Corrección Parcial 1 → Señales y sistemas.

Nombre: Maribel Gómez Manchola.

1. La distancia media entre dos señales periódicas $x_1(t) \in \mathbb{R}, \mathbb{C}$ y $x_2(t) \in \mathbb{R}, \mathbb{C}$; se puede expresar a partir de la potencia media de la diferencia entre ellas.

$$d^2(x_1, x_2) = \overline{P}_{x_1 - x_2} = \lim_{T \to \infty} \frac{1}{T} \int_T |x_1(t) - x_2(t)|^2 dt$$

Sea $x_1(t)$ y $x_2(t)$ dos señales definidas como:

$$x_1(t) = A e^{-jn\omega_0 t}$$

$$x_2(t) = B e^{jm\omega_0 t}$$

con $\omega_0 = \frac{2\pi}{T}$; $T, A, B \in \mathbb{R}^+$ y $n, m \in \mathbb{Z}$. Determine la distancia entre las dos señales. Compruebe sus resultados con python.

$$\overline{P}_e = \overline{P}_{x1} - \frac{2}{T} \int_T x_1(t) x_2^*(t) dt + \overline{P}_{x2}$$

$$\overline{P}_{x1} = \frac{1}{T} \int_T |x_1(t)|^2 dt = \frac{1}{T} \int_T (A e^{-jn\omega_0 t})(A e^{-jn\omega_0 t})^* dt$$

$$= \frac{1}{T} \int_T A^2 e^{-jn\omega_0 t} e^{jn\omega_0 t} dt = \frac{1}{T} \int_T A^2 e^0 dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} A^2 dt = \frac{1}{T} A^2 t \Big|_{t=-T/2}^{t=T/2} = \frac{A^2}{T}\left(\frac{T}{2} - \left(-\frac{T}{2}\right)\right) = \frac{A^2 T}{T} = \boxed{A^2}$$

$$\overline{P}_{x2} = \frac{1}{T} \int_T (B e^{jm\omega_0 t})(B e^{jm\omega_0 t})^* dt = \frac{1}{T} \int_T B^2 e^{jm\omega_0 t - jm\omega_0 t} dt$$

$$= \frac{1}{T} B^2 \int_{-T/2}^{T/2} dt = \frac{B^2}{T}\left[\frac{T}{2} - \left(-\frac{T}{2}\right)\right] = \boxed{B^2}$$

$$-\frac{2}{T} \int_T (A e^{-jn\omega_0 t})(B e^{jm\omega_0 t})^* dt = -\frac{2}{T} \int_T AB e^{-jn\omega_0 t} e^{-jm\omega_0 t} dt$$

$$= -\frac{2AB}{T} \int_T e^{-j\omega_0 t(n+m)} dt$$

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* Caso 1: $n = -m$

Luego.

$$-\frac{2AB}{T}\int_T e^{-jw_0t(n+m)}\,dt = -\frac{2AB}{T}\int_T e^{-jw_0t(m-n)}\,dt$$

$$-\frac{2AB}{T}\int_T e^{-jw_0t(0)}\,dt = -\frac{2AB}{T}\int_{-T/2}^{T/2} 1\,dt$$

$$= -\frac{2AB}{T}\left[\frac{T}{2}-\left(-\frac{T}{2}\right)\right] = -\frac{2AB}{T}[T] = -2AB.$$

* Caso 2: $n \neq -m$

$$-\frac{2AB}{T}\int_T e^{-jw_0t(m+n)}\,dt = -\frac{2AB}{T}\int_{-T/2}^{T/2} e^{-jw_0t(n+m)}\,dt$$

$$-\frac{2AB}{T}\left(\frac{e^{-jw_0t(n+m)}}{-jw_0(n+m)}\right)\Bigg|_{-T/2}^{t=T/2} =$$

$$= -\frac{2AB}{T}\left[\frac{\cos(w_0t(n+m)) - j\,\mathrm{sen}(w_0t(n+m))}{-jw_0(n+m)}\right]\Bigg|_{t=-T/2}^{t=T/2}$$

$m, n \in \mathbb{Z}$

$n+m = p \rightarrow p \in \mathbb{Z}$

$$-\frac{2AB}{T}\left[\frac{\cos\left(\frac{2\pi}{T}\cdot\frac{T}{2}p\right) - j\,\mathrm{sen}\left(\frac{2\pi}{T}\frac{T}{2}p\right) - \cos\left(\frac{2\pi}{T}\left(-\frac{T}{2}\right)p\right) + j\,\mathrm{sen}\left(\frac{2\pi}{T}\left(-\frac{T}{2}\right)p\right)}{-jw_0p}\right]$$

$$= -\frac{2AB}{T}\left[\frac{\cos(\pi p) - j\,\mathrm{sen}(\pi p) - \cos(-\pi p) + j\,\mathrm{sen}(-\pi p)}{-jw_0p}\right]$$

Seno es para $\cos(y) = \cos(-x)$

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Entonces.

$$= -\frac{2AB}{T}\left[\frac{\cos(\pi p)-\cos(\pi p)-j\,sen(\pi p)+j\,sen(-\pi p)}{-jW_0 p}\right]$$

$$= -\frac{2AB}{T}\left[\frac{j\,sen(-\pi p)-j\,sen(\pi p)}{-jW_0 p}\right] \quad ; \quad sen(\pi p) \quad p \in \mathbb{Z} = 0$$

$$= -\frac{2AB}{\cancel{T}}\left[\frac{0}{-j\frac{2\pi}{\cancel{T}}p}\right] = 0$$

$\overline{Pe}$ para estas 2 señales tiene 2 valores posibles.

* Caso 1: $m=-n$

$$\overline{P}_e = A^2+B^2-2AB$$

* Caso 2: $n \neq -m$

$$\overline{Pe} = A^2+B^2$$

$$d^2(x_1,x_2) = Px_1-x_2 = \overline{P}_e$$

$$d^2(x_1,x_2) = \overline{Pe}$$

$$d(x_1,x_2) = \sqrt{\overline{Pe}}$$

* Si $n=-m$

$$d(x_1,x_2) = \sqrt{A^2+B^2-2AB}$$

* Si $n \neq -m$

$$d(x_1,x_2) = \sqrt{A^2+B^2}$$

2) Encuentre la señal en tiempo discreto al utilizar un conversor análogo digital con frecuencia de muestreo de 5 kHz y 4 bits de capacidad de representación, aplicado a la señal continua:

$$x(t) = 3\cos(1000\pi t) + 5\sin(3000\pi t) + 10\cos(4000\pi t).$$

Realizar la simulación del proceso de discretización (incluyendo al menos tres periodos de $x(t)$. En caso de que la discretización no sea apropiada, diseñe e implemente un conversor adecuado

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para la señal estudiada.

$$x(t) = 3\cos(1000\pi t) + 5\,\text{sen}(3000\pi t) + 10\cos(11000\pi t)$$

Primero verificamos que la señal sea wasiperiódica.

$$\left.\begin{array}{l} W_1 = 1000\pi \\ W_2 = 3000\pi \\ W_3 = 11000\pi \end{array}\right\} \frac{W_1}{W_2} = \frac{1000\pi}{3000\pi} = \frac{1}{3}\ ;\ \frac{W_1}{W_3} = \frac{1000\pi}{11000\pi} = \frac{1}{11}$$

$$\frac{W2}{W3} = \frac{3000\pi}{11000\pi} = \frac{3}{11}$$

Como todas las combinaciones dan como resultado un numero racional, es wasiperiódica.

* Discretizamos.

$$t = nts = \frac{n}{Fs} \rightarrow x(T) \rightarrow x(n) \quad n \in \mathbb{Z}$$

$$F_s = 5\,KHz \rightarrow 5000\,Hz$$

Verifico Nyquist con $F_s = 5\,KHz$

$$W_{max} = 11000\pi \rightarrow f_{max} = \frac{W_{max}}{2\pi} = \frac{11000\pi}{2\pi} = 5500\,Hz$$

Segun Nyquist la frecuencia de muestreo, debe ser 2 veces la frecuencia máxima de mi señal wasiperiódica.

$$F_s \text{ debe debe a } 2F(max) = (2)(5500) = 11000\,Hz$$

$F_s > 11000 \rightarrow$ Por lo tanto la frecuencia de muestreo dada en el ejercicio no es apta.

Para que la discretización sea apropiada, vamos a cambiar la frecuencia de muestreo a

$$4f(max) = 4(5500) = 22000\,Hz$$

Periodo de la señal:

$$\left.\begin{array}{l} W_1 = 1000\pi \\ W_2 = 3000\pi \\ W_3 = 11000\pi \end{array}\right\} T_1 = \frac{2\pi}{1000\pi} = \frac{1}{500}\ ;\ T_2 = \frac{2\pi}{3000\pi} = \frac{1}{1500}$$

$$T_3 = \frac{2\pi}{11000} = \frac{1}{5500}$$

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$$T = kT_1 = rT_2 = \ell T_3$$

$$T = \frac{k(1)}{500} = \frac{r(1)}{1500} = \frac{\ell(1)}{5500}$$ multiplicamos por 11000

$$T = \frac{1}{500}$$

3) Sea $x''(t)$ la segunda derivada de la señal $x(t)$, donde $t \in [t_i, t_f]$. Demuestre que los coeficientes de la serie exponencial de Fourier se pueden calcular.

$$C_n = \frac{1}{(t_i - t_f)\, n^2 w_0^2} \int_{t_i}^{t_f} x''(t)\, e^{-jnw_0 t}\, dt; \quad n \in \mathbb{Z}.$$

¿Cómo se pueden calcular los coeficientes a_n y b_n desde $x''(t)$ en la serie trigonométrica de Fourier?

$$C_n = \frac{1}{T} \int_{t_i}^{t_f} x(t)\, e^{-jw_0 nt}\, dt \rightarrow x(t) = \sum_n C_n e^{jnw_0 t}$$

$$x'(t) = \frac{d}{dt} x(t) = \frac{d}{dt} \left\{ \sum_n C_n e^{jnw_0 t} \right\} = \sum_n C_n e^{jnw_0 t}\, jnw_0$$

$$x''(t) = \frac{d}{dt} \left\{ \sum_n C_n e^{jnw_0 t} (jnw_0) \right\} = \sum_n C_n e^{jnw_0 t} (jnw_0)^2$$

$$\tilde{C}_n = \frac{\langle x''(t), e^{jnw_0 t} \rangle}{\| e^{jnw_0 t} \|^2} = \int_{t_i}^{t_f} x''(t)\, e^{-jnw_0 t}\, dt, \quad t = t_f - t_i$$

$$\tilde{C}_n = C_n (jnw_0)^2 = \frac{\int_{t_i}^{t_f} x''(t)\, e^{-jnw_0 t}\, dt}{T}$$

$$C_n = \frac{1}{(t_F - t_i)(jnw_0)^2} \int_{t_i}^{t_f} x''(t)\, e^{-jnw_0 t}\, dt$$

$$C_n = \frac{1}{(t_i - t_f)\, n^2 w_0^2} \int_{t_i}^{t_f} x''(t)\, e^{-jnw_0 t}\, dt$$

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$$x(t) = a_0 + \sum_{n=1}^{N} a_n \cos(n w_0 t) + b_n \operatorname{sen}(n w_0 t)$$

$$x'(t) = \sum_{n=1}^{N} a_n (-n w_0) \operatorname{sen}(n w_0 t) + b_n (n w_0) \cos(n w_0 t)$$

$$x''(t) = \sum_{n=1}^{N} a_n (-n w_0)(n w_0) \cos(n w_0 t) + b_n (n w_0)(-n w_0) \operatorname{sen}(n w_0 t)$$

$$\tilde{a}_n = \frac{2}{T} \int_{t_i}^{t_f} x''(t) \cos(n w_0 t)\, dt \; ; \quad \tilde{b}_n = \frac{2}{T} \int_{t_i}^{t_f} x''(t) \operatorname{sen}(n w_0 t)\, dt$$

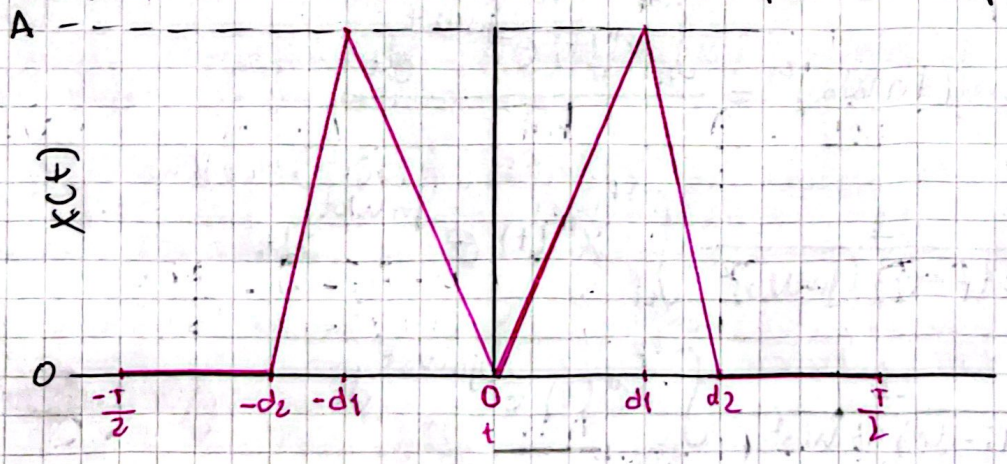
$$a_n (-n^2 w_0^2) = \frac{2}{T} \int_{t_i}^{t_f} x''(t) \cos(n w_0 t)\, dt$$

$$a_n = \frac{2}{-T n^2 w_0^2} \int_{t_i}^{t_f} x''(t) \cos(n w_0 t)\, dt$$

$$b_n (-n^2 w_0^2) = \frac{2}{T} \int_{t_i}^{t_f} x''(t) \operatorname{sen}(n w_0 t)\, dt$$

$$b_n = \frac{2}{-T n^2 w_0^2} \int_{t_i}^{t_f} x''(t) \operatorname{sen}(n w_0 t)\, dt$$

4) Encuentre el espectro de Fourier, su parte real, imaginaria, magnitud, fase y el error relativo para $n \in \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$, apartir de $x''(t)$ para la señal $x(t)$ en la figura. Compruebe el espectro obtenido con la estimación a partir de $x(t)$. Presente las simulaciones de python respectivas.



definimos $x(t)$

$$x(t) = \begin{cases} -T/2 \leq t \leq -d_2 , & x(t) = 0 \quad \#1 \\ -d_2 \leq t \leq -d_1 , & x(t) = \text{recta lineal} \quad \#2 \\ -d_1 \leq t \leq 0 , & x(t) = \text{recta lineal} \quad \#3 \\ 0 \leq t \leq d_1 , & x(t) = \text{recta lineal} \quad \#4 \\ d_1 \leq t \leq d_2 , & x(t) = \text{recta lineal} \quad \#5 \\ d_2 \leq t \leq T/2 , & x(t) = 0 \quad \#6 \end{cases}$$

Ecuaciones
→ lineales

#2

$$m = \frac{0 - A}{-d_2 - (-d_1)} = \frac{-A}{d_1 - d_2} = \frac{A}{d_2 - d_1}$$

$$b = y - mt = 0 - \frac{A(-d_2)}{d_2 - d_1}$$

$$b = \frac{Ad_2}{d_2 - d_1}$$

$$x(t) = \frac{At}{d_2 - d_1} + \frac{Ad_2}{d_2 - d_1}$$

#3

$$m = \frac{A - 0}{-d_1 - 0} = -\frac{A}{d_1} , \quad b = 0 - \frac{A}{d_1}(0) = 0$$

$$x(t) = \frac{At}{d_1}$$

#4 $m = \frac{0 - A}{0 - d_1} = \frac{A}{d_1}$, $b = 0 - \frac{A}{d_1}(0) = 0$, $x(t) = \frac{At}{d_1}$

#5

$$m = \frac{A - 0}{d_1 - d_2} = \frac{A}{d_1 - d_2} , \quad b = 0 - \frac{A(d_2)}{d_1 - d_2} = \frac{Ad_2}{d_2 - d_1}$$

$$x(t) = \frac{A}{d_1 - d_2} t + \frac{Ad_2}{d_2 - d_1}$$

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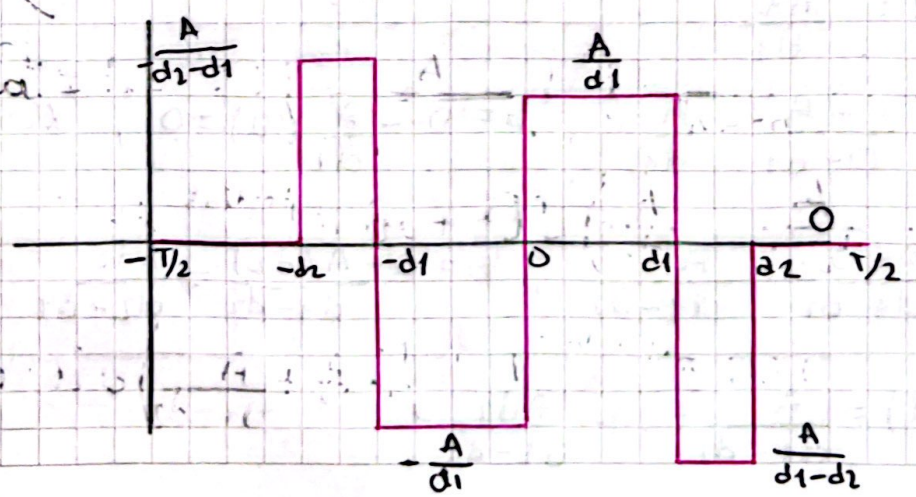
$$x(t)=\begin{cases} -T/2 \le t \le -d_2 , & x(t)=0 \\ -d_2 \le t \le -d_1 , & x(t)=\dfrac{At}{d_2-d_1}+\dfrac{Ad_2}{d_2-d_1} \\ -d_1 \le t \le 0 , & x(t)=\dfrac{-At}{d_1} \\ 0 \le t \le d_1 , & x(t)=\dfrac{At}{d_1} \\ d_1 \le t \le d_2 , & x(t)=\dfrac{At}{d_1-d_2}+\dfrac{Ad_2}{d_2-d_1} \\ d_2 \le t \le T/2 , & x(t)=0 \end{cases}$$

$x'(t)$ de $x(t)$

$$x'(t)\begin{cases} -T/2 < t < -d_2 , & x'(t)=0 \\ -d_2 < t < -d_1 , & x'(t)=\dfrac{A}{d_2-d_1} \\ -d_1 < t < 0 , & x'(t)=-\dfrac{A}{d_1} \\ 0 < t < d_1 , & x'(t)=\dfrac{A}{d_1} \\ d_1 < t < d_2 , & x'(t)=\dfrac{A}{d_1-d_2} \\ d_2 < t < T/2 , & x'(t)=0 \end{cases}$$

Gráfica:



$x''(t)$ de $x'(t)$ de $x(t)$

$$x''(t) = \begin{cases} -T/2 \le t < -d_2 , & x''(t) = 0 \\ t = -d_2 , & x''(t) = \frac{A}{d_2 - d_1}\delta(t + d_2) \\ -d_2 < t < -d_1 , & x''(t) = 0 \\ t = -d_1 , & x''(t) = \left(-\frac{A}{d_2 - d_1} - \frac{A}{d_1}\right)\delta(t + d_1) \\ -d_1 < t < 0 , & x''(t) = 0 \\ t = 0 , & x''(t) = \frac{2A}{d_1}\delta(t) \\ 0 < t < d_1 , & x''(t) = 0 \\ t = d_1 , & x''(t) = \left(-\frac{A}{d_1} + \frac{A}{d_1 - d_2}\right)\delta(t - d_1) \\ d_1 < t < d_2 , & x''(t) = 0 \\ t = d_2 , & x''(t) = \frac{A}{d_2 - d_1}\delta(t - d_2) \\ d_2 < t \le T/2 , & x''(t) = 0 \end{cases}$$

$$C_n = \frac{1}{(t_i - t_f) n^2 w_0^2} \left\{ \int_T \frac{A}{d_2 - d_1} \delta(t + d_2) e^{-jnw_0 t} dt \right.$$

$$+ \int_T \left(-\frac{A}{d_2 - d_1} - \frac{A}{d_1}\right) \delta(t + d_1) e^{-jnw_0 t} dt$$

$$+ \int_T \frac{2A}{d_1} \delta(t) e^{-jnw_0 t} dt + \int_T \left(-\frac{A}{d_1} + \frac{A}{d_1 - d_2}\right) \delta(t - d_1) e^{-jnw_0 t} dt$$

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$$+\int_T \frac{A}{d_2-d_1}\delta(t-d_2)e^{-jnw_0t}\,dt\Bigg\}$$

$$C_n = \frac{1}{-Tn^2\left(\frac{2\pi}{T}\right)^2}\Bigg\{\frac{A}{d_2-d_1}e^{jnw_0d_2} - \frac{A}{d_2-d_1}e^{jnw_0d_1}$$

$$-\frac{A}{d_1}e^{jnw_0d_1} + \frac{2A}{d_1}e^{0} - \frac{A}{d_1}e^{-jnw_0d_1} - \frac{A}{d_2-d_1}e^{-jnw_0d_1}$$

$$+\frac{A}{d_2-d_1}e^{-jnw_0d_2}\Bigg\}$$

$$= -\frac{TAe^{j}}{4\pi^2n^2(d_2-d_1)} + \frac{TAe^{jnw_0d_1}}{4\pi^2n^2(d_2-d_1)} + \frac{TAe^{jnw_0d_1}}{4\pi^2n^2d_1}$$

$$-\frac{TA}{2\pi^2n^2d_1} + \frac{TAe^{-jnw_0d_1}}{4\pi^2n^2d_1} + \frac{TAe^{-jnw_0d_1}}{4\pi^2n^2(d_2-d_1)}$$

$$-\frac{TAe^{-jnw_0d_2}}{4\pi^2n^2(d_2-d_1)}$$

$$= -\frac{TA\cos(nw_0d_2)}{4\pi^2n^2(d_2-d_1)} - \frac{jTA\,\mathrm{Sen}(nw_0d_2)}{4\pi^2n^2(d_2-d_1)}$$

$$+\frac{TA\cos(nw_0d_1)}{4\pi^2n^2(d_2-d_1)} + \frac{jTA\,\mathrm{Sen}(nw_0d_1)}{4\pi^2n^2(d_2-d_1)}$$

$$+\frac{TA\cos(nw_0d_1)}{4\pi^2n^2d_1} + \frac{jTA\,\mathrm{sen}(nw_0d_1)}{4\pi^2n^2d_1} - \frac{TA}{2\pi^2n^2d_1}$$

$$+\frac{TA\cos(nw_0d_1)}{4\pi^2n^2d_1} - \frac{jTA\,\mathrm{sen}(nw_0d_1)}{4\pi^2n^2d_1}$$

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$$+\frac{TA\cos(nW_0 d_1)}{4\pi^2 n^2 (d_2-d_1)} - \frac{jTA\,\mathrm{sen}(nW_0 d_1)}{4\pi^2 n^2 (d_2-d_1)}$$

$$-\frac{TA\cos(nW_0 d_2)}{4\pi^2 n^2 (d_2-d_1)} + \frac{jTA\,\mathrm{sen}(nW_0 d_2)}{4\pi^2 n^2 (d_2-d_1)}$$

$$= -\frac{TA\cos(nW_0 d_2)}{2\pi^2 n^2 (d_2-d_1)} + \frac{TA\cos(nW_0 d_1)}{2\pi^2 n^2 (d_2-d_1)}$$

$$+\frac{TA\cos(nW_0 d_1)}{2\pi^2 n^2 d_1} - \frac{TA}{2\pi^2 n^2 d_1} = C_n$$

Espectro puramente real.

$$|C_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$b_n = 0 \rightarrow$ no tiene parte imaginaria.

$$|C_n| = \frac{\sqrt{a_n^2}}{2} = \frac{|a_n|}{2} = \frac{|2\mathrm{Re}\{C_n\}|}{2}$$

$$|C_n| = \left| -\frac{TA\cos(nW_0 d_2)}{2\pi^2 n^2 (d_2-d_1)} + \frac{TA\cos(nW_0 d_1)}{2\pi^2 n^2 (d_2-d_1)} + \frac{TA\cos(nW_0 d_1)}{2\pi^2 n^2 d_1} - \frac{TA}{2\pi^2 n^2 d_1} \right|$$

Angulo

0 si $a_n > 0$

π si $a_n < 0$

indefinido si $a_n = 0$

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$$C_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T}\left\{ \int_{-d_2}^{-d_1} \frac{At}{d_2-d_1}\, e^{-jn\omega_0 t}\, dt \right.$$

$$+ \int_{-d_2}^{-d_1} \frac{A d_2}{d_2-d_1}\, e^{-jn\omega_0 t}\, dt + \int_{-d_1}^{0} -\frac{A}{d_1}\, t\, e^{-jn\omega_0 t}\, dt$$

$$+ \int_{0}^{d_1} \frac{A}{d_1}\, t\, e^{-jn\omega_0 t}\, dt + \int_{d_1}^{d_2} \frac{At}{d_1-d_2}\, e^{-jn\omega_0 t}\, dt$$

$$\left. + \int_{d_1}^{d_2} \frac{A d_2}{d_2-d_1}\, e^{-jn\omega_0 t}\, dt \right\}$$

$$C_0 = \frac{1}{T}\left\{ \int_{d_2}^{-d_1} \frac{At}{d_2-d_1}\, dt + \int_{-d_2}^{-d_1} \frac{A d_2}{d_2-d_1}\, dt \right.$$

$$+ \int_{-d_1}^{0} -\frac{A}{d_1}\, t\, dt + \int_{0}^{d_1} \frac{A}{d_1}\, t\, dt + \int_{d_1}^{d_2} \frac{At}{d_1-d_2}\, dt$$

$$\left. + \int_{d_1}^{d_2} \frac{A d_2}{d_2-d_1}\, dt \right\}$$

$$= \frac{1}{T}\left\{ \frac{At^2}{2(d_2-d_1)}\Bigg|_{-d_2}^{t=-d_1} + \frac{A d_2 t}{d_2-d_1}\Bigg|_{-d_2}^{t=-d_1} + \left(-\frac{At^2}{2d_1}\right)\Bigg|_{0}^{t=0} \right.$$

$$\left. + \left(\frac{At}{2d_1}\right)_{0}^{t=d_1} + \left(\frac{At^2}{2(d_1-d_2)}\right)_{d_1}^{t=d_2} + \frac{A d_2 t}{d_2-d_1}\Bigg|_{d_1}^{t=d_2} \right\}$$

$$= \frac{1}{T}\left\{ \frac{A(-d_1-(-d_2))^2}{2(d_2-d_1)} + \frac{A d_2(-d_1-(-d_2))}{d_2-d_1} - \frac{A(0-(-d_1))^2}{2d_1} + \right.$$

$$\frac{A(d_1-0)^2}{2d_1} - \left(\frac{A(d_2-d_1)^2}{2(d_2-d_1)}\right) + \frac{A d_2(d_2-d_1)}{d_2-d_1}$$

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