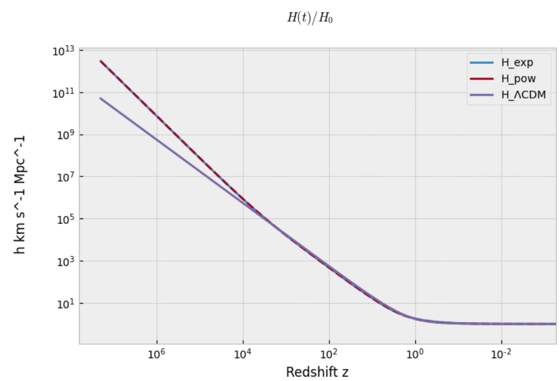

Figure 3. Hubble parameter for both quintessence models and the ΛCDM model plotted against redshift.

11. PROBLEM 11

We calculate the dimensionless age by evaluating the integral

$$H_0t_0 = \int_0^\infty \frac{dz}{(1+z)H/H_0} \tag{11.1}$$

Once again substituting in the dimensionless variable $N = -\ln(1+z)$, using $dz = -(1+z)\ dN$, the integral becomes:

$$H_0t_0 = \int_{-\infty}^0 \frac{dN}{H/H_0} \tag{11.2}$$

Approximating this by integrating over the finite arrays of H/H_0 we get the dimensionless times in Table (I)

Potential	χ^2
Exponential	1.252e+06
Power law	1.29827e+06

Table II.

12. PROBLEM 12

For $k = 0$, the luminosity distance as a function of redshift can be written

$$d_L(z) = \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{H(z')/H_0} \tag{12.1}$$

such that the dimensionless becomes:

$$\frac{H_0 d_L}{c} = (1+z) \int_0^z \frac{dz'}{H(z')/H_0} \tag{12.2}$$

Plotting these for $0 \leq z \leq 2$ in both quintessence models and the λCDM model we get the results shown in figure 4

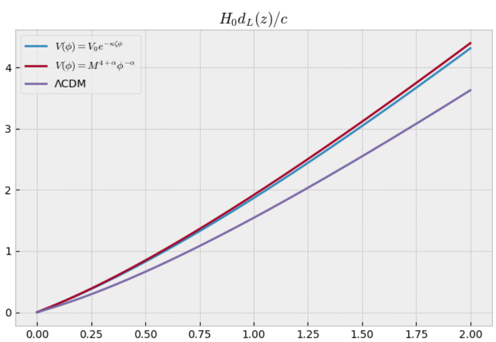

Figure 4. The dimensionless luminosity distance for the two quintessence models and the ΛCDM model, plotted against redshift.

13. PROBLEM 13

Calculating χ^2 for the quintessence models, we get the values shown in Table (II). We see that the exponential potential displays a slightly lower value for χ^2.

14. PROBLEM 14

To determine the optimal value for Ω_{m0} we calculate χ^2 for values in the range $[0, 1]$. With a resolution of 10000 points, we get the optimal value

$$\Omega_{m0} \approx 0.2989 \tag{14.1}$$

with $\chi^2 = 29.74$. (5 orders of magnitude below than the quintessence models!)

ACKNOWLEDGMENTS

I would like to give a huge thanks to Fysikkforeningen for the Bunyanesque[2] amount of coffee they have supplied me with.

REFERENCES

- Reference 1

[2] **Bunyanesque, adjective**: of immense size or stature, as ascribed to Paul Bunyan or to the other characters, exploits, etc., in the legends about him.

(Paul Bunyan is a giant lumberjack and folk hero in American and Canadian folklore)

Appendix A: Name of appendix

This will be the body of the appendix.