Abstract Linear Algebra: Vector Space

A **vector space** over a field $\mathbb{F}$ is a set V equipped with

- vector addition $+ : V \times V \to V$,
- scalar multiplication $\cdot : \mathbb{F} \times V \to V$,

satisfying for all $u, v, w \in V$, $\alpha, \beta \in \mathbb{F}$:

1. $u + v = v + u$ (commutativity)
2. $(u + v) + w = u + (v + w)$ (associativity)
3. There exists $0 \in V$ such that $v + 0 = v$ (additive identity)
4. For each v, there exists $-v$ with $v + (-v) = 0$ (additive inverse)
5. $\alpha(u + v) = \alpha u + \alpha v$ (distributivity I)
6. $(\alpha + \beta)v = \alpha v + \beta v$ (distributivity II)
7. $(\alpha\beta)v = \alpha(\beta v)$ (associativity of scalar mult.)
8. $1 \cdot v = v$ (unit property)

Abstract Linear Algebra: Remark

Remark

A vector space by itself only has the structure provided by addition and scalar multiplication.

- It does *not* include a notion of **length**, **angle**, or **orthogonality**.
- These geometric notions are **induced** once we add an **inner product** and, from it, a norm.

Abstract Linear Algebra: Inner Product

Definition

An **inner product** on a real vector space V is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

such that for all $u, v, w \in V$, $\alpha \in \mathbb{R}$:

1. $\langle u, v \rangle = \langle v, u \rangle$ (symmetry)
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ (linearity in the first slot)
3. $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$ (homogeneity)
4. $\langle v, v \rangle \geq 0$, and $\langle v, v \rangle = 0 \Longleftrightarrow v = 0$ (positivity/definiteness)

Abstract Linear Algebra: Norm

Definition

Given an inner product $\langle \cdot, \cdot \rangle$ on V, the induced **norm** is

$$\|v\| = \sqrt{\langle v, v \rangle}, \quad v \in V.$$

This norm yields geometric notions:

- **Length**: $\|v\|$
- **Angle**: $\cos\theta = \dfrac{\langle u, v \rangle}{\|u\| \, \|v\|}$ (for $u, v \neq 0$)
- **Orthogonality**: $u \perp v \Longleftrightarrow \langle u, v \rangle = 0$

Abstract Linear Algebra: Example

Example

Let $V = \mathbb{R}^2$ with the standard inner product

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2.$$

For $u = (3, 4)$ and $v = (4, -3)$:

$$\langle u, v \rangle = 3 \cdot 4 + 4 \cdot (-3) = 12 - 12 = 0.$$

Thus $u \perp v$. Also $\|u\| = \sqrt{3^2 + 4^2} = 5$.

Exercise

Let $u = (1, 2, 2)$ and $v =$(2,0,1) in $\mathbb{R}^3$ with the standard inner product.

1. Compute $\langle u, v \rangle$.
2. Find $\|u\|$ and $\|v\|$.
3. Determine the cosine of the angle between u and v.

Abstract Linear Algebra: Exploration

Task for Exploration

Work with the standard inner product on $\mathbb{R}^2$ or $\mathbb{R}^3$.

1. Pick two vectors u and v. Compute

$$|\langle u, v\rangle|\,, \quad \|u\|\,, \quad \|v\|\,, \quad \|u+v\|\,.$$

2. Compare $|\langle u, v\rangle|$ with $\|u\|\;\|v\|$. Try several pairs.
3. Compare $\|u+v\|$ with $\|u\| + \|v\|$. Does it always hold?
4. Explore special cases: u and v linearly dependent; $u \perp v$.

You are rediscovering the Cauchy–Schwarz inequality and the triangle inequality. Equality holds when u and v are linearly dependent (Cauchy–Schwarz), and when u and v point in the same direction (triangle).

Abstract Linear Algebra: Beyond the Curriculum

⚠ **Not part of curriculum**

Linear algebra has powerful real-world applications:

- **Cryptography**: RSA and lattice methods use vector spaces and linear maps.
- **Engineering**: Signal processing and compression use inner products and orthogonal bases.
- **Machine Learning**: Vector spaces model data; norms and inner products measure similarity.