TTK4190 Guidance and Control of Vehicles

Assignment 1

Written Fall 2020 By Magnus Dyre-Moe, Patrick Nitschke and Siawash Naqibi

Problem 1 - Attitude Control of Satellite

You can answer Problem 1 in this file/section. One subsection for each part of the problem might be a good solution.

For Problem 1, the equations of motion for the satellite are:

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{T}_q(\mathbf{q})\boldsymbol{\omega} \\ \mathbf{I}_{CG}\dot{\boldsymbol{\omega}} - \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega} &= \boldsymbol{\tau} \end{aligned} \tag{1}$$

Problem 1.1

For the equilibrium point, we start by setting $\dot{\mathbf{q}} = \mathbf{0}$ and $\dot{\omega} = \mathbf{0}$. Next, $\mathbf{I}_g$ is 3x3 matrix with 720 on the diagonal, i.e. $\mathbf{I}_g = 720\mathbf{I}_3$. Also,

$$\mathbf{T_q}(\mathbf{q}) = \begin{bmatrix} 0 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \tag{2}$$

For the rest of this exercise, the top row from the matrix above will not be included in the analysis. This yields the equilibrium point:

$$\mathbf{x_0} = [\mathbf{q^T}, \omega^\mathbf{T}]^\mathbf{T} = [0, 0, 0, 0, 0, 0]^T \tag{3}$$

To linearise the spacecraft model, we calculate the jacobian using Matlab and insert $\mathbf{x_0}$:

$$\mathbf{A} = \begin{bmatrix} 0 & w_3/2 & -w_2/2 & q_1/2 & -q_4/2 & q_3/2 \\ -w_3/2 & 0 & w_1/2 & q_4/2 & q_1/2 & -q_2/2 \\ w_2/2 & -w_1/2 & 0 & -q_3/2 & q_2/2 & q_1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

And

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/720 & 0 & 0 \\ 0 & 1/720 & 0 \\ 0 & 0 & 1/720 \end{bmatrix} \tag{5}$$

Problem 1.2

Let $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\tau$ where A and B are as in assignment 1.1. Furthermore, let our control law be:

$$\tau = -\mathbf{K_d}\omega - k_p\epsilon \tag{6}$$

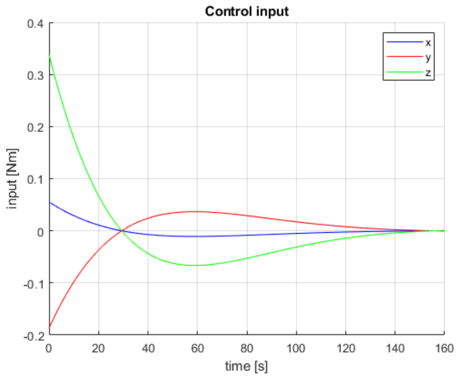
The closed loop system then gives $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$ where the new $\mathbf{A} - \mathbf{BK}$ matrix yields:

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \\ -k_p/720 & 0 & 0 & -k_d/720 & 0 & 0 \\ 0 & -k_p/720 & 0 & 0 & -k_d/720 & 0 \\ 0 & 0 & -k_p/720 & 0 & 0 & -k_d/720 \end{bmatrix} \tag{7}$$

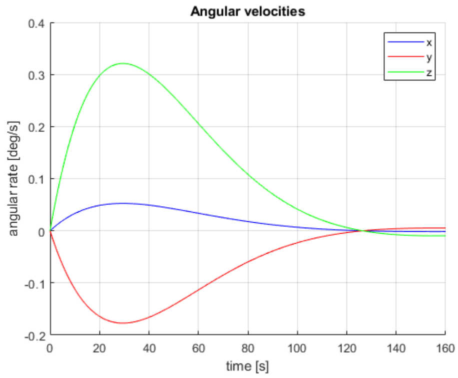
All eigenvalues of this system have a negative real part, and thus the system is stable as long as $k_p, k_d > 0$. We do not want complex poles, as this gives rise to oscillations. Ideally, we want real poles in the left half plane for critical damping. The poles for the satelite is complex, meaning an underdamped process. The poles we acheived are,

$$\lambda_{1,2,3,4,5,6} = -0.0278 \pm i0.0248$$

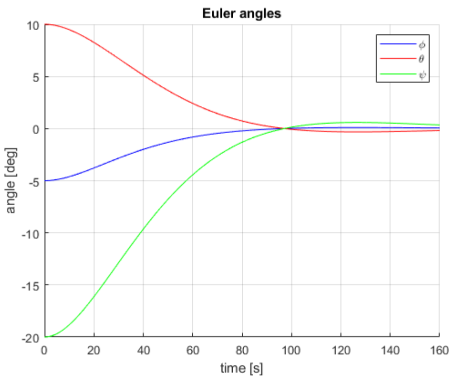
Problem 1.3



(a) Control input



(b) Angular velocities



(c) Euler angles

Figure 1: Simulating the satelite with the control law given in (6).

The euler angles converges to zero for ϕ, θ and ψ. From the closed loop analysis in Problem 1.2 we found the system to be stable with complex eigenvalues. Hence, the system converging was to be expected. Also, with a slight imaginary part we also expected some slight oscillations, which were also apparent in our observations. Here it is worth noting that we have expanded the time horizon from 400 to 1600 iterations. This is largely due to the reason that our system is quite slow, with the poles of our system being very close to zero.

Convergence to a non zero reference can easily be done by modifying the states to contain a constant reference, i.e $\tilde{\theta} = \theta - \theta_{ref}$.

Problem 1.4

We are given the control law,

$$\tau = K_d \omega - k_p \tilde{\epsilon} \tag{8}$$

The quaternion error can be written as,

$$\tilde{\mathbf{q}} := \begin{bmatrix} \tilde{\eta} \\ \tilde{\epsilon} \end{bmatrix} = \bar{\mathbf{q}}_d \otimes \mathbf{q} \tag{9}$$

where,

$$\bar{\mathbf{q}}_d \otimes \mathbf{q} = \begin{bmatrix} \eta_d \eta - \epsilon_d^T \epsilon \\ \eta_d \epsilon + \eta \epsilon_d + S(\epsilon_d)\epsilon \end{bmatrix} \tag{10}$$

When $\tilde{q}$ convereges, i.e. $q = q_d$, we get $\eta = \eta_d$ and $\epsilon = \epsilon_d$, resulting in the following matrix:

$$\tilde{q} = \begin{bmatrix} \eta^2 - \epsilon^T \epsilon \\ 2\eta\epsilon + S(\epsilon)\epsilon \end{bmatrix} = \begin{bmatrix} 1 - 2\epsilon^T \epsilon \\ 2\eta\epsilon \end{bmatrix} \tag{11}$$

By equalities $\eta = \sqrt{1 - \epsilon^T \epsilon}$ and $S(\epsilon)\epsilon = 0$ where $S(\cdot)$ is the skew symetric matrix.

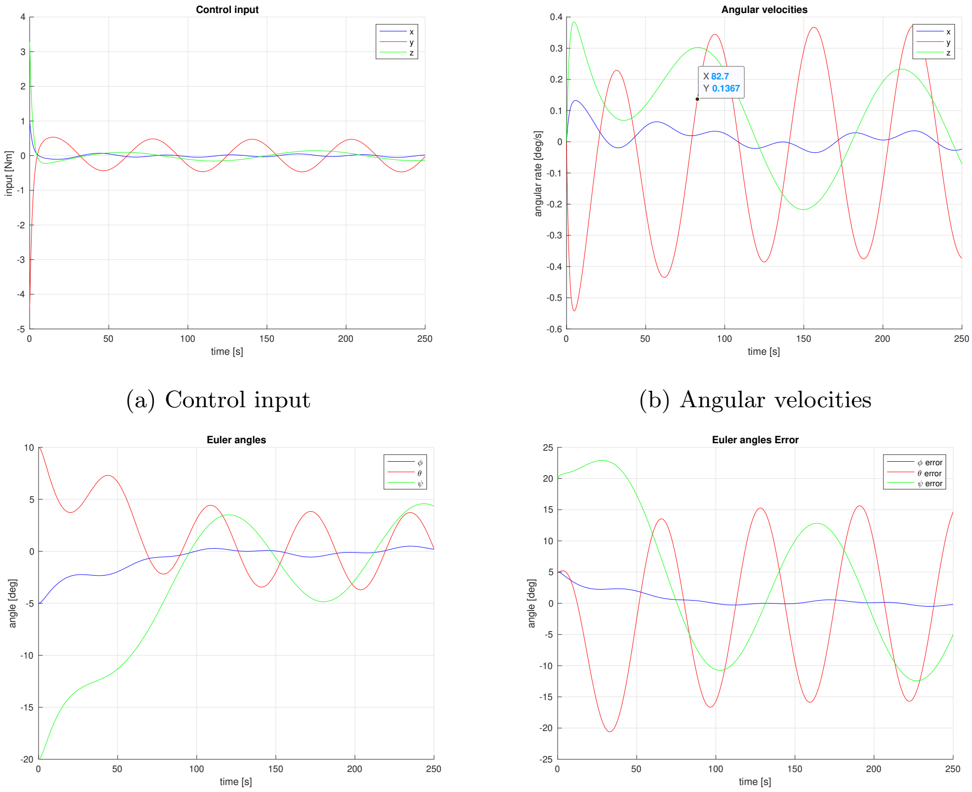
One could assume that by the convergence of $q = q_d$, we would expect that the quaternion error $\tilde{q}$, also becomes zero. However, this is not directly clear from equation (11), where we are slightly uncertain on how the result should be interpreted. (Considering that this could be an important result, any explanation would be appreciated.)

Problem 1.5

Implementing the control law given (8) with time varying references to ψ, θ, ϕ given by

$$\begin{aligned} \phi(t) &= 0 \\ \theta(t) &= 15cos(0.1t) \\ \psi(t) &= 10sin(0.05t) \end{aligned} \tag{12}$$

Yields the following response:



(a) Control input (b) Angular velocities

(c) Euler angles (d) Error in angles

Figure 2: Task 1.5: Simulating the satelite with time varying reference in position

With our reference being sinusoidal, our system behaves to a small degree of what we expected. Initially, this was unclear with a simulation time of 160 seconds, but by expanding the time axis to 250 seconds we saw that the system did in fact follow the sinusoidal pattern quite well, however it misses significantly on the time and on the desired amplitudes. This time delay and miss on amplitude can be seen on figure 2, subplot (d), where we observe significant errors in the angle of the satellite.

Our guess for this error is the fact that our system is incredibly slow, with poles very close to zero. As a consequence, the satellite struggles to match fast changes in the reference position, which is why the satellite cannot reach its desired amplitude and a time delay arises.

Problem 1.6

The control law in this problem can be written as,

$$\boldsymbol{\tau} = -\mathbf{K}_d\tilde{\boldsymbol{\omega}} - k_p\tilde{\boldsymbol{\epsilon}} \tag{13}$$

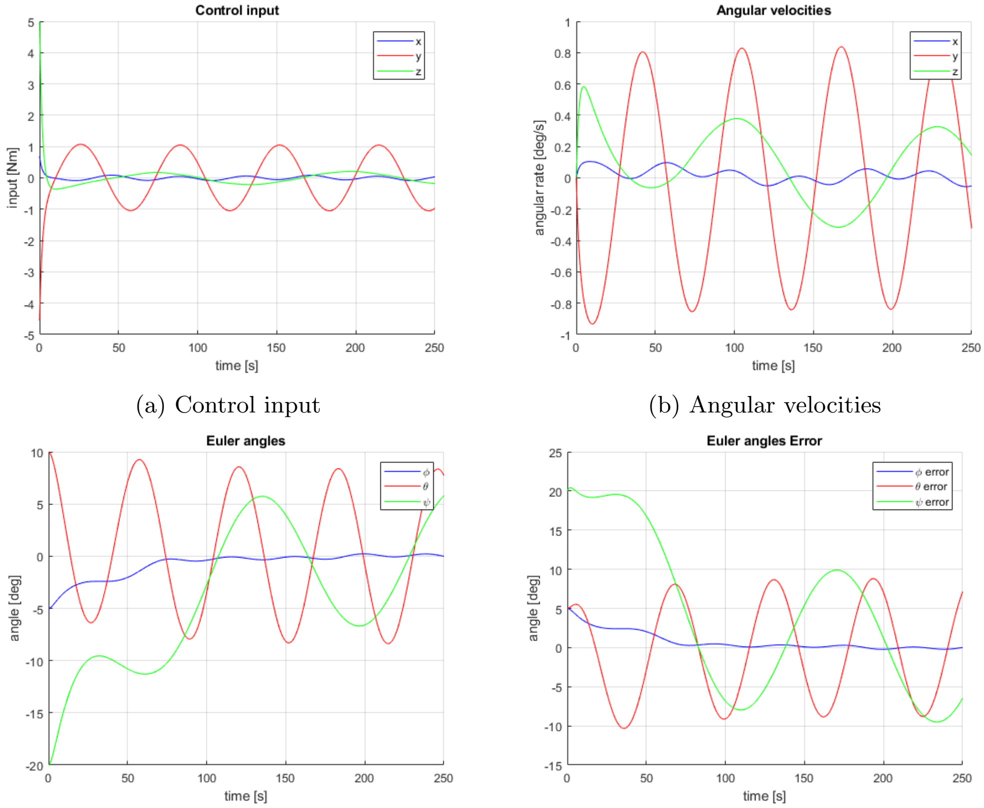
and the desired angular velocity as,

$$\boldsymbol{\omega}_d = \mathbf{T}_{\Theta_d}^{-1}(\Theta_d)\dot{\Theta}_d \tag{14}$$

Where $\mathbf{T}_{\Theta_d}^{-1}$ is the transformation from BODY to NED, and $\dot{\Theta}_d$ is the time differentiated reference in position, which yields:

$$\dot{\Theta}_d = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 \\ -1.5sin(0.1t) \\ 0.5cos(0.05t) \end{bmatrix} \tag{15}$$

Implementing the control law (13) gives us the following response:



(a) Control input

(b) Angular velocities

(c) Euler angles

(d) Error in angles

Figure 3: Task 1.6: Simulating the satelite with time varying reference in position and angular velocity

We see that the behaviour of the satellite is much more promising than that in figure 2, with a clear decrease in angle error. This is because the satellite manages to rotate to much larger angles, being more faster in general.

Yet, there still is large degree of error. This could possibly be improved by moving the system poles away from the origin, consequently making the system faster. However, testing simply with higher k_p and lower $\mathbf{K}_d$ values result in a unstable system, so essentially we are unsure. A PD-controller is the ideal controller for this system as well.

Problem 1.7

The Lyapunov function can be written as

$$V = \frac{1}{2}\tilde{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\tilde{\boldsymbol{\omega}} + 2k_p(1-\tilde{\eta}) \tag{16}$$

Where V is positive given the quadratic term always being positive or zero and $1-\tilde{\eta}$ being between zero and one by definition. V is also radially unbounded given V approaches infinity when omega approaches infinity.

Deriving $\dot{V}$ we are given $\omega_d = 0$ meaning $\tilde{\omega} = \omega$. We will derive $\dot{V}$ by using the system equation for $om\dot{e}ga$ and control law (8):

$$
\begin{aligned}
\dot{V} &= I_g \tilde{\omega} \dot{\tilde{\omega}} - 2k_p \dot{\tilde{\eta}} \\
&= I_g \omega \dot{\omega} + k_p \tilde{\epsilon}^T \omega \\
&= I_g \omega \dot{\omega} + (-K_d \omega - \tau)^T \omega \\
&= I_g \omega \dot{\omega} - K_d \omega^T \omega - \tau \omega \\
&= S(I_g \omega)\omega - k_d \omega^T \omega \\
&= -k_d \omega^T \omega
\end{aligned}
\tag{17}
$$

The Lyapunov global asymptotic stability states that if we have an energy function V which is positive definite for all inputs and the time differentiated energy function $\dot{V}$ is negative for all non-zero inputs and $\dot{V}(0) = 0$ the system is globally asymptotically stable.

We have a positive energy function V and $\dot{V}$ is indeed negative for all ω non-zero and $\dot{V}(0) = 0$. Hence the system is globally asymptotically stable.

A Matlab code

A.1 Prob1.m

```
display('Magnus has a')
pp = 'small pp'
display(pp)
```

References