AST3220 - Project 3:
Inflation without approximation

Candidate nr. 14

(Dated: June 14, 2023)

1. PROBLEM A)

Using the convention of represent As H_i neccesarily has the same dimensions as H it is clear that h must be dimensionless:

$$[h] = [H][H_i]^{-1} = 1 \tag{1.1}$$

The hubble parameter has dimension velocity per distance, which can be written

$$[H] = (\mathsf{L}\mathsf{T}^{-1})\mathsf{L}^{-1} = \mathsf{T}^{-1} \tag{1.2}$$

so that τ is dimensionless as well:

$$[\tau] = [H][t] = \mathsf{T}^{-1}\mathsf{T} = 1 \tag{1.3}$$

Both ϕ and the Planck energy has units of energy, making ψ dimensionless as well:

$$[\psi] = [\phi][E_p]^{-1} = \mathsf{M}\mathsf{L}^2\mathsf{T}^{-2}(\mathsf{M}\mathsf{L}^2\mathsf{T}^{-2})^{-1} = 1 \tag{1.4}$$

Lastly, we rewrite the potential v using the definiton $E_p^2 = \hbar c^5/G$, so that

$$v = \frac{\hbar c^3}{H_i^2 E_p^2} = \frac{1}{H_i^2}\frac{G}{c^2}V \tag{1.5}$$

Using the definiton of H_i, we then see that

$$[v] = [H_i]^{-1}[Gc^{-2}V] = [H_i]^{-1}[H_i] = 1 \tag{1.6}$$

so v is dimensionless as well

2. PROBLEM B)

A. Hubble parameter and scale factor

Using the definitons of the dimensionless variables and applying the chain rule, we see that

$$\begin{align}
\frac{d}{d\tau}\left(\ln\frac{a}{a_i}\right) &= \frac{dt}{d\tau}\frac{d}{dt}\left(\ln\frac{a}{a_i}\right) \tag{2.1}\\
&= \frac{1}{H_i}\frac{\dot{a}}{a} \tag{2.2}\\
&= \frac{H}{H_i} \tag{2.3}\\
&= h \tag{2.4}
\end{align}$$

B. Continuity equation (name???)

Similarly we continue using the chain rule to rewrite $\dot{\phi}$, $\ddot{\phi}$ and V' in terms of τ and ψ:

$$\begin{align}
\frac{d\phi}{dt} &= \frac{d\tau}{dt}\frac{d\phi}{d\psi}\frac{d\psi}{d\tau} = H_i E_p \frac{d\psi}{d\tau} \tag{2.5}\\
\frac{d^2\phi}{dt^2} &= \frac{d\tau}{dt}\frac{d}{d\tau}\frac{d\phi}{dt} = H_i^2 E_p \frac{d^2\psi}{d\tau^2} \tag{2.6}\\
\frac{dV}{d\phi} &= \frac{d\psi}{d\phi}\frac{dV}{dv}\frac{dv}{d\psi} = \frac{1}{E_p}\frac{H_i^2 E_p^2}{\hbar c^3}\frac{dv}{d\psi} \tag{2.7}
\end{align}$$

Which can be inserted to the (????) equation, giving

$$H_i^2 E_p \frac{d^2\psi}{d\tau^2} + 3HH_i E_p \frac{d\psi}{d\tau} + H_i^2 E_p \frac{dv}{d\psi} = 0 \tag{2.8}$$

Dividing both sides by $H_i^2 E_p$ and using the definition $h = H/H_i$, this reduces to

$$\frac{d^2\psi}{d\tau^2} + 3h\frac{d\psi}{d\tau} + \frac{dv}{d\psi} = 0 \tag{2.9}$$

1. *Hubble parameter*

Lastly, we rewrite the Hubble parameterin terms of the dimensionless quantities. We have

$$\begin{align}
H^2 &= \frac{8\pi G}{3c^2}\left[\frac{1}{2\hbar c^3}\dot{\phi} + V(\phi)\right] \tag{2.10}\\
&= \frac{8\pi G}{3c^2}\left[\frac{H_i^2 E_p^2}{2\hbar c^3}\left(\frac{d\psi}{d\tau}\right)^2 + V(\phi)\right] \tag{2.11}\\
&= \frac{8\pi G}{3c^2}\frac{H_i^2 E_p^2}{\hbar c^3}\left[\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v(\psi)\right] \tag{2.12}
\end{align}$$

Where we used equation (2.5) and the definition of v. Next, inserting the definition of the planck energy

$$E_p^2 = \frac{\hbar c^5}{G} \tag{2.13}$$

and dividing both sides by H_i, we finally get:

$$h^2 = \frac{8\pi}{3}\left[\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v(\psi)\right] \tag{2.14}$$

3. PROBLEM C)

Read up on this shit

4. PROBLEM D)

During slow-roll, the number of remaining e-folds until inflation ends can be calculated as

$$N = \frac{8\pi}{E_p^2}\int_{\phi_{\text{end}}}^{\phi} \frac{V}{V'}\, d\phi \tag{4.1}$$

Where we make the substitution $d\phi = E_p d\psi$ and have

$$\frac{V}{V'} = \frac{1}{2}\phi = \frac{E_p}{2}\psi \tag{4.2}$$

Giving

$$N = 4\pi\int_{\psi_{\text{end}}}^{\psi} \psi\, d\psi \tag{4.3}$$

$$= 2\pi\left(\psi^2 - \psi_{end}^2\right) \tag{4.4}$$

where ψ_{end} is given by taking $\epsilon(\phi_{\text{end}}) = 1$ as the end of inflation:

$$\epsilon(\phi_{\text{end}}) = \frac{E_p^2}{16\pi}\left(\frac{V'}{V}\right)^2 = \frac{1}{16\pi}\left(\frac{dv}{d\psi}\right) \tag{4.5}$$

$$= \frac{1}{4\pi\psi_{\text{end}}^2} \tag{4.6}$$

$$= 1 \tag{4.7}$$

$$\tag{4.8}$$

Solving for ψ_end this gives

$$\Rightarrow \qquad \psi_{\text{end}}^2 = \frac{1}{4\pi} \tag{4.9}$$

Inserting this, the number of remaining e-folds is

$$N = 2\pi\psi^2 - \frac{1}{2} \tag{4.10}$$

The number of remaining e-folds at the initial time $t = t_i$ is the total number of e-folds N_tot. Evaluating $N(t)$ at t_i and solving for the initial field value ψ_i we then get:

$$\psi_i = \sqrt{\frac{1}{2\pi}\left(N_{tot} + \frac{1}{2}\right)} \tag{4.11}$$

Which evaluates to $\psi_i \approx 8.925$ for 500 total e-folds of inflation.

5. PROBLEM E) & F)

To solve the equations of motion of the field, we rename $\xi = \frac{d\psi}{d\tau}$ so equation 2.9 can instead be written as a set of two first order equations:

$$\frac{d\psi}{d\tau} = \xi \tag{5.1}$$

$$\frac{d\xi}{d\tau} = -3h(\xi,\psi)\xi - \frac{d}{d\psi}v(\psi) \tag{5.2}$$

where h is then given by

$$h^2 = \frac{8\pi}{3}\left[\frac{1}{2}\xi^2 + v(\psi)\right] \tag{5.3}$$

and $v(\psi)$ is given by our choice of potential. Equations 5.1 and 5.2 are then easily solved using a numerical integrator. A plot of the resulting field ψ plotted against τ is attached in Figure (1), along with the slow-roll solution.

With the solutions ξ and ψ we can easily find $\ln\left(\frac{a}{a_i}\right)$ by numerically evaluating the integral

$$\ln\left(\frac{a}{a_i}\right) = \int_0^t H dt \tag{5.4}$$

$$= \int_0^\tau h(\xi,\psi) d\tau \tag{5.5}$$

The results are plotted in Figure (5), along with the slow-roll solution

Based on the lectures, we expect the field to decay until it reaches ψ_{end}. When it reaches ψ_{end} it should then start behaving like a damped oscillator centered around $\psi = 0$. This is in agreement with the numerical solution (Figure 1), which starts oscillating at $\tau \approx 1000$. This is also when the slow-roll approximation starts deviating significantly from the exact solution, as it instead continues decaying at a constant rate.

This is around the same time the slow-roll approximation for the scale factor deviates from the exact solution (Figure 5), as one might expect. BLABLABLABLA

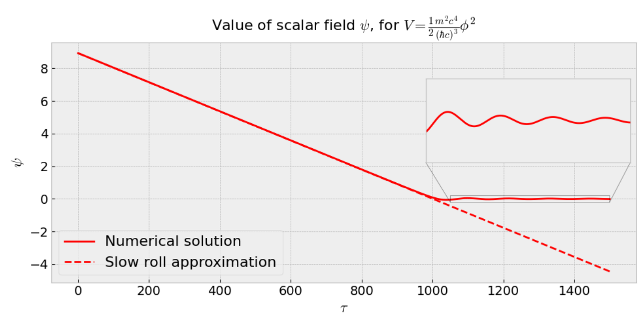

Figure 1. The dimensionless scalar field ψ for the quadratic potential, along with it's approximation in the slow-roll regime. Plotted against the dimensionless time τ.

6. PROBLEM G)

SLOW ROLL PARAMETERS AGAINST TAU

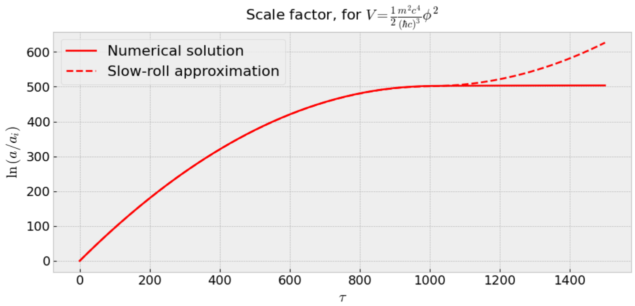

Figure 2. The logarithm of the scale factor ln $\left(\frac{a}{a_i}\right)$ $foraquadraticpotential, alongwiththeslow - rollapproximation.Plottedasafunctionofthedimensionlesstime\tau$

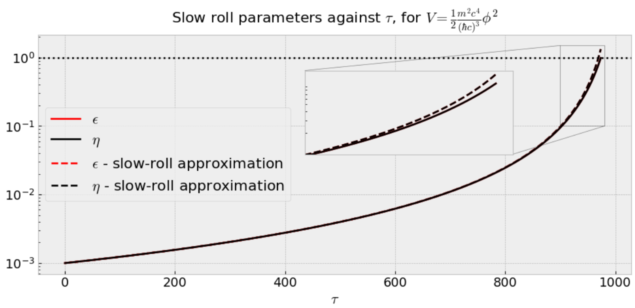

Figure 3. The slow-roll parameters ϵ and η for a quadratic potential, plotted against dimensionless time τ.

7. PROBLEM H)

We now want to express the equation of state parameter in terms of the dimensionless variables. The equation of state of the scalar field is given by

$$w_\phi = \frac{p_\phi}{\rho_\phi c^2} = \frac{\frac{1}{2\hbar c^3}\left(\frac{d\phi}{dt}\right)^2 - V}{\frac{1}{2\hbar c^3}\left(\frac{d\phi}{dt}\right)^2 + V} \tag{7.1}$$

From before, we know that

$$\frac{d\phi}{dt} = H_i E_p \frac{d\psi}{d\tau} \tag{7.2}$$

and by definition we have

$$V = \frac{H_i^2 E_p^2}{\hbar c^3} v \tag{7.3}$$

Inserting these into the equation of state, we then get

$$w_\phi = \frac{\frac{H_i^2 E_p^2}{2\hbar c^3}}{\frac{H_i^2 E_p^2}{2\hbar c^3}} \frac{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 - v}{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v} = \frac{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 - v}{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v} \tag{7.4}$$

8. PROBLEM I)

In the slow-roll regime we expect a small $\frac{d\psi}{d\tau}$, giving $\left(\frac{d\psi}{d\tau}\right)^2 \approx 0$. The equation of state then becomes

$$w_\phi \approx \frac{-v}{v} = -1 \tag{8.1}$$

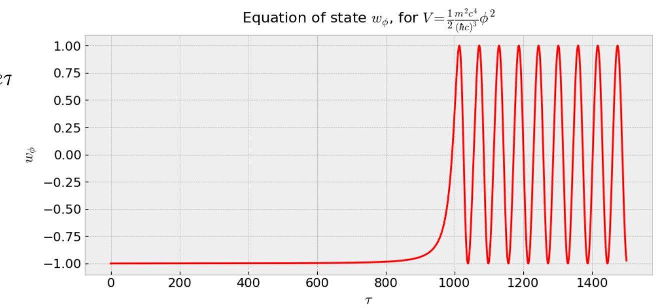

Figure 4. The scalar field ϕ's equation of state, given a quadratic potential, plotted against the dimensionless time τ.

9. PROBLEM J)

Plotting the slowroll parameters against N we get the plot shown in Figure (5).

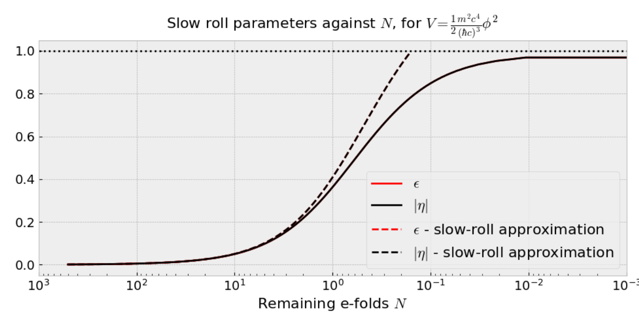

Figure 5.

10. PROBLEM K)

The predicted curve in the n-r plane for $N \in [50, 60]$ can be seen in Figure (6)

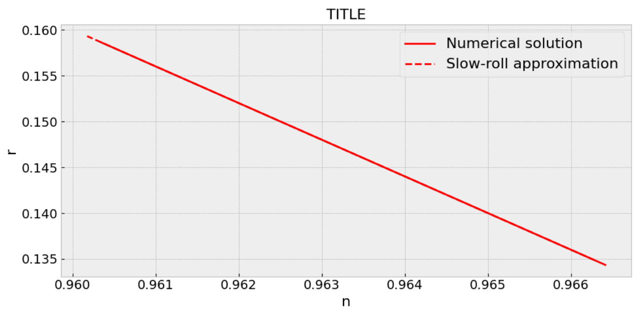

Figure 6. The spectral index n and the tensor-to-scalar ratio of the quadratic potential, for $N \in [50, 60]$.

11. PROBLEM L)

Substituting $y = -\sqrt{\frac{2}{3}}\frac{\phi}{M_p}$, we can write

$$V(\phi) = \frac{3M^2M_P^2}{4}\left(1 - e^y\right)^2 \tag{11.1}$$

We also have

$$y' = -\sqrt{\frac{2}{3}}\frac{1}{M_p}, \quad y'' = 0 \tag{11.2}$$

We then get

$$V' = -\frac{3M^2M_p^2}{2}\left(1 - e^y\right) \cdot e^y \cdot y' \tag{11.3}$$

$$= \sqrt{\frac{2}{3}}\frac{2}{M_p}\frac{e^y}{1 - e^y}V \tag{11.4}$$

And, differentiating equation (11.3), we find

$$V'' = y'\frac{3M^2M_p^2}{2}\frac{d}{d\phi}\left(e^{2y} - e^y\right) \tag{11.5}$$

$$= y'^2\frac{3M^2M_p^2}{2}\left(2e^{2y} - e^y\right) \tag{11.6}$$

$$= \frac{2}{3}\frac{3M^2}{2}\left(2e^{2y} - e^y\right) \tag{11.7}$$

$$= \frac{4}{3}\frac{1}{M_p^2}\frac{\left(2e^{2y} - e^y\right)}{\left(1 - e^y\right)^2}V \tag{11.8}$$

We can then find the slow roll parameters. Using $M_p = \frac{E_p^2}{8\pi}$, these can be written

$$\epsilon = \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2 \tag{11.9}$$

$$\eta = M_p^2\frac{V''}{V} \tag{11.10}$$

By insertion, we then get

$$\epsilon = \frac{4}{3}\frac{e^{2y}}{\left(1 - e^y\right)^2} \tag{11.11}$$

$$\eta = \frac{4}{3}\frac{\left(2e^{2y} - e^y\right)}{\left(1 - e^y\right)^2} \tag{11.12}$$

12. PROBLEM M)

The governing equations are then solved just like before, only exchanging the quadratic potentiakl with the Starobinsky potential. The resulting dimensionless field ψ is then shown in Figure (7), and the scale factor can be seen in Figure (8).

From Figure (8) we note that the logarithm of the scale factor goes as

$$\ln\left(\frac{a}{a_i}\right) = \tau \tag{12.1}$$

Giving an exponential scale factor during inflation

$$a = a_i e^{\tau} = a_i e^{H_i t} \tag{12.2}$$

In other words the Starobinsky potential gives rise to inflation following the de Sitter model.

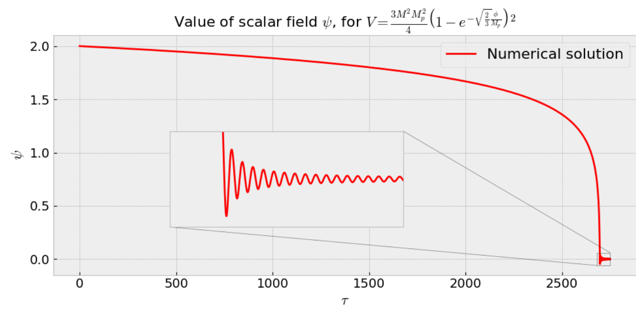

Figure 7. The value of the dimensionless field ψ for Starobinsky inflation, plotted as a function of the dimensionless time τ.

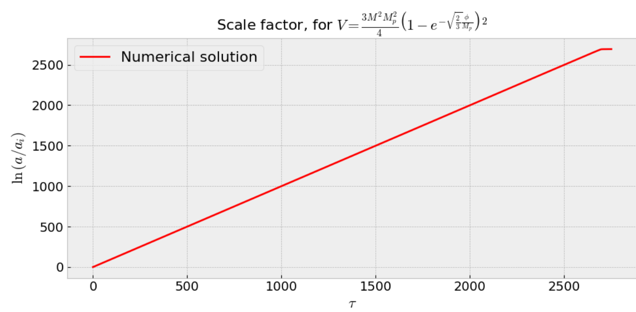

Figure 8. The logarithm of the scale factor $\ln\left(\frac{a}{a_i}\right)$ for Starobinsky inflation. Plotted against the dimensionless time τ.

13. PROBLEM N)

Plotting the slowroll parameters against N we get the plot shown in Figure (9). The predicted curve in the n-r plane for $N \in [50, 60]$ can be seen in Figure (10)

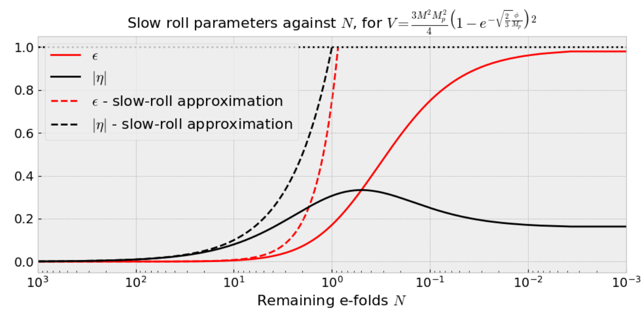

Figure 9. The slow-roll parameters ϵ and η plotted against N, the number e-folds remaining before the end of inflation.

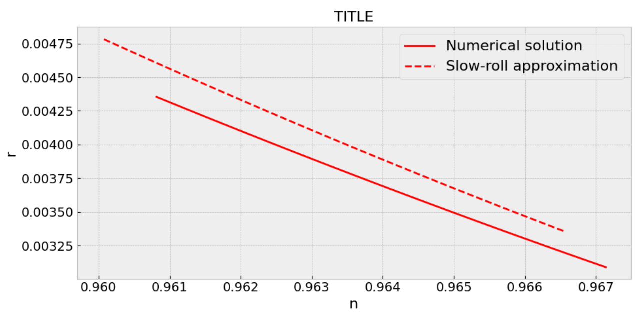

Figure 10. The spectral index n and the tensor-to-scalar ratio of the Starobinsky potential, for $N \in [50, 60]$.

14. PROBLEM O)

A. Remaining e-folds N

From before, we have that

$$\frac{V}{V'} = \sqrt{\frac{3}{2}}\frac{M_p}{2}\frac{1-e^y}{e^y} \tag{14.1}$$

$$= \sqrt{\frac{3}{16\pi}}\frac{E_p}{2}\left(e^{-y}-1\right) \tag{14.2}$$

$$\tag{14.3}$$

Using $y = -\sqrt{\frac{16\pi}{3}}\frac{\phi}{E_p}$, the remaining number of e-foldings becomes

$$N = \frac{8\pi}{E_p^2}\int_{\phi_{end}}^{\phi}\frac{V}{V'}\,d\phi \tag{14.4}$$

$$= -\frac{8\pi}{E_p}\sqrt{\frac{3}{16\pi}}\int_{y_{end}}^{y}\frac{V}{V'}\,d\tilde{y} \tag{14.5}$$

$$= -\frac{3}{4}\int_{y_{end}}^{y} e^{-\tilde{y}}-1\,d\tilde{y} \tag{14.6}$$

$$= \frac{3}{4}\left[e^{-\tilde{y}}+\tilde{y}\right]_{\tilde{y}=y_{end}}^{y} \tag{14.7}$$

Where we on the last line applied the slow-roll criterion

$$\phi \gg \phi_{end} \quad \Rightarrow \quad y \gg y_{end} \tag{14.8}$$

Such that

$$N \approx \frac{3}{4}\left(e^{-y}+y\right) \tag{14.9}$$

Assuming the field is on the order $\phi \sim \phi_i = 2$, we have $y \approx -8$, giving

$$e^{-y} \gg y \tag{14.10}$$

With which we use to further approximate N as

$$N \approx \frac{3}{4}e^{-y} \tag{14.11}$$

B. Slow-roll parameters

The slow-roll parameters are found in a similar manner. Using

$$e^y \ll 1 \tag{14.12}$$

we have that $1-e^y \approx 1$, so that ϵ can be approximated as

$$\epsilon = \frac{4}{3}\frac{e^{2y}}{(1-e^y)^2} \approx \frac{4}{3}e^{2y} = \frac{3}{4}\frac{1}{N^2} \tag{14.13}$$

Similarly, η becomes

$$\eta = \frac{4}{3}\frac{(2e^{2y}-e^y)}{(1-e^y)^2} \approx \frac{4}{3}(2e^{2y}-e^y) \tag{14.14}$$

WeMultiplying equation (14.12) by e^y on both sides, we get

$$e^{2y} \ll e^y \tag{14.15}$$

and η is further simplified to

$$\eta \approx -\frac{4}{3}e^y = -\frac{1}{N} \tag{14.16}$$

C. Tensor-to-scalar ratio and spectral index

To approximate the tensor-to-scalar ratio r we simply insert our approximation of ϵ, giving

$$r = 16\epsilon \approx \frac{12}{N^2} \tag{14.17}$$

Similarly, the spectral index is approximated

$$n = 1-6\epsilon+2\eta \approx 1-\frac{9}{2}\frac{1}{N^2}-\frac{2}{N} \tag{14.18}$$

$$\approx 1-\frac{2}{N} \tag{14.19}$$

where we used that N will be a large number during slow-roll, making the second order term negligble.

15. PROBLEM P)

The latest constraints from observations (REFERENCE) suggests an upper limit on the tensor-to-scalar ratio $r < 0.044$. This is clearly not consistent with the ϕ^2 potential (10) READ STUFF

ACKNOWLEDGMENTS

I would like thank myself for writing this beautiful document.

REFERENCES

[1] reference