NTNU

PROJECT

Least Squares Finite Element Method

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Abstract

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Project

Least Squares Finite Element Method

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The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too...

Notation

LAH **L**ist **A**bbreviations **H**ere

Chapter 1

Theory

1.1 Basis of LSFEM

Let us look at a general boundary value problem where $f \in Y(\Omega)$, $g \in B(\partial\Omega)$, $B : X(\partial\Omega) \to B(\partial\Omega)$ and $L : X(\Omega) \to Y(\Omega)$. Find $u \in X(\Omega)$ such that

$$Lu = f \quad \text{in } \Omega \tag{1.1}$$

$$Bu = g \quad \text{on } \partial\Omega. \tag{1.2}$$

Whenever this BVP has a unique solution, a least-squares functional can be defined as

$$J(u; f, g) = ||Lu - f||_Y^2 + ||Bu - g||_B^2 \tag{1.3}$$

and the corresponding minimization problem is then given as

$$\min_{u \in X} J(u; f, g) \tag{1.4}$$

For any well-posed problem $\exists \alpha, \beta > 0$ such that

$$\beta ||u||_X \leq J(u; 0, 0) \leq \alpha ||u||_X. \tag{1.5}$$

The fact that our functional is norm-equivalent is of crucial importance to a successfull LS-method. Minimizing this functional is equivalent to solving the Euler-lagrange equations formulated as

$$\text{find } u \in X \text{ such that } Q(u,v) = F(v) \;\; \forall v \in X \tag{1.6}$$

Where $Q(u,v)$ is the continous bilinear form given as $(Lu, Lv)_Y$ and $F(v)$ is the bounded linear functional given as $(Lv, f)_Y$.

1.2 Example - Poisson problem

The poisson problem is defined as

$$-\Delta u = f \text{ in } \Omega \tag{1.7}$$

$$u = g \text{ on } \partial\Omega \tag{1.8}$$

The straight forward LSFEM approach is to define $w = -\nabla u$ and solve the system of equations

$$w + \nabla u = 0 \text{ in } \Omega \tag{1.9}$$

$$\nabla w = f \text{ in } \Omega \tag{1.10}$$

$$u = 0 \text{ on } \partial\Omega. \tag{1.11}$$

which can be written in the same form as 1.2 with $u = w \oplus u$, $f = (0,0,f)$, $g = 0$, $B = (0,0,1)^T$ and L given as

$$L = \begin{bmatrix} 1 & 0 & \partial/\partial x \\ 0 & 1 & \partial/\partial y \\ \partial/\partial x & \partial/\partial y & 0 \end{bmatrix} \tag{1.12}$$

We define the search space $X = H^1(\Omega; \mathrm{div}) \times H_0^1(\Omega)$ and the solution space $Y \times B = [L^2(\Omega)]^3 \times L^2(\Omega)$ and the functional can then be defined as in 1.3. The variational formulation of the problem can be stated. Find $u \in X$ s.t.

$$Q(u,\phi) = F(\phi) \;\; \forall \;\; \phi \in X. \tag{1.13}$$

We require that $f \in Y$. Notice that the spaces X and Y chosen as described above fullfill the condition 1.5.

1.3 Example - Diffusion convection problem

The diffusion convection problem to be analyzed is given as

$$-\Delta u + b \cdot \nabla u = f \text{ in } \Omega \tag{1.14}$$

$$u = g \text{ on } \partial\Omega \tag{1.15}$$

The straight forward LSFEM approach is to define $w = -\nabla u$ and solve the system of equations

$$w + \nabla u = 0 \text{ in } \Omega \tag{1.16}$$

$$\nabla \cdot w - b \cdot w = f \text{ in } \Omega \tag{1.17}$$

$$u = 0 \text{ on } \partial\Omega. \tag{1.18}$$

which can be written in the same form as 1.2 with $z = w \oplus u$, $f = (0, 0, f)$, $B = (0, 0, 1)^T$ and A given as

$$A = \begin{bmatrix} 1 & 0 & \partial/\partial x \\ 0 & 1 & \partial/\partial y \\ \partial/\partial x - b_1 & \partial/\partial y - b_2 & 0 \end{bmatrix} \tag{1.19}$$

By defining the residual and functional as in 1.24 and 1.25 the variational formulation of the problem can be stated. Find $z \in W \times H_0^1$ s.t.

$$B(z, \phi) = (F, \phi) \ \ \forall \ \ \phi \in W \times H_0^1. \tag{1.20}$$

With $W := \{w \in [L_2(\Omega)]^{n_d}\}$.

1.4 basis due to jiang

[1] The Least Squares Finite Element Method is a numerical method similar to mixed galerkin, however it assures a symmetric problem. Let us look at a system of first order differential equations on the form

$$Au = f \text{ in } \Omega \tag{1.21}$$

$$Bu = g \text{ on } \partial\Omega. \tag{1.22}$$

Where A is a partial differential operator defined as

$$A = \sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i} + A_0. \tag{1.23}$$

n beeing the number of dimensions of the domain Ω. Let us initially assume that $g = 0$. Further we require $f \in L_2(\Omega)$ and choose $V = \{v \in L_2(\Omega) | v = 0 \text{ on } \partial\Omega\}$. A residual is defined

$$R(v) = Av - f, \tag{1.24}$$

and a functional

$$J(v) = \frac{1}{2}||R(v)||_0^2. \tag{1.25}$$

The solution u is restricted to the space $H_0^1(\Omega)$. By minimizing J we obtain

$$\lim_{t\to 0} \frac{d}{dt} I(u + tv) = \int_\Omega (Av)^T (Au - f) d\Omega = 0 \text{ , } \forall v \in V. \tag{1.26}$$

We can now write a variational formulation of the least-squares method: Find $u \in V$ such that

$$B(u, v) = F(v) \quad , \quad \forall v \in V, \tag{1.27}$$

where

$$B(u,v) = (Au, Av), \tag{1.28}$$

$$F(v) = (f, Av). \tag{1.29}$$

[1] Notice that the bilinear form B is symmetric. The bilinear form that surged from a first-order problem by the LSFEM leads us to a variational formulation similar to the one obtained from a second order problem by regular FEM. Generally the bilinear form from LSFEM will correspond to a bilinear form of a problem of twice the order obtained using FEM. In order to avoid problems of large complexity a higher order PDE should therefore be transformed to a system of first order PDE's before applying the LSFEM-method.

In order to apply a numerical algorithm the domain Ω needs to be discretized, we name this discretization Ω_h. A set of basis functions $\{N\}_i$ is defined for $V_h = H_0^1(\Omega_h)$ such that the discrete variational formulation can be stated. Find $u_h \in V_h$ such that

$$B(u_h, v_h) = F(v_h) \quad , \quad \forall v_h \in V_h, \tag{1.30}$$

Appendix A

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Bibliography

[1] Bo-nan Jiang. *The Least-Squares Finite Element Method*. Springer Berlin Heidelberg, 1998. ISBN http://id.crossref.org/isbn/978-3-662-03740-9. doi: 10.1007/978-3-662-03740-9. URL http://dx.doi.org/10.1007/978-3-662-03740-9.