Alternating Direction Implicit (ADI) method for Parabolic equation

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Consider the following system in the rectangular domain Ω :

$$\begin{cases} \dfrac{\partial U}{\partial t} = D\left(\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2}\right) + f(t,x,y), \quad (x,y) \in \Omega \\ \partial\Omega = \partial\Omega_R \cup \partial\Omega_L \cup \partial\Omega_T \cup \partial\Omega_B \\ \vec{n}_B \cdot \nabla U = 0, \quad (x,y) \in \partial\Omega_N \subseteq \partial\Omega \\ U(x,y) = \mu(x,y), \quad (x,y) \in \partial\Omega_D \subseteq \partial\Omega \\ U(0,x,y) = g(x,y) \end{cases}$$

where $\vec{n}_B$ is a vector normal to the boundary and D is a thermal diffusivity constant. Symbols $\partial\Omega_R,\ \partial\Omega_L,\ \partial\Omega_T,\ \partial\Omega_B$ denote right, left, top and bottom boundary respectively of the domain Ω.

1 Numerical scheme

Lets denote Ω_Δ as the discretization of our domain (fig. (1)) with mesh points $(x_i, y_i),\ i = 1,\dots,M_x, j = 1,\dots,M_y$. The idea behind the ADI method is to split the finite difference equations into two, one with the x-derivative taken implicitly and the next with the y-derivative taken implicitly (fig. (2)),

$$\frac{u_{i,j}^{n+1/2} - u_{i,j}^{n}}{\Delta t/2} = D\Big(\Lambda_1 u_{i,j}^{n+1/2} + \Lambda_2 u_{i,j}^{n}\Big) + f_{i,j}^{n} \tag{1}$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+1/2}}{\Delta t/2} = D\Big(\Lambda_1 u_{i,j}^{n+1/2} + \Lambda_2 u_{i,j}^{n+1}\Big) + f_{i,j}^{n} \tag{2}$$

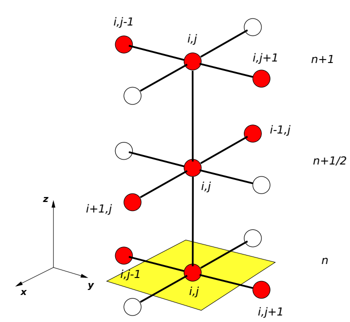

Figure 2: Two steps stencil.

where discrete operators Λ_1 and Λ_2 are defined as follows

$$\Lambda_1 u_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}, \quad \Lambda_2 u_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}$$

The equation (1) can be rewritten as

$$-\alpha u_{i-1,j}^{n+1/2} + (1+2\alpha)u_{i,j}^{n+1/2} - \alpha u_{i+1,j}^{n+1/2} = \beta u_{i,j-1}^{n} + (1-2\beta)u_{i,j}^{n} + \beta u_{i,j+1}^{n} + \Delta t/2 \cdot f_{i,j}^{n}, \tag{3}$$

$$\alpha = \frac{\Delta t/2 \cdot D}{\Delta x^2}, \quad \beta = \frac{\Delta t/2 \cdot D}{\Delta y^2},$$

and the equation (2)

$$-\beta u_{i,j-1}^{n+1} + (1+2\beta)u_{i,j}^{n+1} - \beta u_{i,j+1}^{n+1} = \alpha u_{i-1,j}^{n+1/2} + (1-2\alpha)u_{i,j}^{n+1/2} + \alpha u_{i+1,j}^{n+1/2} + \Delta t/2 \cdot f_{i,j}^{n}, \tag{4}$$

for $i = 2,\dots,M_x - 1,\ j = 2,\dots,M_y - 1$.

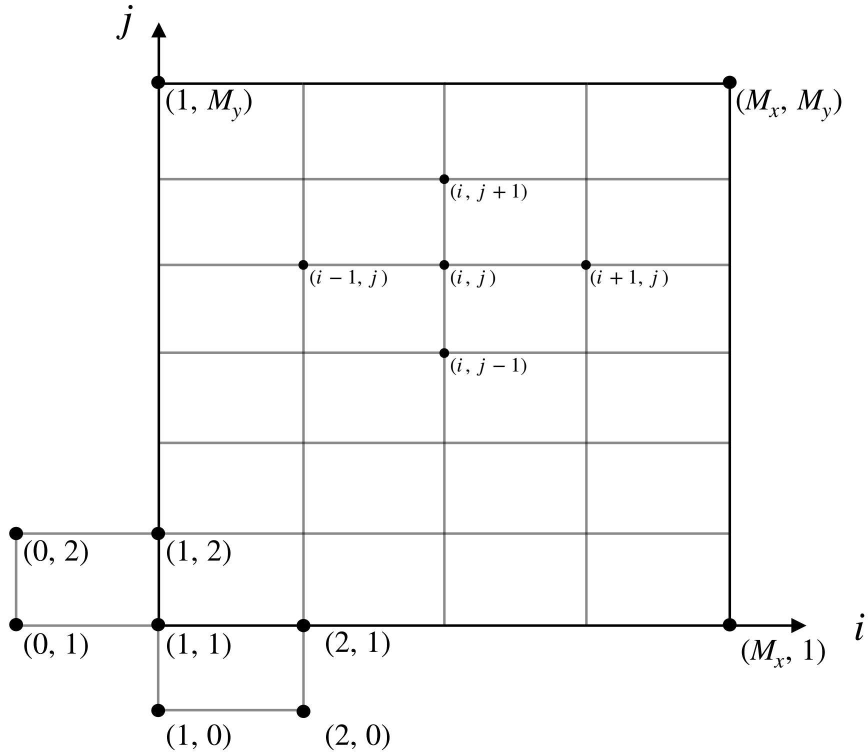

Figure 1: Rectangular grid with "fictitious" grid points.

2 Neumann boundary conditions (zero flux)

2.1 Step 1

2.1.1 Equations

Boundary condition approximation for scheme (3).

We have zero flux boundary condition. Along the bottom boundary when this is applied at the grid point $(k, 1)$ we obtain

$$\frac{u_{k,2} - u_{k,0}}{2\Delta x} = 0, \quad k = 1, 2, \ldots, M_x, \tag{5}$$

This approximation involves the value of U at the "fictitious" grid point $(k, 0)$ which lies outside the domain Ω. We write the approximation (3) at the boundary points

$$-\alpha u_{k-1,1}^{n+1/2} + (1+2\alpha)u_{k,1}^{n+1/2} - \alpha u_{k+1,1}^{n+1/2} = \beta u_{k,0}^{n} + (1-2\beta)u_{k,1}^{n} + \beta u_{k,2}^{n} + \Delta t/2 \cdot f_{k,1}^{n}, \tag{6}$$

We eliminate $u_{k,0}^n$ between (5) and (6) to get equations for **bottom boundary**

$$-\alpha u_{k-1,1}^{n+1/2} + (1+2\alpha)u_{k,1}^{n+1/2} - \alpha u_{k+1,1}^{n+1/2} = (1-2\beta)u_{k,1}^{n} + 2\beta u_{k,2}^{n} + \Delta t/2 \cdot f_{k,1}^{n},$$

for $k = 2, 3, \ldots, M_x - 1$.

Considering also the boundary conditions at the left and right boundaries

$$\frac{u_{2,1} - u_{0,1}}{2\Delta y} = 0, \quad \frac{u_{M_x+1,1} - u_{M_x-1,1}}{2\Delta y} = 0, \tag{7}$$

We will get expressions at the **corner points** $(1,1)$ **and** $(M_x,1)$ respectively

$$(1+2\alpha)u_{1,1}^{n+1/2}-2\alpha u_{2,1}^{n+1/2}=(1-2\beta)u_{1,1}^{n}+2\beta u_{1,2}^{n}+\Delta t/2\cdot f_{1,1}^{n},$$

$$-2\alpha u_{M_x-1,1}^{n+1/2}+(1+2\alpha)u_{M_x,1}^{n+1/2}=(1-2\beta)u_{M_x,1}^{n}+2\beta u_{M_x,2}^{n}+\Delta t/2\cdot f_{M_x,1}^{n},$$

Similar to the bottom boundary, the approximation at the **top boundary points** (k,M_y) becomes

$$-\alpha u_{k-1,M_y}^{n+1/2}+(1+2\alpha)u_{k,M_y}^{n+1/2}-\alpha u_{k+1,M_y}^{n+1/2}=2\beta u_{k,M_y-1}^{n}+(1-2\beta)u_{k,M_y}^{n}+\Delta t/2\cdot f_{k,M_y}^{n},$$

for $k=2,3,\dots,M_x-1$.

And for the **corner points** $(1,M_y)$ **and** (M_x,M_y) respectively

$$(1+2\alpha)u_{1,M_y}^{n+1/2}-2\alpha u_{2,M_y}^{n+1/2}=2\beta u_{1,M_y-1}^{n}+(1-2\beta)u_{1,M_y}^{n}+\Delta t/2\cdot f_{1,M_y}^{n},$$

$$-2\alpha u_{M_x-1,M_y}^{n+1/2}+(1+2\alpha)u_{M_x,M_y}^{n+1/2}=2\beta u_{M_x,M_y-1}^{n}+(1-2\beta)u_{M_x,M_y}^{n}+\Delta t/2\cdot f_{M_x,M_y}^{n}.$$

Left boundary approximation (points $(1,k)$, $\quad k=2,3,\dots,M_y-1$)

$$(1+2\alpha)u_{1,k}^{n+1/2}-2\alpha u_{2,k}^{n+1/2}=\beta u_{1,k-1}^{n}+(1-2\beta)u_{1,k}^{n}+\beta u_{1,k+1}^{n}+\Delta t/2\cdot f_{1,k}^{n},$$

Right boundary approximation (points (M_x,k), $\quad k=2,3,\dots,M_y-1$)

$$-2\alpha u_{M_x-1,k}^{n+1/2}+(1+2\alpha)u_{M_x,k}^{n+1/2}=\beta u_{M_x,k-1}^{n}+(1-2\beta)u_{M_x,k}^{n}+\beta u_{M_x,k+1}^{n}+\Delta t/2\cdot f_{M_x,k}^{n},$$

2.1.2 Matrix form

We have to solve the system of linear equations with tridiagonal matrix A of the following form:

$$A\cdot X_j=b_j,$$

for $j=1,2,\dots,M_y$

$$A=\begin{pmatrix} (1+2\alpha) & -2\alpha & 0 & & 0 \\ -\alpha & (1+2\alpha) & -\alpha & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha & (1+2\alpha) & -\alpha \\ 0 & & 0 & -2\alpha & (1+2\alpha) \end{pmatrix}, \quad X_j=\begin{pmatrix} u_{1,j}^{n+1/2} \\ u_{2,j}^{n+1/2} \\ \vdots \\ u_{M_x,j}^{n+1/2} \end{pmatrix}$$

$$b_1=\begin{pmatrix} (1-2\beta)u_{1,1}^{n}+2\beta u_{1,2}^{n}+\Delta t/2\cdot f_{1,1}^{n} \\ (1-2\beta)u_{2,1}^{n}+2\beta u_{2,2}^{n}+\Delta t/2\cdot f_{2,1}^{n} \\ \vdots \\ (1-2\beta)u_{M_x,1}^{n}+2\beta u_{M_x,2}^{n}+\Delta t/2\cdot f_{M_x,1}^{n} \end{pmatrix}$$

$$b_j=\begin{pmatrix} \beta u_{1,j-1}^{n}+(1-2\beta)u_{1,j}^{n}+\beta u_{1,j+1}^{n}+\Delta t/2\cdot f_{1,j}^{n} \\ \beta u_{2,j-1}^{n}+(1-2\beta)u_{2,j}^{n}+\beta u_{2,j+1}^{n}+\Delta t/2\cdot f_{2,j}^{n} \\ \vdots \\ \beta u_{M_x,j-1}^{n}+(1-2\beta)u_{M_x,j}^{n}+\beta u_{M_x,j+1}^{n}+\Delta t/2\cdot f_{M_x,j}^{n} \end{pmatrix}, \quad j=2,3,\dots,M_y-1$$

$$b_{M_y}=\begin{pmatrix} 2\beta u_{1,M_y-1}^{n}+(1-2\beta)u_{1,M_y}^{n}+\Delta t/2\cdot f_{1,M_y}^{n} \\ 2\beta u_{2,M_y-1}^{n}+(1-2\beta)u_{2,M_y}^{n}+\Delta t/2\cdot f_{2,M_y}^{n} \\ \vdots \\ 2\beta u_{M_x,M_y-1}^{n}+(1-2\beta)u_{M_x,M_y}^{n}+\Delta t/2\cdot f_{M_x,M_y}^{n} \end{pmatrix}$$

2.2 Step 2

To get computational formulas for second step we should swap indices $u_{m,l}$: $m \leftrightarrow l$, $M_x \leftrightarrow M_y$, $\alpha \leftrightarrow \beta$.

3 Dirichlet boundary conditions

3.1 Step 1

Dirichlet boundary condition on the right side of the domain

$$U\Big|_{\partial\Omega_R} = a \quad \Rightarrow \quad u_{M_x,j} = a,\ j = 1, 3, \ldots, M_y \tag{8}$$

The equation (3) for $i = M_x - 1$ takes the form

$$-\alpha u^{n+1/2}_{M_x-2,j} + (1+2\alpha) u^{n+1/2}_{M_x-1,j} - \alpha u^{n+1/2}_{M_x,j} = \beta u^n_{M_x-1,j-1} + (1-2\beta) u^n_{M_x-1,j} + \beta u^n_{M_x-1,j+1} + \Delta t/2 \cdot f^n_{M_x-1,j}, \tag{9}$$

$$-\alpha u^{n+1/2}_{M_x-2,j} + (1+2\alpha) u^{n+1/2}_{M_x-1,j} = \beta u^n_{M_x-1,j-1} + (1-2\beta) u^n_{M_x-1,j} + \beta u^n_{M_x-1,j+1} + \Delta t/2 \cdot f^n_{M_x-1,j} + \alpha a, \tag{10}$$