Robotics, Lab 1

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1 Introduction

This report addresses the work done in Lab 1, in the course Robotics at Instituto Superior Técnico. The main objective is to investigate direct and inverse kinematics of a Serial Manipulator. The objectives have been to develop/improving skills in use of homogeneous matrix transformations to represent positions and orientations of the rigid bodies forming a robot, and synthesis of kinematic models for serial manipulators. The work is divided in the two respective parts: direct- and inverse kinematics.

2 Serial Manipulator

The serial manipulator used in this lab is shown in figure 1. It has 6 degrees-of-freedom (dof). Measurement used in the lab has been carried out by inspecting the manipulator directly. Every joint is assumed not to have physical limits regarding the rotation.

3 Direct kinematics

A function written in MATLAB has been created. The input of the function is a set of 6 angles, one for each dof. The output is the position given in Cartesian coordinates, and the orientation given in Euler angles of the end-effector shown in figure 2, relative to a reference frame with the origin located at the base of the robot.

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27th of March, 2019

Figure 1. Serial manipulator

3.1 Allocation of reference frames

The frames are illustrated in figure 3. The rotation around each axis is assumed to be positive counter clockwise. To model the links and joints, and the transformation between them, the Denavit-Hertenberg convention (DH) [1] is used. It is assumed that dimensions not shown in figure 2 can be safely assumed within reasonable, nonzero, bounds.

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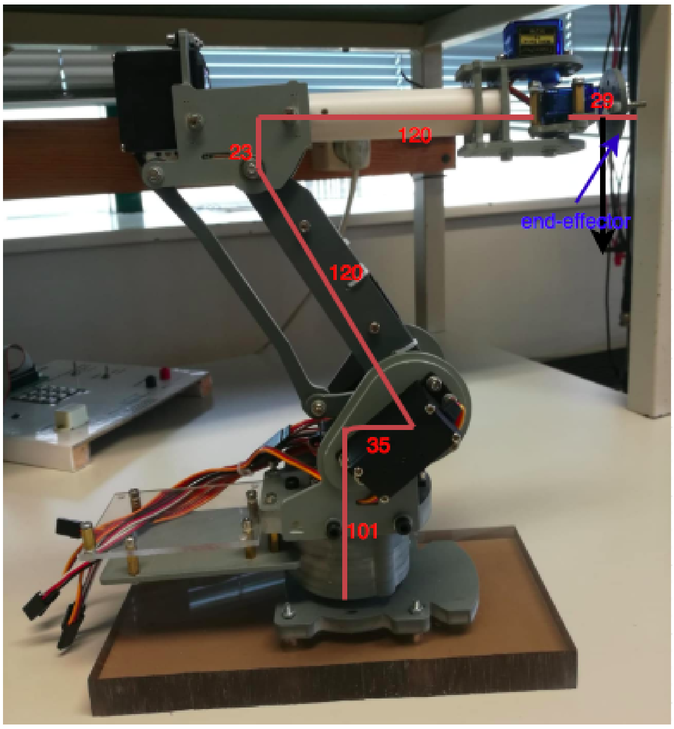

Figure 2. Measurements and end-effector

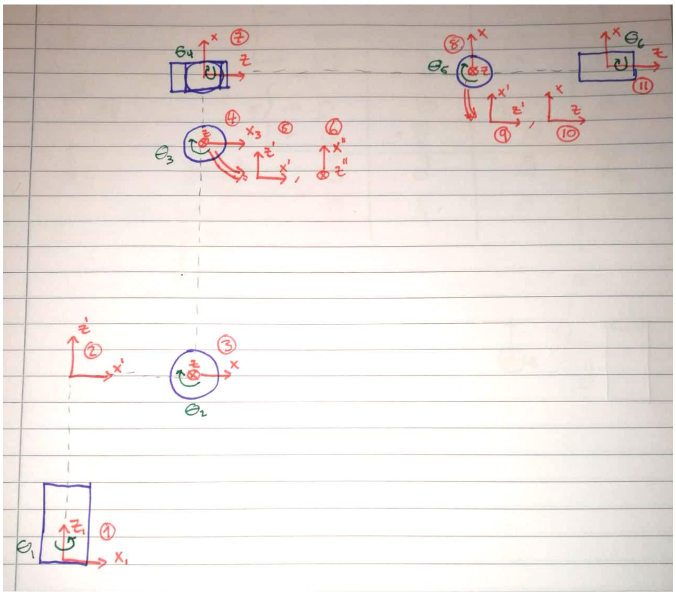

Figure 3. The figure illustrates the assigned reference frame to each joint and the auxiliary ones. The auxiliary frames are marked with dots. θ_i is the input variables.

3.2 Denavit-Hertenberg parameters

In order to find the pose of the end-effector, the DH approach assigns a reference frame to each joint. The frames has been assigning by assuming zero value for the joints. The z-axis will always be the axis of rotation for each joint. DH uses four parameters to describe each frame relative to the previous frame. The parameters are:

- a_i - distance from z_i to z_{i+1} along x_i
- α_i - angle from z_i to z_{i+1} around x_i
- d_i - distance from x_{i-1} to x_i along z_i
- θ_i - angle from x_{i-1} to x_i around z_i

In addition to the frame of each joint, four auxiliary frames where applied to ease the complexity of deciding the DH parameters, and the computations needed to be done. The auxiliary frames are marked with dots and are illustrated in figure 3. The DH parameters are shown in table 1.

i	a_i [mm]	α_i [rad]	d_i [mm]	θ_i [rad]
1′	0	0	101	θ_1
2	35	$\frac{\pi}{2}$	0	θ_2
3′	0	$-\frac{\pi}{2}$	120	0
4	0	$\frac{\pi}{2}$	0	θ_3
5′	0	0	0	$\frac{\pi}{2}$
6	23	$\frac{\pi}{2}$	0	θ_4
7′	0	0	120	0
8	0	$-\frac{\pi}{2}$	0	θ_5
9′	0	$\frac{\pi}{2}$	0	0
10	0	0	29	θ_6

Table 1
Denavit-Hertenberg parameters. Auxiliary frames are marked with a dot.

3.3 Link transformation

The motion of the robot and the relation between different parts of the serial manipulator can be expressed through a series of homogeneous transformation matrices that links the last reference frame to the first. The transformation matrix are:

$$T_i^{i-1} = \begin{pmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & a_{i-1} \\ s_{\theta_i}c_{\alpha_{i-1}} & c_{\theta_i}c_{\alpha_{i-1}} & -s_{\alpha_{i-1}} & -s_{\alpha_{i-1}}d_i \\ s_{\theta_i}s_{\alpha_{i-1}} & c_{\theta_i}s_{\alpha_{i-1}} & c_{\alpha_{i-1}} & c_{\alpha_{i-1}}d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (1)$$

where, α_{i-1}, θ_i, a_i and d_i is the DH parameters described in 3.2, and s and c is the sinus and cosinus of the given angle. By directly replacing the DH parameters, the transformation matrix between each reference frame is obtained. The transformation T_{i-1}^{i} from the first to the last reference frame are thus:

$$T_6^0 = T_1^0 T_2^1 T_3^2 T_4^3 T_5^4 T_6^5 \tag{2}$$

The product between the transformation matrix T_{i-1}^{i} and the previous position:

$$p_i = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \tag{3}$$

where x, y and z are the Cartesian coordinates, will give the new obtained position:

$$p_i = T_i^{i-1} p_{i-1} \tag{4}$$

The Euler angles can be found from the transformation matrix T_1^6. The upper left 3x3 matrix constitute the rotation matrix, and by inspection the angle offset of the end-effector can be found. α is the rotation around the z-axis, β around the y-axis, and γ around the x-axis. Thus we have:

$$\beta = atan2(-T_{31}, \pm\sqrt{T_{11}^2 + T_{21}^2}) \tag{5}$$

$$\alpha = atan2(\frac{T_{21}}{cos(\beta)}, \frac{T_{11}}{cos(\beta)}) \tag{6}$$

$$\gamma = atan2(\frac{T_{32}}{cos(\beta)}, \frac{T_{33}}{cos(\beta)}) \tag{7}$$

where T_{ij} represent the element at row i and column j in the transformation matrix.

4 INVERSE KINEMATICS

The second part of the Lab was to write a MATLAB function capable of computing the inverse kinematics of the serial manipulator. The function accepts a position (x, y, z) and the orientation (α, β, γ), of the end-effector and should return all the corresponding solutions in the space of the joint angles. The first 3 angles are found using the geometric method, while the last 3 angles are found using the algebraic method. But first off all, we need to calculate the transformation matrix from the base-tool to the provided input. The matrix are:

$$T_{bt} = \begin{pmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma & x \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - s_\alpha c_\gamma & y \\ -s_\beta & s_\beta c_\gamma & c_\beta c_\gamma & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

where x, y, z, α, β and γ is the given input parameters. The "XYZ" convention is used.

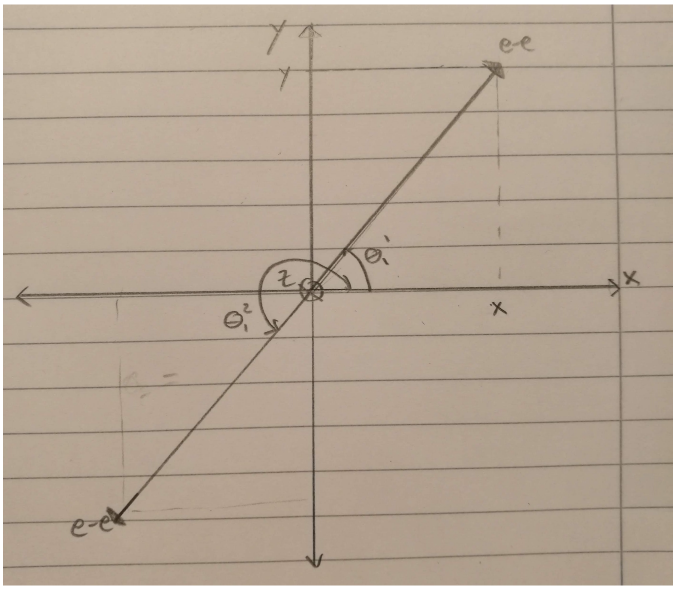

Figure 4. Example of positioning for robot arm. Different measures used to compute angle θ_1 is visualized.

4.1 θ_1 ,θ_2 and θ_3 using the geometric method

We start by finding the first angle, θ_1, using the input of x and y, which results in two solutions for the first angle:

$$\theta_1^1 = \arctan(x/y) \tag{9}$$

$$\theta_1^2 = \arctan(x/y) + \pi \tag{10}$$

The remaining angles will depend on the solution of the first joint. To simplify the calculations, we move our coordinate system to the second joint and discard the tip of the end-effector, resulting in only considering joint 2 to 5. This is done through a coordinate change on the input. (px, py, pz) will be the solution where joint 2 is closest, and (qx, qy, qz) furthest

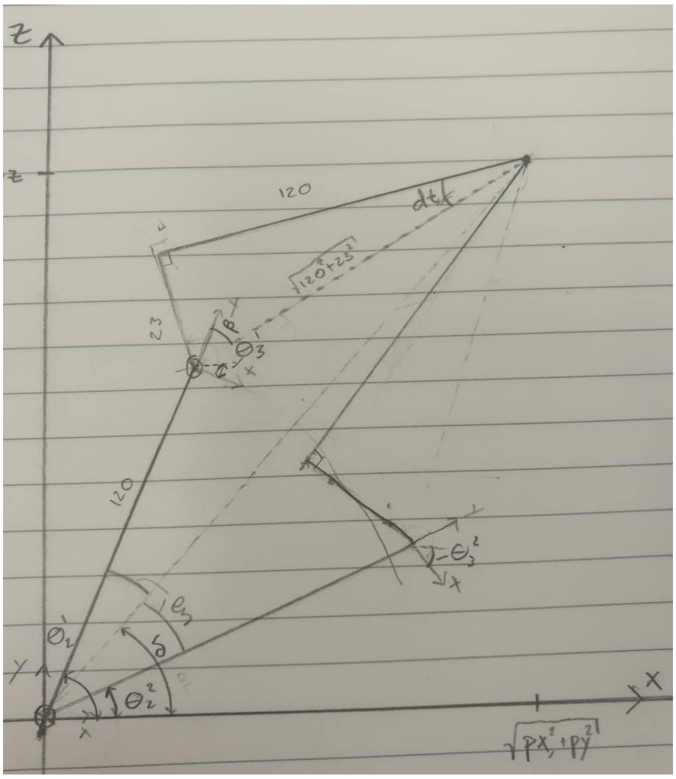

Figure 5. Example of positioning for robot arm. Different measures used to compute angle θ_2 and θ_3 is visualized.

away, relative to (x, y, z). It is assumed that the small distance from joint 3 to 4 is perpendicular to the upper arm.

$$\begin{aligned} px &= x - 23 * T_{bt}(1,3) - 35 * \cos(\theta_1^1), \\ py &= y - 23 * T_{bt}(2,3) - 35 * \sin(\theta_1^1), \\ pz &= z - 23 * T_{bt}(3,3) - 101 \end{aligned} \tag{11}$$

$$\begin{aligned} qx &= x - 23 * T_{bt}(1,3) - 35 * \cos(\theta_1^2), \\ qy &= y - 23 * T_{bt}(2,3) - 35 * \sin(\theta_1^2), \\ qz &= z - 23 * T_{bt}(3,3) - 101 \end{aligned} \tag{12}$$

where $T_{bt}(x, y)$ is the respective element in T_{bt}. Used measures are visualized in figure 2. The small angle provided by the distance from joint 3 to 4 is compensated for by dt. See figure 5. θ_3 can be found from the cosine sentence, revealing two solutions for θ_3 with respect to (px, py ,pz):

$$\theta_3^1 = \frac{\pi}{2} + \arccos \frac{px^2 + py^2 + pz^2 - 2 * 120^2 - 23^2}{2 * 120 * \sqrt{120^2 + 23^2}} - dt \tag{13}$$

$$\theta_3^2 = \frac{\pi}{2} - \arccos \frac{px^2 + py^2 + pz^2 - 2 * 120^2 - 23^2}{2 * 120 * \sqrt{120^2 + 23^2}} - dt \tag{14}$$

where arccos is the inverse cosine and dt is the compensated angle described above. We will have 2 equal solutions for (qx, qy, qz), resulting in 4 different solutions for θ_3. θ_2 is found by considering the angle from joint 2 to (px, py, pz). This angle can be found as:

$$\delta = \arctan \frac{pz^2}{\sqrt{px^2 + py^2}} \tag{15}$$

where arctan is the inverse tangens. Depending on θ_1 and θ_3, we will have four solutions for θ_2. By adding or subtracting a angle ζ from β, we will obtain the four solutions for θ_2. See figure 5 for a visualization. ζ can be found by using the cosine sentence:

$$\zeta = \arccos \frac{120^2 + 23^2 - 120^2 - px^2 + py^2 + pz^2}{-2 * 120 * \sqrt{px^2 + py^2 + pz^2}} \tag{16}$$

θ_2 can now be expressed as:

$$\theta_2^1 = delta + \zeta - \pi/2 \tag{17}$$

$$\theta_2^2 = delta - \zeta - \pi/2 \tag{18}$$

for the case where $\theta_3 < 0$. For the case where $\theta_3 >= 0$ we will have:

$$\theta_2^3 = delta - \zeta - \pi/2 \tag{19}$$

$$\theta_2^4 = delta + \zeta - \pi/2 \tag{20}$$

There will be 4 equal solutions with respect to (qx, qy, qz).

4.2 θ_4 ,θ_5 and θ_6 using the algebraic method

Given that we know the homogeneous link transformation, T_{bt} , representing the desired position and orientation for the end-effector, we have that:

$$T_{bt} = T_1^0(\theta_1) T_2^1(\theta_2) T_3^2(\theta_3) T_4^3(\theta_4) T_5^4(\theta_5) T_6^5(\theta_6) \tag{21}$$

Since we already have found θ_1, θ_2 and θ_3, we only have 3 unknown angles. By Taking the inverse of the first 3 angles we obtain the equation:

$$T_3^0(\theta_1,\theta_2,\theta_2)^{-1}T_{bt} = T_6^3(\theta_4,\theta_5,\theta_6) \tag{22}$$

where $T_3^0(\theta_1,\theta_2,\theta_2)^{-1}$ is found using the DH parameters, and $T_6^3(\theta_4,\theta_5,\theta_6)$ is found by multiplying out equation 1, using symbolab.com's matrix calculator [2]. We obtain the following matrix for $T_6^3(\theta_4,\theta_5,\theta_6)$:

$$T_6^3 = \begin{pmatrix} -c_{\theta_6}s_{\theta_5} & s_{\theta_6}s_{\theta_5} & c_{\theta_5} & 27c_{\theta_5}+120 \\ c_{\theta_6}c_{\theta_4}c_{\theta_5} - s_{\theta_4}s_{\theta_6} & -c_{\theta_4}c_{\theta_5}s_{\theta_6} - c_{\theta_6}s_{\theta_4} & c_{\theta_4}s_{\theta_5} & 27c_{\theta_5}s_{\theta_5}+23 \\ c_{\theta_4}s_{\theta_6} + c_{\theta_6}c_{\theta_5}s_{\theta_4} & c_{\theta_6}c_{\theta_4} - c_{\theta_5}c_{\theta_4}s_{\theta_6} & s_{\theta_4}s_{\theta_5} & 29s_{\theta_4}s_{\theta_5} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{23}$$

By comparing the elements (1,2), (1,3) and (2,3) we obtain the equations for the remaining angles, as follows:

$$\theta_5^1 = \arccos T_{bt}(1,3) \tag{24}$$

$$\theta_5^2 = -\theta_3^1 \tag{25}$$

$$\theta_4^1 = \arccos \frac{T_{bt}(2,3)}{\sin\theta_5} \tag{26}$$

$$\theta_4^2 = \theta_4^1 \tag{27}$$

$$\theta_6^1 = \arcsin \frac{T_{bt}(1,2)}{\sin\theta_5} \tag{28}$$

$$\theta_6^2 = \arcsin \frac{T_{bt}(1,2)}{\sin -\theta_5} \tag{29}$$

We obtain two solution for θ_4, θ_5 and θ_6. One can note that the manipulator is in a singularity and it is not possible to distinguish the effect of motion by joints 4 and 6, when $\theta_5 = 2\pi k$ where $k \in Z$.

5 Results and discussion

The results obtained from the written functions for direct and inverse kinematic are listed in table 2 and 3.

Transformation matrices of end-effector for the results in table 2 in corresponding order is:

$$T_6^0 = \begin{pmatrix} 0 & 0 & 1 & 184 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 244 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{30}$$

$$T_6^0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 184 \\ 1 & 0 & 0 & 244 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{31}$$

$$T_6^0 = \begin{pmatrix} 1 & 0 & 0 & 178 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -48 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{32}$$

$$T_6^0 = \begin{pmatrix} 0 & 0 & 1 & -304 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 124 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{33}$$

input $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)$	**Euler angles output** (α,β,γ)	**Cart. coord. output** (x,y,z)
(0,0,0,0,0,0)	(-135,-90,-45)	(184,0,244)
(-90,0,0,0,0,0)	(-135,-90,-45)	(0,-184,244)
(0,-90,0,0,0,0)	(0,0,180)	(178,0,-48)
(180,-90,90,0,0,0)	(45,-90,-45)	(-304,0,124)

Table 2
Results for direct kinematic. Input- and Euler angles are given in degrees. Cartesian coordinates are given in millimetres.

For the inverse kinematics we will have at most 8 solutions, depending on how far the serial manipulator is stretched out and on singularity in θ_5. The first row in table 3 we have a situation where $\theta_5 = 0$, and we have a singularity. We will then have infinite solutions by rotating θ_4 and θ_6 in different positions. In table 3 they are set to 0. The last row in table 3 we have a situation where serial manipulator is stretched out as far as possible in positive x-direction. This results in that we only have 4 solutions, because the arm is not long enough for the situation where $\theta_1 = 180°$. From this

input (x,y,z,α,β,γ)	angle output ($\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$)
(0,0,0,-135,-90,-45)	(0,0,0,0,0,0)
	(0,0,0,0,0,0)
	(0,-80,158,0,0,0)
	(0,-80,158,0,0,0)
	(180,-42,57,0,0,0)
	(180,-42,57,0,0,0)
	(180,-64,101,0,0,0)
	(180,-64,101,0,0,0)
(0,-184,244,-135,-90,-45)	(-90,0,0,180,90,-90)
	(-90,0,0,-180,-90,90)
	(-90,-80,158,180,90,-90)
	(-90,-80,158,-180,-90,90)
	(90,-42,57,180,90,-90)
	(90,-42,57,-180,-90,90)
	(90,-64,101,180,90,-90)
	(90,-64,101,-180,-90,90)
(174,0,266,0,-40,180)	(0,-80,158,90,50,0)
	(0,-80,158,-90,-50,0)
	(0,0,0,90,50,0)
	(0,0,0,-90,-50,0)
	(180,-64,101,90,50,0)
	(180,-64,101,-90,-50,0)
	(180,-42,57,90,50,0)
	(180,-42,57,-90,-50,0)
(-304,0,124,45,-90,45)	(180,-90,90,90,180,0)
	(180,-90,90,-90,-180,0)
	(180,-79,68,90,180,0)
	(180,-79,68,-90,-180,0)
	(360,-86-43i,79+84i,90,180,0)
	(360,-86-43i,79+84i,-90,-180,0)
	(360,-86+43i,79-84i,90,180,0)
	(360,-129+43i,79-84i,-90,-90,0)

Table 3
Results for inverse kinematic. Input- and Output angles are given in degrees. Cartesian coordinates are given in millimetres. The results have been rounded of to nearest integer.

it follows that θ_2 and θ_3 will be imaginary numbers, and thus not a valid solution.

6 Conclusion

We have found the direct- and inverse kinematics for a serial manipulator with 6 dof. Physical constraints have been discarded, resulting in at most 8 possible solutions for the inverse kinematics. There will be situation where there will be less than 8 possible solutions. It has also been detected a singularity in the fifth joint.

7 Matlab user manual

Two .m-files are provided. One for direct and one for inverse kinematics. The function for direct kinematics accepts six angles in degrees and returns the transformation matrix T, and the pose of the robot. Example:

$$\begin{aligned} &forward\_kinematics([\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6]) \\ &\qquad = T, [x, y, z, \alpha, \beta, \gamma] \end{aligned} \tag{34}$$

The function for the inverse kinematics accepts the pose of the end-effector and returns a 6x8 matrix, where each column corresponds to one solution containing the six angles. Example:

$$\begin{aligned} &inverse\_kinematics([x, y, z, \alpha, \beta, \gamma]) \\ &\qquad = [s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8] \end{aligned} \tag{35}$$

where each element in the output is in the form:

$$s_i = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{pmatrix} \tag{36}$$

References

[1] M. Shahinpoor, *A Robot Engineering Textbook*. Harper Row Publishers, Inc, 1987.
[2] "Matrix Calculator," https://www.symbolab.com/solver/matrix-calculator, [Online; accessed 20-Mars-2019].