Simplified Energy Transfer Model

This file contains the authors solution to a problem given in the course TMA4195 Mathematical Modelling at NTNU.

Canonical model. By the Stefan–Boltzmann law, the total energy radiated from a body is given by $\varepsilon\sigma T^4$, where ε is the emissivity of the body relative to a black body and σ is the Stefan–Boltzmann constant. We assume energy-equilibrium in the earth-atmosphere system, so that the total energy radiated has to equal the incoming energy for both the earth and the atmosphere. This yields the system

$$\varepsilon_E \sigma T_E^4 = R_0^E \overset{\text{equil.}}{=} P_{\text{in}}^E + R_{\text{in}}^E - P_{\text{latent}} - P_{\text{E}\to\text{A}}, \tag{1}$$

$$\varepsilon_A \sigma T_A^4 = R_0^A \overset{\text{equil.}}{=} P_{\text{in}}^A + R_{\text{in}}^A + P_{\text{latent}} + P_{\text{E}\to\text{A}}. \tag{2}$$

where

$$P_{\text{latent}} + P_{\text{E}\to\text{E}} = (\alpha+\beta)(T_E - T_A).$$

To simplify the testing of different approaches and different transfer coefficients, we formulate a canonical model. Let

$$\begin{aligned} P_{\text{in}}^E &= \gamma_0 P_0^S, \\ P_{\text{in}}^A &= \gamma_1 P_0^S. \end{aligned}$$

and

$$\begin{aligned} R_{\text{in}}^E &= \lambda_0 R_0^E + \mu_0 R_0^A, \\ R_{\text{in}}^A &= \lambda_1 R_0^E + \mu_1 R_0^A, \end{aligned} \tag{3}$$

Equations (1) and (2) can be written as a system for the temperatures as follows.

$$a_0 T_E^4 + b_0 T_A^4 + c_0 T_E + d_0 T_A = e_0, \tag{4}$$

$$a_1 T_E^4 + b_1 T_A^4 + c_1 T_E + d_1 T_A = e_1. \tag{5}$$

In the above notation, the coefficients are given by

$$\begin{aligned} a_0 &= \sigma\varepsilon_{\text{E}}(1-\lambda_0) & a_1 &= -\sigma\varepsilon_{\text{E}}, \\ b_0 &= -\sigma\varepsilon_{\text{A}}\mu_0 & b_1 &= \varepsilon_{\text{A}}\sigma(1-\mu_1), \\ c_0 &= \alpha+\beta & c_1 &= -(\alpha+\beta), \\ d_0 &= -(\alpha+\beta) & d_1 &= \alpha+\beta, \\ e_0 &= \gamma_0 P_0^S & e_1 &= \gamma_1 P_0^S \end{aligned}$$

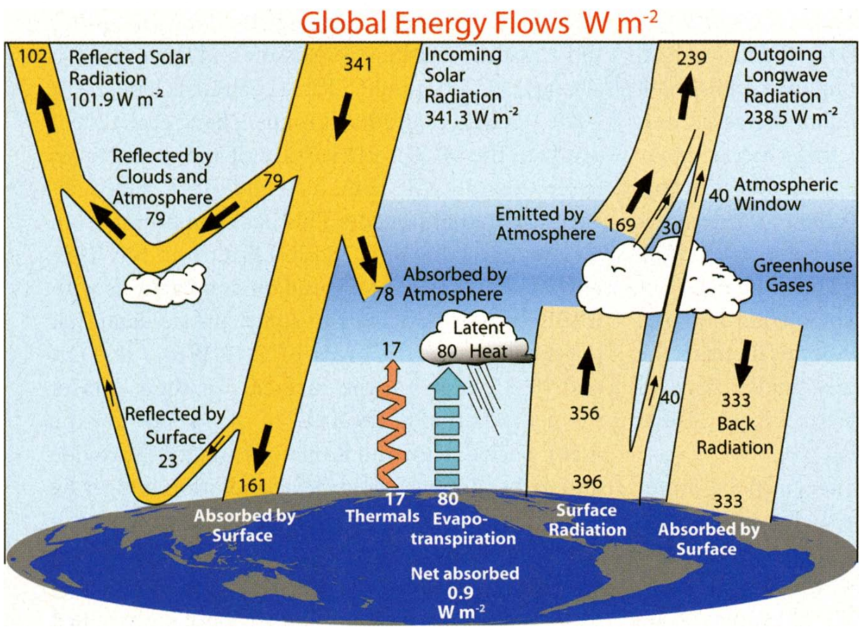

Figure 1: Empirical energy flows from [1].

In order to solve this system we need to express the coefficients γ_0, γ_1, λ_0, λ_1, μ_0, μ_1 in terms of the transfer coefficients provided in the problem description.

Transfer coefficients. We use the energy transfer approach. Ideally, we should be able to reproduce the empirical energy flows as given in Figure 1. in the problem description, and to find an equilibrium temperature for the earth that is close to some measured mean global surface temperature.

We assume that the atmosphere is a single layer. For incoming high-frequency solar radiation, the earth and the atmosphere absorbs energy according to

$$\begin{aligned}
\gamma_0 =&(1-r_{\mathrm{SM}})(1-a_{\mathrm{SW}})(1-a_{\mathrm{O_3}})(1-C_{\mathrm{C}}r_{\mathrm{SC}})(1-C_{\mathrm{C}}a_{\mathrm{SC}})(1-r_{\mathrm{SE}}),\\
\gamma_1 =&\big((1-r_{\mathrm{SM}})a_{\mathrm{SW}}+(1-r_{\mathrm{SM}})(1-a_{\mathrm{SW}})a_{\mathrm{O_3}}\\
&+(1-r_{\mathrm{SM}})(1-a_{\mathrm{SW}})(1-a_{\mathrm{O_3}})(1-C_{\mathrm{C}}r_{\mathrm{SC}})C_{\mathrm{C}}a_{\mathrm{SC}}\big).
\end{aligned}$$

Using $P_0^S = 341\ Wm^{-2}$, we get absorbed high-frequency radiation equal to 156 Wm^{-2} and 82 Wm^{-2} for the earth and the atmosphere, respectively. This is in close agreement with the energy flows given in the project description.

The real challenge lies in determining appropriate coefficients for the low-frequency dynamics governed by (3). A given fraction of the atmosphere is covered by clouds, which absorb and reflect low-frequency radiation, but no emissivity is given in the project description. Moreover, it is unclear exactly how to model clouds using transfer coefficients since we assumed the atmosphere to be a single, one-dimensional layer.

To overcome this difficulty, we propose new aggregate reflection and absorption coefficients for the fraction of the atmosphere where clouds are present. Assume for the moment that the atmosphere has a vertical height, and that the clouds are on average in the middle of the atmosphere. If we assume

$$\begin{aligned}a_{\mathrm{LAC}} =&\frac{1}{2}a_{\mathrm{LW}} + (1-\frac{1}{2}a_{\mathrm{LW}})(1-r_{\mathrm{LC}})a_{\mathrm{LC}} + (1-\frac{1}{2}a_{\mathrm{LW}})(1-r_{\mathrm{LC}})(1-a_{\mathrm{LC}})a_{\mathrm{LW}},\\ r_{\mathrm{LAC}} =&(1-\frac{1}{2}a_{\mathrm{LW}})^2 r_{\mathrm{LC}},\end{aligned} \tag{6}$$

then it follows that

$$\begin{aligned}\lambda_0 =&C_C(1-r_{\mathrm{LE}})r_{\mathrm{LAC}},\\ \mu_0 =&f_A(1-r_{\mathrm{LE}}),\\ \lambda_1 =&C_C(1-r_{\mathrm{LAC}})a_{\mathrm{LAC}} + (1-C_C)a_{\mathrm{LW}},\\ \mu_1 =&0.\end{aligned} \tag{7}$$

This system has solution

$$T_E = 8.5\ C^\circ, \qquad T_A = -8.2\ C^\circ,$$

found numerically by Newtons method. The low-frequency energy flows at equilibrium are $R^E_{\mathrm{in}} = 318\ Wm^{-2}$, $R^A_{\mathrm{in}} = 290\ Wm^{-2}$ and $P_{\mathrm{latent}} + P_{\mathrm{E\to E}} = 117$ Wm^{-2}. This system therefore seem to underestimate the temperature difference between the earth and the atmosphere.

To find the sensitivity of the equilibrium temperature to changes in the transfer coefficients, we differentiate the system with respect to the different parameters. It is possible to find exact expressions for the partial derivatives by implicit partial differentiation. However, since the sensitivity depends on the equilibrium temperature, which is found numerically with newtons method, and since we are only interested in the values of the derivatives around current equilibrium values, we find the partial derivatives numerically. Values of derivatives, in addition to change in temperature resulting from a 2% change in parameter value are shown in Table 1.

Comments and improvements. The transfer coefficients for the low-frequency system was determined by a combination of reasoning and trial-and-error. Model improvements could have been pursued using at least two different strategies.

Firstly, using the same model as above, the values for model sensitivity could have been used as a reference for tuning the transfer parameters as given in the project description, so that the model would more accurately mimic the empirical energy flows.

Secondly, one could have considered the clouds to have a different temperature than the atmosphere and to be a transfer layer in itself, yielding a system with three equations instead of two. The situation however becomes increasingly more complex when adding transfer layers.

Parameter	∂T_E	∂T_A	ΔT_E (2%)	ΔT_E (2%)
r_{SM}	-87.27	-73.93	-1.75	-1.48
r_{SC}	-46.34	-37.00	-0.93	-0.74
r_{SE}	-67.58	-52.66	-1.35	-1.48
$a_{\mathrm{O_3}}$	7.97	22.88	0.16	0.46
a_{SC}	-1.20	8.52	-0.02	0.17
a_{SW}	8.43	24.2	0.17	0.48
r_{LC}	-27.46	-34.66	-0.60	-0.69
r_{LE}	-120.18	-93.65	-2.40	-1.87
a_{LC}	-3.42	3.49	0.07	0.07
a_{LW}	52.79	56.22	1.06	1.12
f_A	182.32	142.06	3.65	2.84
α & β	-1.18	0.26	-0.02	0.01

Table 1: Model sensitivity [K].

Code. All code written to produce the above results are available at https://github.com/magnusco/simple_climate_model.

References

[1] Kevin E Trenberth, John T Fasullo, and Jeffrey Kiehl. *Earth's global energy budget*. Bulletin of the American Meteorological Society 90.3 (2009), pages 311–324.