Convex Optimization Project Report

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1 Single-User MIMO

We assume a communications scenario, where the transmitter and receiver use N and M antennas, respectively. This yields a channel matrix $\mathbf{H} \in \mathbb{C}^{M\times N}$, whose entries $h_{\textbf{row, column}}$ encode the channel gain at receiver 'row' when receiving a signal from transmitter 'column'. Overall, the transmission behavior yields the formula

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \quad \mathbf{n} \sim \mathcal{N}_C(\mathbf{0}, \mathbf{R}_\mathbf{n}),$$

where $\mathbf{n} \in \mathbb{C}^M$ represents the coloured noise with positive semi-definite covariance matrix $\mathbf{R}_\mathbf{n} \in \mathbb{C}^{M\times M}$. Every product of a row from the matrix $\mathbf{H}$ with the signal vector computes the superposition of the transmitted symbols that is received at a single receiving antenna. For our MIMO setting, we want to optimize the channel capacity given by

$$\begin{aligned} C &= \log\left((\pi e)^N \det(\mathbf{H}\mathbf{P}\mathbf{H}^H + \mathbf{R}_\mathbf{n})\right) - \log\left((\pi e)^N \det(\mathbf{R}_\mathbf{n})\right) \\ &= \log\left(\det(\mathbf{H}\mathbf{P}\mathbf{H}^H\mathbf{R}_\mathbf{n}^{-1} + \mathbb{1}^{N\times N})\right) \end{aligned}$$

where $\mathbf{P} = \mathbb{E}[\mathbf{x}\mathbf{x}^H]$ is the signal covariance. For simplicity, we assumed normalized powers and a bandwidth of $B = 1\,\mathrm{Hz}$. Given the provided problem description, we want to optimize the capacity over this signal covariance. This results in a first natural conic constraint, as this desired matrix has to be positive semi-definite. In addition, we want to enforce arbitrary affine constraints on the individual antenna powers $P_i = \mathbb{E}[x_i x_i^*]$ as well as the sum-power $P_{sum} = \sum_{i=1}^N P_i$.
Altogether, we arrive at the optimization problem

$$\begin{aligned} \max_{\mathbf{P}} \quad & C \\ \text{subject to} \quad & \mathbb{E}[x_i x_i^*] = [\mathbf{P}]_{ii} = P_i \quad i \subseteq \{1, N\}, \\ & \sum_{j=1}^N [\mathbf{P}]_{jj} = \operatorname{trace}(\mathbf{P}) \le P_{\text{sum}}, \\ & \mathbf{P} \succeq \mathbf{0} \end{aligned}$$

Attention has to be paid regarding the different indices i and j for the constraints: the sum power concerns all transmitting antennas, while only a subset of them is further constrained. Additionally, CVXPy only provides the logarithm of base 10, which is why we need to divide all capacities by $\log_1 0(2)$ to get results with the unit bits.
The notebook added to this submission contains all relevant code. For the variables, we use a sum-power of 10 and randomly choose a subset of antennas with individual constraints. Additional information are printed in the notebook. We report the average capacity, rounded to the fifth decimal place in two ways:

- Average capacity over 100 runs for random channel matrix $\mathbf{H}$:

$$\mathbb{E}_\mathbf{H}[C] \approx \frac{1}{100}\sum_{i=1}^{100} C_i = 102.19314\,\text{bits}.$$

- Average capacity over 100 runs for random channel matrix $\mathbf{H}$ and random noise covariance $\mathbf{R}$:

$$\mathbb{E}_{\mathbf{H},\mathbf{R}}[C] \approx \frac{1}{100}\sum_{i=1}^{100} C_i = 102.22830\,\text{bits}.$$

2 Addition of a Co-Channel

As an extension to the above problem, we now add a single-antenna co-channel with channel vector $\mathbf{h}_c \in \mathbb{C}^N$, yielding a received symbol

$$y_c = \mathbf{h}_c^H \mathbf{x} + n_c,$$

where n_c is now a noise scalar drawn from a normal distribution $n_c \sim \mathcal{N}(0, \sigma_c^2)$. The objective is for this receiver to not receive any signal in expectation, only noise. Mathematically, we derive

$$\begin{aligned}
\mathbb{E}[||y_c||^2] = \mathbb{E}[||\mathbf{h}_c^H\mathbf{x} + n_c||^2] &= \mathbb{E}[\mathbf{h}_c^H\mathbf{x}\mathbf{x}^H\mathbf{h}_c + \mathbf{h}_c^H\mathbf{x}n_c + (n_c)^2] \\
&= \mathbb{E}[\mathbf{h}_c^H\mathbf{x}\mathbf{x}^H\mathbf{h}_c] + \mathbb{E}[\mathbf{h}_c^H\mathbf{x}n_c] + \mathbb{E}[n_c^2] \\
&\overset{(1)}{=} \mathbb{E}[\mathbf{h}_c^H\mathbf{P}\mathbf{h}_c] + \mathbb{E}[\mathbf{h}_c^H\mathbf{x}]\mathbb{E}[n_c] + \mathbb{E}[n_c^2] \\
&\overset{(2)}{=} \mathbb{E}[\mathbf{h}_c^H\mathbf{P}\mathbf{h}_c] + 0 + \sigma_c^2,
\end{aligned}$$

where in (1) we used the fact that $\mathbf{h}_c$ is independent of n_c to write the expectation as a product. (2) then follows from the definition of the noise distribution. The result shows that the give additional constraint for the co-channel is in fact equivalent to

$$\mathbb{E}[\mathbf{h}_c^H\mathbf{P}\mathbf{h}_c] = 0.$$

In order to realize the expectation objective, we have to adapt the initial setting to include multiple samples of channel matrices and vectors. In order to approximate the required expectations, we use Monte Carlo approximations. For k samples of channel matrix and co-channel vector, we define

$$\begin{aligned}
\max_{\mathbf{P}} \quad & \mathbb{E}_{\mathbf{H}}[C] \approx \frac{1}{k}\sum_{\ell=1}^{k} C_\ell \\
\text{subject to} \quad & \mathbb{E}[x_i x_i^*] = [\mathbf{P}]_{ii} = P_i \quad i \subseteq \{1, N\}, \\
& \sum_{j=1}^{N} [\mathbf{P}]_{jj} = \mathrm{trace}(\mathbf{P}) \leq P_{\text{sum}}, \\
& \mathbf{P} \succeq \mathbf{0}, \\
& \mathbb{E}_{\mathbf{h}_c}[\mathbf{h}_c^H\mathbf{P}\mathbf{h}_c] \approx \frac{1}{k}\sum_{\ell=1}^{k} \mathbf{h}_{c,\ell}^H\mathbf{P}\mathbf{h}_{c,\ell} = 0.
\end{aligned}$$