Abstract Linear Algebra: Vector Space

A **vector space** over a field $\mathbb{F}$ is a set V equipped with

- vector addition $+ : V \times V \to V$,
- scalar multiplication $\cdot : \mathbb{F} \times V \to V$,

satisfying for all $u, v, w \in V$, $\alpha, \beta \in \mathbb{F}$:

1. $u + v = v + u$ (commutativity)
2. $(u + v) + w = u + (v + w)$ (associativity)
3. There exists $0 \in V$ such that $v + 0 = v$ (additive identity)
4. For each v, there exists $-v$ with $v + (-v) = 0$ (additive inverse)
5. $\alpha(u + v) = \alpha u + \alpha v$ (distributivity I)
6. $(\alpha + \beta)v = \alpha v + \beta v$ (distributivity II)
7. $(\alpha\beta)v = \alpha(\beta v)$ (associativity of scalar mult.)
8. $1 \cdot v = v$ (unit property)

Abstract Linear Algebra: Remark

Remark

A vector space by itself only has the structure provided by addition and scalar multiplication.

- It does *not* include a notion of **length**, **angle**, or **orthogonality**.
- These geometric notions are **induced** once we add an **inner product** and, from it, a norm.

Abstract Linear Algebra: Inner Product

Definition

An **inner product** on a real vector space V is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

such that for all $u, v, w \in V$, $\alpha \in \mathbb{R}$:

1. $\langle u, v \rangle = \langle v, u \rangle$ (symmetry)
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ (linearity in the first slot)
3. $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$ (homogeneity)
4. $\langle v, v \rangle \geq 0$, and $\langle v, v \rangle = 0 \iff v = 0$ (positivity/definiteness)

Abstract Linear Algebra: Norm

Definition

Given an inner product $\langle \cdot, \cdot \rangle$ on V, the induced **norm** is

$$\|v\| = \sqrt{\langle v, v \rangle}, \quad v \in V.$$

This norm yields geometric notions:

- **Length**: $\|v\|$
- **Angle**: $\cos\theta = \dfrac{\langle u, v \rangle}{\|u\| \, \|v\|}$ (for $u, v \neq 0$)
- **Orthogonality**: $u \perp v \iff \langle u, v \rangle = 0$

Abstract Linear Algebra: Example

Example

Let $V = \mathbb{R}^2$ with the standard inner product

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2.$$

For $u = (3, 4)$ and $v = (4, -3)$:

$$\langle u, v \rangle = 3 \cdot 4 + 4 \cdot (-3) = 12 - 12 = 0.$$

Thus $u \perp v$. Also $\|u\| = \sqrt{3^2 + 4^2} = 5$.

Exercise

Let $u = (1, 2, 2)$ and $v = (2, 0, 1)$ in $\mathbb{R}^3$ with the standard inner product.

1. Compute $\langle u, v \rangle$.
2. Find $\|u\|$ and $\|v\|$.
3. Determine the cosine of the angle between u and v.

Abstract Linear Algebra: Task for Exploration

Work with the standard inner product on $\mathbb{R}^2$ or $\mathbb{R}^3$.

1. Pick two vectors u and v. Compute

$$|\langle u, v\rangle|, \quad \|u\|, \quad \|v\|, \quad \|u+v\|.$$

2. Compare $|\langle u, v\rangle|$ with $\|u\|\,\|v\|$. What do you notice? Try several different pairs.
3. Compare $\|u+v\|$ with $\|u\|+\|v\|$. Does the inequality always hold?
4. Explore special cases: – when u and v are multiples of each other, – when u and v are orthogonal.

You are discovering the Cauchy–Schwarz inequality and the triangle inequality.

Abstract Linear Algebra: Motivation

⚠ **Not part of curriculum**

Linear algebra has powerful real-world applications:

- **Cryptography:** Modern encryption methods (RSA, lattice-based cryptography) rely on vector spaces, modular arithmetic, and transformations.
- **Engineering:** Signal processing and image compression (e.g., JPEG) use inner products, norms, and orthogonal bases (Fourier, wavelets).
- **Machine Learning & Data Science:** High-dimensional vector spaces model data; norms and inner products measure similarity and optimize algorithms.

Even though this is not part of the exam curriculum, these examples show why the abstract theory of vector spaces is central in science and technology.