NTNU

Project

Least Squares Finite Element Method

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Abstract

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Least Squares Finite Element Method

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The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too...

Notation

LAH **L**ist **A**bbreviations **H**ere

Chapter 1

Theory

1.1 Informal introduction to LSFEM

The Least Squares Finite Element Method is a numerical method similar to mixed galerkin. But with a different approach regarding defining the bilinear functional. Let us look at a system of first order differential equations on the form

$$Au = f \text{ in } \Omega \tag{1.1}$$

$$u = g \text{ on } \partial\Omega. \tag{1.2}$$

Where A is a partial differential operator defined as

$$A = \sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i} + A_0. \tag{1.3}$$

n beeing the number of dimensions of the domain Ω. If u happens to be a vector function of say k dimensions then A_i will be a matrix with k columns and k or more rows. Let us initially assume without loss of generality that $g = 0$. Further we require $f \in L_2(\Omega)$ and choose $V = \{v \in L_2(\Omega) | v = 0 \text{ on } \partial\Omega\}$. A residual is defined

$$R(v) = Av - f, \tag{1.4}$$

and a functional

$$J(v) = \frac{1}{2}||R(v)||_0^2. \tag{1.5}$$

The solution u and it's gradient needs to be in L^2 for the functional to make sence, hence u is restricted to the space $H_0^1(\Omega)$. The homogenous boundary condition is now baked into the definition of the search space. By minimizing J we obtain

$$\lim_{t\to 0}\frac{d}{dt}J(u+tv) = \int_\Omega (Av)^T(Au-f)d\Omega = 0 \ , \forall v \in V. \tag{1.6}$$

We can now write a variational formulation of the least-squares method: Find $u \in V$ such that

$$Q(u,v) = F(v) \quad , \quad \forall v \in V, \tag{1.7}$$

where

$$Q(u,v) = (Au, Av), \tag{1.8}$$

$$F(v) = (f, Av). \tag{1.9}$$

[1] Notice that the bilinear form Q is symmetric, this is an important advantage LSFEM has over regular galerkin methods. The bilinear form that surged from a first-order problem by the LSFEM leads us to a variational formulation similar to the one obtained from a second order problem by regular FEM. Generally the bilinear form from LSFEM will correspond to a bilinear form of a problem of twice the order obtained using FEM. In order to avoid problems of large complexity a higher order PDE should therefore be transformed to a system of first order PDE's (similar to a mixed galerkin approach) before defining the least squares functional.

[1]

1.2 Formal formulation of LSFEM

Let us look at a general boundary value problem where $f \in Y(\Omega)$, $g \in B(\partial\Omega)$, $\mathcal{B}\colon X(\partial\Omega) \to B(\partial\Omega)$ and $\mathcal{L}\colon X(\Omega) \to Y(\Omega)$. Find $u \in X(\Omega)$ such that

$$\mathcal{L}u = f \quad \text{in } \Omega \tag{1.10}$$

$$\mathcal{B}u = g \quad \text{on } \partial\Omega. \tag{1.11}$$

Whenever this BVP has a unique solution, a least-squares functional can be defined as

$$J(u; f, g) = ||\mathcal{L}u - f||_Y^2 + ||\mathcal{B}u - g||_B^2 \tag{1.12}$$

and the corresponding minimization problem is then given as

$$\min_{u \in X} J(u; f, g) \tag{1.13}$$

For any well-posed problem $\exists \alpha, \beta > 0$ such that

$$\alpha||u||_X^2 \leq J(u; 0, 0) = (\mathcal{L}u, \mathcal{L}u)_Y + (\mathcal{B}u, \mathcal{B}u)_B \leq \beta||u||_X^2. \tag{1.14}$$

The fact that our functional is norm-equivalent is of crucial importance to a successfull LS-method. It is therefore important that the spaces X, Y and B is chosen such that the LS-functional defines a norm is equivalent to $||\cdot||_X$. Need to show this ? Minimizing this functional is equivalent to solving the Euler-lagrange equations formulated as

$$\text{find } u \in X \text{ such that } Q(u, v) = F(v) \;\; \forall v \in X \tag{1.15}$$

Where $Q(u, v)$ and $F(v)$ are defined as

$$Q(u, v) = (\mathcal{L}u, \mathcal{L}v)_Y + (\mathcal{B}u, \mathcal{B}v)_B, \tag{1.16}$$

$$F(v) = (f, \mathcal{L}v)_Y + (g, \mathcal{L}v)_B. \tag{1.17}$$

Notice that $Q(u, v)$ defines an inner product and $Q(u, u)^{1/2} = J(u; 0, 0)^{1/2}$ defines the corresponding norm.

In order to solve this numerically we define the function spaces X^h, Y^h and B^h and the corresponding variational formulation is then written as

$$\text{find } u^h \in X^h \text{ such that } Q(u^h, v^h) = F(v^h) \ \ \forall v^h \in X^h. \tag{1.18}$$

[2]

1.2.1 Error analysis

Let u be the analytical solution of a problem of the type 1.11, u^h is our numerical solution to 1.18 and $u^h_\perp$ is the orthogonal projection of u in X_h.

$$Q(u - u^h, u - u^h) = Q(u - u^h, u - u^h_\perp) + Q(u - u^h, u^h_\perp - u^h) \tag{1.19}$$

$$= Q(u - u^h, u - u^h_\perp) \tag{1.20}$$

$$\leq \beta ||u - u^h||_{X_h} \, ||u - u^h_\perp||_{X_h}. \tag{1.21}$$

The first equality is due to adding and subtracting $u^h_\perp$, because both u^h and $u^h_\perp$ solves the variatonal formulation we can cancel the last term, and by using the norm-equivalency from 1.14 and schwartz inequality we get the last expression. Now by applying the first inequality of 1.14 we end up with

$$||u - u^h||_{X_h} \leq \frac{\beta}{\alpha} ||u - u^h_\perp||_{X_h} = \min_{w^h \in X_h} \frac{\beta}{\alpha} ||u - w^h||_{X_h}. \tag{1.22}$$

Hence we can show that the Least squares method provides a convergence result of similar order as the Finite element method.

1.3 Example - Poisson problem

The poisson problem is defined as

$$-\Delta u = f \text{ in } \Omega \tag{1.23}$$

$$u = g \text{ on } \partial\Omega \tag{1.24}$$

Let us first consider the homogenous case. The straight forward LSFEM approach is to define $\vec{w} = -\nabla u$ and solve the system of equations

$$\vec{w} + \nabla u = 0 \text{ in } \Omega \tag{1.25}$$

$$\nabla \cdot \vec{w} = f \text{ in } \Omega \tag{1.26}$$

$$u = 0 \text{ on } \partial\Omega. \tag{1.27}$$

which can be written in the same form as 1.11 with $\mathbf{u} = \vec{w} \oplus u$, $\vec{f} = (0, 0, f)$, $g = 0$, $\mathcal{B} = (0, 0, 1)^T$ and $\mathcal{L}$ given as

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & \partial/\partial x \\ 0 & 1 & \partial/\partial y \\ \partial/\partial x & \partial/\partial y & 0 \end{bmatrix} \tag{1.28}$$

We define the search space $X = H^1(\Omega; \text{div}) \times H_0^1(\Omega)$ and the solution space $Y \times B = [L^2(\Omega)]^3 \times L^2(\Omega)$ and the functional can then be defined as in 1.12. The variational formulation of the problem can be stated. Find $\mathbf{u} \in X$ s.t.

$$Q(\mathbf{u}, \phi) = F(\phi) \ \ \forall \ \ \phi \in X. \tag{1.29}$$

We require that $\vec{f} \in Y$. Should I state the numerical variational formulation instead?? Notice that the spaces X and Y chosen as described above fullfill the condition 1.14.

1.4 Example - Diffusion convection reaction problem

The diffusion convection problem to be analyzed is given as

$$-\Delta u + \vec{b} \cdot \nabla u + \sigma u = f \text{ in } \Omega \tag{1.30}$$

$$u = g \text{ on } \partial\Omega \tag{1.31}$$

where $\vec{b} = [b_1, b_2]$ is a vector field, and σ is some reaction constant. By following the same approach as for the poisson problem we end up with $\mathcal{L}$ on the form

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & \partial/\partial x \\ 0 & 1 & \partial/\partial y \\ \partial/\partial x - b_1 & \partial/\partial y - b_2 & \sigma \end{bmatrix} \tag{1.32}$$

which leads to a similar but slightly different linear system than the one created by the poisson problem.

1.5 Boundary conditions

1.5.1 non-homogenous Dirichlet boundary conditions

If $g \neq 0$ then we simply define a lifting function $R_g \in X$ such that $R_g(\partial\Omega) = g(\partial\Omega)$. By defining $\tilde{\mathbf{u}} = \mathbf{u} - R_g$ we can replace $\mathbf{u}$ in the variation formulation and get

$$Q(\tilde{\mathbf{u}} + R_g, \phi) = F(\phi) \tag{1.33}$$

$$Q(\tilde{\mathbf{u}}, \phi) + Q(R_g, \phi) = F(\phi) \tag{1.34}$$

$$Q(\tilde{\mathbf{u}}, \phi) = F(\phi) - Q(R_g, \phi) \tag{1.35}$$

$$Q(\tilde{\mathbf{u}}, \phi) = \tilde{F}(\phi) \tag{1.36}$$

1.5.2 non-homogenous Neumann boundary conditions

Because of the geometry of our problem and the fact that we define the flux as an extra variable we can trasform the neumann conditions to a dirichlet condition on the flux.

$$\frac{\partial u}{\partial \vec{n}} = h \text{ on } \partial\Omega \tag{1.37}$$

$$\nabla u \cdot \vec{n} = h \tag{1.38}$$

$$\vec{w} \cdot \vec{n} = -h. \tag{1.39}$$

Let us define $\hat{x}$ and $\hat{y}$ as the unit vectors in each direction. Notice that for the west ($x = 0$) and east ($x = 1$) edges the normal vector $\vec{n} = \pm\hat{x}$, and at the north ($y = 1$) and south($y = 0$) edges $\vec{n} = \pm\hat{y}$. This way we can write the Neumann conditions as a

dirichlet condition on the first and second component of $\vec{w} = [w_1 \ w_2]$.

$$w_1 = \pm h \text{ for } y = 0 \text{ and } y = 1 \tag{1.40}$$

$$w_2 = \pm h \text{ for } x = 0 \text{ and } x = 1 \tag{1.41}$$

Chapter 2

New Theory

2.1 Using the Least squares method to gain stability

In regular galerkin FEM for the Diffusion-Transport equation you end up with the variational formulation

$$a(u,v) = (f,v)_0 \quad \forall v \in V \tag{2.1}$$

where the bilinear functional is given as

$$a(u,v) = \mu \int_\Omega \nabla u \cdot \nabla v d\Omega + \int_\Omega v(b \cdot \nabla u) d\Omega \tag{2.2}$$

We can define a new norm from this functional in the following manner

$$a(u,u) = \mu \int_\Omega \nabla u \cdot \nabla u d\Omega + \int_\Omega u(b \cdot \nabla u) d\Omega \tag{2.3}$$

$$= \mu ||\nabla u||_0^2 + \frac{1}{2} \int_\Omega b \cdot \nabla u^2 d\Omega \tag{2.4}$$

$$= \mu ||\nabla u||_0^2 - \frac{1}{2} \int_\Omega u^2 (\nabla \cdot b) d\Omega \tag{2.5}$$

$$\tag{2.6}$$

Let $\gamma_0 \leq -\frac{1}{2}\nabla \cdot b \leq \gamma_1$, we can now make a lower and upper bound for the bilinear form

$$\mu ||\nabla u||_0^2 + \gamma_0 ||u||_0^2 \leq a(u,u) \leq \mu ||\nabla u||_0^2 + \gamma_1 ||u||_0^2 \tag{2.7}$$

should I change signs in the equation??

It is clear that for small μ and large γ_1 the bilinear form is no longer coercive and thus our convergence requirements are no longer valid.Remember that the functional in the LSFEM-formulation is chosen such that it is norm-equivalent with $||\cdot||_1$, hence we can find α, β such that $\alpha||u||_1^2 \leq Q(u,u) \leq \beta||u||_1^2$. Now, let us explore what happens if we add the variational formulation obtained from the LSFEM to the standard FEM formulation

$$a(u,v) + \delta Q(u,v) = (f,v) + \delta F(v) \tag{2.8}$$

$$\mathring{a}(u,v) = \mathring{f}(v) \tag{2.9}$$

Let us study the coerciveness of this functional,

$$\mathring{a}(u,u) \geq \mu||\nabla u||_0^2 + \gamma_0||u||_0^2 + \delta\alpha||u||_1^2 \tag{2.10}$$

$$\geq \mu||\nabla u||_0^2 + \gamma_0||u||_0^2 + \delta\alpha||u||_0^2 \tag{2.11}$$

$$\geq \mu||\nabla u||_0^2 + \mu||u||_0^2 \tag{2.12}$$

$$= \mu||u||_1^2 \tag{2.13}$$

In the third inequality we make the assumption that $\gamma_0 + \delta\alpha \geq \mu$ in other words δ, (the amount of smoothing from LS) has to be chosen such that $\delta \geq (\mu - \gamma_0)/\alpha$. We can also prove stability for a discrete solution of the variatonal formulation,

$$\mathring{a}(u_h, u_h)_h \leq C||f||_0 \tag{2.14}$$

The proof for a similar method can be found in [3] Ch.12.

2.1.1 estimation of the constant α

By performing the proof of the poincaré inequality from [3] for our domain Ω we end up with showing that $|v|_1 \leq \sqrt{2/3}||v||_1 \;\; \forall v \in H_0^1$. By using this inequality twice we end up showing for the poisson problem that

$$|||v||| \geq \frac{2}{3}||v||_1, \tag{2.15}$$

Hence $\alpha = \frac{2}{3}$. should find this constant for the difftrans problem

2.2 Nonlinear diffusion transport problem

Now let the vector field b be a function of u. This gives us a nonlinear variational formulation that does not allow us to solve our system of equations directly. In order to find a solution Newtons method was used. For a regular galerkin approach the jacobian of the vector generated by Q needs to be calculated or at least approximated. For the least squares setting one needs to add the jacobian of F as well since this depends on b and thus also u. Let us consider the diffusion-transport equation with non-homogenous boundary conditions and a non-linear gradient term with a LS-approach. This corresponds to solving the equation

$$Q(\tilde{\mathbf{u}}, \phi; b) = \tilde{F}(\phi; b) \tag{2.16}$$

$$Q(\tilde{\mathbf{u}}, \phi; b) - \tilde{F}(\phi; b) = 0 \tag{2.17}$$

$$\mathcal{F}(\tilde{\mathbf{u}}, \phi; b) = 0 \tag{2.18}$$

Where the dirichlet homogenous boundary conditions on $\tilde{\mathbf{u}}$ are required by the search space.

Newtons method is then applied, you start by guessing an initial $\tilde{\mathbf{u}}_h^0$ and then repeating

1. $r^k = \mathcal{F}(\tilde{\mathbf{u}}_h^k)$, Calculating the residual
2. $\hat{e}^k = \mathcal{J}_k^{-1} r^k$, Calculating the error
3. $\tilde{\mathbf{u}}_h^{k+1} = \tilde{\mathbf{u}}_h^k - \hat{e}^k$, updating the solution

until you reach your solution. $\mathcal{J}_k$ is the jacobian matrix of $\mathcal{F}(\mathbf{u_h}^k)$.

Chapter 3

Implementation

For the general problem 1.11 the functional Q will take the form

$$Q(u,v) = \int_{\Omega} (\mathcal{L}v)^T (\mathcal{L}u) d\Omega. \tag{3.1}$$

Implementing Q requires two sets of basis functions $\{N_i\}$ that describes the search and solution space. In this project assignment the search and solution space will be described by the the same set of basis functions which will depend on the method applied. u is discretized as

$$u_h = \sum_{I=0}^{K} a_I N_I. \tag{3.2}$$

Since equation 1.15 requires equality for all test functions in the search space we simply solve the equation for each basis function. We are therefore left with a system of K

equations. Equation 3.1 can then be written for each test function as

$$Q(u_h, N_I) = \int_\Omega (\mathcal{L}N_I)^T(\mathcal{L}u_h)d\Omega \tag{3.3}$$

$$= \int_\Omega (\mathcal{L}N_I)^T(\mathcal{L}\sum_{J=1}^{K} a_J N_J)d\Omega \tag{3.4}$$

$$= \sum_{J=1}^{K} \int_\Omega (\mathcal{L}N_I)^T(\mathcal{L}a_J N_J)d\Omega \tag{3.5}$$

$$= \sum_{J=1}^{K} \int_\Omega (\mathcal{L}N_I)^T(\mathcal{L}a_J N_J)d\Omega \tag{3.6}$$

$$= \sum_{J=1}^{K} \int_\Omega (\mathcal{L}N_I)^T(\mathcal{L}N_J)d\Omega \ \cdot a_J. \tag{3.7}$$

The total system of equation for all test functions can then be written as a matrix equation

$$Au = F. \tag{3.8}$$

Where $A^{LS}_{I,J} = \int_\Omega (\mathcal{L}N_I)^T(\mathcal{L}N_J)d\Omega$, written explicitly it will be a 3-by-3 matrix on the form

$$A^{LS}_{I,J} = \int_\Omega \begin{bmatrix} N_I N_J + \partial_x N_I \partial_x N_J & \partial_x N_I \partial_y N_J & N_I N_{J,x} \\ \partial_y N_I \partial_x N_J & N_I N_J + \partial_y N_I \partial_y N_J & N_I N_{J,y} \\ N_{I,x} N_J & N_{I,y} N_J & N_{I,x} N_{J,x} + N_{I,y} N_{J,y} \end{bmatrix} d\Omega \text{ where}$$

$\partial_x = \partial/\partial x$ for the poisson problem and $\mu\partial/\partial x - b_1$ for the diffusion transport problem. Similarly F_I will be given as

$$F_I = \int_\Omega (\mathcal{L}N_I)^T \vec{f} d\Omega = \int_\Omega \begin{bmatrix} \partial_x N_I \\ \partial_x N_I \\ 0 \end{bmatrix} f d\Omega \tag{3.9}$$

Notice that by doing the splitting of variables we obtain a system of equations three times as big as if we were to solve the equation directly.

3.1 LSFEM for poisson

finite element space X_h^1 ? quadrature...

3.2 LS spectral method for poisson

The spectral implementation is done using Gauss Lobatto nodes and quadrature and the lagrange functions based on the GL nodes as basis functions. Notice that the discrete solution u_h consist of the discretizations of both u and $w = -\nabla u$. u_h can be structured blockwise such that or nodewise. By choosing a blockwise representation the final system of equations can be written as

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} \begin{bmatrix} u^h \\ w_1^h \\ w_2^h \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix}.$$

Where each block $A_{i,j}$ corresponds to calculating element i, j in the matrix 3 for all the indices I, J. In order to implement this matrix it is convenient to write it in a compact form using the kronecker tensor product. The components needed for this formulation is the $n \times n$ diagonal matrix W with the GLL-weigths along the diagonal and the $n \times n$ matrix $(L)_{i,j} = l_j'(x_i)$ where l_j is the jth lagrange polynomial and x_i is the ith node in either x or y direction. Note that the formulation is based on a grid of GLL-nodes in both x and y direction.

$$A^{LS} = \begin{bmatrix} W \otimes (L^T W L + W) & WL \otimes L^T W & W \otimes WL \\ L^T W \otimes WL & (L^T W L + W) \otimes W & WL \otimes W \\ W \otimes L^T W & L^T W \otimes W & L^T W L \otimes W + W \otimes L^T W L \end{bmatrix}$$

do I need to show how this is derivated?

Similarly without the reformulation of the PDE and with regular galerkin formulation the stiffness matrix will simply be $A_{3,3} = W \otimes L^T W L + L^T W L \otimes W$

3.3 LS spectral method for Diffusion transport

The matrix corresponding to the discretized variational formulation can be divided into two parts where one dependends on μ alone and the other one contains the contributions from the gradient term in the original equation.

$$A^{LS} = \begin{bmatrix} W \otimes (\mu L^T W L + W) & \mu W L \otimes L^T W & W \otimes W L \\ \mu L^T W \otimes W L & (\mu L^T W L + W) \otimes W & W L \otimes W \\ W \otimes L^T W & L^T W \otimes W & L^T W L \otimes W + W \otimes L^T W L \end{bmatrix}$$

in addition it will be added some extra terms dependent on b to the upper left 2×2 block matrix. Let B_1 and B_2 be diagonal $n^2 \times n^2$ matrices with the values of the first and second component of b evaluated in each spacial node along the diagonal.

$$G_{1,1} = -\mu B_1(W \otimes WL) - \mu(W \otimes L^T W)B_1 + B_1(W \otimes W)B_1 \tag{3.10}$$

$$G_{1,2} = -\mu B_2(W \otimes WL) - \mu(L^T W \otimes W)B_1 + B_1(W \otimes W)B_2 \tag{3.11}$$

$$G_{2,1} = G_{1,2}^T \tag{3.12}$$

$$G_{2,2} = -\mu B_2(WL \otimes W) - \mu(L^T W \otimes W)B_2 + B_2(W \otimes W)B_2. \tag{3.13}$$

We can then define the total matrix for the diffusion transport problem K^{LS} as

$$K^{LS} = A^{LS} + G^{LS} \tag{3.14}$$

Where G^{LS} is the matrix given as $G^{LS} = \begin{bmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Adding the gradient term in our equation also affects the loading function. In a compact notation the discretized loading vector from the variational formulation can be written as $F^{LS} = \begin{bmatrix} \mu(W \otimes L^T W)F - (W \otimes W)B_1 F \\ \mu(L^T W \otimes W)F - (W \otimes W)B_2 F \\ 0 \end{bmatrix}$, where F is the vector with the loading function evaluated in each spacial node.

3.4 non-linear diffusion transport problem

The stepwise algorithm to solve the nonlinear equation is descriped in chapter 2. However there are several computational steps that needs to be taken care of, both with regular galerking and least squares. In both cases we obtain two matrices which we will name A and G and with superscript LS if they refer to the least squares formulation. In both cases only G will depend on the numerical solution u_h. An important difference however is that in the LS setting the F vector will depend on u_h while in the straight forward galerkin setting it will not. For regular galerkin spectral approach we obtain

$$A\tilde{u} + AR_g + G(\tilde{u} + R_g)(\tilde{u} + R_g) - F = 0 \tag{3.15}$$

Notice that for each iteration the matrix $G(\tilde{u} + R_g)$ needs to be evaluated, the homogenous boundary conitions on $\tilde{u}$ needs to be imposed and the jacobian needs to be calculated. The jacobian $\mathcal{F}$ will for this setting be given as

$$\mathcal{J}_{i,j} = A_{i,j} + G(\tilde{u} + R_g)_{i,j} + (\tilde{u} + R_g)_i \frac{\partial}{\partial \tilde{u}_j}(G(\tilde{u} + R_g))_i. \tag{3.16}$$

With the LS formulation we obtain

$$A^{LS}\tilde{u} + A^{LS}R_g + G^{LS}(\tilde{u} + R_g)(\tilde{u} + R_g) - F^{LS}(\tilde{u} + R_g) = 0 \tag{3.17}$$

$$A^{LS}\tilde{u} + A^{LS}R_g + G^{LS}(\tilde{u} + R_g)\tilde{u} - F^{LS}(\tilde{u} + R_g) = 0 \tag{3.18}$$

It is clear from this equation that the term surging from the loading function also needs to be handled when calculating the jacobian. The lifting function R_g does only have nonzero values in the third "block", hence it belongs to the kernel of the G-matrix.

$$\mathcal{J}_{i,j} = A_{i,j} + G(\tilde{u} + R_g)_{i,j} + [\frac{\partial}{\partial \tilde{u}_j}(G(\tilde{u} + R_g))_{i,:}]\tilde{u} \; - \frac{\partial}{\partial \tilde{u}_j}F(\tilde{u} + R_g)_i. \tag{3.19}$$

Let us first consider the F-vector. The terms are given in equation 3.3 and it is clear that only the last terms in each block depends on u, notice that it does not depend on the components of the gradient $[w_1 \; w_2]$. Hence the contribution to the total Jacobi matrix will only be in block $(1, 3)$ and $(2, 3)$. Further since B_1 and B_2 are both diagonal matrices where $B_{i,i}(u) = B_{i,i}(u_i)$ the jacobian can be calculated efficiently by creating

the matrices dB_1 and dB_2 which has the partial derivative of B_1, B_2 wrt. u evaluated in each node.

Chapter 4

Results

4.1 The main differences

Using least squares will always give you a SPD system of equations which can be advantageous. However for second order equations this system is three times as big as if we were to solve it using more standard methods. Comparing the correctness of the solution as done in figure **??** shows that the convergence rate is the same as for standard methods, but the the value of the residual is slightly higher for the least squares method. This can be explained by the functional that is minimized. Notice that in the least squares methods you minimize the square of the residual. since the correctness of both methods are restricted by the smoothness of the solution and the number of discrete points LS-methods will minimize the residual squared down to a given precision and hence the residual itself to a slightly higher value. The condition number is also worth comparing ...

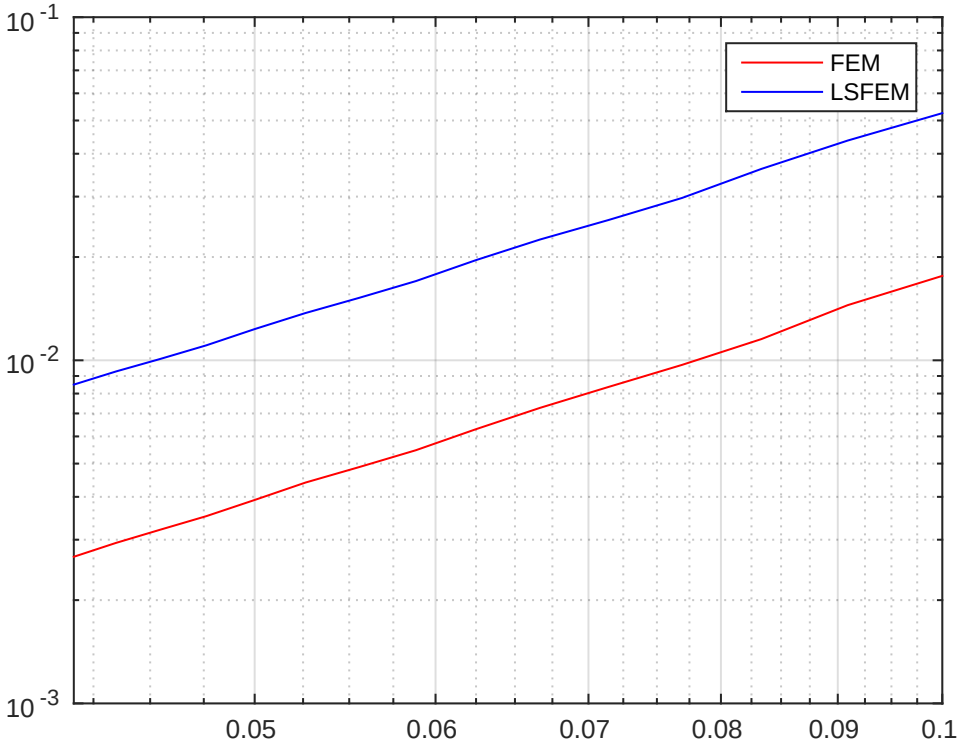

FIGURE 4.1: convergence of LSFEM and FEM

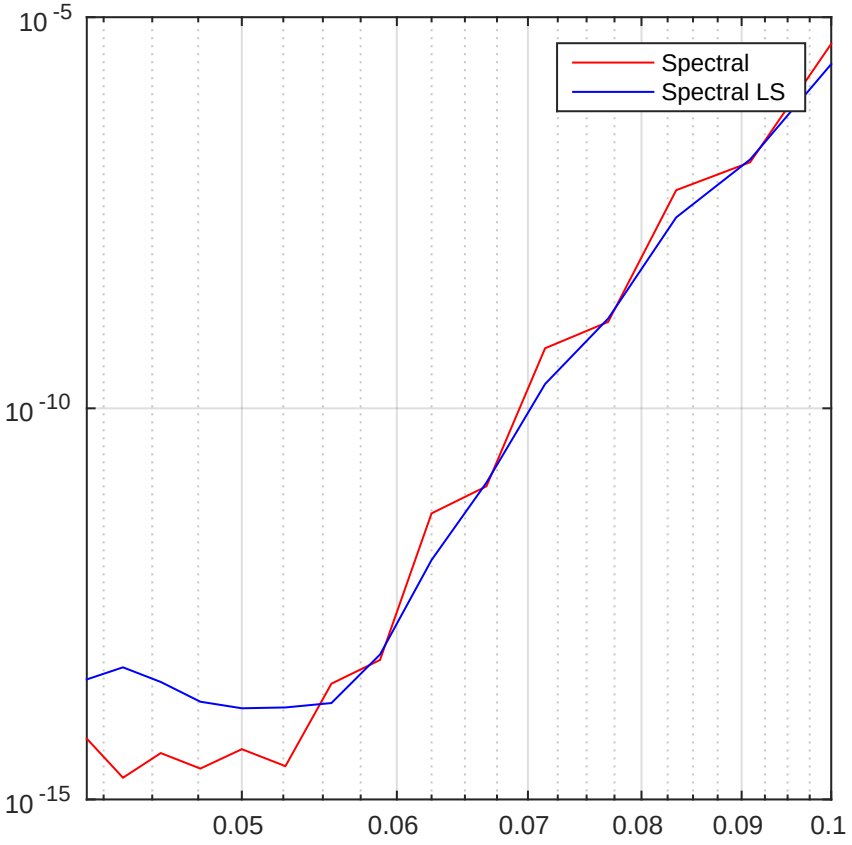

FIGURE 4.2: convergence of Spectral and LS-Spectral method

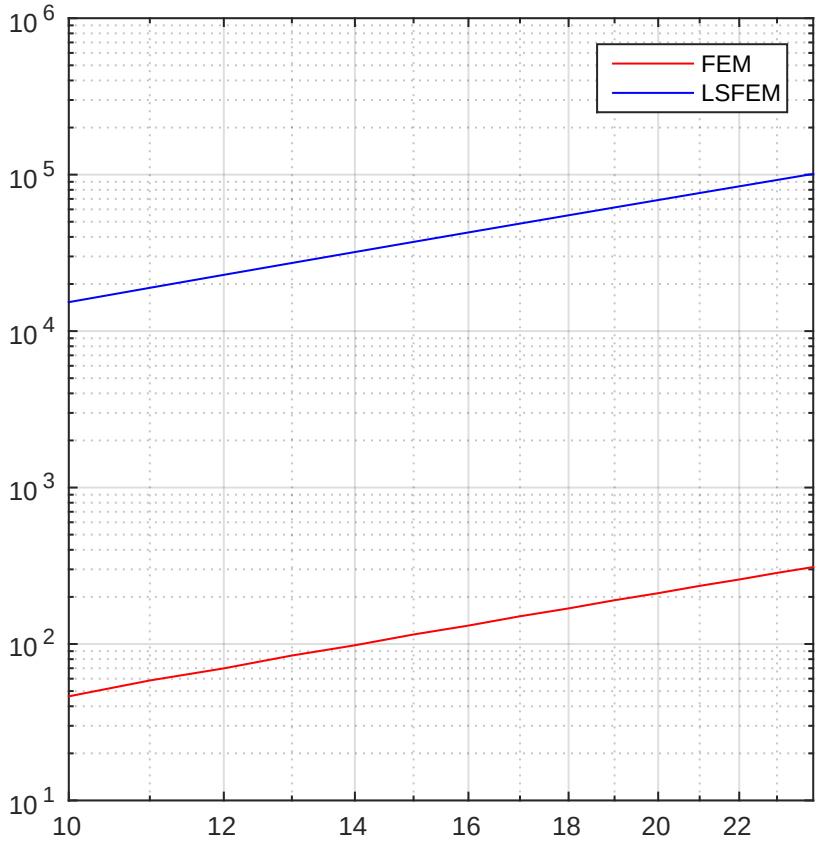

FIGURE 4.3: condition number of LSFEM and FEM

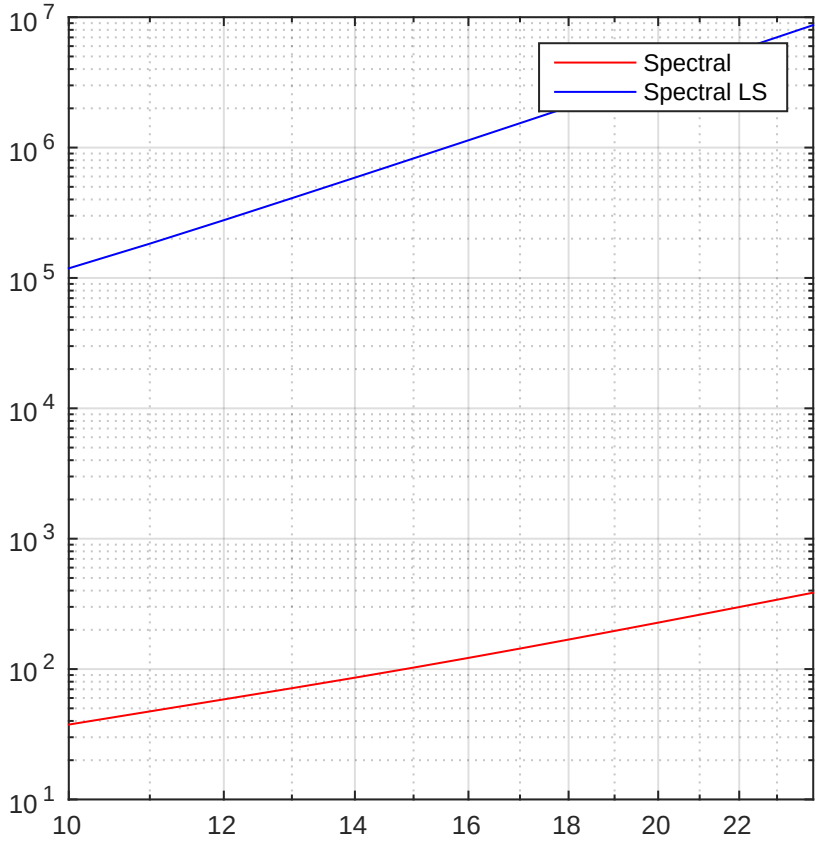

FIGURE 4.4: condition number of Spectral and LS-Spectral method

Appendix A

Appendix Title Here

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