HOMEWORK # 11

04/12/2025

1 Question 1

This question revolves around approximating the following function in (1).

$$\int_{-5}^{5} \frac{1}{1+s^2} ds \tag{1}$$

We will first approximate the function using a *Composite Trapezoidal Rule* and a *Composite Simpsons Rule*. We will then use methods derived in class to select n such that certain error methods are met. Finally, we will approximate using the developed codes and the predicted value of n and compare to the output of the `Scipy` quadrature function.

1.1 Writing Composite Codes

For this part, we are asked to write code to approximate the integral in (1) using a *Composite Trapezoidal Rule* and a *Composite Simpson's Rule*. For the first approximation, we are instructed to do this by partitioning the interval $[-5, 5]$ into equally spaced points $t_0, t_1, t_2, \ldots, t_n$. For the second approximation using a *Composite Simpson's Rule*, we are instructed to do this by again partitioning the interval $[-5, 5]$ into equally spaced points $t_0, t_1, t_2, \ldots, t_n$ where $n = 2k$ is even. The code that was produced for both approximation methods can be found on the GitHub repository under `Homework_11`.

1.2 Using Error Estimates to Choose n

We know the following error estimate for the *Trapezoidal Rule* in (2) and the error estimate for the *Simpson's Rule* in (3) where $|E|$ represents the absolute error.

$$|E| \leq \frac{(b-a)^3}{12n^2}[\max|f''(x)|] \tag{2}$$

$$|E| \leq \frac{(b-a)^5}{180n^4}[\max|f^{(4)}(x)|] \tag{3}$$

Using the above error estimates, we are to find the values of n such that $|E| \leq 10^{-4}$. In the case of the *Composite Trapezoidal Rule*, we have the following for $|E|$.

$$|E| = \left| \int_{-5}^{5} \frac{1}{1+s^2} ds - T_n \right|$$

In the above equation, T_n denotes the approximation using the trapezoidal rule. Similarly, in the case of the *Composite Simpson's Rule*, we have the following for $|E|$.

$$|E| = \left| \int_{-5}^{5} \frac{1}{1+s^2} ds - S_n \right|$$

In the above equation, S_n denotes the approximation using the Simpson's rule. To find the corresponding values of n to reach the required absolute error, we use `Sympy` to find the necessary derivatives and maximums. Using `Sympy`, the following information was obtained for the trapezoidal error estimate.

$$f''(x) = \frac{6x^2 - 2}{(1+x^2)^3}, \text{ and } \max|f''(x)| \approx 2$$

And for the Simpson's error estimate, the following values were found.

$$f^{(4)}(x) = \frac{24\left(\frac{16x^4}{(x^2+1)^2} - \frac{12x^2}{(x^2+1)} + 1\right)}{(x^2+1)^3}, \text{ and } \max|f^{(4)}(x) \approx 24$$

With this information, we can find the value of n for the trapezoidal error estimate to be less than 10^{-4} as shown below.

$$\begin{aligned}
\frac{(b-a)^3}{12n^2}\max|f''(x)| &\leq 10^{-4} \\
\frac{(5-(-5))^3}{12n^2}2 &\leq 10^{-4} \\
\frac{2000}{12n^2} &\leq 10^{-4} \\
\frac{2000}{10^{-4}} &\leq 12n^2 \\
\frac{2000}{12 * 10^{-4}} &\leq n^2 \\
\sqrt{\frac{2000}{12 * 10^{-4}}} &\leq n \\
\sqrt{\frac{2000}{12 * 10^{-4}}} &\leq n \\
1291 &\leq n
\end{aligned}$$

Now finding the value of n for the Simpsons error estimate to be less than 10^{-4} as shown below.

$$\begin{aligned}
\frac{(b-a)^5}{180n^4}\max|f^{(4)}(x)| &\leq 10^{-4} \\
\frac{10^5}{180n^4}24 &\leq 10^{-4} \\
\frac{2400000}{180n^4} &\leq 10^{-4} \\
\frac{2400000}{180 * 10^{-4}} &\leq n^4 \\
\frac{2400000}{180 * 10^{-4}} &\leq n^4 \\
\left(\frac{2400000}{180 * 10^{-4}}\right)^{1/4} &\leq n \\
108 &\leq n
\end{aligned}$$

1.3 Comparisons

For this part, we are to use the values of n we found in the previous part to find the approximations T_n and S_n. We are to then compare these values to the approximation using `Scipy` Quadrature function with the error requirements set to 10^{-6} and 10^{-4}. The following values were found.

Approximation Method	Value	Absolute Error
Composite Trapezoidal $n = 1291$	2.7468013859623697	1.4792766211968456e-07
Composite Simpson's $n = 108$	2.7468015287482044	5.14182740829483e-09
`Scipy.quad` $err = 1e^{-6}$	2.7468015338900327	-8.881784197001252e-16
`Scipy.quad` $err = 1e^{-4}$	2.746801533909586	1.9554136088117957e-11

Table 1: Approximation Values.

2 Question 2

For this question, we are asked to use the transformation $t = x^{-1}$ to approximate the following integral in (4) using *Composite Simpson's Rule* with 5 nodes.

$$\int_1^{\infty} \frac{\cos(x)}{x^3} dx \tag{4}$$

Making the transformation $t = x^{-1}$, we get the following integral as shown below.

$$\begin{aligned} \int_1^{\infty} \frac{\cos(x)}{x^3} dx &= \int_1^{0} \frac{\cos\left(\frac{1}{t}\right)}{\left(\frac{1}{t}^3\right)} \left(-\frac{1}{t^2}\right) dt \\ &= \int_0^1 \cos(1/t)\, t dt \end{aligned}$$

To approximate using a *Composite Simpson's Rule*, we will use the code that we developed for Question 1. In order to use Simpson's with five nodes, we will pass $n = 4$ into the function to denote four sub-intervals. It is also important to note that this particular integrand begins to fail for values close to 0. To remedy this, we can let $\epsilon = 1e^{-6}$ denote the lower bound of the interval. With this, the integral becomes the following.

$$\int_{\epsilon}^{1} \cos(1/t)\, t dt$$

Using the code from Question 1 with the mentioned modifications, the integral was approximated to be 0.018773431522191356.

3 Question 3

For this question, we are told to assume that the error in an integration formula is given by the following asymptotic expansion.

$$I - I_n = \frac{C_1}{n\sqrt{n}} + \frac{C_2}{n^2} + \frac{C_3}{n^3\sqrt{n}} + \frac{C_4}{n^3} + \dots \tag{5}$$

We are then asked to generalize the *Richardson Extrapolation Process* to obtain the estimate I has an error of $\frac{1}{n\sqrt{n}}$. We are to assume that I_n, $I_{n/2}$, and $I_{n/4}$ have been computed. From the given formula in (5), we can see that the dominant error term is as follows where $E(n)$ is the error.

$$E(n) \approx \frac{C_1}{n\sqrt{n}} = \frac{C_1}{n^{3/2}}$$

To generalize the Richardson extrapolation process, we need to eliminate the leading error term $\frac{1}{n^{3/2}}$, and subsequently the next dominant term $\frac{1}{n^2}$. To do so, let I_n, $I_{n/2}$, and $I_{n/4}$ be approximations with step sizes h, $h/2$, and $h/4$, respectively. Also let $p = \frac{3}{2}$, which corresponds to the exponent on the leading-order error term. The first level Richardson extrapolation removes the $\mathcal{O}(1/n^{3/2})$ term as shown below.

$$I_n^{(1)} = \frac{2^p I_{n/2} - I_n}{2^p - 1} = \frac{2^{3/2} I_{n/2} - I_n}{2^{3/2} - 1}$$

We apply the same extrapolation to $I_{n/2}$ and $I_{n/4}$ to obtain the following.

$$I_{n/2}^{(1)} = \frac{2^{3/2} I_{n/4} - I_{n/2}}{2^{3/2} - 1}$$

This step cancels the leading-order $\mathcal{O}(1/n^{3/2})$ error from both estimates. Now we can apply a second level of Richardson extrapolation to $I_n^{(1)}$ and $I_{n/2}^{(1)}$ to remove the next dominant error term, $\mathcal{O}(1/n^2)$ as shown below.

$$I^{(2)} = \frac{2^p I_{n/2}^{(1)} - I_n^{(1)}}{2^p - 1} = \frac{2^{3/2} I_{n/2}^{(1)} - I_n^{(1)}}{2^{3/2} - 1}$$

The result $I^{(2)}$ is an approximation to I with an improved asymptotic error of order $\mathcal{O}\left(\frac{1}{n^2\sqrt{n}}\right)$.