Oblig05 — Fys2160 — 2015

Exercise 0.6. Ideal gas in low dimensions

In this project we will address the behavior of ideal gas in a one- or two-dimensional system for a system with given N, V, and T. First, we address the behavior of a one-dimensional gass of N particles. The particles can move freely along the x-axis over a distance L. Ignore the y- and z-directions. The motion of the particle is described by the state $n_x = 0, 1, 2, \ldots$, and the energy of state n_x is $\varepsilon(n_x) = an_x^2$, where $a = h^2/(8mL^2)$.

a) Show that the partition function for one particle in one dimension is

$$Z_{1,x} = \left(\frac{2\pi mkT}{h^2}\right)^{1/2} L \; . \tag{0.16}$$

You may need the integral

$$\int_0^\infty e^{-\lambda^2 x^2} dx = \frac{\sqrt{\pi}}{2\lambda} \; . \tag{0.17}$$

b) Find the partition function for a gas of N particles moving in one dimension.

c) Find the energy of the one-dimensional gas as a function of T.

d) Find the entropy for the gas.

Assume that the particles may move freely in the x and y plane over an area from 0 to L along each axis.

e) Find the partition function, energy and entropy of the gas.

By limiting the dimension of the gas, we also limit what aspects of the gas we consider. In a quantum-mechanical consideration of the system, both the wave function and the Hamilton-operator may be separated in x, y, and z-components by $\psi(x)\psi(y)\psi(z)$. The energy eigenstates can therefore be written as $E = E_{xy} + E_z$. We may consider the two-dimensional system as a special case of a three-dimensional system where one edge have be reduced to a size L_z. In this case, the rest energy in the z-direction becomes large, and the spacing of the levels for the z-direction becomes large. We can therefore assume that all the particles will be in the ground state in the z-direction.

f) Show that a reasonable partition function for the two-dimensional system, when we consider motion in the x, y, and z-direction will be

$$Z = Z_{2d} \left(\exp(-\beta\varepsilon_{z0})\right)^N \; , \tag{0.18}$$

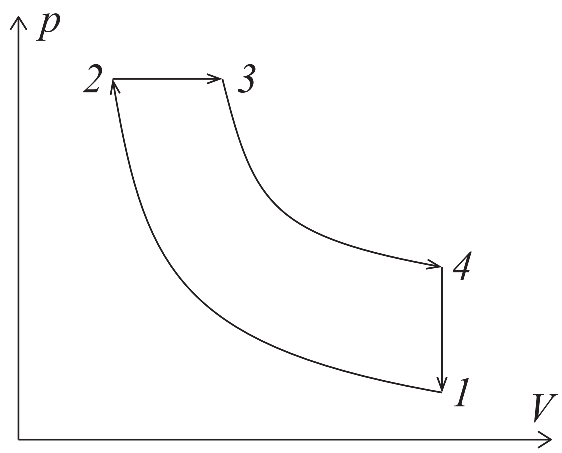

Fig. 0.2 Illustration of a cycle in the engine.

where Z_{2d} is the partition function found above, and ε_{z0} is the ground state for motion in the z-direction. You can assume that the temperature is low.

g) Find the energy of the system with this partition function.

Exercise 0.7. Diesel Engine

In this project we will address the idealized Otto engine. The engine cycle consists of the following steps:

- Air is sucked into a cylinder (not part of the sketch)
- Air is compressed adiabatically ($1 \to 2$)
- At 2 fuel is injected into the cylinder. Since the temperature is high the fuel ignites immediately.
- The fuel burns slowly, and in the first part of the process the gas is expanding at approximately constant pressure ($2 \to 3$)
- The remaining of the effect-phase is adiabatic ($3 \to 4$)
- Finally, during the exhaust phase, air is ejected from the cylinder ($4 \to 1$).

This process is shown in Fig. 0.2.

You may assume that the engine behaves as a reversible, quasistatic engine going through the idealized cycle with an ideal gas. The adiabatic constant is γ and is given as $\gamma = C_P/C_V$. Use the numbers as subscripts to describe the states of the gas at various points along the cycle (p_1, V_1 etc). Let W_{12} be defined as the work the gas does on its surroundings from $1 \to 2$. Let ΔS_{12} be the entropy change and Q_{12} be the thermal energy (heat) transferred to the gas.

a) If you study a single cycle, what is the change in the entropy of the gas through a whole cycle?

b) Show that $\Delta S_{23} = C_p \ln T_3/T_2$.

c) Show that $\Delta S_{41} = C_V \ln T_1/T_4$.

d) Show that $T_3^\gamma T_1 = T_2^\gamma T_4$.

e) Show that $T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$ and $T_3 V_3^{\gamma-1} = T_4 V_1^{\gamma-1}$.

f) Find Q_{41} and Q_{23}.

g) Show that the ideal efficiency is

$$e = 1 - \frac{C_V(T_4 - T_1)}{C_P(T_3 - T_2)} . \tag{0.19}$$

h) (Not compulsory) Show that the ideal efficiency is given as:

$$e = 1 - \frac{r_l^{-\gamma} - r_k^{-\gamma}}{\gamma\left(r_l^{-1} - r_k^{-1}\right)} \tag{0.20}$$

where $r_k = V_1/V_2$ is the compression ratio and $r_l = V_1/V_3$ is the expansion ratio.

i) (Question for class discussion) How could you in practice produce the highest possible efficiency?