HOMEWORK # 09

03/21/2025

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1 Question 1

In this question, we are asked to find the least squares solution to the following overdetermined linear system.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

1.1 Least Squares Solution

To find the least squares solution to the above linear system, we need to solve $\mathbf{A}^T\mathbf{A}\vec{x} = \mathbf{A}^T\vec{b}$ where:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \vec{x} = \begin{bmatrix} u \\ v \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Given $\mathbf{A}$, we also know $\mathbf{A}^T$. If we let $\mathbf{A}' = \mathbf{A}^T\mathbf{A}$ and $\vec{b'} = \mathbf{A}^T\vec{b}$, then we have the following.

$$\mathbf{A}' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } \vec{b'} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Plugging into the equation $\mathbf{A}^T\mathbf{A}\vec{x} = \mathbf{A}^T\vec{b}$, we get the following.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

From above, we have $u = 1$ and $2v = 1$ which implies $v = \frac{1}{2}$. Therefore, the least squares solution of the overdetermined linear system is $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$.

2 Question 2

For this question, we are asked to find the vector $\vec{x}$ that minimizes the quantitiy $E^2 = b_1^2 + 4b_2^2 + 25b_3^2 + 9b_4^2$ when the following holds.

$$\begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

2.1 Finding $\vec{x}$

In the original question, we are given the hint to multiply the system above with a suitable diagonal matrix, so that the problems becomes a regular least squares problem. With that in mind, let us define the matrices $\mathbf{A}$ and $\vec{c}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} \text{ and } \vec{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Looking at E^2, we can see that it is a linear combination of the form $E^2 = \vec{b}^T\mathbf{W}\vec{b}$ where $\mathbf{W}$ is defined as follows.

$$\mathbf{W} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Since $E^2 = \vec{b}^T\mathbf{W}\vec{b}$, we know that $E = \vec{b}^T\mathbf{D}\vec{b}$ where $\mathbf{D} = diag(\sqrt{1}, \sqrt{4}, \sqrt{25}, \sqrt{9})$. Using this, we can now convert the original problem to that of a regular least squares problem. To do so, we will left-multiply the original equation by our diagonal matrix $\mathbf{D}$. That is, we now have $\mathbf{D}(\mathbf{A}\vec{x} - \vec{c}) = \mathbf{D}\vec{b}$. Written as a minimization problem, we have $min\|\mathbf{D}(\mathbf{A}\vec{x} - \vec{c})\|$. We know then, that the solution to the least squares problem is the solution to $\mathbf{A'}^T\mathbf{A'}\vec{x} = \mathbf{A'}\vec{b'}$ where $\mathbf{A'} = \mathbf{DA}$ and $\vec{c'} = \mathbf{D}\vec{c}$ as shown below.

$$\mathbf{A'} = \begin{bmatrix} 1 & 3 \\ 12 & -2 \\ 20 & 0 \\ 6 & 21 \end{bmatrix} \text{ and } \vec{c'} = \begin{bmatrix} 1 \\ 4 \\ 15 \\ 12 \end{bmatrix}$$

As done in Question 1, let $\tilde{\mathbf{A}} = \mathbf{A'}^T\mathbf{A'}$ and $\tilde{c} = \mathbf{A'}\vec{c'}$. So we have the following system

$$\tilde{\mathbf{A}}\vec{x} = \tilde{c'}$$

where $\tilde{\mathbf{A}}$ and $\tilde{c}$ are as follows.

$$\tilde{\mathbf{A}} = \begin{bmatrix} 581 & 105 \\ 105 & 454 \end{bmatrix} \text{ and } \tilde{c} = \begin{bmatrix} 421 \\ 247 \end{bmatrix}$$

So we have the following.

$$\begin{bmatrix} 581 & 105 \\ 105 & 454 \end{bmatrix} \vec{x} = \begin{bmatrix} 421 \\ 247 \end{bmatrix}$$

Using Gaussian elimination, we get the following.

$$\left[\begin{array}{cc|c} 1 & 0 & 0.653608916355752 \\ 0 & 1 & 0.392887805688648 \end{array}\right]$$

That is, our solution to the original problem is $x_1 = 0.653608916355752$ and $x_2 = 0.392887805688648$.

3 Question 3

This question mainly involves proving linear independence for different sets. We are also given the fact that if a function is identically zero over an interval, the derivatives of the function must also be identically zero over the same interval. With this in mind, we are to prove linear independence for the following sets.

3.1 Set (a)

For this part, we are to prove linear independence for the set $\{1, x, x^2, \ldots, x^n\}$. We know that in order for a set to be linearly independent, then the set must satisfy the following.

$$c_0 + c_1x + c_2x^2 + \cdots + c_nx^n = 0, \text{ for all } x \in I, \text{ where } I \text{ is the interval considered}$$

We will let $f(x)$ be equal to this. That is

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n = 0$$

So in our case, we have the following.

$$f(x) = 1 + x + x^2 + \cdots + x^n = 0$$

If we continually take the derivative of $f(x)$, we get the following.

$$\begin{aligned} f(x) &= 1 + x + x^2 + \cdots + x^n = 0 \\ f'(x) &= 0 + 1 + 2x + \cdots + nx^{n-1} = 0 \\ f''(x) &= 0 + 0 + 2 + \cdots + n(n-1)x^{n-2} = 0 \\ &\vdots \\ f^{(n)}(x) &= 0 + 0 + 0 + \cdots + n! = 0 \end{aligned}$$

Since each term in $f(x)$ contains a constant c_n, the equation is as follows.

$$f^{(n)}(x) = 0 + 0 + 0 + \cdots + n!c_n = 0$$

Since we have $n!c_n = 0$, we know c_n must be 0. By continuously integrating $f(x)$, we can see that $c_0, c_1, c_2, \ldots, c_n$ must all equal 0. Since each constant is equal to 0, we know the equation $c_0+c_1x+c_2x^2+\cdots+c_nx^n = 0$ is satisfied. Therefore, the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent.

3.2 Set (b)

For this part, we are to prove linear independence for the set $\{1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx\}$.

First, let us assume that there exists a linear combination of the functions that equals zero for all x:

$$f(x) = c_0 + \sum_{k=1}^{n} c_k \cos(kx) + \sum_{k=1}^{n} d_k \sin(kx) = 0 \quad \text{for all } x \in [-\pi, \pi].$$

We know that the functions $\{1, \cos(kx), \sin(kx)\}$ are orthogonal over $[-\pi, \pi]$. Additionally, we know the following.

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx &= 0 \quad \text{if } m \neq n, \\ \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx &= 0 \quad \text{if } m \neq n, \\ \int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx &= 0 \quad \text{for all } m, n. \end{aligned}$$

To find the coefficients, we can take inner products/integrals of both sides of $f(x) = 0$ with each basis function. That is, we will multiply vy $\cos(mx)$ for $1 \leq m \leq n$ as shown below.

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = 0.$$

Only the term $c_m \cos(mx)$ survives due to orthogonality:

$$c_m \int_{-\pi}^{\pi} \cos^2(mx)\,dx = 0 \quad \Rightarrow \quad c_m = 0.$$

We can now do the same for $\sin(mx)$ as shown below.

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = 0 \quad \Rightarrow \quad d_m = 0.$$

And finally, we will multiply by 1.

$$\int_{-\pi}^{\pi} f(x) \cdot 1 \, dx = 0 \quad \Rightarrow \quad c_0 = 0.$$

As shown above, all coefficients are zero meaning that the function set is linearly independent over any interval.

4 Question 4

For this question, we are asked to prove the three-term recursion formula for orthogonal polynomials which is given below.

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x) \text{ where } b_k = \frac{\langle x\phi_{k-1}, \phi_{k-1}\rangle}{\langle \phi_{k-1}, \phi_{k-1}\rangle} \text{ and } c_k = \frac{\langle x\phi_{k-1}, \phi_{k-2}\rangle}{\langle \phi_{k-2}, \phi_{k-2}\rangle}$$

4.1 Proving Three Term Recursion

For this part, we are given a very useful hint which tells us that since $\phi_k(x)$ is a polynomial of degree k and of the form $\phi_k(x) = x^k +$[lower order terms], we can re-write the original three-term recursion as follows.

$$\begin{aligned}\phi_k(x) &= (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x) - \{a_{k-3}\phi_{k-3}(x) + a_{k-4}\phi_{k-4}(x) + \cdots + a_0\phi_0\} \\ &= (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x) - \sum_{j}^{k-3} a_j\phi_j(x)\end{aligned}$$

With this, the hint compels us to show that each $a_j = 0$ which can be done by taking the inner product of $\vec{\phi_k}$ and $\phi_j(x)$. So we have the following.

$$\langle \phi_{k-1}\rangle = \langle (x - b_k)\phi_{k-1}, \phi_j\rangle - c_k\langle \phi_{k-3}\rangle + \sum_{i=0}^{k-3} a_i\langle \phi_i, \phi_j\rangle$$

Using orthogonality, we know that $\langle \phi_k, \phi_j\rangle = 0$, $\langle \phi_{k-1}, \phi_j\rangle = 0$, and $\langle \phi_{k-2}, \phi_j\rangle = 0$. In the case of the last term, we know that $\langle \phi_i, \phi_j\rangle = 0$ when $i \neq j$ which leaves $\langle \phi_i, \phi_j\rangle = \langle \phi_j, \phi_j\rangle$ when $i = j$. Using this, we have the following.

$$0 = 0 - 0 + a_j\langle \phi_j, \phi_j\rangle \text{ for all } j \leq k - 3$$

As shown above, we can see that a_j must equal 0 for the equality to hold. With this, the hint suggests that we should calculate the values for b_k and c_k. To solve for b_k, we will first take the inner product of the original recursion equation and $\phi_{k-1}(x)$ which gives us the following.

$$\langle \phi_k, \phi_{k-1}\rangle = \langle (x - b_k)\phi_{k-1}, \phi_{k-1}\rangle - c_k\langle \phi_{k-2}, \phi_{k-1}\rangle$$

Like before, using orthogonality, we know that $\langle \phi_k(x), \phi_{k-1}(x)\rangle = 0$ and $\langle \phi_{k-2}, \phi_{k-1}\rangle = 0$. Plugging in we get the following.

$$0 = \langle (x - b_k)\phi_{k-1}, \phi_{k-1}\rangle - 0$$

And expanding and simplifying gives us the following.

$$\begin{aligned}0 &= \langle (x - b_k)\phi_{k-1}, \phi_{k-1}\rangle \\ 0 &= \langle (x\phi_{k-1} - b_k\phi_{k-1}), \phi_{k-1}\rangle \\ 0 &= \langle x\phi_{k-1}, \phi_{k-1}\rangle - b_k\langle \phi_{k-1}, \phi_{k-1}\rangle \\ b_k &= \frac{\langle x\phi_{k-1}, \phi_{k-1}\rangle}{\langle \phi_{k-1}, \phi_{k-1}\rangle}\end{aligned}$$

Now to calculate the value of c_k, we can take the inner product of the original recursion equation and ϕ_{k-2} which gives the following.

$$\langle \phi_k, \phi_{k-2} \rangle = \langle (x - b_k)\phi_{k-1}, \phi_{k-2} \rangle - c_k \langle \phi_{k-2}, \phi_{k-2} \rangle$$

Using similar techniques as done for b_k, we can proceed as shown below.

$$\begin{aligned}
\langle \phi_k, \phi_{k-2} \rangle &= \langle (x - b_k)\phi_{k-1}, \phi_{k-2} \rangle - c_k \langle \phi_{k-2}, \phi_{k-2} \rangle \\
0 &= \langle (x - b_k)\phi_{k-1}, \phi_{k-2} \rangle - c_k \langle \phi_{k-2}, \phi_{k-2} \rangle \\
0 &= \langle x\phi_{k-1}, \phi_{k-2} \rangle - b_k \langle \phi_{k-1}, \phi_{k-2} \rangle - c_k \langle \phi_{k-2}, \phi_{k-2} \rangle \\
0 &= \langle x\phi_{k-1}, \phi_{k-2} \rangle - 0 - c_k \langle \phi_{k-2}, \phi_{k-2} \rangle \\
0 &= \langle x\phi_{k-1}, \phi_{k-2} \rangle - c_k \langle \phi_{k-2}, \phi_{k-2} \rangle
\end{aligned}$$

$$c_k = \frac{\langle x\phi_{k-1}, \phi_{k-2} \rangle}{\langle \phi_{k-2}, \phi_{k-2} \rangle}$$

5 Question 5

This question pertains to the computing of the Chebyshev polynomials $T_n(x)$. For this question, we are given one of the many formulas for computing the Chebyshev polynomials which can be seen below.

$$T_n(x) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$$

In the above formula, z is implicitly defined through x as $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$. With this information, we are asked to verify that the formula produces the same polynomials as the standard definition of Chebyshev polynomials.

5.1 Verification

As hinted at in the original question, we will verify the given formula by first verifying the results for T_0 and T_1. We will then use the two values in the three-term recursion to fully validate the formula. We know that the first two Chebyshev polynomials are given as follows.

$$\begin{aligned}
T_0 &= 1 \\
T_1 &= x
\end{aligned}$$

Using the formula with $n = 0$, we get the following

$$\begin{aligned}
T_0(x) &= \frac{1}{2}\left(z^0 + \frac{1}{z^0}\right) \\
&= \frac{1}{2}\left(1 + \frac{1}{1}\right) \\
&= \frac{1}{2}(1 + 1) \\
&= \frac{1}{2}(2) \\
&= 1
\end{aligned}$$

which satisfies the classical definition of T_0. Now doing the same with $n = 1$, we get

$$\begin{aligned}
T_1(x) &= \frac{1}{2}\left(z^1 + \frac{1}{z^1}\right) \\
&= \frac{1}{2}\left(z + \frac{1}{z}\right)
\end{aligned}$$

We are given that $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$ so we have $T_1 = x$ which satisfies the classical definition of the Chebyshev polynomial T_1. Now that T_0 and T_1 are verified, we will use the three-term recursion as shown below.

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

On the LHS, we have

$$T_{n+1}(x) = \frac{1}{2}\left(z^{n+1} + \frac{1}{z^{n+1}}\right).$$

While on the right, we have

$$\begin{aligned}
2xT_n(x) - T_{n-1}(x) &= 2\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)\left(\frac{1}{2}\left(z^n + \frac{1}{z^n}\right)\right) - \frac{1}{2}\left(z^{n-1} + \frac{1}{z^{n-1}}\right) \\
&= \frac{1}{2}\left(z + \frac{1}{z}\right)\left(z^n + \frac{1}{z^n}\right) - \frac{1}{2}\left(z^{n-1} + \frac{1}{z^{n-1}}\right) \\
&= \frac{1}{2}\left(z^{n+1} + \frac{1}{z^{n-1}} + z^{n-1} + \frac{1}{z^{n+1}}\right) - \frac{1}{2}\left(z^{n-1} + \frac{1}{z^{n-1}}\right) \\
&= \frac{1}{2}\left(z^{n+1} + \frac{1}{z^{n-1}} + z^{n-1} + \frac{1}{z^{n+1}} - z^{n-1} - \frac{1}{z^{n-1}}\right) \\
&= \frac{1}{2}\left(z^{n+1} + \frac{1}{z^{n+1}}\right)
\end{aligned}$$

As you can see, the LHS and the RHS are equivalent. Therefore, the given formula for $T_n(x)$ correctly produces the Chebyshev polynomials.