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Least Squares Method

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Abstract

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In this project some basic properties of the least squares method is analysed. Both finite element and spectral methods are discussed, and the implementation is mainly performed using spectral basis functions. The problems analysed in this project are diffusion convection reaction PDEs, both linear and non-linear. The results are compared to standard Galerkin method where both error and condition number is considered. The least squares method is also applied to Galerkin methods to gain stability with great results.

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Notation

CONVENTION we let subscript h denote the discretized variables

Chapter 1

Numerical theory

1.1 Galerkin formulation

Throughout this thesis all numerical methods will be based on the Galerkin formulation. Let us consider a general boundary value problem (BVP)

$$\begin{aligned}\mathcal{L}u &= f \quad \text{in } \Omega \\ \mathcal{B}u &= g \quad \text{on } \partial\Omega.\end{aligned}\tag{1.1}$$

The domain Ω is a closed subspace of \mathbb{R}^d , $\mathcal{L} : X(\Omega) \rightarrow Y(\Omega)$ and $\mathcal{B} : X(\partial\Omega) \rightarrow B(\partial\Omega)$ are two linear operators, $f \in Y(\Omega)$ and $g \in B(\partial\Omega)$ are known functions and $u \in X(\Omega)$ is the wanted solution. The space $X(\Omega)$ will be denoted as the search space. A weak formulation can now be obtained by multiplying the first equation in 1.1 by a test function $v \in X^t(\Omega)$ and integrating over the domain Ω . By choosing $X^t(\Omega) = X(\Omega)$ the Galerkin formulation is obtained. For more examples and information on this subject the first chapters in [1] are recommended.

By the Lax-Milgram theorem it is known that a BVP is well-posed if the Operator \mathcal{L} is both bounded and coersive.

Solving the BVP numerically involves choosing discrete subsets of X, X^t, Y, B . These will be denoted X_h, X_h^t, Y_h, B_h . The discrete subspaces can be chosen in a number of ways and the defining basis functions vary from one numerical method to another. In

this thesis the spectral and finite element basis will be shortly stated and the spectral element basis will be viewed in more detail.

- a common BVP, weak formulation
- search and solution space, test and basis functions
- error estimation

1.2 Finite element method

Finite element method is one of the most widely used numerical methods applied on problems within construction, flow simulation and many other areas. It offers a precise mathematical foundation and due to the connectivity properties of the elements it guaranties a sparse system. The decomposition of the geometrical domain into a finite amount of elements chosen according to the problem wanted to solve, makes it possible to create general algorithms applicable to all kinds of geometries. For the full mathematical foundation of FEM it will be referred to [1], but some of the key propertie will be stated here in order to provide a thourough understanding of the spectral element method.

FEM provides an alorithm for solving any well-posed BVP 1.1 and the mathematical formulation is obtained by first finding the Galerkin formulation and choosing a discrete subset $X_h^p(\Omega) \subset X(\Omega)$ spanned by the finite element basis functions ϕ_i^p . p denotes the polynomial degree of the basis-functions, in 1D and for $p = 1$ the basis functions are defined as

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{if } x_{i-1} \leq x \leq x_i, \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & \text{if } x_i \leq x \leq x_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\text{supp}(\phi_i) = [x_{i-1}, x_{i+1}]$ and as a consequence of this $(\phi_i, \phi_j)_\Omega = 0$ if $|i-j| > 1$. These qualities is what gives rise to the resulting sparse linear system. By increasing the polynomial order the number of gridpoints used to define the polynomial will need to increase as well. This implies either reducing the distance between the gridpoints or increasing the support of each basis function. Both approaches will reduce the sparsity of

the final matrix. Another key aspect of FEM is the treatment of the domain Ω , on which a triangulization $\{\mathcal{T}_h\}$ is defined such that the original domain is divided into elements. By defining a reference element ($[-1, 1]$ in 1D) and a general mapping function, all the local contributions can be calculated by a generalized quadrature rule before being added to the global system of equations. This is a process tailored for parallelization, and can be generalized for a wide range of problems.

FEM is called a projection method since the solution $u_h \in X^h$ is a projection of the actual solution u of the BVP onto the discrete space X^h . Provided that the initial BVP is well-posed there exists to constants $M, \alpha > 0$ known as the bounded and coercivity constant such that the error of the solution can be reduced to a pure interpolation error. The result is known as Cea's lemma,

$$\|u - u_h\|_X \leq \frac{M}{\alpha} \|u - I_h u\|_X. \quad (1.2)$$

Where I_h is the projection operator.

Before this section ends it is important to understand the two ways to improve the error and the effects these two ways have on the algorithm. Assume the solution of the BVP to be infinitely smooth and let h denote the general size of the elements and p the order of the polynomial basis that defines X^h . Roughly speaking the error is given as $e = Ch^p$ with C being some constant. This is not a formal truth but rather a guideline as to how the error behaves, factors such as geometric complexity, condition-number, non-linear operators and the regularity of the solution will all provide slightly more complicated error estimates. However for a simpler BVP such as $u, f, g \in C^\infty(\Omega)$, $\Omega = [-1, 1]^d$, $\mathcal{L} = -\Delta$, $\mathcal{B} = 1$ the error estimate is valid. performing a h -refinement will lead to an algebraic convergence, while the sparsity of the system is conserved and the total algorithm does not change in any other way than increasing the number of elements. Keeping h constant and increasing p will provide spectral convergence, but the sparsity will be reduced and all integrals solved will require quadrature rules of higher order. A more formal statement and numerical validation can be found in [2] chapter 2.6.

1.3 Spectral methods

Spectral methods (SM) share some of the mathematical ideas as FEM, but are not as widely used in real life problems. There are many ways to apply SM, and in this thesis only the Galerkin version with numerical integration (known as G-NI) will be considered and will be referred to only as SM. For a full introduction to SM and its applications to BVP see [?]. SM can be reduced to an interpolation problem such as FEM, and are very interesting from a theoretical point of view due to its spectral convergence rate which allows you to obtain solutions of extremely high accuracy. The most important draw-back of SM are the difficulties with applications to complex geometries. Although the system of equations arising from a BVP can be constructed in an elegant way it is rarely sparse and often result in expensive calculations.

For a BVP in one dimension SM defines a set of basis functions $\{\psi_i\}_N$ which spans the whole domain Ω . The discrete space $X_h(\Omega)$ spanned by the basis functions involves all polynomials up to degree N . A function u is projected onto X_h by the relation

$$u_h(x) = \sum_{i=0}^N a_i \psi_i(x). \quad (1.3)$$

Where the coefficients a_i are called the expansion coefficients. There are many possible choices for the basis and the belonging coefficients, in this thesis and the algorithms used the functions ψ_i will be the Lagrange polynomials based on the Gauss-Lobatto-Legendre (GLL) nodes. The GLL-nodes are given as the solutions of the equation

$$(1 - \xi^2)L'_N(\xi) = 0. \quad (1.4)$$

L_N being the Legendre polynomial of degree N , defined from the Sturm-Liouville problem

$$\frac{d}{dx} \left[(1 - x^2) \frac{d}{dx} L_n(x) \right] + n(n+1)L_n(x) = 0. \quad (1.5)$$

With equations 1.5 and 1.4 the basis functions ϕ_j can be stated as

$$\phi_j = \prod_{i \neq j}^N \frac{x - x_i}{x_j - x_i}. \quad (1.6)$$

This should be taken a bit more thoroughly

$\{x_i\}$ being the solutions to 1.4. Note that $\phi_j(x_i) = 0$ when $i \neq j$, the expansion coefficients in 1.3 are then chosen as $a_i = u(x_i)$ to minimize the projection error in $L^2(\Omega)$. expanding a basis such that the coefficients are simply the evaluation of the function in that particular point, is known as a nodal SM. Creating a basis for 2 and 3 dimensions is done simply by taking the cross product of the basis functions in each direction.

$$\Phi_{ijk}(x, y, z) = \phi_i(x)\phi_j(y)\phi_k(z) \quad (1.7)$$

- quadrature rule
- the choice of nodes and basis functions
- the role of the jacobian
- cross product formulations

[?]

1.4 Spectral element method

In the early 1980's the the idea to combine FEM and SM came along in order to obtain the robustness and resulting sparse system of FEM and the spectral convergence rate provided by SM. The result was the Spectral element method. Read articles... Several formulations was investigated and the development of super computers has played an important role in deciding the method applied today. The basic idea is to divide the domain of the BVP wanted to solve into elements as in FEM and then use spectral basis functions of higher degree with support only within one element.

- READ LITERATURE ON THIS !!

Chapter 2

Fluid dynamics and problem specific mathematics

2.1 Navier-Stokes equation

The physics regarding fluids in motion are described mathematically by the Navier-Stokes equation. The equations can be derived in many ways, and it is referred to [?] for a complete explication. The general idea is to conserve momentum and mass in a control domain and by applying Reynolds transport theorem and the result is the following two equations referred to as the N-S equations.

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}(\nabla \cdot \mathbf{u}) = -\nabla p + \nu \nabla \cdot \tau \quad (2.1)$$

$$\nabla \cdot (\rho \mathbf{u}) = 0 \quad (2.2)$$

These equations have been studied for centuries and different physical situations lead to different simplifications and sets of equations. Examples are the Euler equations, Stokes problem, darcy flow and the incompressible navier-stokes equations. Without doing the derivation of the equations a short discussion of each term in equation N-S will be given here in order to understand there physical origin and their mathematical attributes.

The term $\partial \mathbf{u} / \partial t$ is the time-derivative of the flow, for a steady state flow this term will be equal zero. The discretization of this term is often based on some implicit scheme.

The convective term $\mathbf{u}(\nabla \cdot \mathbf{u})$ describes the transport due to the flow itself on each of its components. This term is non-linear and does therefore require the equations to be solved by some iteration procedure such as Newton iterations.

The diffusive term $\nu \Delta \mathbf{u}$ describes the natural diffusion of the fluid. The effect of diffusion is determined by the viscosity of the fluid. Mathematically this term makes the equations numerically stable and it is therefore generally easier to solve the N-S equations for high-viscosity fluids.

The pressure gradient ∇p is the only term involving the pressure, ...DISCUST THE MEANING OF PRESSURE...in modern solution techniques it is common to solve eq. 2.2 in a decoupled manner.

The reynolds stress tensor $\tau = \dots$ is the turbulent term of the N-S equation. In laminar flow this term can be neglected.

The second equation in 2.2 imposes a divergence free constraint on the solution \mathbf{u} if the fluid is assumed to be incompressible. The density ρ will be a constant and the simplification follows trivially.

The coefficient ν is the inverse Reynolds number $1/Re$. The reynolds number can be understood in many different ways, but perhaps the most import role is that it describes the relation between the biggest length scales of the flow and the viscous length scales. Turbulent flows are characterized by high reynolds number. The huge span in length scales requires an extremely fine mesh if the equations 2.2 are to be solved exactly. Because a fine mesh implies a high computational cost a direct numerical solution (DNS) is not feasible for problems of a certain complexity.

- The time-derivative
- The convective term
- The diffusive term
- The pressure gradient

- The reynolds stress tensor
- The reynolds number
- The mass equation

2.2 Resolving the turbulent term

Depending on the wanted accuracy of your solution and the number of computational units at hand the turbulent term can be solved in different ways. A brief overview of the different approaches to the turbulent term are discussed in the following subsections

- Laminar flow
- RANS
- LES
- K-epsilon and K-omega
- DNS

2.3 Solution methods of incombressible N-S

A non-linear set of equations requires a non-trivial solution method, and when the domain of the problem can be anything from a simple channel to a moving turbine there are many considerations that needs to be made. Although the equations have been known for over 200 years no one has been able to prove or disprove the well-posedness of the problem. FINAL ELEMENT FORMULATION WELL-POSED?? Some of the most common algorithms will be discussed in the following subsections

- coupled versus decoupled
- The Uzawa algorithm
- Pressure correction method
- fractional step method

2.4 nek5000

Nek5000 is a turbulent flow solver developed mainly by Paul Fischer and has through the past 20 years had several contributors. It is an open-source code applicable to many different flow situations.

- Application areas
- Description of the numerical algorithm
- usage of the program

Chapter 3

Application of nek5000 and the creation of grid

3.1 Nek5000

There are many numerical solvers for turbulent flows available on the market. From large commercial softwares such as fluent which runs as a black-box solver, to CDP which is a bit more specialized solver with the possibility to modify the code to apply it to the wanted usage and to full open-source codes such as nek5000 and openFOAM. The solvers can vary both in the fundamental numerical method (FV,FD,FEM,SEM) the implemented algorithm (Fractional Step, Poisson pressure, Uzawa) and the type of simulation (RANS,LES,DNS). The software is chosen depending on the users requirement, but the work flow does follow a certain structure in all cases.

- Practical problem – Mathematical formulation – Generation of mesh – Simulation – post-processing

An example is waterflow in a channel, which would be the practical problem. The mathematical formulation would be the incompressible N-S eq, with LES dynamical SGS. The mesh could consist of tetrahedros or hexahedrons. The simulation is the performed on some cluster before the wanted data in some sub-domain is visualized and presented.

Nek provides a basic tool for generation of mesh. For more complex geometries this tool cannot compare with more visualized-based softwares such as ICEM from ANSYS which exports mesh to several numerical solvers such as Fluent and nastran. It is therefore very useful to have an automatic way of converting a mesh created in ICEM to the format required by nek5000. In order for nek to run optimally the elements should be as homogenous and as similar to the reference element as possible. It is therefore of great interest to be able to propagate curved geometries into the neighbour-elements in order to have a smooth as possible transition from a boundary with high curvature.

3.2 Gas dispersion in a simplified urban area

The problem investigated in this work is gas dispersion of neutral gas in a velocity field through four cubic blocks. Similar simulations have been done in CDP and Fluent which are compared to data from a wind-tunnel experiment performed by ALAN.

- Description of problem and domain
- mesh and fluent/CDP mesh

3.3 Creation of the mesh-conversion script

- initial script
- changes and modifications
- performance testing
- pitfalls

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