Oblig08 — Fys2160 — 2015

Exercise 0.12. Relativistic Fermi Gas

For relativistic electrons - that is for electrons where $\varepsilon \gg mc^2$ - the energy is given as $\varepsilon \simeq pc$, where p is the momentum. For a particle in a square box of size $L \times L \times L$, the momentum is

$$p = \frac{\pi\hbar}{L}\left(n_x^2 + n_y^2 + n_z^2\right)^{1/2} , \tag{0.32}$$

just as for non-relativistic electrons.

a) Show that the density of states has the form

$$D(\varepsilon) = \frac{\pi}{a^3}\varepsilon^2 , \tag{0.33}$$

where $a = c\pi\hbar/L$.

b) Show that the Fermi energy of a gas of N electrons is

$$\varepsilon_F = \hbar c\pi \left(3n/\pi\right)^{1/3} , \tag{0.34}$$

where $n = N/V$.

c) Show that the total energy when $T = 0$ is

$$U_0 = \frac{3}{4}N\varepsilon_F . \tag{0.35}$$

Exercise 0.13. Chemical potential in a Fermi Gas

For a Fermi-gas with N particles and volume V we can define the chemical potential μ by solving the following equation with respect to μ:

$$N = \int_0^\infty D(\varepsilon) f(\varepsilon,\mu,T)\,d\varepsilon , \tag{0.36}$$

where the density of states, $D(\varepsilon)$ for particles with spinn 1/2 is given as

$$D(\varepsilon) = \frac{3N}{2\varepsilon_F^{3/2}}\varepsilon^{1/2} = \frac{4V}{\sqrt{\pi}}\left(\frac{2\pi m}{h^2}\right)^{3/2}\varepsilon^{1/2} , \tag{0.37}$$

and f is given as

$$f(\varepsilon,\mu,T) = \frac{1}{1+\exp\left(\frac{\varepsilon-\mu}{kT}\right)} . \tag{0.38}$$

Eq. (0.36) can be solved numerically to find $\mu(T)$, and we will here show how to proceed to do this.

In order to solve the equations numerically, it is useful to non-dimensionalize the equations. We do this by introducing a characteristic energy, ε_F, corresponding to the Fermi energy:

$$\varepsilon_F = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3} . \tag{0.39}$$

Based on this energy, we introduce new dimensionless variables $t = kT/\varepsilon_F$, $c = \mu/\varepsilon_F$ and $x = \varepsilon/\varepsilon_F$.

a) Show that (0.36) can be written as

$$1 = \frac{3}{2}\int_0^\infty \frac{x^{1/2}}{\exp((x-c)/t)+1}\,dx , \tag{0.40}$$

using the dimensionless variables.

b) What happens when $t = 0$? What is c in this case? Explain.

c) Now we will find $c(t)$ numerically by varying c while holding t fixed until the integral gives the desired value. You should do this for t in the range from 0.1 to 2 and plot the results. (Hint: You can use the function `integral(F,0,Inf)` to find the integral of the function for a given set of values for t and c. You then need to adjust c until the integral becomes $2/3$. The function `F` is formulated as an alpha-function. For example, the function $F(x) = 2cx^2$ is formulated as `F = @(x)2*c*x.^2;`).

d) Use your calculated values for $\mu(T)$ to find the energy $U(T)$ numerically for temperatures up to $t = 2$. Plot the result.

e) Find the heat capacity as a function of temperature from your numerical calculation of the energy as a function of temperature.

f) Plot the distribution function $f(\varepsilon,\mu(T),T)$ as a function of ε for a range of values of T using your calculated values for $\mu(T)$. Comment on the results.

g) (For discussion in class) Explain graphically why the initial curvature of $\mu(T)$ is upward in 1d and downward in 3d. (Hint: Use the integral for N and use the graphs to consider the behavior of the integrand from $T = 0$ to a finite temperature. You have found the density of states in 1d previously.)