NTNU

Project

Least Squares Finite Element Method

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April 2015

NTNU

Abstract

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The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too...

Notation

LAH **L**ist **A**bbreviations **H**ere

Chapter 1

Theory

1.1 Basis of LSFEM

Let us look at a general boundary value problem where $f \in Y(\Omega)$, $g \in B(\partial\Omega)$, $\mathcal{B} \colon X(\partial\Omega) \to B(\partial\Omega)$ and $\mathcal{L} \colon X(\Omega) \to Y(\Omega)$. Find $u \in X(\Omega)$ such that

$$\mathcal{L}u = f \quad \text{in } \Omega \tag{1.1}$$

$$\mathcal{B}u = g \quad \text{on } \partial\Omega. \tag{1.2}$$

Whenever this BVP has a unique solution, a least-squares functional can be defined as

$$J(u; f, g) = ||\mathcal{L}u - f||_Y^2 + ||\mathcal{B}u - g||_B^2 \tag{1.3}$$

and the corresponding minimization problem is then given as

$$\min_{u \in X} J(u; f, g) \tag{1.4}$$

For any well-posed problem $\exists \alpha, \beta > 0$ such that

$$\beta||u||_X \leq J(u; 0, 0) \leq \alpha||u||_X. \tag{1.5}$$

The fact that our functional is norm-equivalent is of crucial importance to a successfull LS-method. Need to show this ? Minimizing this functional is equivalent to solving the

Euler-lagrange equations formulated as

$$\text{find } u \in X \text{ such that } Q(u,v) = F(v) \;\; \forall v \in X \tag{1.6}$$

Where $Q(u,v)$ is the continous bilinear form given as $(\mathcal{L}u, \mathcal{L}v)_Y$ and $F(v)$ is the bounded linear functional given as $(\mathcal{L}v, f)_Y$. [1]

1.2 Example - Poisson problem

The poisson problem is defined as

$$-\Delta u = f \text{ in } \Omega \tag{1.7}$$

$$u = g \text{ on } \partial\Omega \tag{1.8}$$

The straight forward LSFEM approach is to define $w = -\nabla u$ and solve the system of equations

$$w + \nabla u = 0 \text{ in } \Omega \tag{1.9}$$

$$\nabla w = f \text{ in } \Omega \tag{1.10}$$

$$u = 0 \text{ on } \partial\Omega. \tag{1.11}$$

which can be written in the same form as 1.2 with $u = w \oplus u$, $f = (0, 0, f)$, $g = 0$, $\mathcal{B} = (0, 0, 1)^T$ and $\mathcal{L}$ given as

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & \partial/\partial x \\ 0 & 1 & \partial/\partial y \\ \partial/\partial x & \partial/\partial y & 0 \end{bmatrix} \tag{1.12}$$

We define the search space $X = H^1(\Omega; \mathrm{div}) \times H_0^1(\Omega)$ and the solution space $Y \times B = [L^2(\Omega)]^3 \times L^2(\Omega)$ and the functional can then be defined as in 1.3. The variational formulation of the problem can be stated. Find $u \in X$ s.t.

$$Q(u, \phi) = F(\phi) \;\; \forall \;\; \phi \in X. \tag{1.13}$$

We require that $f \in Y$. Notice that the spaces X and Y chosen as described above fullfill the condition 1.5.

1.3 Example - Diffusion convection problem

The diffusion convection problem to be analyzed is given as

$$-\Delta u + b \cdot \nabla u = f \text{ in } \Omega \tag{1.14}$$

$$u = g \text{ on } \partial\Omega \tag{1.15}$$

The straight forward LSFEM approach is to define $w = -\nabla u$ and solve the system of equations

$$w + \nabla u = 0 \text{ in } \Omega \tag{1.16}$$

$$\nabla \cdot w - b \cdot w = f \text{ in } \Omega \tag{1.17}$$

$$u = 0 \text{ on } \partial\Omega. \tag{1.18}$$

which can be written in the same form as 1.2 with $u = w \oplus u$, $f = (0, 0, f)$, g $= 0$, $\mathcal{B} = (0, 0, 1)^T$ and L given as

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & \partial/\partial x \\ 0 & 1 & \partial/\partial y \\ \partial/\partial x - b_1 & \partial/\partial y - b_2 & 0 \end{bmatrix} \tag{1.19}$$

We define the search space $X = H^1(\Omega; \text{div}) \times H_0^1(\Omega)$ and the solution space $Y \times B = [L^2(\Omega)]^3 \times L^2(\Omega)$ and the functional can then be defined as in 1.3. The variational formulation of the problem can be stated. Find $u \in X$ s.t.

$$Q(u, \phi) = F(\phi) \;\; \forall \;\; \phi \in X. \tag{1.20}$$

We require that $f \in Y$. Notice that the spaces X and Y chosen as described above fullfill the condition 1.5.

1.4 basis due to jiang

[2] The Least Squares Finite Element Method is a numerical method similar to mixed galerkin, however it assures a symmetric problem. Let us look at a system of first order differential equations on the form

$$Au = f \text{ in } \Omega \tag{1.21}$$

$$Bu = g \text{ on } \partial\Omega. \tag{1.22}$$

Where A is a partial differential operator defined as

$$A = \sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i} + A_0. \tag{1.23}$$

n beeing the number of dimensions of the domain Ω. Let us initially assume that $g = 0$. Further we require $f \in L_2(\Omega)$ and choose $V = \{v \in L_2(\Omega) | v = 0 \text{ on } \partial\Omega\}$. A residual is defined

$$R(v) = Av - f, \tag{1.24}$$

and a functional

$$J(v) = \frac{1}{2}||R(v)||_0^2. \tag{1.25}$$

The solution u is restricted to the space $H_0^1(\Omega)$. By minimizing J we obtain

$$\lim_{t\to 0} \frac{d}{dt} I(u + tv) = \int_\Omega (Av)^T (Au - f) d\Omega = 0 \text{ , } \forall v \in V. \tag{1.26}$$

We can now write a variational formulation of the least-squares method: Find $u \in V$ such that

$$B(u, v) = F(v) \quad , \quad \forall v \in V, \tag{1.27}$$

where

$$B(u, v) = (Au, Av), \tag{1.28}$$

$$F(v) = (f, Av). \tag{1.29}$$

[2] Notice that the bilinear form B is symmetric. The bilinear form that surged from a first-order problem by the LSFEM leads us to a variational formulation similar to the one obtained from a second order problem by regular FEM. Generally the bilinear form from LSFEM will correspond to a bilinear form of a problem of twice the order obtained using FEM. In order to avoid problems of large complexity a higher order PDE should therefore be transformed to a system of first order PDE's before applying the LSFEM-method.

In order to apply a numerical algorithm the domain Ω needs to be discretized, we name this discretization Ω_h. A set of basis functions $\{N\}_i$ is defined for $V_h = H_0^1(\Omega_h)$ such that the discrete variational formulation can be stated. Find $u_h \in V_h$ such that

$$B(u_h, v_h) = F(v_h) \quad , \quad \forall v_h \in V_h, \tag{1.30}$$

Chapter 2

Implementation

For the general problem 1.2 the functional Q will take the form

$$Q(u,v) = \int_{\Omega} (\mathcal{L}v)^T (\mathcal{L}u) d\Omega. \tag{2.1}$$

Implementing Q requires two sets of basis functions $\{N_i\}$ that describes the search and solution space. In this project assignment the search and solution space will be described by the the same set of basis functions which will depend on the method applied. u is discretized as

$$u_h = \sum_{I=0}^{K} a_I N_I. \tag{2.2}$$

Since equation 1.6 requires equality for all test functions in the search space we simply solve the equation for each basis function. We are therefore left with a system of K

equations. Equation 2.1 can then be written for each test function as

$$Q(u_h, N_I) = \int_\Omega (\mathcal{L}N_I)^T (\mathcal{L}u_h) d\Omega \tag{2.3}$$

$$= \int_\Omega (\mathcal{L}N_I)^T (\mathcal{L}\sum_{J=1}^{K} a_J N_J) d\Omega \tag{2.4}$$

$$= \sum_{J=1}^{K} \int_\Omega (\mathcal{L}N_I)^T (\mathcal{L}a_J N_J) d\Omega \tag{2.5}$$

$$= \sum_{J=1}^{K} \int_\Omega (\mathcal{L}N_I)^T (\mathcal{L}a_J N_J) d\Omega \tag{2.6}$$

$$= \sum_{J=1}^{K} \int_\Omega (\mathcal{L}N_I)^T (\mathcal{L}N_J) d\Omega \ \cdot a_J. \tag{2.7}$$

The total system of equation for all test functions can then be written as a matrix equation

$$Au = F. \tag{2.8}$$

Where $A_{I,J} = \int_\Omega (\mathcal{L}N_I)^T (\mathcal{L}N_J) d\Omega$. For the poisson equation $A_{I,J}$ will be a 3-by-3 block matrix on the form

$$A_{I,J} = \int_\Omega \begin{bmatrix} N_I N_J + N_{I,x} N_{J,x} & N_{I,x} N_{J,y} & N_I N_{J,x} \\ N_{I,y} N_{J,x} & N_I N_J + N_{I,y} N_{J,y} & N_I N_{J,y} \\ N_{I,x} N_J & N_{I,y} N_J & N_{I,x} N_{J,x} + N_{I,y} N_{J,y} \end{bmatrix} d\Omega$$

Notice that by doing the splitting of variables we obtain a system of equations three times as big as if we were to solve the equation directly.

2.1 LSFEM for poisson

2.2 LS spectral method for poisson

The spectral implementation is done using Gauss Lobatto nodes and quadrature and the lagrange functions based on the GL nodes as basis functions. The final system of equations can be written as

Chapter 3

Results

3.1 The main differences

Using least squares will always give you a SPD system of equations which can be advantageous. However for second order equations this system is three times as big as if we were to solve it using more standard methods. Comparing the correctness of the solution as done in figure 3.1 shows that the convergence rate is the same as for standard methods, but the the value of the residual is slightly higher for the least squares method. This can be explained by the functional that is minimized. Notice that in the least squares methods you minimize the square of the residual. since the correctness of both methods are restricted by the smoothness of the solution and the number of discrete points LS-methods will minimize the residual squared down to a given precision and hence the residual itself to a slightly higher value. The condition number is also worth comparing . . .

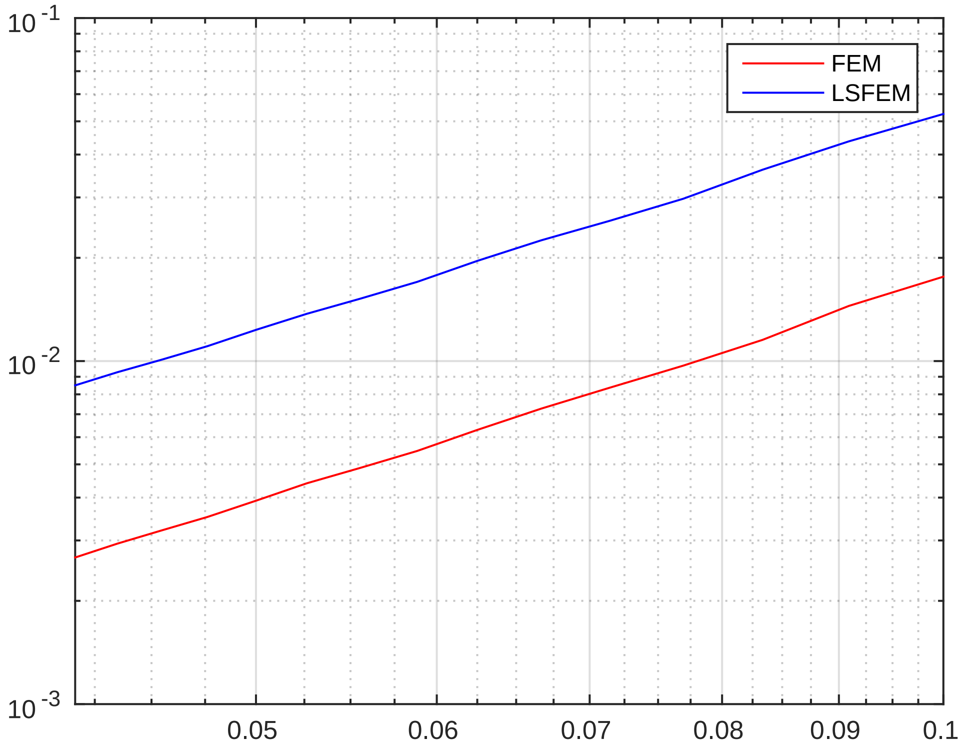

FIGURE 3.1: convergence of LSFEM and FEM

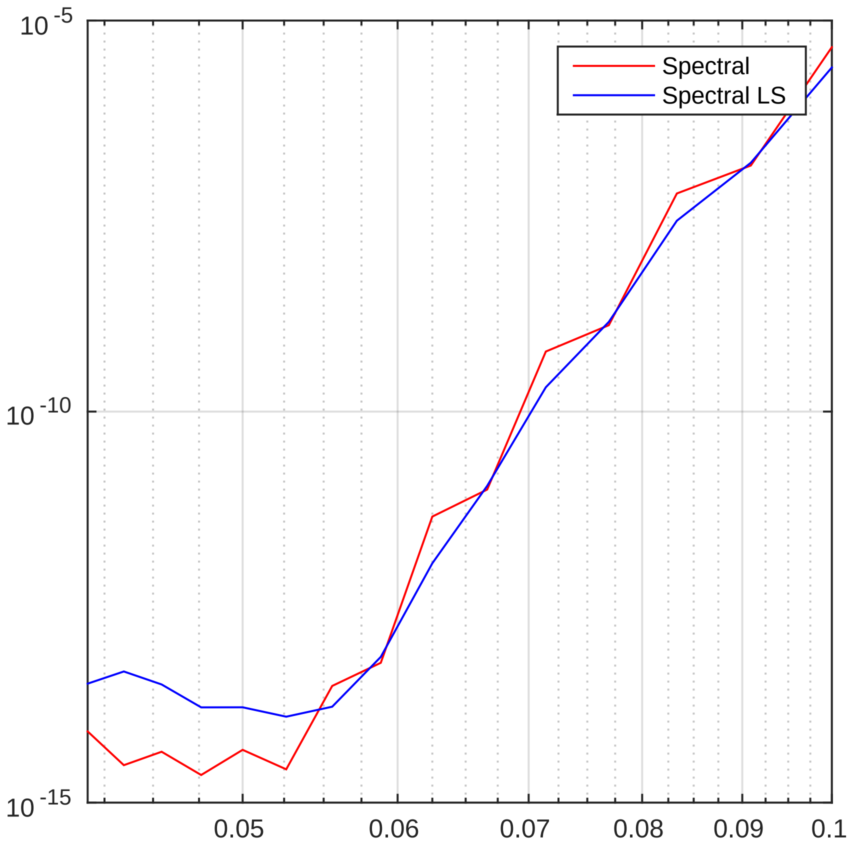

FIGURE 3.2: convergence of Spectral and LS-Spectral method

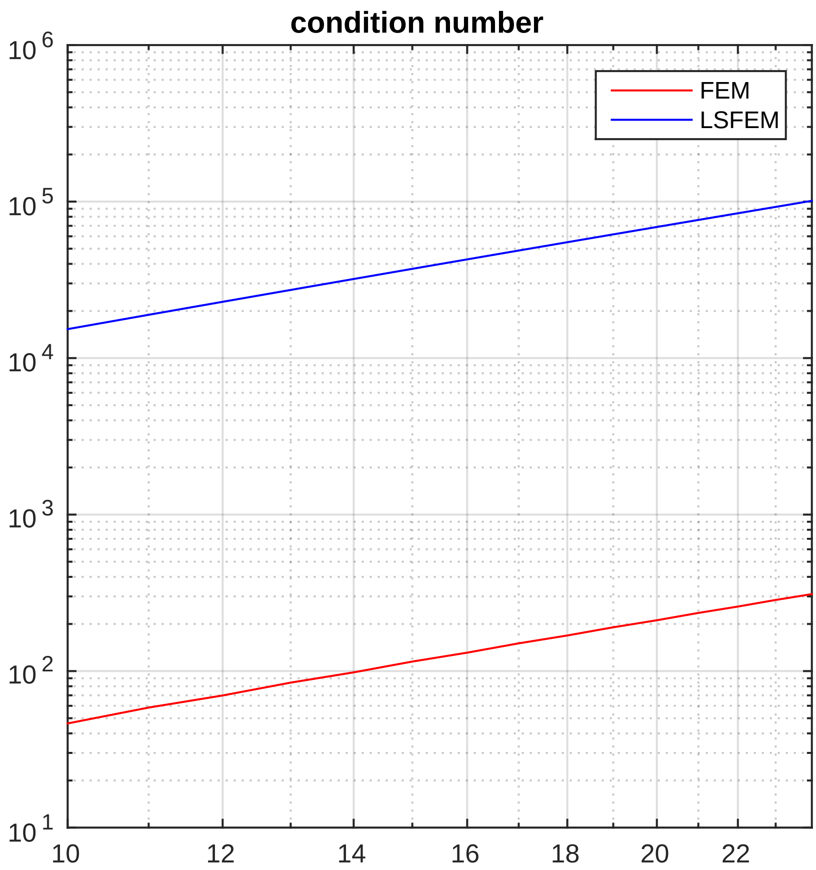

FIGURE 3.3: condition number of LSFEM and FEM

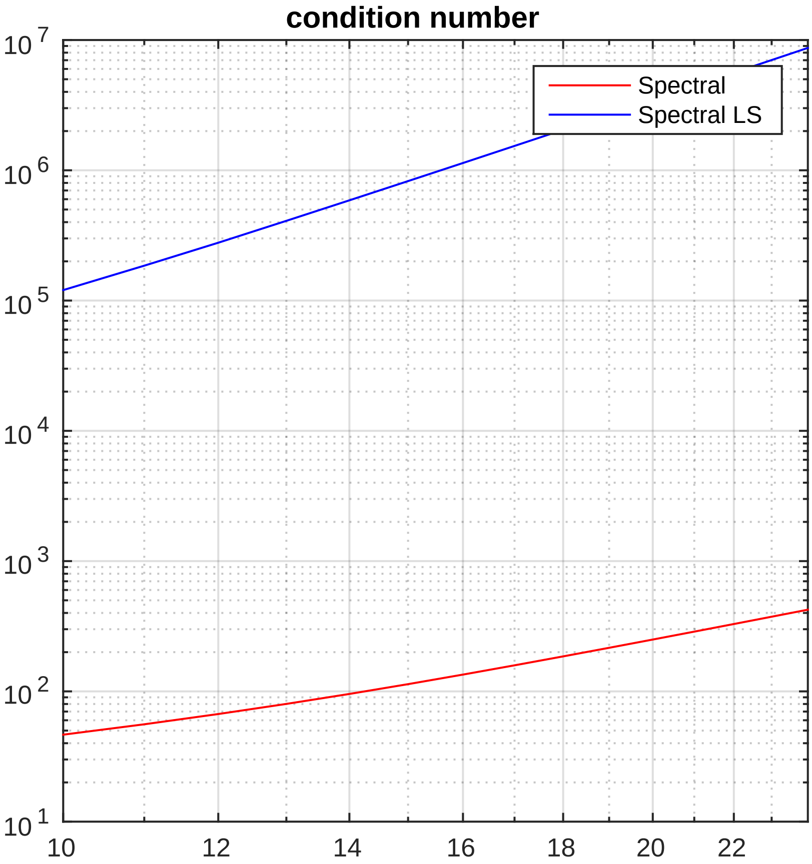

FIGURE 3.4: condition number of Spectral and LS-Spectral method

Appendix A

Appendix Title Here

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Bibliography

[1] Max D. Gunzburger Pavel B. Bochev. *Least-Squares Finite Element Methods*. Springer, 2009.

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