Vector Calculus

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August 24, 2025

Contents

0 Introduction

This is mostly a summary of the lectures given by Mats Ehrnström for MA1103 Vector Calculus spring 2022. Might also be relevant for TMA4105 Mathematics 2, idk

0.1 Notation

(Ignore this section unless some notation is unclear later)

- The notation used for vectors will be both $\vec{x}$ and $\mathbf{x}$, and we will consider only real vectors, thus:

$$\vec{x}, \mathbf{y} \in \mathbb{R}^n \quad \text{for } n \geq 2$$

$$\vec{x} = \mathbf{x}$$

- $C^m(\mathcal{D}, \mathcal{C})$ is the class of functions $f : \mathcal{D} \to \mathcal{C}$ which are (at least) m times continuously differentiable on the domain $\mathcal{D}$, and with codomain $\mathcal{C}$

- A set U has border ∂U. We will consider sets $U \subseteq \mathbb{R}^k, \quad k \geq 1$. A point $\mathbf{x} \in \partial U$ if

  $$\forall \epsilon > 0 : \quad \{\mathbf{y} \in \mathbb{R}^k; |\mathbf{x} - \mathbf{y}| < \epsilon\} \cap U \neq \emptyset \quad \wedge \quad \{\mathbf{y} \in \mathbb{R}^k; |\mathbf{x} - \mathbf{y}| < \epsilon\} \not\subseteq U$$

  If $U = U \cup \partial U$, then U is *closed*, if $\partial U \cap U = \emptyset$ then U is *open*

  - Note $\{\mathbf{y} \in \mathbb{R}^k; |\mathbf{x} - \mathbf{y}| < \epsilon\}$ is just a k-dimentional ball with radius ϵ and center in $\mathbf{x}$

1 Geometry

1.1 Curves

Definition 1 (Parametric curve). *A parametric curve γ is a continuous function*

$$\gamma : I \to \mathbb{R}^n$$
$$t \mapsto (\gamma_1(t), \gamma_2(t), ..., \gamma_n(t))^{\mathrm{T}}$$

where $I \subseteq \mathbb{R}$ is an interval

Definition 2 (Smooth curve). *$\gamma(t)$ is smooth if*

$$|\dot{\gamma}(t)| = \sqrt{(\dot{\gamma}_1(t))^2 + (\dot{\gamma}_2(t))^2 + ... + (\dot{\gamma}_n(t))^2} \neq 0$$

Definition 3 (Arc length). *The arc length on $[a, b]$ is the integral*

$$L(\gamma) = \int_a^b ds = \int_a^b |\dot{\gamma}(t)| dt$$

Note the element $ds := |\dot{\gamma}(t)| dt$

Theorem 1 (differentiation of curves). *let u, v differentiable $I \to \mathbb{R}^n$*

1. $(u \cdot v)' = u' \cdot v + u \cdot v'$
2. $(u \times v)' = u' \times v + u \times v'$
3. *For $\lambda : \mathbb{R} \to I$; $[u(\lambda(t))]' = u'(\lambda(t))\lambda'(t)$*

1.2 Some special vectors

Assume $\gamma : I \mapsto \mathbb{R}^n$ is smooth

Definition 4 (unit tangent)**.**

$$T := \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}$$

Definition 5. *(Curvature)*

$$\kappa := |\frac{dT}{ds}|$$

Lemma 1 (computation of curvature)**.**

$$\kappa = |\frac{dT}{ds}| = |\frac{dT}{dt}\frac{dt}{ds}| = |\frac{dT}{dt}| \cdot \frac{1}{|\dot{\gamma}(t)|}$$

Definition 6 (unit normal)**.**

$$N := \frac{dT/ds}{|dT/ds|} = \frac{1}{\kappa}\frac{dT}{ds} = \frac{\frac{d}{dt}T}{|\frac{d}{dt}T|}$$

Note: As $|T(t)| = 1$, $\frac{d}{ds}[T(t)T(t)] = 2\frac{d}{ds}T(t) \cdot T(t) = \frac{d}{ds}1 = 0$ Thus we have found a vector, $\frac{d}{ds}T(t)$ which must be perpendicular to $T(t)$. The unit normal also happens to tell us in which direction the curve is bending towards at location $(t, \gamma(t))$.

Note also that κ is to N, as $|\dot{\gamma}(t)|$ is to T

Definition 7 (binormal vector)**.** *For $n = 3$, as in $\mathbb{R}^n$*

$$B = T \times N$$

Then $B \perp T, B \perp N, |B| = 1$

1.3 Polar coordinates

2 Differentiation

2.1 Schwarz' theorem

Theorem 2 (Schwarz)**.** *If all of*

2.2 Jacobi matrix

2.3 Hesse matrix

2.4 Taylor formulas

Theorem 3 (Intermediate value theorem)**.** *Let $F : U \subseteq \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in U$ $\gamma : t \mapsto (1-t)\mathbf{x} + t\mathbf{y}$ such that $t \in [0,1] \Rightarrow \gamma(t) \in U$*

Then: $\exists \tau \in [0,1]$;

$$F(\mathbf{y}) - F(\mathbf{x}) = \nabla F(\gamma(\tau)) \cdot (\mathbf{y} - \mathbf{x}) \tag{1}$$

Proof. Skipped $\square$

In the intermediate value theorem, Theorem 3 we simply choose two points in the domain U, and draw a line γ between them. Then there must exist a point on that line ($\gamma(\tau)$) at which F has exactly the average slope of the line between $F(\mathbf{y})$ and $F(\mathbf{x})$.

Rearanging Equation 1 gives

$$F(\mathbf{y}) = F(\mathbf{x}) + \nabla F(\mathbf{c}) \cdot (\mathbf{y} - \mathbf{x}), \quad \mathbf{c} = \gamma(\tau)$$

which we might consider the taylor polynomial of 0^{th} degree

Theorem 4 (Taylor polynomial degree 2)**.**

$$F(\mathbf{x} + \mathbf{h}) = F(\mathbf{x}) + \mathbf{h} \cdot \nabla F(\mathbf{x}) + \mathbf{h}^{\mathrm{T}}[D^2 F(\mathbf{x})]\mathbf{h} + \mathcal{O}(|\mathbf{h}|^3) \tag{2}$$

Note that $[D^2 F(\mathbf{x})]$ *is a Hesse matrix, such that* $\mathbf{h}^{\mathrm{T}}[D^2 F(\mathbf{x})]\mathbf{h}$ *is a regular matrix quadratic form with column vector* $\mathbf{h}$*, while* $\mathbf{h} \cdot \nabla F(\mathbf{x})$ *is the regular dot product between two* $\mathbb{R}^n$*-vectors*

Proof. Skipped $\square$

3 Vector functions

3.1 Implicit function theorem

3.2 Inverse function theorem

Theorem 5 (Inverse function theorem)**.** *For* $\vec{F} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ *where* U *is open,* $\mathbf{x} \in U$ *is fixed and the* $n \times n$ *Jacobi matrix* $[\vec{F}(\mathbf{x})]$ *is invertible:*

$$\Rightarrow \exists \tilde{U} \ni \mathbf{x}; \quad \vec{F} : \tilde{U} \to \vec{F}(\tilde{U})$$

is invertible.

Further:

$$[D(\vec{F}^{-1})](\mathbf{y}) = [D\vec{F}]^{-1} \circ \vec{F}^{-1}(\mathbf{y})$$

is continuous on its domain $F(\tilde{U})$

Note that $[D\vec{F}]$ *is a Jacobi matrix to be inverted at* $\vec{F}^{-1}(\mathbf{y})$*, which is a* $\mathbb{R}^n$*-vector*

Corollary 1 (one-domentional inverse function theorem)**.** $f : \mathbb{R} \to \mathbb{R}$ *Let* $y = f(x)$*. If* $\dfrac{\mathrm{d}f}{\mathrm{d}x} = f'(x) \neq 0$*:*

$$\frac{\mathrm{d}f^{-1}}{\mathrm{d}y}(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}$$

3.3 Lagrange multiplier

Let:

$$F : U \subseteq \mathbb{R}^n \to \mathbb{R}$$
$$G : U \subseteq \mathbb{R}^n \to \mathbb{R}$$

We want to find the n-vector $\mathbf{x} \in U$ which minimizes $F(\mathbf{x})$ while guaranteeing $G(\mathbf{x}) = 0$

Theorem 6 (the Lagrange multiplier)**.** *Any local minimum of $F \in C^1(U, \mathbb{R})$ subject to $\{\mathbf{x} \in U; G(\mathbf{x}) = 0\}$, for a $G \in C^1(U, \mathbb{R})$ on an open $U \subseteq \mathbb{R}$ must be where*

$$\nabla F = \lambda \cdot \nabla G \tag{3}$$

for a real number λ

Proof. Let the C^1 curve $\gamma(t)$ be a parameterization of $\{\mathbf{x}; G(\mathbf{x}) = 0\}$, where the minimum of F along $\gamma(t)$ is $\mathbf{x}_0 = \gamma(t_0)$. A minimum of F requires:

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t))\Big|_{t=t_0} = 0 = \nabla F(\gamma(t_0)) \cdot \dot{\gamma}(t_0) \quad \text{(chain rule)}$$

Thus the gradient ∇F is orthogonal to the tangent of every parameterization $\gamma(t)$ of $\{\mathbf{x}; G(\mathbf{x}) = 0\}$ at the minimum $\mathbf{x}_0$: $F(\mathbf{x}_0) \perp \gamma(\dot{t}_0)$

The minimum is by definition required to satisfy

$$G(\mathbf{x}_0) = 0 \Rightarrow \nabla G(\gamma(t_0)) \cdot \dot{\gamma}(t_0) = 0 \quad \text{(using the same } \gamma(t))$$

Which also gives $G(\mathbf{x}_0) \perp \gamma(\dot{t}_0)$

Thus:

$$\nabla F \parallel \nabla G$$

which we write using a real number λ:

$$\nabla F = \lambda \cdot \nabla G$$

□

4 Integrals

4.1 Tornelli-Fubini

4.2 Variable substitution

4.3 Conservative Fields

4.4 Greens theorem

4.5 Gauß' theorem

4.6 Kelvin-Stokes